BRANCHES OF PERIODIC ORBITS FOR THE PLANAR RESTRICTED 3-BODY PROBLEM

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Abstract. We describe a method for studying the existence and the linear stability of branches of periodic solutions for a dynamical system with a parameter. We apply the method to the planar restricted 3-body problem extending the results of [A]. More precisely, we prove the existence of some continuous branches of periodic orbits with the energy or the masses of the primaries as parameters, and provide an approximation of the orbits with rigorous bounds. We prove the linear stability or instability of the orbits.

1. Introduction

The literature about periodic solutions for the planar restricted 3-body problem is extremely rich, so that any attempt to make a list of references is futile. We only cite the well known books [MH, SM, S] as standard references. By a standard derivation, see e.g. [A], it can be shown that when the primaries orbit around each other clockwise with period 2π, then the motion of the third body is described in a rotating reference frame, i.e. in synodical coordinates, by the following system of second order differential equations:

$$\begin{aligned} \ddot{x} + 2\dot{y} &= \Omega_x \\ \ddot{y} - 2\dot{x} &= \Omega_y, \end{aligned} \tag{1}$$

where

$$\Omega(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{m_1}{\sqrt{(x-R_1)^2+y^2}} + \frac{m_2}{\sqrt{(x+R_2)^2+y^2}} + C, \tag{2}$$

m_1 and m_2 are the masses of the primaries, C is an arbitrary constant, $R_1 = \frac{m_2}{(m_1+m_2)^{2/3}}$, $R_2 = \frac{m_1}{(m_1+m_2)^{2/3}}$ and $P_1 = (R_1, 0)$, $P_2 = (-R_2, 0)$ are the coordinates of the primaries. For convenience we choose $C = -\frac{m_1}{R_1} - \frac{m_2}{R_2}$, so that $\Omega(0,0) = 0$.

In the recent paper [A], periodic and chaotic solutions of the planar restricted 3-body problem were investigated by means of computer assisted methods and topological theorems; a rigorous proof of the existence of a class of periodic orbits at different energy levels with a very narrow estimate of the locations of the intersection of such periodic orbits with the line connecting the primaries was provided. Here we extend those results in two directions. First, we prove the existence of continuous branches of solutions, with either the energy or the ratio between the

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masses of the primaries as a parameter. Second, we study the (linear) stability of some orbits.

More precisely, we describe three continuous branches of periodic orbits with the energy as a parameter and three continuous branches of periodic orbits with m_1 as a parameter. Moreover, we conjecture that the solutions on two of such branches (in both cases) are linearly stable, while the other one is unstable. Furthermore we provide explicit functions $\hat{x}(h)$ and $\hat{x}(m_1)$ describing the intersection of the orbits with the x-axis in dependence of the energy or m_1 respectively.

For three of these six branches, we provide a computer assisted proof of their existence and we provide a rigorous L^∞ bound on the accuracy of the functions $\hat{x}(h)$ and $\hat{x}(m_1)$. For the other three branches we can only provide a partial proof. Although in principle we could provide a full proof for all branches, this has not been possible because it would have required an extremely long computation time.

Concerning the linear stability, we provide a rigorous proof only for a small section of the branch. As above, a proof for all branches would require too much computation time.

These results highlight some important features of computer assisted proofs. On the positive side, the technique is extremely versatile and may be applied to a large class of problems for which no analytical technique is available. Their main drawback, besides aesthetic considerations, consist in the difficulty of providing results when the parameters entering the problem assume an arbitrary value.

We point out that, although we feel that these results are interesting in themselves, the main purpose of the paper is to describe the computer assisted methods and to show how their application leads to results otherwise unavailable. All the proofs can be easily reproduced on any recent computer. To this purpose, in the last section we provide all necessary information, mostly by referring to other papers, and we also provide on the web the Mathematica-C++ program we used.

The layout of the paper is as follows: in Section 2 we introduce some basic facts concerning the system; in Section 3 we summarize some result obtained in a previous paper which are relevant to the following sections; in Section 4 we state our results and in section 5 we explain the computer assisted proofs.

2. Basics

It is well known that $H(x,\dot{x},y,\dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\Omega(x,y)$ is an integral of the motion called the *Jacobi integral*, therefore the motion takes place on the manifold $H(x,\dot{x},y,\dot{y}) = h$. With some abuse of language, we call the Jacobi integral the *energy* of a solution.

