TRACE FORMULAS FOR NON-SELF-ADJOINT PERIODIC SCHRÖDINGER OPERATORS AND SOME APPLICATIONS

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Abstract. Recently, a trace formula for non-self-adjoint periodic Schrödinger operators in $L^2(\mathbb{R})$ associated with Dirichlet eigenvalues was proved in [9]. Here we prove a corresponding trace formula associated with Neumann eigenvalues.

In addition we investigate Dirichlet and Neumann eigenvalues of such operators. In particular, using the Dirichlet and Neumann trace formulas we provide detailed information on location of the Dirichlet and Neumann eigenvalues for the model operator with the potential Ke^{2ix}, where $K \in \mathbb{C}$.

1. Introduction

Consider the differential expression L

$$L = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R},$$

where V is a continuous *complex-valued* periodic function on $\mathbb{R}$ of period π. The *Hill operator* H in $L^2(\mathbb{R})$ generated by the differential expression L is defined by

$$(Hf)(x) = L(f(x)),\ x \in \mathbb{R},\ f \in dom(H) = H^{2,2}(\mathbb{R}),$$

where $H^{2,2}(\mathbb{R})$ denotes the usual Sobolev space. Then H is a densely defined closed operator in $L^2(\mathbb{R})$ (see, e.g., [7, Ch. 5]).

In addition, we define families of differential operators $H^D_{x_0}$ and $H^N_{x_0}$ in $L^2([x_0, x_0+\pi])$, $x_0 \in \mathbb{R}$, as follows:

$$(H^D_{x_0}f)(x) = L(f(x)),\ x \in [x_0, x_0+\pi],\ f \in dom(H^D_{x_0}),$$
$$(H^N_{x_0}f)(x) = L(f(x)),\ x \in [x_0, x_0+\pi],\ f \in dom(H^N_{x_0}),$$

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where

$$\begin{aligned} dom(H_{x_0}^D) = \big\{ f \in L^2([x_0, x_0+\pi]) \,\big|\, f,\, f' \in AC([x_0, x_0+\pi]); \\ f(x_0+) = 0 = f(x_0+\pi-) \big\}, \\ dom(H_{x_0}^N) = \big\{ f \in L^2([x_0, x_0+\pi]) \,\big|\, f,\, f' \in AC([x_0, x_0+\pi]); \\ f'(x_0+) = 0 = f'(x_0+\pi-) \big\}. \end{aligned}$$

The $L^2(\mathbb{R})$ spectrum of the Hill operator H is purely continuous and it is the union of countably many analytic arcs in the complex plane [18]. On the other hand, the spectra of $H_{x_0}^D$ and $H_{x_0}^N$ are purely discrete. We will denote the *Dirichlet eigenvalues* (i.e., the eigenvalues of $H_{x_0}^D$) by $\mu_j(x_0)$, $j \in \mathbb{N}$, and the *Neumann eigenvalues* (i.e., the eigenvalues of $H_{x_0}^N$) by $\nu_k(x_0)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where we number these eigenvalues in the order of non-decreasing magnitudes.

When the potential V is real-valued and periodic, the operators H, $H_{x_0}^D$ and $H_{x_0}^N$ are all self-adjoint, and hence the spectra of these operators are subsets of the real line. In this case, the spectrum of H is a countable union of closed intervals $[E_{2m}, E_{2m+1}]$, $m \in \mathbb{N}_0$, on the real line, where E_m are such that $L\phi = E_m\phi$ has a nontrivial (i.e., nonzero) periodic solution of period 2π. Thus, E_m are eigenvalues of the self-adjoint operator associated with L under periodic boundary conditions at 0 and 2π, and they are all real. Moreover,

$$\nu_0(x_0) \le E_0, \quad E_{2m-1} \le \mu_m(x_0),\, \nu_m(x_0) \le E_{2m},\; m \in \mathbb{N},\; x_0 \in \mathbb{R} \tag{1.1}$$

(see, e.g., [7, Theorem 3.1.1], [14]).

Moreover, the following trace formulas hold.

Lemma 1.1. *Suppose that $V \in C^1(\mathbb{R})$ is a periodic function on $\mathbb{R}$ and let $x \in \mathbb{R}$. Then,*

$$V(x) = E_0 + \sum_{m=1}^{\infty} \big(E_{2m-1} + E_{2m} - 2\mu_m(x) \big), \tag{1.2}$$

$$V(x) = 2\nu_0(x) - E_0 + \sum_{m=1}^{\infty} \big(2\nu_m(x) - E_{2m-1} - E_{2m} \big). \tag{1.3}$$

Under the hypothesis that $V \in C^3(\mathbb{R})$ is a real-valued periodic function on $\mathbb{R}$, Trubowitz [22] proved the Dirichlet trace formula (1.2). The Neumann trace formula (1.3) for real-valued potentials is due to McKean and Trubowitz [15]. In 2001, Gesztesy [9] extended (1.2) to complex-valued periodic $C^1(\mathbb{R})$ potentials. In Section 3, we will prove the trace formula (1.3) for complex-valued periodic $C^1(\mathbb{R})$ potentials, by closely following the methods in [9]. We refer to [10, p. 121] (cf. also [11]) for further references and a detailed history of trace formulas.

In the self-adjoint case, Borg [4] showed that $V = E_0$ is the only real-valued periodic potential, for which the Hill operator H has the spectrum $[E_0, \infty)$. One can see that if $V \in C^1(\mathbb{R})$ is real-valued and periodic, the Dirichlet trace formula (1.2) yields $V(x) = E_0$ since then $E_{2m-1} = E_{2m}$ and $E_{2m-1} \leq \mu_m(x) \leq E_{2m}$ for $m \in \mathbb{N}$. Extensions of this simple observation to reflectionless potentials can be found in [5].

Unlike the self-adjoint case, there are infinitely many complex-valued potentials that generate a half-line spectrum $[0, \infty)$ as shown by Gasymov [13] (also, see [17], [20]). In these cases, even though all E_m are real and $E_{2m-1} = E_{2m}$ for all $m \in \mathbb{N}$, the Dirichlet and Neumann eigenvalues $\mu_m(x)$, $\nu_m(x)$ are not trapped between E_{2m-1} and E_{2m}, due to non-self-adjointness of the corresponding operators H_x^D and H_x^N. From the inverse spectral point of view, this explains the highly non-uniqueness property of complex-valued periodic potentials (see, e.g., [10, p. 113]). As a concrete example, the potential $V(x) = e^{2ix}$ generates the spectrum $[0, \infty)$ (cf. [13]) and one infers $E_0 = 0$, $E_{2m-1} = E_{2m} = m^2$, $m \in \mathbb{N}$. Marchenko [16] provides the *asymptotic* location of $\mu_j(x)$ and $\nu_k(x)$ (see (3.6), (3.7) and (3.8) below). However, up to date, the precise location of $\mu_j(x)$ and $\nu_k(x)$ for small j, k, is not known for this example. The principal application of the trace formulas proved in this paper will be a determination of the location of these eigenvalues. This complements the asymptotic results of Marchenko [16].

In Section 2, we will briefly review elements of Floquet theory and show that algebraic and geometric multiplicities of the eigenvalues E_m

are different for some cases in Theorem 1.4. Our principal new result, the Neumann trace formula for periodic potentials will be proved in Section 3. In Section 4, we will prove the following result:

Theorem 1.2. *If $\overline{V(-x)} = V(x)$, $x \in \mathbb{R}$, then for all $x_0 \in \mathbb{R}$, $j \in \mathbb{N}$, and $k \in \mathbb{N}_0$, $\mu_j(\pi - x_0) = \overline{\mu_j(x_0)}$ and $\nu_k(\pi - x_0) = \overline{\nu_k(x_0)}$.*

In Section 5, we will recall some unpublished work by Deift [6] and review some basic facts on Bessel functions that we use in our final Section 6. In Section 6, we will use the trace formulas (1.2) and (1.3) to investigate the location of $\mu_j(x_0)$ and $\nu_k(x_0)$ for the concrete example $V(x) = Ke^{2ix}$, $K \in \mathbb{C}$. More precisely, we will prove the following theorems.