This system admits five equilibrium points. Three such points are aligned with the primaries and are called L_1, L_2 and L_3 (the collinear equilibrium solutions). One of the collinear points lies between the primaries; call that point L_1. All the collinear points are saddles for Ω. The remaining two equilibrium points are called L_4 and L_5; they are the absolute minima of Ω and they both sit at the vertices of an equilateral triangle with the primaries at the other vertices (the equilateral equilibrium solutions).

In order to study the system at some fixed energy h we consider the half Poincaré return map $H_1 : D(H_1) \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows. Given (x, p_x) such that $x \neq \{R_{1,} - R_2\}$ and $2\Omega(x,0) + h - p_x^2 > 0$, there exists a unique positive value of

$p_y = p_y(x, p_x) = \sqrt{2\Omega(x,0) + h - p_x^2}$ such that $H(x, p_x, 0, p_y) = h$. We let

$$H_1(x, p_x) = (\varphi_1(x, p_x; T_1), \varphi_2(x, p_x; T_1)),$$

where $\varphi(x, p_x; \cdot) : \mathbb{R} \to \mathbb{R}^4$ is the solution of the equation (1) with initial conditions $x(0) = x$, $\dot{x}(0) = p_x$, $y(0) = 0$, $\dot{y}(0) = p_y$ and φ_i, $i = 1, \ldots, 4$ are its components. By definition $\varphi_3(x, p_x; 0) = 0$ and $\varphi_4(x, p_x; 0) > 0$, therefore $\varphi_3(x, p_x; t) > 0$ for all positive and small t. If there exists a time T such that $\varphi_3(x, p_x; T) = 0$ and $\varphi_3(x, p_x; t) > 0$ for all $t \in (0, T)$, then we have $(x, p_x) \in D(H_1)$ and we set $H_1(x, p_x) = (\varphi_1(x, p_x; T), \varphi_2(x, p_x; T))$.

By inspection it is easy to see that, if $(x(t), y(t))$ is a solution of (1), then $(\tilde{x}(t), \tilde{y}(t)) := (x(-t), -y(-t))$ is also a solution. As it is well known, if there exist x_1, x_2 such that $H_1(x_1, 0) = (x_2, 0)$, then there exists a periodic orbit of (1) whose intersections with the x-axis occur orthogonally at $(x_1, 0)$ and $(x_2, 0)$, respectively with positive and negative $y-$velocity. On the other hand, if the system admits a periodic orbit which crosses orthogonally the x-axis at some points x_1 and x_2 (resp. with positive and negative $y-$velocity), then $H_1(x_1, 0) = (x_2, 0)$. It follows that, in order to study the periodic orbits of (1) it is convenient to study the second component of $H_1(x, 0)$ as described in Section 5.

3. Previous results

The case considered in [A] is usually referred to as "the Copenhagen orbits", from the results of the Observatory of Copenhagen (see [S] and references therein). We recall that, by our choice of the constant C in (2), $h = 0$ is the energy of the stationary solution at the Lagrangian point L_1 (the origin). Since the problem has the additional symmetry consisting in switching the primaries, we only looked for orbits that cross the portion of the x-axis between the primaries with positive speed in the y direction. The lowest value of h we considered was 0, when the admissible region is split in two parts touching at the origin and no trajectory can enter both regions. The highest value of h we considered is 0.8, since at slightly larger value the bounded admissible region touches the unbounded part and some trajectory starting close to the primaries may be unbounded. A subset of the results on periodic orbits given in [A] is the following Theorem:

Theorem 3.1. *Let $m_1 = m_2 = 1$. For all energy values displayed in column h of (3), system (1) admits at least a periodic solution for each value printed in the remaining columns. Such solution crosses the x-axis orthogonally twice and only twice. The x-coordinate of the intersection with positive y-velocity lies in the interval centered in the position given in the table with width 10^{-3}. The orbits in the column S_2 are retrograde around P_2; the orbits in the classes S_1 are direct around P_1; the orbits in the class L are retrograde around L_1.*

(3)

h	S_1	S_2	L
0	0.3158	-0.4399	
.1	0.3028	-0.4365	0.02697
.2	0.2883	-0.4330	0.03894
.3	0.2712	-0.4294	0.04873
.4	0.2443	-0.4257	0.05751
.5		-0.4218	0.06575
.6		-0.4178	0.07369
.8		-0.4093	0.08922

In [A] we left open a few questions: are the orbits unique in the interval of the $x-$axis of width 10^{-3} mentioned above? Do orbits described by the same column belong to a continuous branch of periodic solutions as the numerical experiments hint? Are the orbits linearly stable?

4. Results

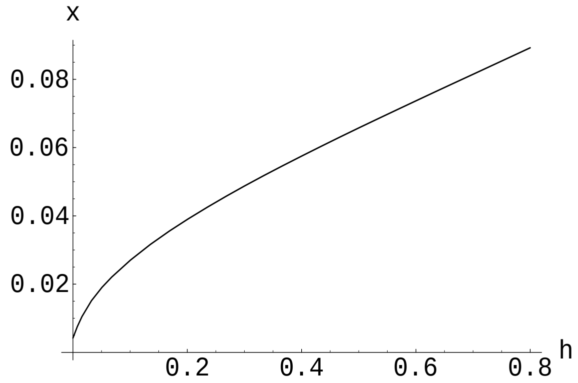
In order to simplify the statements of the result we introduce the following notation: if $x \in \mathbb{R}$ and $\delta > 0$ let $x \pm \delta := [x-\delta, x+\delta]$. The first results concern branches of orbits for varying energy.

4.1. **Branches of Copenhagen orbits with the energy as a parameter.** In this subsection we prove that the values printed in the columns L, S_2 of Table (3) actually belong to single branches of solutions and we provide a partial result concerning S_1. Let

$$\begin{aligned}
\hat{x}_L(h) &= 0.00421194 + 0.964456\,h - 25.5237\,h^2 + 607.141\,h^3 \\
&\quad - 9765.68\,h^4 + 105786\,h^5 - 779487\,h^6 + 3.9051\,10^6\,h^7 \\
&\quad - 1.30082\,10^7\,h^8 + 2.69484\,10^7\,h^9 - 2.8188\,10^7\,h^{10} - 183369.\,h^{11} \\
&\quad + 2.25854\,10^7\,h^{12} + 4750983\,h^{13} - 16231191\,h^{14} - 17340947\,h^{15} \\
&\quad - 4.27086\,10^6\,h^{16} + 9.87905\,10^6\,h^{17} + 1.79343\,10^7\,h^{18} \\
&\quad + 1.9389\,10^7\,h^{19} + 1.67368\,10^7\,h^{20}, \\
\hat{x}_{S_1}(h) &= 0.315762 - 0.222075\,h - 1.96366\,h^2 + 43.5272\,h^3 \\
&\quad - 501.934\,h^4 + 2634.33\,h^5 - 5256.63\,h^6, \\
\hat{x}_{S_2}(h) &= -0.439928 + 0.0664862\,h + 0.0208835\,h^2 + 0.00925382\,h^3.
\end{aligned}$$

The following theorems hold:

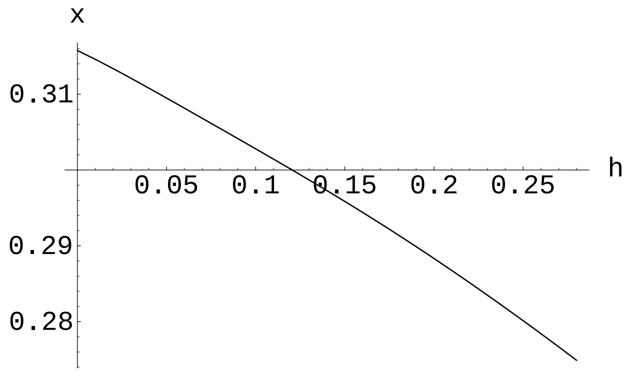
Theorem 4.1. *Let $m_1 = m_2 = 1$, let $I_h = [.01, .8]$ and let $\delta = .0002$. There exists a continuous function $x_L : I_h \to [.008, .09]$ such that for all $h \in I_h$ there exists a periodic orbit of (1) of energy h crossing the $x-$axis at $x = x_L(h)$ with positive y-velocity. Such orbit is retrograde around L_1. Furthermore $x_L(h) \in \hat{x}_L(h) \pm \delta$ for all $h \in I_h$. For all $h \in I_h$ and for all $\bar{x} \in \hat{x}_L(h) \pm \delta$, $\bar{x} \neq x_L(h)$ there is no orbit of energy h crossing the $x-$axis orthogonally at $\bar{x}$.*



Graph of $\hat{x}_L(h)$

Next we consider the orbits in the class $S_{1,2}$.