Theorem 1.3. *Suppose that $V(x) = Ke^{2ix}$ and $K = |K|e^{2i\varphi_0}$ for some $\varphi_0 \in \mathbb{R}$. Then for all $j \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $\mu_j(K, x_0) = \mu_j(|K|, x_0 + \varphi_0)$ and $\nu_k(K, x_0) = \nu_k(|K|, x_0 + \varphi_0)$.*

Theorem 1.4. *Suppose that $V(x) = Ke^{2ix}$, $K > 0$. Then*

(i) $(j-1)^2 \le \mu_j(0) \le j^2$ *for all* $j \in \mathbb{N}$.

(ii) *If* $0 < K \le 1$, *then for all* $j \in \mathbb{N}$, $j^2 - \frac{K}{2} < j^2 - \frac{K}{2} + \sum_{m=1}^{j-1}(m^2 - \mu_m(0)) < \mu_j(0) < j^2$, *and hence* $\mu_j(0) \neq E_m$ *for all* $j \in \mathbb{N}$.

(iii) *If* $K > 1$ *then* $M_1 = K - 2\sum_{m=1}^{\lfloor\sqrt{K}\rfloor}(m^2 - \mu_m(0)) > 0$ *and* $j^2 - \frac{M_1}{2} < \mu_j(0) < j^2$ *for all* $j \ge \sqrt{K} + 1$, *where* $\lfloor\sqrt{K}\rfloor$ *denotes the largest integer that is less than or equal to* $\sqrt{K}$. *In particular,* $\mu_j(0) \neq E_m$ *if* $j \ge \sqrt{K} + 1$ *and* $m \in \mathbb{N}_0$.

(iv) $k^2 \le \nu_k(0) \le (k+1)^2$ *for all* $k \in \mathbb{N}_0$.

(v) *If* $0 < K \le 1$, *then for all* $k \in \mathbb{N}_0$, $k^2 < \nu_k(0) < k^2 + \frac{K}{2} - \nu_0(0) - \sum_{m=1}^{k-1}(\nu_m(0) - m^2) < k^2 + \frac{K}{2} - \nu_0(0)$, *and hence* $\nu_k(0) \neq E_m$ *for all* $k, m \in \mathbb{N}_0$.

(vi) *If* $K > 1$ *then* $M_2 = K - 2\nu_0(0) - 2\sum_{m=1}^{\lfloor\sqrt{K}\rfloor}(\nu_m(0) - m^2) > 0$ *and* $k^2 < \nu_k(0) < k^2 + \frac{M_2}{2}$ *for all* $k \ge \sqrt{K}$. *In particular,* $\nu_k(0) \neq E_m$ *for* $k \ge \sqrt{K}$ *and* $m \in \mathbb{N}_0$.

(vii) *If* $K > 0$ *then* $(2j-1)^2 < \mu_{2j-1}(\pi/2) \le \mu_{2j}(\pi/2) < (2j)^2$ *for all* $j \in \mathbb{N}$*, and hence* $\mu_j(\pi/2) \neq E_m$ *for all* $j \in \mathbb{N}$ *and* $m \in \mathbb{N}_0$*. Moreover, if* $0 < K \le 1$ *then for all* $j \in \mathbb{N}$*,*

$$(2j-1)^2 < \mu_{2j-1}(\pi/2) < (2j-1/2)^2 < \mu_{2j}(\pi/2) < (2j)^2.$$

(viii) *If* $K > 0$ *then* $(2k)^2 < \nu_{2k}(\pi/2) \le \nu_{2k+1}(\pi/2) < (2k+1)^2$ *for all* $k \in \mathbb{N}_0$*, and hence* $\nu_k(\pi/2) \neq E_m$ *for all* $k \in \mathbb{N}_0$*. Moreover, if* $0 < K \le \frac{1}{2}$ *then for all* $k \in \mathbb{N}_0$*,*

$$(2k)^2 < \nu_{2k}(\pi/2) < (2k+1/2)^2 < \mu_{2k+1}(\pi/2) < (2k+1)^2.$$

(ix) *For every* $x_0 \in (0,\pi)$*,* $\mu_j(x_0)$*,* $\nu_k(x_0) \neq E_m$*.*

2. Background

In this section, we introduce some definitions and basic facts on Floquet theory. In addition, we investigate algebraic and geometric multiplicities of the eigenvalues E_m.

We study

$$L\psi(x) = -\psi''(x) + V(x)\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where $\lambda \in \mathbb{C}$ and $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a periodic function of period π.

For each $\lambda \in \mathbb{C}$ and $x_0 \in \mathbb{R}$, there exists a fundamental system of solutions $c(\lambda,\, x_0, x)$, $s(\lambda,\, x_0, x)$ of equation (2.1) such that

$$c(\lambda,\, x_0, x_0) = 1, c'(\lambda,\, x_0, x_0) = 0;$$
$$s(\lambda,\, x_0, x_0) = 0, s'(\lambda,\, x_0, x_0) = 1.$$

One can check that these two solutions satisfy the Volterra integral equations

$$\begin{aligned} c(\lambda, x_0, x) =& \cos[\sqrt{\lambda}(x-x_0)] \\ &+ \int_{x_0}^{x} dx_1 \frac{\sin[\sqrt{\lambda}(x-x_1)]}{\sqrt{\lambda}} V(x_1) c(\lambda, x_0, x_1), \end{aligned} \tag{2.2}$$

$$\begin{aligned} s(\lambda, x_0, x) =& \frac{\sin[\sqrt{\lambda}(x-x_0)]}{\sqrt{\lambda}} \\ &+ \int_{x_0}^{x} dx_1 \frac{\sin[\sqrt{\lambda}(x-x_1)]}{\sqrt{\lambda}} V(x_1) s(\lambda, x_0, x_1). \end{aligned} \tag{2.3}$$

From these integral equations, along with uniqueness of their solutions, one can show that

$$\frac{\partial}{\partial x_0}\Big(c(\lambda, x_0, x)\Big) = (\lambda - V(x_0))s(\lambda, x_0, x), \tag{2.4}$$

$$\frac{\partial}{\partial x_0}\Big(s(\lambda, x_0, x)\Big) = -c(\lambda, x_0, x). \tag{2.5}$$

The *monodromy matrix* M associated with (2.1) is defined by

$$M(\lambda, x_0) = \begin{pmatrix} c(\lambda, x_0, x_0+\pi) & s(\lambda, x_0, x_0+\pi) \\ c'(\lambda, x_0, x_0+\pi) & s'(\lambda, x_0, x_0+\pi) \end{pmatrix},$$

and the corresponding *Floquet discriminant* $\Delta(\lambda)$ (half the trace of M) is defined by

$$\Delta(\lambda) = \frac{1}{2}\Big(c(\lambda,\, x_0, x_0+\pi) + s'(\lambda,\, x_0, x_0+\pi)\Big). \tag{2.6}$$

Using (2.4) and (2.5), it can be shown that Δ is independent of $x_0 \in \mathbb{R}$.

The Floquet discriminant $\Delta(\lambda)$ is an entire function of order $\frac{1}{2}$, and E_m, $m \in \mathbb{N}_0$, are the zeros of $\Delta(\lambda)^2 - 1 = 0$ (see, e.g., [7, Ch. 4]). Moreover, since for each $x_0 \in \mathbb{R}$ the entire function $\lambda \mapsto s(\lambda, x_0, x_0+\pi)$ is of order $\frac{1}{2}$ (see, e.g., [7, Ch. 4], [21, Ch. 21]), the function has infinitely many zeros $\mu_j(x_0)$, $j \in \mathbb{N}$, by the Picard little theorem (see, e.g., [2, Ch. 5]). Similarly, there are infinitely many zeros $\nu_k(x_0)$, $k \in \mathbb{N}_0$, of the entire function $c'(\lambda, x_0, x_0+\pi)$ of order $\frac{1}{2}$.