Conjecture 4.2. *Let* $m_1 = m_2 = 1$*, let* $I_h = [0, .28]$ *and let* $\delta = .0004$*. There exists a continuous function* $x_{S_1} : I_h \to [.27, .32]$ *such that for all* $h \in I_h$ *there exists a periodic orbit of (1) of energy* h *crossing the* $x-$*axis at* $x = x_{S_1}(h)$ *with positive* y*-velocity. Such orbit is direct around* P_2*. Furthermore* $x_{S_1}(h) \in \hat{x}_{S_1}(h) \pm \delta$ *for all* $h \in I_h$*. For all* $h \in I_h$ *and for all* $\bar{x} \in \hat{x}_{S_1}(h) \pm \delta$*,* $\bar{x} \neq x_{S_1}(h)$ *there is no orbit of energy* h *crossing the* $x-$*axis orthogonally at* $\bar{x}$*.*



Graph of $\hat{x}_{S_1}(h)$

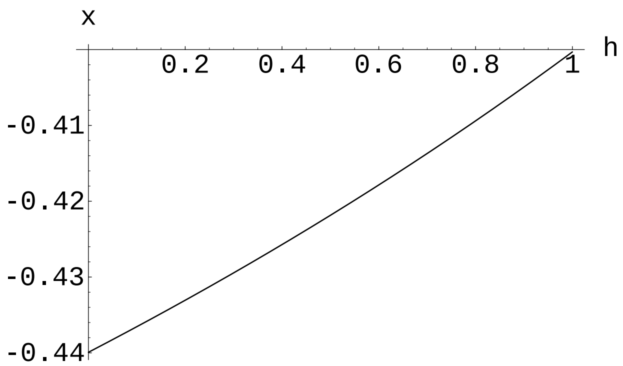
In principle we have the means to prove Conjecture 4.2, and we believe it to be true. But a full proof with our method would require about 10 years of computational time. This is due to the fact that this orbit appears to be very sensitive to the value of h, therefore with a single step we can only prove the statement for a very narrow interval in h. With about a week of computation time we could prove the following:

Theorem 4.3. *The statement of Conjecture 4.2 hold if* I_m *is substituted with the set* $\tilde{I}_m := \cup_{i=0}^{28}[10^{-2}i, 10^{-2}i + .00004]$.

For the orbits S_2 we could get a full result in a reasonable time:

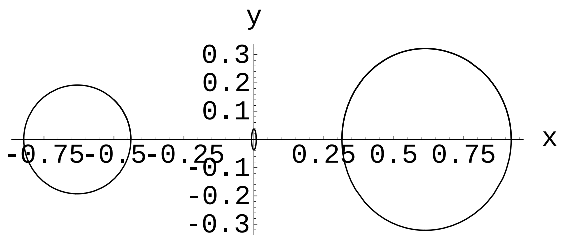
Theorem 4.4. *Let* $m_1 = m_2 = 1$*, let* $I_h = [0, 1]$ *and let* $\delta = .00025$*. There exists a continuous function* $x_{S_2} : I_h \to [-.44, -.4]$ *such that for all* $h \in I_h$ *there exists a periodic orbit of (1) of energy* h *crossing the* $x-$*axis at* $x = x_{S_2}(h)$ *with positive* y*-velocity. Such orbit is retrograde around* P_1*. Furthermore* $x_{S_2}(h) \in \hat{x}_{S_2}(h) \pm \delta$

for all $h \in I_h$. For all $h \in I_h$ and for all $\bar{x} \in \hat{x}_{S_2}(h) \pm \delta$, $\bar{x} \neq x_{S_2}(h)$ there is no orbit of energy h crossing the $x-$axis orthogonally at $\bar{x}$.

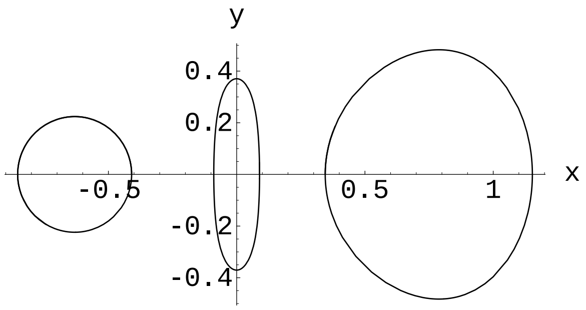


Graph of $\hat{x}_{S_2}(h)$

The following pictures display the orbits belonging to the classes S_2, L and S_1 (from left to right) at $h = .01$ and at $h = .8$.



Orbits at $h = .01$

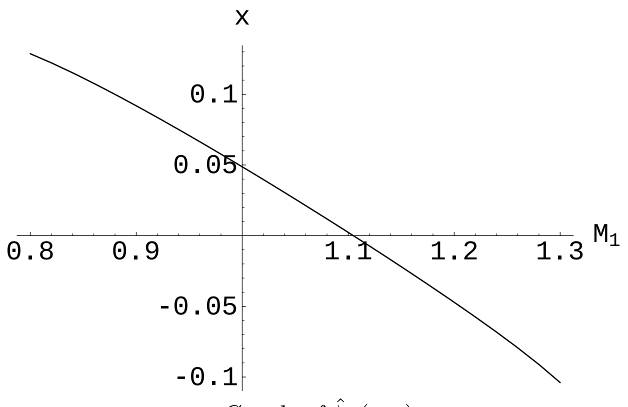


Orbits at $h = .8$

4.2. **Branches of orbits with the mass of a primary as a parameter.** We describe how the periodic orbits vary when the ratio of the masses of the primaries is changed. More precisely, we vary m_1 while keeping m_2 (and h) fixed. Let $\hat{\xi}_L(m_1) =$,

$$\begin{aligned}
\hat{\xi}_L(m_1) &= -4956.55 + 48949.4\, m_1 - 217113\, m_1^2 + 569529\, m_1^3 \\
&\quad - 978391\, m_1^4 + 1150070\, m_1^5 - 936769\, m_1^6 + 522068\, m_1^7 \\
&\quad - 190515\, m_1^8 + 41108.2\, m_1^9 - 3982.74\, m_1^{10}, \\
\hat{\xi}_{S_1}(m_1) &= -41.5749 + 484.357\, m_1 - 2433.07\, m_1^2 + 7057.59\, m_1^3 \\
&\quad - 13035.9\, m_1^4 + 15787.6\, m_1^5 - 12175.2\, m_1^6 + 4981.48\, m_1^7 \\
&\quad + 327.841\, m_1^8 - 1739.8\, m_1^9 + 1076.09\, m_1^{10} - 343.997\, m_1^{11} \\
&\quad + 59.1689\, m_1^{12} - 4.34767\, m_1^{13}, \\
\hat{\xi}_{S_2}(m_1) &= -0.00291 - 0.471637\, m_1 + 0.191508\, m_1^2 - 0.290152\, m_1^3 \\
&\quad + 0.196462\, m_1^4 - 0.0596989\, m_1^5 + 0.0069742\, m_1^6.
\end{aligned}$$

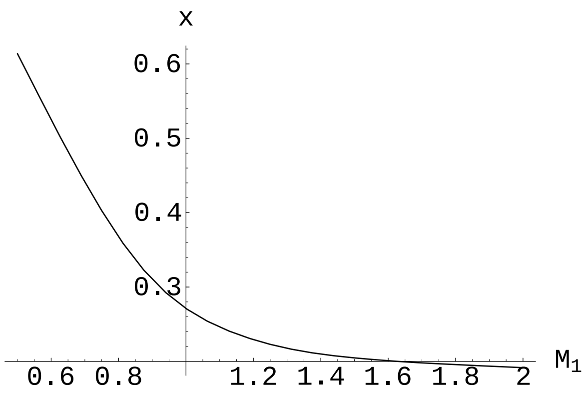
Theorem 4.5. *Let $m_2 = 1$, let $I_m = [.8, 1.3]$ and let $\delta = .0001$. There exists a continuous function $\xi_L : I_m \to [-.11, .13]$ such that for all $m_1 \in I_m$ there exists a periodic orbit of (1) of energy $h = .3$ crossing the $x-$axis at $x = \xi_L(h)$ with positive y-velocity. Such orbit is retrograde around L_1. Furthermore $\xi_L(m_1) \in \hat{\xi}_L(m_1) \pm \delta$ for all $m_1 \in I_m$. For all $m_1 \in I_m$ and for all $\bar{x} \in \hat{\xi}_L(m_1) \pm \delta$, $\bar{x} \neq \xi_L(m_1)$ there is no orbit of energy $h = .3$ crossing the $x-$axis orthogonally at $\bar{x}$.*