Algebraic and geometric multiplicities of the eigenvalues E_m. When V is real-valued and periodic, the Hill operator associated with L in $L^2([0, 2\pi])$ with periodic boundary conditions at 0 and 2π (which can easily be defined also for $V \in L^1_{\text{loc}}(\mathbb{R})$) is self-adjoint, and hence the algebraic and geometric multiplicities are the same. However, these two multiplicities of the eigenvalues E_m do not necessarily agree in the context of non-self-adjoint Hill operators. In particular, for the concrete example $V(x) = Ke^{2ix}$ that generates a non-self-adjoint Hill operator in $L^2([0, 2\pi])$ with periodic boundary conditions at 0 and 2π, we will explain below why these two multiplicities are different for all E_m, $m \in \mathbb{N}$ if $|K| \le 1$.

The geometric multiplicity of an eigenvalue is the number of linearly independent eigenfunctions corresponding to the eigenvalue. In addition, the geometric multiplicity of the eigenvalues E_m corresponding to periodic or anti-periodic boundary conditions at 0 and π agrees with the number of linearly independent eigenvectors of the monodromy matrix $M(E_m, x_0)$ (see, e.g., [7, Ch. 1]).

Since the Wronskian of $c(E_m, x_0, \cdot)$ and $s(E_m, x_0, \cdot)$ is one (i.e., the determinant of the monodromy matrix equals one), and since the trace of the monodromy matrix $M(E_m, x_0)$ is 2 (or -2) (i.e., $\Delta(E_m) = \pm 1$), we see that 1 (or -1) is the only eigenvalue of the matrix $M(E_m, x_0)$. In this case, the only way the monodromy matrix can have two linearly independent eigenvectors occurs when $M(E_m, x_0)$ equals the 2×2 identity matrix I_2 (or $-I_2$). Thus, the geometric multiplicity of E_m is 2 if and only if $c(E_m, x_0, x_0 + \pi) = s'(E_m, x_0, x_0 + \pi) = \pm 1$ and $c'(E_m, x_0, x_0 + \pi) = 0 = s(E_m, x_0, x_0 + \pi)$.

In the special case where V is real-valued and periodic, the condition $E_{2m-1} = E_{2m}$ forces $E_{2m} = \mu_m = \nu_m$, $m \in \mathbb{N}$ (see, e.g., [7, §2.3]) and hence $c'(E_{2m}, x_0, x_0 + \pi) = 0 = s(E_{2m}, x_0, x_0 + \pi)$. So $M(E_{2m}, x_0) = \pm I_2$. Thus, if $E_{2m-1} = E_{2m}$ for some $m \in \mathbb{N}$ then the geometric multiplicities of such E_{2m} are all 2. However, when V is complex-valued and periodic, the condition $E_{2m-1} = E_{2m}$ does not imply $c'(E_{2m}, x_0, x_0+\pi) = 0 = s(E_{2m}, x_0, x_0+\pi)$, and hence the geometric multiplicity of E_{2m} could be 1.

When $V(x) = Ke^{2ix}$, $K \in \mathbb{C}$, the Floquet discriminant takes on the special form $\Delta(\lambda) = \cos(\pi\sqrt{\lambda})$ (see [13], [17], [20, Theorem 2]). Thus, $E_{2m-1} = E_{2m} = m^2$ for all $m \in \mathbb{N}$, but as we see from Theorem 1.4, if $|K| \leq 1$ then $s(E_{2m}, x_0, x_0 + \pi) \neq 0$ since $\mu_j(x_0) \neq m^2$ for any $j \in \mathbb{N}$. Thus, if $|K| \leq 1$, the geometric multiplicity of each E_m, $m \in \mathbb{N}$, equals 1. On the other hand, the algebraic multiplicity of the eigenvalues E_m could be higher than one. In general, the algebraic multiplicity of the eigenvalue E_m is the multiplicity of the zero of $\Delta(\lambda) \mp 1 = 0$ at $\lambda = E_m$ (see, e.g., [12, Theorem 3.3]). So except for $E_0 = 0$, the eigenvalues $E_{2m-1} = E_{2m} = m^2$ have algebraic multiplicities 2 and geometric multiplicities 1. This result is mentioned in [3, p. 5] without proof.

3. Trace formulas

In this section, we prove a trace formula for V associated with the Neumann eigenvalues ν_k, $k \in \mathbb{N}_0$.

In [9], Gesztesy proved the following trace formula for Dirichlet eigenvalues μ_j, $j \in \mathbb{N}$ in the general case where $V \in C^1(\mathbb{R})$ is complex-valued and periodic (actually, a larger class of potentials was considered in [9]). For simplicity we assume that V has period π.

Theorem 3.1. *Suppose that the potential $V \in C^1(\mathbb{R})$ is periodic of period π. Then*

$$V(x) = E_0 + \sum_{m=1}^{\infty} \big(E_{2m-1} + E_{2m} - 2\mu_m(x)\big), \quad x \in \mathbb{R}. \tag{3.1}$$

Next, we will prove the corresponding trace formula associated with Neumann eigenvalues $\nu_k(x)$. In Section 6, these two trace formulas will be used to investigate location of $\mu_j(x)$ and $\nu_k(x)$ for the example $V(x) = Ke^{2ix}$, $K \in \mathbb{C}$.

Theorem 3.2. *Suppose that the potential $V \in C^1(\mathbb{R})$ is periodic of period π. Then*

$$V(x) = 2\nu_0(x) - E_0 + \sum_{m=1}^{\infty} \big(2\nu_m(x) - E_{2m-1} - E_{2m}\big), \quad x \in \mathbb{R}. \tag{3.2}$$

Proof. We will closely follow the proof in the Dirichlet case in [9].

Let $G(\lambda, x_1, x_2)$ be the Green's function of the Hill operator H in $L^2(\mathbb{R})$. Then the diagonal Green's function $g(\lambda, x) = G(\lambda, x, x)$ of H becomes

$$g(\lambda, x) = -\frac{s(\lambda, x, x+\pi)}{2\sqrt{\Delta^2(\lambda) - 1}} \tag{3.3}$$

(see, e.g., [9, eq. (3.58)]), where the branch of the square root is chosen so that the positive real axis maps onto itself and the square root is analytically continued to the complex λ-plane away from the spectrum of H. Next, using (2.5) and (3.3), one can show by straightforward computations that

$$\frac{1}{2} g_{xx}(\lambda, x) + (\lambda - V(x)) g(\lambda, x) = \frac{c'(\lambda, x, x+\pi)}{2\sqrt{\Delta^2(\lambda) - 1}}. \tag{3.4}$$

Together with (2.5) this implies

$$-2g_{xx}(\lambda, x) g(\lambda, x) + g_x(\lambda, x)^2 + 4(V(x) - \lambda) g(\lambda, x)^2 = 1. \tag{3.5}$$

Next, we recall that $\{E_m\}_{m\in\mathbb{N}_0}$, $\{\mu_j(x)\}_{j\in\mathbb{N}}$ and $\{\nu_k(x)\}_{k\in\mathbb{N}_0}$ are the zeros of the functions $\Delta(\lambda)^2 - 1$, $s(\lambda, x, x+\pi)$ and $c'(\lambda, x, x+\pi)$, respectively. Thus,

$$E_{\substack{2m-1\\2m}} \underset{m\to\infty}{=} \left(m + \frac{c_1}{m} + \frac{c_2}{m^2} \pm \frac{\delta_m}{m^2} + \frac{\varepsilon_m^{\pm}}{m^3}\right)^2, \tag{3.6}$$

$$\mu_m(x) \underset{m\to\infty}{=} \left(m + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{s_m(x)}{m^2}\right)^2, \tag{3.7}$$

$$\nu_m(x) \underset{m\to\infty}{=} \left(m + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{c_m(x)}{m^2}\right)^2, \tag{3.8}$$

where $\{\delta_m\}_{m\in\mathbb{N}}$, $\{\varepsilon_m^{\pm}\}_{m\in\mathbb{N}}$, $\{s_m(x)\}_{m\in\mathbb{N}}$, $\{c_m(x)\}_{m\in\mathbb{N}} \in \ell^2(\mathbb{N})$ (see, e.g., [16, Chs. 1, 3]). (Actually, Marchenko [16] did not explicitly work out the proof of (3.8), but using his equation (1.5.3) along with ideas in the proof of [16, Theorem 1.5.1], one can directly prove (3.8). When V is real-valued, one can use the interlacing property (1.1) and (3.6).)