Graph of $\hat{\xi}_L(m_1)$

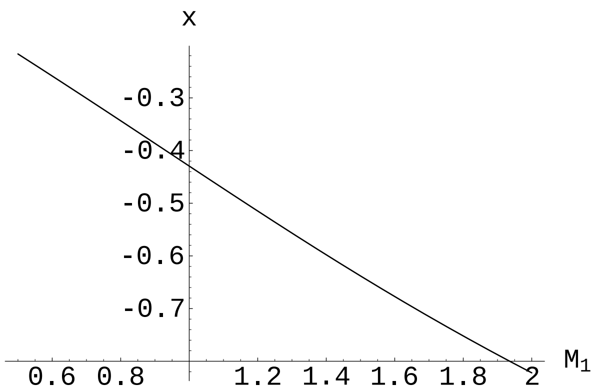
Concerning the orbits in the classes S_1 and S_2 we have the following:

Conjecture 4.6. *Let $m_2 = 1$, let $I_m = [.5, 2]$ and let $\delta = .0001$. There exists a continuous function $\xi_{S_1} : I_m \to [.2, .62]$ such that for all $m_1 \in I_m$ there exists a periodic orbit of (1) of energy $h = .3$ crossing the $x-$axis at $x = \xi_{S_1}(h)$ with positive y-velocity. Such orbit is direct around P_2. Furthermore $\xi_{S_1}(m_1) \in \hat{\xi}_{S_1}(m_1) \pm \delta$ for all $m_1 \in I_m$. For all $m_1 \in I_m$ and for all $\bar{x} \in \hat{\xi}_{S_1}(m_1) \pm \delta$, $\bar{x} \neq \xi_{S_1}(m_1)$ there is no orbit of energy $h = .3$ crossing the $x-$axis orthogonally at $\bar{x}$.*



Graph of $\hat{\xi}_{S_1}(m_1)$.

Conjecture 4.7. *Let $m_2 = 1$, let $I_m = [.5, 2]$ and let $\delta = .0001$. There exists a continuous function $\xi_{S_2} : I_m \to [-.83, -.21]$ such that for all $m_1 \in I_m$ there exists a periodic orbit of (1) of energy $h = .3$ crossing the $x-$axis at $x = \xi_{S_2}(h)$ with positive y-velocity. Such orbit is retrograde around P_1. Furthermore $\xi_{S_2}(m_1) \in \hat{\xi}_{S_2}(m_1) \pm \delta$ for all $m_1 \in I_m$. For all $m_1 \in I_m$ and for all $\bar{x} \in \hat{\xi}_{S_2}(m_1) \pm \delta$, $\bar{x} \neq \xi_{S_2}(m_1)$ there is no orbit of energy $h = .3$ crossing the $x-$axis orthogonally at $\bar{x}$.*

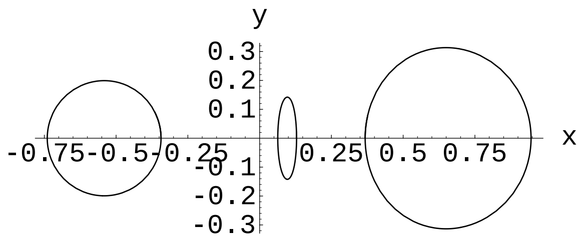


Graph of $\hat{\xi}_{S_2}(m_1)$.

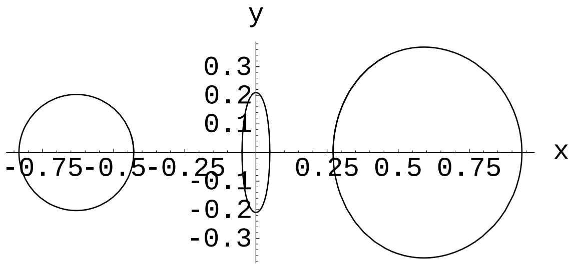
As before, the lack of the proof is only due to unreasonable computation time. In about a week we could prove the following:

Theorem 4.8. *The statements of Conjectures 4.6 and 4.7 hold if I_m is substituted with the set $\tilde{I}_m := \cup_{i=50}^{200}[.01i, .01i + 10^{-8}]$.*

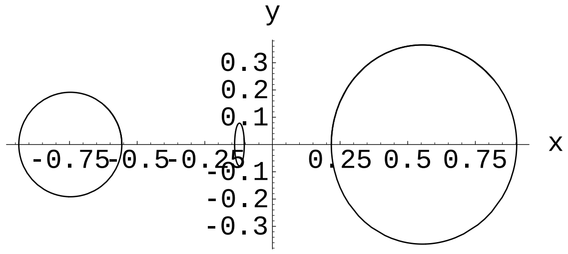
The following pictures display the orbits at different values of m_1.



Orbits at $h = .3$ with $m_1 = .8$ and $m_2 = 1$.



Orbits at $h = .3$ with $m_1 = 1$ and $m_2 = 1$.



Orbits at $h = .3$ with $m_1 = 1.3$ and $m_2 = 1$.

4.3. **Linear stability of the orbits.** We have numerical evidence of the following:

Conjecture 4.9. *All periodic orbits in obtained in Theorems 4.1 and 4.5 are unstable. All periodic orbits obtained in Theorems 4.3, 4.4 and 4.8 are linearly stable.*

Although we have a metod to prove the above conjecture by computer assistance, it is not really practical because it would require a very long computation time. Therefore we chose to prove the conjecture only for a small interval of the parameters m_1 and h, and we provide the algorithm for the verification of further intervals.

Theorem 4.10. *Let $m_2 = 1$. For all $m_1 \in 1 \pm 10^{-5}$ and for all $h \in .3 \pm 10^{-5}$ there exists one and only one periodic orbit of 1 crossing the the $x-$axis with positive y-velocity in the interval $.0487323 \pm 10^{-4}$ and such orbit is unstable.*

Remark 4.1. *The uniqueness result, together with the estimate on the location of the orbit, imply that this is the orbit obtained in Theorems 4.1 and 4.5 for the common values of the parameters, therefore the orbit is retrograde around L_1.*

Theorem 4.11. *Let $m_2 = 1$. For all $m_1 \in 1 \pm 10^{-6}$ and for all $h \in .3 \pm 10^{-6}$ there exists one and only one periodic orbit of 1 crossing the the $x-$axis with positive y-velocity in each of the intervals $\hat{x}_{S_1}(.3) \pm 10^{-3}$ and $\hat{x}_{S_2}(.3) \pm 10^{-3}$. Such orbits are linearly stable.*

Remark 4.2. *As before, the orbits obtained in Theorem 4.11 have to coincide with the orbits obtained in Theorems 4.3, 4.4, 4.8.*

5. The proofs

5.1. **The technique.** The proofs of existence of periodic solutions are obtained by a combination of the method of reversing symmetries, see [La1, La2, LQ], and rigorous computations; we summarize here the procedure, see [A] for details. Consider at first a fixed value for h and m_1. Let $f(x)$ be the second component of $H_1(x, 0)$. According to the results in Section 2, we have to look for zeros of the function f, therefore we first look for an approximate zero, call it $\bar{x}$. We choose x_1 and x_2 such that $x_1 < \bar{x} < x_2$ and both x_1 and x_2 are very close to $\bar{x}$. Then we compute the rigorous half Poincaré map H_1 at $(x_1, 0)$ and $(x_2, 0)$. When we can prove that the second component of $H_1(x_1, 0)$ has opposite sign with respect to the second component $H_1(x_2, 0)$ and that the segment joining the two points belongs entirely to the domain of H_1, then by the continuity of the half Poincaré map we have a proof that there exists at least a point $x_1 < \tilde{x} < x_2$ such that $H_1(\tilde{x}, 0)$ lies on the x axis. Furthermore, we can prove the uniqueness of the orbit in the interval $[x_1, x_2]$ by showing that for all $x \in [x_1, x_2]$ the derivative of the second component of the map $x \mapsto H_1(x, 0)$ does not change sign. Note that the computation of the derivatives of the half-Poincaré map was not necessary in [A].

In order to obtain the proof with the parameters m_1 and/or h varying in some range, we give to such parameters some interval values. More precisely, we partition the interval I_h (resp. I_m) in a certain number of subintervals of equal size (the actual number varies from 10^3 for the proof of Theorem 4.1 to 10^7 for the proof of Theorem 4.8). Then we solve the differential equation with the Lohner algorithm (see below) with the parameters h or m_1 having an interval value and we repeat the procedure for all subintervals.