We write

$$g(\lambda, x) = \frac{i\, f_0(\lambda, x)}{2\sqrt{\lambda}} \quad \text{for some function } f_0(\lambda, \cdot) \in C^3(\mathbb{R}). \tag{3.9}$$

Then (3.6) and (3.7) together with (3.3) and the Hardamard factorization theorem (see, e.g., [2, Ch. 5]) imply that

$$f_0(\lambda, x) \underset{\lambda\to i\infty}{=} O(1) \quad \text{for each } x \in \mathbb{R}.$$

Next, we infer from (3.5) and (3.9) that $f_0(\lambda, x)^2 \underset{\lambda\to i\infty}{=} 1$ for every $x \in \mathbb{R}$. Moreover, one can see that $f_0(\lambda, x) = f_0(x) + f_1(\lambda, x)/\lambda$ for some $f_0(x)^2 = 1$ and $f_1(\lambda, \cdot) \in C^3(\mathbb{R})$, where $f_1(\lambda, x) \underset{\lambda\to i\infty}{=} O(1)$ for each $x \in \mathbb{R}$. Thus,

$$g(\lambda, x) = \frac{i\, f_0(x)}{2\sqrt{\lambda}} + \frac{i\, f_1(\lambda, x)}{2(\sqrt{\lambda})^3}, \tag{3.10}$$

$$g_x(\lambda, x),\ g_{xx}(\lambda, x) \underset{\lambda \to i\infty}{=} O(|\lambda|^{-3/2}). \tag{3.11}$$

Next, from (3.5), (3.10) and (3.11) we infer that

$$f_1(\lambda, x) \underset{\lambda \to i\infty}{=} f_0(x)V(x)/2 \ \text{ for every } x \in \mathbb{R},$$

and hence $g(\lambda, x)$ has the following asymptotic expression

$$g(\lambda, x) \underset{\lambda \to i\infty}{=} f_0(x) \left(\frac{i}{2\sqrt{\lambda}} + \frac{i}{4(\sqrt{\lambda})^3} V(x) + O(|\lambda|^{-5/2}) \right). \tag{3.12}$$

(Actually, it is known that $f_0(x) = 1$ but $f_0(x)^2 = 1$ suffices for our argument below.) The asymptotic expressions (3.11) and (3.12) along with (3.4) imply

$$\left(\frac{c'(\lambda, x, x+\pi)}{2\sqrt{\Delta^2(\lambda) - 1}} \right)^2 \underset{\lambda \to i\infty}{=} -\frac{\lambda}{4} + \frac{1}{4} V(x) + O(|\lambda|^{-1}). \tag{3.13}$$

Next, one can choose $n \in \mathbb{N}$ so that the infinite product

$$\prod_{m=n}^{\infty} \frac{\nu_m(x)^2}{E_{2m-1}E_{2m}} \tag{3.14}$$

converges absolutely to a non-zero complex number.

If $c'(\lambda, x, x+\pi) = 0$ for some $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, then it is clear that $t \mapsto c(\lambda, x, t)$ is an eigenfunction of the Neumann problem in $L^2([x, x+\pi])$ and $\lambda = \nu_k(x)$ for some $k \in \mathbb{N}_0$. Since $\lambda \mapsto c'(\lambda, x, x+\pi)^2$ and $4(\Delta(\lambda)^2 - 1)$ are entire functions of order $\frac{1}{2}$, the Hardamard factorization theorem implies that

$$c'(\lambda, x, x+\pi)^2 = A_1(x) \prod_{m=0}^{N-1} (\nu_m(x) - \lambda)^2 \prod_{m=N}^{\infty} \left(1 - \frac{\lambda}{\nu_m(x)}\right)^2,$$

$$4(\Delta(\lambda)^2 - 1) = B_1 \prod_{m=0}^{2N-2} (E_m - \lambda) \prod_{m=N}^{\infty} \left[\left(1 - \frac{\lambda}{E_{2m-1}}\right) \left(1 - \frac{\lambda}{E_{2m}}\right) \right],$$

for some function A_1 on $\mathbb{R}$ and a constant $B_1 \in \mathbb{C}$. Here we chose $N \geq n$ so that for all $m \geq N$, $E_m \neq 0$, $\nu_m(x) \neq 0$ and

$$|\lambda - E_m| \leq |\lambda| \ \text{ for all large } |\lambda| \text{ with } \arg(\lambda) = \frac{\pi}{2}. \tag{3.15}$$

This is possible because $|\mathrm{Im}\,(E_m)| \leq \max_{0\leq x\leq\pi} |\mathrm{Im}\,(V(x))|$ for $m \in \mathbb{N}_0$.

Thus,

$$\begin{aligned}
&\left(\frac{1}{\lambda}\right) \frac{c'(\lambda,x,x+\pi)^2}{4(\Delta^2(\lambda)-1)} \\
&= -A(x)\frac{(\nu_0(x)-\lambda)^2}{\lambda(E_0-\lambda)} \prod_{m=1}^{\infty}\left[\frac{(\nu_m(x)-\lambda)^2}{(E_{2m-1}-\lambda)(E_{2m}-\lambda)}\right] \\
&= A(x)\frac{\left(1-\frac{\nu_0(x)}{\lambda}\right)^2}{1-\frac{E_0}{\lambda}} \prod_{m=1}^{\infty}\left[\frac{\left(1-\frac{\nu_m(x)}{\lambda}\right)^2}{\left(1-\frac{E_{2m-1}}{\lambda}\right)\left(1-\frac{E_{2m}}{\lambda}\right)}\right],
\end{aligned} \tag{3.16}$$

where

$$A(x) = -\frac{A_1(x)}{B_1} \prod_{m=N}^{\infty} \frac{E_{2m-1}E_{2m}}{\nu_m(x)^2}. \tag{3.17}$$

Next, we have

$$\begin{aligned}
&\ln\left(\prod_{m=1}^{N}\left[\frac{(\nu_m(x)-\lambda)^2}{(E_{2m-1}-\lambda)(E_{2m}-\lambda)}\right]\right) \\
&= \sum_{m=1}^{N}\left[2\ln\left(1-\frac{\nu_m(x)}{\lambda}\right) - \ln\left(1-\frac{E_{2m-1}}{\lambda}\right) - \ln\left(1-\frac{E_{2m}}{\lambda}\right)\right] \\
&\underset{\lambda\to i\infty}{=} -\frac{1}{\lambda}\sum_{m=1}^{N}\left[2\nu_m(x) - E_{2m-1} - E_{2m}\right] + O(|\lambda|^{-2}),
\end{aligned} \tag{3.18}$$

where we used $\ln(1-t) \underset{t\to 0}{=} -t + O(t^2)$. Moreover,

$$\begin{aligned}
&\sum_{m=N+1}^{\infty}\left|\frac{(2\nu_m(x)-E_{2m-1}-E_{2m})}{\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}\right| \\
&\leq \sum_{m=N+1}^{\infty} \frac{|2\nu_m(x)-E_{2m-1}-E_{2m}|}{|\lambda|^{-2}\,|(E_{2m-1}-\lambda)\,(E_{2m}-\lambda)|} \\
&\leq \sum_{m=N+1}^{\infty} \frac{|2\nu_m(x)-E_{2m-1}-E_{2m}|}{|\lambda|^{-2}\,|\lambda^2|} < +\infty,
\end{aligned}$$