More precisely, consider Theorem 4.1, the proof of the other theorems being similar. Set $I_h = \cup_{i=50}^{3999} I_h^i$, where $I_h^i = [.0002i, .0002(i+1)]$. For all i we check by the numerical rigorous procedure described below that the second component of the half Poincaré map at $(\hat{x}_L(.0002i) - \delta, 0)$ is negative and and at $(\hat{x}_L(.0002i) + \delta, 0)$ is positive for all $h \in I_h^i$. Then we partition the segment $(\hat{x}_L(.0002i) \pm \delta, 0)$ in 10 subinterval, and we check that the derivative with respect to x of the second component of the half Poincaré map is strictly positive in each subinterval, which yields the result of existence and uniqueness of a fixed point for the half Poincaré map in I_h^i. The existence of a continuous function x_L satisfying the statement of the theorem follows by the continuity of the half poincaré map and the implicit function theorem.

To prove Theorems 4.10 and 4.11 we compute the gradient

$$D(x, p_x) = \begin{pmatrix} \frac{\partial H_1(x,p_x)}{\partial x} \\ \frac{\partial H_1(x,p_x)}{\partial p_x} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

of the half Poincaré map with both the energy and m_1 taking interval values in the region where the fixed point has been proved to exist. It is easy to check that, if $(D_{11}+D_{22})^2 - 4D_{11}D_{22} + 4D_{12}D_{21} > 0$, then the eigenvalues λ_1, λ_2 of $D(x, p_x)$ are real and satisfy $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (recall that $\lambda_1 \lambda_2 = 1$), in which case the orbit is unstable, while if $(D_{11} + D_{22})^2 - 4D_{11}D_{22} + 4D_{12}D_{21} < 0$, then the eigenvalues are complex conjugate of modulus 1 and the orbit is linearly stable.

5.2. **Computational details.** We describe here briefly the algorithm used in the computer assisted proofs. Consider first the existence result. Since no analytical solution of equation 1 is available, all we can do in order to prove that the second component of $H_1(x, 0)$ is positive or negative is to estimate the trajectories and compute the intersections with the Poincaré plane with rigorous error bounds. We use the Lohner algorithm with interval arithmetics. We start with a Taylor method of order 25, i.e. we estimate the trajectory of an interval by using the Taylor expansion of order 25 and we estimate the error by the Lagrange remainder. More precisely, if τ is the time step, we compute a rough but rigorous enclosure D of the trajectory at times $[0, \tau]$, that is an interval set D such that the solution of the equation lies in D for all times between 0 and τ, and by Lagrange theorem we estimate the error we make neglecting the remaining terms of the Taylor expansion by computing $x^{(26)}(D)\frac{\tau^{26}}{26!}$, where $x^{(26)}(D)$ (which is an interval enclosing all possible values assumed by the 26th derivative of the trajectory, therefore enclosing the Lagrange remainder) is computed using a recursive algorithm for the time derivatives of the solutions.

The computation of the variational equation are obtained similarly, see [Z] for details.

The round-off errors are taken care directly by suitable C++ libraries (see [CAPD]) and by Mathematica. Such errors may vary by changing computer and/or operating system, but since all proofs go through with a relatively large margin with respect to the round-off errors, we expect that all the proofs can be easily reproduced on any recent computer.

To perform the proofs the author implemented a version of the whole algorithms in a combination of Mathematica and C++ under the Linux O.S. More precisely, Mathematica has been only used to handle all the data and to perform a few algorithms which are less demanding for the CPU, but more complicated to implement. Furthermore Mathematica has been used to make all numerical experiments used to compute the approximate branches and to draw the pictures. On the other hand C++ has been used for the heavy interval arithmetic computations, where it offered a much better speed. The connection between the two languages is obtained by MathLink. We wish to point out that the full proof took almost a month of CPU time on a machine equipped with a 1.5GHz Pentium IV processor. The reader who desires to reproduce the computer assisted proofs in this paper without writing the program can download the Mathematica notebook and the C++ sources for the full program from http://www1.mate.polimi.it/~gianni/r3b.tar.Z. The notebook is provided with comments and instructions.

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