and by (3.6), (3.8) and (3.15),

$$\begin{aligned}&\sum_{m=N+1}^{\infty}\frac{|\nu_m(x)^2-E_{2m-1}E_{2m}|}{\left|\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)\right|}\\&\le\sum_{m=N+1}^{\infty}\left|\nu_m(x)^2-E_{2m-1}E_{2m}\right|<+\infty.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\ln\left(\prod_{m=N+1}^{\infty}\left[\frac{(\nu_m(x)-\lambda)^2}{(E_{2m-1}-\lambda)(E_{2m}-\lambda)}\right]\right)\\&=\sum_{m=N+1}^{\infty}\ln\left[1-\frac{(E_{2m-1}-\lambda)\,(E_{2m}-\lambda)-(\nu_m(x)-\lambda)^2}{(E_{2m-1}-\lambda)\,(E_{2m}-\lambda)}\right]\\&=\sum_{m=N+1}^{\infty}\ln\left[1-\frac{(2\nu_m(x)-E_{2m-1}-E_{2m})\lambda+(E_{2m-1}E_{2m}-\nu_m(x)^2)}{\lambda^2\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}\right]\\&=-\sum_{m=N+1}^{\infty}\left[\frac{(2\nu_m(x)-E_{2m-1}-E_{2m})\lambda+(E_{2m-1}E_{2m}-\nu_m(x)^2)}{\lambda^2\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}\right]\\&+O\left(\sum_{m=N+1}^{\infty}\left|\frac{(2\nu_m(x)-E_{2m-1}-E_{2m})\lambda+(E_{2m-1}E_{2m}-\nu_m(x)^2)}{\lambda^2\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}\right|^2\right)\\&=-\frac{1}{\lambda}\sum_{m=N+1}^{\infty}\frac{(2\nu_m(x)-E_{2m-1}-E_{2m})}{\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}\\&-\frac{1}{\lambda^2}\sum_{m=N+1}^{\infty}\frac{(E_{2m-1}E_{2m}-\nu_m(x)^2)}{\left(\frac{E_{2m-1}}{\lambda}-1\right)\left(\frac{E_{2m}}{\lambda}-1\right)}+O(|\lambda|^{-2}),\end{aligned}\tag{3.19}$$

where we used again (3.15) to bound the error term. Next, we use the dominated convergence theorem (with the discrete counting measure) in (3.19) to derive

$$\begin{aligned}&\ln\left(\prod_{m=N+1}^{\infty}\left[\frac{(\nu_m(x)-\lambda)^2}{(E_{2m-1}-\lambda)(E_{2m}-\lambda)}\right]\right)\\&\underset{\lambda\to i\infty}{=}-\frac{1}{\lambda}\sum_{m=N+1}^{\infty}(2\nu_m(x)-E_{2m-1}-E_{2m})+O\left(\frac{1}{|\lambda|^2}\right).\end{aligned}\tag{3.20}$$

Finally, (3.16), (3.18) and (3.20) along with the fact $e^t \underset{t\to 0}{=} 1 + O(t)$ yield

$$\frac{1}{\lambda}\frac{c'(\lambda,x,x+\pi)^2}{4(\Delta^2(\lambda)-1)} \underset{\lambda\to i\infty}{=} A(x)\Big[1 - \frac{2\nu_0(x)-E_0}{\lambda} - \frac{1}{\lambda}\sum_{m=1}^{\infty}[2\nu_m(x) - E_{2m-1} - E_{2m}] + O\left(\frac{1}{|\lambda|^2}\right)\Big]. \tag{3.21}$$

Thus, comparing (3.21) with (3.13), we infer $A(x) = -\frac{1}{4}$ and the desired trace formula (3.2). □

4. The case when $\overline{V(-x)} = V(x)$

In this section, we prove some general results on the location of Dirichlet and Neumann eigenvalues.

The following lemma on periodicity of the eigenvalues is well-known and hence we omit the proof.

Lemma 4.1. *Suppose that $V \in L^1_{loc}(\mathbb{R})$ is periodic of period π. Then $\mu_j(x_0)$, $j \in \mathbb{N}$, and $\nu_k(x_0)$, $k \in \mathbb{N}_0$, are all periodic functions of period π.*

Next, we prove the following theorem, regarding a certain symmetry of the eigenvalues $\mu_j(x_0)$ and $\nu_k(x_0)$.

Theorem 4.2. *Suppose that $V \in L^1_{loc}(\mathbb{R})$ is periodic of period π and that $\overline{V(-x)} = V(x)$ for all $x \in \mathbb{R}$, or equivalently, $\overline{V(\pi - x)} = V(x)$ for all $x \in [0,\pi]$. Then for every $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $x_0 \in [0,\pi]$,*

$$\overline{\mu_j(\pi - x_0)} = \mu_j(x_0) \quad \textit{and} \quad \overline{\nu_k(\pi - x_0)} = \nu_k(x_0). \tag{4.1}$$

Proof. Let $y(\mu_j(x_0),x) = s(\mu_j(x_0),x_0,x)$. Then since $\overline{V(-x)} = V(x)$, we see that $\overline{y(\mu_j(x_0),-x)}$ is also a solution of

$$-\psi''(x) + V(x)\psi(x) = \mu_j(x_0)\psi(x) \tag{4.2}$$

with $\mu_j(x_0)$ replaced by $\overline{\mu_j(x_0)}$. (To see this, we take the complex conjugate of equation (4.2) and replace x by $-x$.) We write $\overline{y(\mu_j(x_0),-x)} = y(\overline{\mu_j(x_0)},x)$.

Thus, $y(\mu_j(x_0), x_0) = 0 = y(\mu_j(x_0), x_0 + \pi)$ yields

$$y(\overline{\mu_j(x_0)}, -x_0 - \pi) = 0 = y(\overline{\mu_j(x_0)}, -x_0) = y(\overline{\mu_j(x_0)}, (-x_0 - \pi) + \pi).$$

This implies

$$\mu_j(-x_0 - \pi) = \overline{\mu_j(x_0)}.$$

By Lemma 4.1 we then have

$$\mu_j(\pi - x_0) = \mu_j((\pi - x_0) - 2\pi) = \overline{\mu_j(x_0)}.$$

Similarly, we can show that $\nu_k(\pi - x_0) = \overline{\nu_k(x_0)}$. □

Thus, for all $x_0 \in [0, \pi/2]$, $\mu_j(x_0)$ and $\nu_k(x_0)$ contain all information of the eigenvalues for $x_0 \in \mathbb{R}$. Moreover, we have the following corollary regarding reality of the eigenvalues $\mu_j(x_0)$, $\nu_k(x_0)$ for $x_0 = 0$, $\frac{\pi}{2}$.

Corollary 4.3. *Suppose that $V \in L^1_{loc}(\mathbb{R})$ is periodic of period π and that $\overline{V(-x)} = V(x)$ for all $x \in \mathbb{R}$. Then for all $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\mu_j(0)$, $\mu_j(\frac{\pi}{2})$, $\nu_k(0)$ and $\nu_k(\frac{\pi}{2})$ are all real.*

Proof. Since $\mu_j(0) = \mu_j(\pi) = \overline{\mu_j(\pi - \pi)} = \overline{\mu_j(0)}$, $\mu_j(0)$ are all real. Similarly, $\nu_k(0)$ are all real. Moreover,

$$\mu_j(\frac{\pi}{2}) = \overline{\mu_j(\pi - \frac{\pi}{2})} = \overline{\mu_j(\frac{\pi}{2})}, \quad \nu_k(\frac{\pi}{2}) = \overline{\nu_k(\pi - \frac{\pi}{2})} = \overline{\nu_k(\frac{\pi}{2})}.$$

□

5. Bessel functions and some unpublished work of Deift

In this section, we reconsider some results in an unpublished manuscript by Deift [6], where he explicitly expressed $c'(\lambda, x_0, x_0 + \pi)$ and $s(\lambda, x_0, x_0 + \pi)$ for $V(x) = Ke^{2ix}$, $K \in \mathbb{C}$, in terms of Bessel functions. In addition we will introduce some useful facts on Bessel functions.

Consider the Schrödinger equation

$$-\psi''(x) + Ke^{2ix}\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}, \tag{5.1}$$

where λ, $K \in \mathbb{C}$. Gasymov [13] (also, see [17], [20, Theorem 2]) showed that if $V(x) = Ke^{2ix}$, $K \in \mathbb{C}$, then

$$\Delta(\lambda) = \cos(\pi\sqrt{\lambda}).$$

From this fact it is clear that

$$E_0 = 0, \quad E_{2m-1} = E_{2m} = m^2, \quad m \in \mathbb{N}. \tag{5.2}$$

Next, the Bessel function $J_\nu(u)$ of the first kind is given by

$$J_\nu(u) = \left(\frac{u}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{u}{2}\right)^{2m}, \quad u, \nu \in \mathbb{C}, \tag{5.3}$$

where we choose the negative real axis as its branch cut for $\nu \notin \mathbb{Z}$. The Bessel function J_ν solves the following differential equation

$$\frac{d^2}{du^2} J_\nu(u) + \frac{1}{u}\frac{d}{du} J_\nu(u) + \frac{u^2 - \nu^2}{u^2} J_\nu(u) = 0. \tag{5.4}$$

One can show that $x \mapsto J_{\sqrt{\lambda}}(\sqrt{K}e^{ix})$ is a solution of (5.1) (see, e.g., [8, p. 196]). Moreover,

$$c(\lambda, 0, x) = \frac{\pi\sqrt{K}}{2}\left(Y'_{\sqrt{\lambda}}(\sqrt{K}) J_{\sqrt{\lambda}}(\sqrt{K}e^{ix}) - J'_{\sqrt{\lambda}}(\sqrt{K}) Y_{\sqrt{\lambda}}(\sqrt{K}e^{ix})\right),$$
$$s(\lambda, 0, x) = \frac{\pi i}{2}\left(Y_{\sqrt{\lambda}}(\sqrt{K}) J_{\sqrt{\lambda}}(\sqrt{K}e^{ix}) - J_{\sqrt{\lambda}}(\sqrt{K}) Y_{\sqrt{\lambda}}(\sqrt{K}e^{ix})\right),$$

where Y_ν is the Bessel function of the second kind (also a solution of (5.4)), defined by

$$Y_\nu(u) = \frac{J_\nu(u)\cos(\nu\pi) - J_{-\nu}(u)}{\sin(\nu\pi)}. \tag{5.5}$$

In his unpublished manuscript [6], Deift showed that

$$s(\lambda, 0, \pi) = \pi J_{\sqrt{\lambda}}(\sqrt{K}) J_{-\sqrt{\lambda}}(\sqrt{K}), \tag{5.6}$$
$$c'(\lambda, 0, \pi) = \pi K J'_{\sqrt{\lambda}}(\sqrt{K}) J'_{-\sqrt{\lambda}}(\sqrt{K}), \tag{5.7}$$

where he used (5.5) with

$$J_\nu(ue^{i\pi}) = e^{\nu\pi i} J_\nu(u),$$
$$Y_\nu(ue^{i\pi}) = e^{-\nu\pi i} Y_\nu(u) + 2i\cos(\nu\pi) J_\nu(u).$$

In Section 6, we will extensively use equations (5.6) and (5.7), along with the trace formulas (3.1), (3.2) to investigate the location of μ_j and ν_k.

Next, we list a number of basic facts on Bessel functions that will be used in the next section.

Lemma 5.1.

(1) *If ν is real, then $J_\nu(u)$ is real for all $u \in \mathbb{R}$.*
(2) *If $\nu \geq -1$, then all zeros of $J_\nu(u)$ are real and if $\nu \geq 0$, then all zeros of $J'_\nu(u)$ are real.*
(3) *If $\nu \geq 0$, then the smallest positive zeros of $J_\nu(u)$ and $J'_\nu(u)$ are greater than ν.*
(4) *If $n \in \mathbb{Z}$, then $J_{-n}(u) = (-1)^n J_n(u)$ and $J'_{-n}(u) = (-1)^n J'_n(u)$, $u \in \mathbb{C}$.*
(5) *J_ν and $J_{-\nu}$ are linearly independent if and only if $\nu \notin \mathbb{Z}$.*
(6) *J_n and J_m do not have a common zero if $m, n \in \mathbb{N}_0$ with $m \neq n$.*
(7) *J'_{n-1} and J'_n do not have a common zero for $n \in \mathbb{N}$.*

Proof. See, for example, [23, Ch. 15] and [1, Ch. 9] for proofs of these results. We also note that assertions (1), (4) and (5) can be derived directly from (5.3). □

6. Applications of the trace formulas

In this section, we will prove a number of results regarding the location of the Dirichlet and Neumann eigenvalues for the potential $V(x) = Ke^{2ix}$, the collection of which becomes Theorem 1.4.

In the following theorem, we will use $\mu_j(K, x_0)$ and $\nu_k(K, x_0)$ to explicitly indicate the K-dependence of these eigenvalues.

Theorem 6.1. *Suppose that $V(x) = Ke^{2ix}$ and $K = |K|e^{2i\varphi_0}$ for some $\varphi_0 \in \mathbb{R}$. Then for each $x \in \mathbb{R}$, $j \in \mathbb{N}$, and $k \in \mathbb{N}_0$,*

$$\mu_j(K, x) = \mu_j(|K|, x + \varphi_0) \text{ and } \nu_k(K, x) = \nu_k(|K|, x + \varphi_0).$$

Proof. With $K = |K|e^{2i\varphi_0}$, (5.1) becomes

$$-\psi''(x) + |K|e^{2i(x+\varphi_0)}\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}. \tag{6.1}$$

Next, we consider the equation

$$-\psi''(x) + |K|e^{2ix}\psi(x) = \lambda\psi(x), \quad x \in \mathbb{R}. \tag{6.2}$$

Clearly, $y(x)$ is a solution of (6.1) if and only if $y_1(x) = y(x - \varphi_0)$ is a solution of (6.2). Moreover, $y(x) = 0 = y(x + \pi)$ if and only if

$y_1(x+\varphi_0) = 0 = y_1(x+\varphi_0+\pi)$. This implies that $\mu_j(|K|e^{2i\varphi_0}, x) = \mu_j(|K|, x+\varphi_0)$, $j \in \mathbb{N}$. Similarly one proves that $\nu_k(|K|e^{2i\varphi_0}, x) = \nu_k(|K|, x+\varphi_0)$, $k \in \mathbb{N}_0$. $\square$

Remark. (i) From the proof of Theorem 6.1, we conclude that for all $x \in \mathbb{R}$, $j \in \mathbb{N}$, and $k \in \mathbb{N}_0$,

$$\mu_j(K,x) = \mu_j(e^{2ix}K, 0) \text{ and } \nu_k(K,x) = \nu_k(e^{2ix}K, 0). \tag{6.3}$$

(ii) By Theorem 6.1, the periodic curves in the complex plane, generated by $x \mapsto \mu_j(x)$ and $x \mapsto \nu_k(x)$ remain the same for all $K \in \mathbb{C}$ with the same magnitude. Thus, we will focus on the case $K > 0$.

Next, we will provide more precise location of the Dirichlet eigenvalues $\mu_j(0)$.

Theorem 6.2. *Suppose that $V(x) = Ke^{2ix}$ and $K > 0$. Then*

(i) $(j-1)^2 \le \mu_j(0) \le j^2$ *for all* $j \in \mathbb{N}$.

(ii) *If* $0 < K \le 1$, *then for all* $j \in \mathbb{N}$,
$j^2 - \frac{K}{2} < j^2 - \frac{K}{2} + \sum_{m=1}^{j-1}(m^2 - \mu_m(0)) < \mu_j(0) < j^2$, *and hence* $\mu_j(0) \ne E_m$ *for all* $j \in \mathbb{N}$, $m \in \mathbb{N}_0$.

(iii) *If* $K > 1$ *then* $M_1 = K - 2\sum_{m=1}^{\lfloor\sqrt{K}\rfloor}(m^2 - \mu_m(0)) > 0$ *and* $j^2 - \frac{M_1}{2} < \mu_j(0) < j^2$ *for all* $j \ge \sqrt{K}+1$, *where* $\lfloor\sqrt{K}\rfloor$ *denotes the largest integer that is less than or equal to* $\sqrt{K}$. *In particular,* $\mu_j(0) \ne E_m$ *if* $j \ge \sqrt{K}+1$ *and* $m \in \mathbb{N}_0$.

Proof of (ii). We recall that $\mu_j(0)$, $j \in \mathbb{N}$, are the zeros of $s(\lambda, 0, \pi)$ and from (5.6),

$$s(\lambda, 0, \pi) = \pi J_{\sqrt{\lambda}}(\sqrt{K}) J_{-\sqrt{\lambda}}(\sqrt{K}). \tag{6.4}$$

Also, by Corollary 4.3, $\mu_j(0) \in \mathbb{R}$ for all $j \in \mathbb{N}$.

Below, we will show the existence of $\mu_j(0)$ in certain intervals applying the intermediate value theorem. Subsequently the convergence of the series in the trace formula in Theorem 3.1 will be used to show that there exist no other $\mu_j(0)$.

First, we will show that if $0 < K \le 1$, then $s((2n)^2, 0, \pi) > 0$ and $s((2n-1)^2, 0, \pi) < 0$, $n \in \mathbb{N}$. The continuous function $s(\lambda, 0, \pi)$ is real for $\lambda \geq 0$ (in fact, it is real for all $\lambda \in \mathbb{R}$ because if $\lambda < 0$ then the two Bessel functions in (6.4) at $\sqrt{K} \in \mathbb{R}$ are complex conjugates of each other since $\mathrm{Re}\,(\sqrt{\lambda}) = 0$).

By Lemma 5.1 (4),

$$J_{-n}(u) = (-1)^n J_n(u), \quad n \in \mathbb{Z}$$

and hence

$$s(n^2, 0, \pi) = (-1)^n \left(J_n(\sqrt{K}) \right)^2 . \tag{6.5}$$

By Lemma 5.1 (2) and (3), for each $\sqrt{\lambda} \geq 0$, the zeros of $u \mapsto J_{\sqrt{\lambda}}(u)$ are all real, and positive zeros of these functions are all greater than $\sqrt{\lambda}$. Thus, if $0 < K \leq 1$, then for every $n \geq 1$, $J_n(\sqrt{K}) \neq 0$, and $J_0(\sqrt{K}) \neq 0$ by (5.3) since the terms in the series (5.3) have alternating signs and since absolute values of these terms are strictly decreasing if $\sqrt{K} \leq 2$. Thus, the sequence $n \mapsto s(n^2, 0, \pi) = (-1)^n \left(J_n(\sqrt{K}) \right)^2$ alternates its sign for all $n \geq 0$. By the intermediate value theorem at least one of $\mu_j(0)$ lies in every open interval $((n-1)^2, n^2)$, $n \in \mathbb{N}$. Next, using the trace formula (3.1), we will show that there is precisely one $\mu_j(0)$ in every interval $((n-1)^2, n^2)$.

The trace formula (3.1) at $x_0 = 0$ reads

$$\begin{aligned} K = Ke^{2ix_0}\Big|_{x_0=0} &= E_0 + \sum_{j=1}^{\infty} (E_{2j-1} + E_{2j} - 2\mu_j(0)) \\ &= 0 + \sum_{j=1}^{\infty} \left(2j^2 - 2\mu_j(0)\right) . \end{aligned} \tag{6.6}$$

This implies that if there were more than one $\mu_j(0)$ in some $((n-1)^2, n^2)$ or if there were one $\mu_j(0)$ on the negative real axis, then $j^2 - \mu_j(0) > 1$ for all $j \geq n+1$, and hence the sum in (6.6) would be divergent. Thus, there is exactly one $\mu_j(0)$ in every interval $((n-1)^2, n^2)$, $n \in \mathbb{N}$. Since the sum is K and since $2j^2 - 2\mu_j(0) > 0$ for all $j \in \mathbb{N}$, we have

$$j^2 - \frac{K}{2} < j^2 - \frac{K}{2} + \sum_{m=1}^{j-1} \mu_m(0) < \mu_j(0) < j^2, \quad j \in \mathbb{N}.$$

Proof of (i) and (iii). There is at least one $\mu_j(0)$ in each closed interval $[(n-1)^2, n^2]$, $n \in \mathbb{N}$. Otherwise, $s((n-1)^2, 0, \pi) \neq 0$ and $s(n^2, 0, \pi) \neq 0$.

By (6.4), there would be at least one $\mu_j(0)$ in the interior of the interval by the intermediate value theorem. Next, since the smallest positive zero of $J_\nu(u)$, $\nu \geq 0$, is greater than ν, the sequence $n \mapsto s(n^2, 0, \pi)$ alternates its sign for $n \geq \sqrt{K}$. However, it is possible that $s(n^2, 0, \pi) = 0$ for some $0 \leq n < \sqrt{K}$.

Suppose that $s(n_0^2, 0, \pi) = 0$ for some $n_0 \in \mathbb{N}$, $0 < n_0 < \sqrt{K}$. Then by (6.4) either $J_{n_0}(\sqrt{K}) = 0$ or $J_{-n_0}(\sqrt{K}) = 0$. However, since J_{n_0} and J_{-n_0} are linearly dependent, by Lemma 5.1 (4), one concludes that $J_{n_0}(\sqrt{K}) = 0$ and $J_{-n_0}(\sqrt{K}) = 0$. Next, we will show that $s(\lambda, 0, \pi)$ has at least double zeros at $\lambda = n_0^2$ if $s(n_0^2, 0, \pi) = 0$.

Since $J_{n_0}(\sqrt{K}) = 0$, one infers that $\sqrt{\lambda} \mapsto J_{\sqrt{\lambda}}(\sqrt{K}) = (\sqrt{\lambda} - n_0)f(\sqrt{\lambda})$ for some entire function f. Since $J_{-n_0}(\sqrt{K}) = 0$, also $J_{\sqrt{\lambda}}(\sqrt{K}) = (\sqrt{\lambda} - n_0)(\sqrt{\lambda} + n_0)f_1(\lambda)$ for some entire function f_1. Thus,

$$\begin{aligned} s(\lambda, 0, \pi) &= \pi J_{\sqrt{\lambda}}(\sqrt{K}) J_{-\sqrt{\lambda}}(\sqrt{K}) \\ &= \pi(\lambda - n_0^2)^2 f_1(\sqrt{\lambda}) f_1(-\sqrt{\lambda}) \end{aligned}$$

and hence $s(\lambda, 0, \pi)$ has at least a double zero at $\lambda = n_0^2$. If $n_0 = 0$ then since $n_0 = -n_0$, we have for some entire function f_1,

$$s(\lambda, 0, \pi) = \pi \lambda f_1(\sqrt{\lambda}) f_1(-\sqrt{\lambda}),$$

and hence the zero can be simple or it can be of higher order.

By Lemma 5.1 (6), if $s(n_0^2, 0, \pi) = 0$ for some integer $0 < n_0 < \sqrt{K}$, then $s(n^2, 0, \pi) \neq 0$ for $n \in \mathbb{N}_0$ with $n \neq n_0$. If $n_0 = 0$, then the zero must be simple, due to the convergence of the sum in (6.6) since $(j-1)^2 < \mu_j(0) < j^2$ for all $j \geq 2$. If $n_0 \neq 0$ then the zero is not simple from the above argument. In fact, the zero must be of order 2; otherwise, the sum in (6.6) would diverge.

If $s(n^2, 0, \pi) \neq 0$ for all $n \in \mathbb{N}_0$, then $(j-1)^2 < \mu_j(0) < j^2$ for all $j \in \mathbb{N}$ like in the case (ii). Thus, we proved (i).

The proof of (iii) is analogous to that of (ii). □

Next, we study the location of Neumann eigenvalues $\nu_k(0)$.

Theorem 6.3. *Suppose that $V(x) = Ke^{2ix}$ and $K > 0$. Then*

(i) $k^2 \le \nu_k(0) \le (k+1)^2$ *for all* $k \in \mathbb{N}_0$.
(ii) *If* $0 < K \le 1$, *then for all* $k \in \mathbb{N}_0$,
$k^2 < \nu_k(0) < k^2 + \frac{K}{2} - \nu_0(0) - \sum_{m=1}^{k-1}(\nu_m(0) - m^2) < k^2 + \frac{K}{2} - \nu_0(0)$, *and hence* $\nu_k(0) \ne E_m$ *for* $k \in \mathbb{N}_0$, $m \in \mathbb{N}_0$.
(iii) *If* $K > 1$ *then* $M_2 = K - 2\nu_0(0) - 2\sum_{m=1}^{\lfloor\sqrt{K}\rfloor}(\nu_m(0) - m^2) > 0$ *and* $k^2 < \nu_k(0) < k^2 + \frac{M_2}{2}$ *for all* $k \ge \sqrt{K}$. *In particular,* $\nu_k(0) \ne E_m$ *for* $k \ge \sqrt{K}$, $m \in \mathbb{N}_0$.

Proof of (ii). The arguments are very similar to that in the proof of Theorem 6.2 (ii). However, in the trace formula (3.2), $\nu_0(x_0)$ is paired with E_0, unlike in the Dirichlet case (3.1). Hence, a more careful analysis is needed in the present Neumann case.

We recall that $\nu_k(0)$, $k \in \mathbb{N}_0$, are the zeros of

$$c'(\lambda, 0, \pi) = \pi K J'_{\sqrt{\lambda}}(\sqrt{K}) J'_{-\sqrt{\lambda}}(\sqrt{K}), \tag{6.7}$$

a real-valued continuous function on the real line. Moreover, by Corollary 4.3, $\nu_k(0) \in \mathbb{R}$ for all $k \in \mathbb{N}_0$.

If $0 < K \le 1$, as in the case of Dirichlet eigenvalues, the sequence $n \mapsto c'(n^2, 0, \pi) = (-1)^n \left(J'_n(\sqrt{K})\right)^2$ alternates its sign for all $n \ge 1$. By (5.3), we see that $J'_0(\sqrt{K}) \ne 0$ as in the case of $J_0(\sqrt{K}) \ne 0$ for $\sqrt{K} \le 2$. Hence, by the intermediate value theorem there is at least one $\nu_k(0)$ in the interval $((n-1)^2, n^2)$ for every $n \in \mathbb{N}$. Moreover, $\nu_k(0) \ne n^2$ for any $n \in \mathbb{Z}$. Next, we will show that $\nu_0(0) > 0$, that is, there is no negative Neumann eigenvalue since $c'(0,0,\pi) \ne 0$.

The trace formula (3.2) at $x_0 = 0$ reads

$$\begin{aligned} K = Ke^{2ix_0}\Big|_{x_0=0} &= 2\nu_0(0) - E_0 + \sum_{k=1}^{\infty}(2\nu_k(0) - E_{2k-1} - E_{2k}) \\ &= 2\nu_0(0) + \sum_{k=1}^{\infty}\left(2\nu_k(0) - 2k^2\right). \end{aligned} \tag{6.8}$$

Next, suppose that $J'_{\sqrt{\lambda}}(\sqrt{K}) = 0$ at $\lambda = \nu_0(0) < 0$. Then since $J'_{\sqrt{\lambda}}(\sqrt{K})$ is an entire function of $\sqrt{\lambda}$, we can write

$$J'_{\sqrt{\lambda}}(\sqrt{K}) = (\sqrt{\lambda} - i\sqrt{|\nu_0(0)|})f(\sqrt{\lambda})$$

for some entire function f. Since

$$J'_{-\sqrt{\nu_0(0)}}(\sqrt{K}) = \overline{J'_{\sqrt{\nu_0(0)}}(\sqrt{K})} = 0,$$

we see that $f(-i\sqrt{|\nu_0(0)|}) = 0$, and hence

$$J'_{\sqrt{\lambda}}(\sqrt{K}) = (\sqrt{\lambda} - i\sqrt{|\nu_0(0)|})(\sqrt{\lambda} + i\sqrt{|\nu_0(0)|})f_1(\sqrt{\lambda})$$

for some entire function f_1. Thus,

$$\begin{aligned} c'(\lambda, 0, \pi) &= \pi K J'_{\sqrt{\lambda}}(\sqrt{K}) J'_{-\sqrt{\lambda}}(\sqrt{K}) \\ &= \pi K(\lambda - \nu_0(0))^2 f_1(\sqrt{\lambda}) f_1(-\sqrt{\lambda}), \end{aligned}$$

implying $\nu_1(0) < 0$. But then (6.8) would diverge, since there exists at least one $\nu_k(0)$ in the open interval $((n-1)^2, n^2)$ for every $n \in \mathbb{N}$. This is a contradiction, and hence there is no negative Neumann eigenvalue.

The previous argument also shows that if $c'(n_0^2, 0, \pi) = 0$ for some $n_0 \in \mathbb{N}$, then $c'(\lambda, 0, \pi)$ has at least a double zero at $\lambda = n_0^2$, while $c'(\lambda, 0, \pi)$ could have a simple zero at $\lambda = 0$.

Moreover, still assuming $0 < K \le 1$, if there were more than one Neumann eigenvalue $\nu_k(0)$ in some interval $((n-1)^2, n^2)$, there would be at least three $\nu_k(0)$ in this interval, counting multiplicity since $\nu_k(0) \neq m^2$. This would violate the convergence of the sum in (6.8). Hence, there is exactly one $\nu_k(0)$ in each interval $((k-1)^2, k^2)$, $k \in \mathbb{N}$. Since $\nu_k(0) - k^2 > 0$ for all $k \in \mathbb{N}_0$, we infer that

$$\begin{aligned} &0 < \nu_0(0) < \frac{K}{2}, \\ &k^2 < \nu_k(0) < k^2 + \frac{K}{2} - \nu_0(0) - \sum_{m=1}^{k-1} \nu_m(0) < k^2 + \frac{K}{2} - \nu_0(0), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \mathbb{N}. \end{aligned}$$

Proof of (i) and (iii). Suppose that $K > 1$. We recall that $\nu_k(x_0)$ is numbered according to $k \in \mathbb{N}_0$. In proving Theorem 6.2 (i) and (iii), we used Lemma 5.1 (6), while here we need Lemma 5.1 (7). The point of

Lemma 5.1 (7) in the proof is that if $c'(n^2,0,\pi)=0$ then $c'(k^2,0,\pi)\neq 0$ for $k=(n-1)$ and $k=(n+1)$. The rest of the proof is similar to that of Theorem 6.2 (i) and (iii). Hence, we omit further details. □

Next, we investigate $\mu_j(\pi/2)$, $j\in\mathbb{N}$.

Theorem 6.4. *Suppose that $V(x)=Ke^{2ix}$ and $K>0$. Then,*

$$(2j-1)^2<\mu_{2j-1}(\pi/2)\le\mu_{2j}(\pi/2)<(2j)^2,\quad j\in\mathbb{N},$$

and hence $\mu_j(\pi/2)\neq E_m$, $m\in\mathbb{N}_0$.

Moreover, if $0<K\le 1$, then

$$(2j-1)^2<\mu_{2j-1}(\pi/2)<(2j-\frac{1}{2})^2<\mu_{2j}(\pi/2)<(2j)^2,\quad j\in\mathbb{N}.$$

Proof. By (5.6) and (6.3), $\mu_j(\pi/2)$ are the values of λ for which

$$s(\lambda,\pi/2,3\pi/2)=\pi J_{-\sqrt{\lambda}}(e^{i\pi/2}\sqrt{K})J_{\sqrt{\lambda}}(e^{i\pi/2}\sqrt{K})=0.$$

If $\lambda\ge 0$, then all the zeros u of $J_{\sqrt{\lambda}}(u)$ are real and one concludes that $J_{\sqrt{\lambda}}(e^{i\pi/2}\sqrt{K})\neq 0$. Thus, the zeros of $s(\lambda,\pi/2,3\pi/2)$ agree with those of $J_{-\sqrt{\lambda}}(i\sqrt{K})$ and hence with those of $\left[\left(i\sqrt{K}\,2^{-1}\right)^{\sqrt{\lambda}}J_{-\sqrt{\lambda}}(i\sqrt{K})\right]$.

Let

$$f(\alpha)=\left(\frac{i\sqrt{K}}{2}\right)^{-\alpha}J_\alpha(i\sqrt{K})=\sum_{m=0}^{\infty}\frac{\left(\frac{\sqrt{K}}{2}\right)^{2m}}{m!\Gamma(m+\alpha+1)}.\tag{6.9}$$

We will show below that if $0<K\le 1$, then

$$f(-n)>0\text{ and }f(-(2n-1/2))<0\text{ for all }n\in\mathbb{N}.\tag{6.10}$$

By the intermediate value theorem this then implies the existence of at least one $\mu_j(\pi/2)$ in every interval of the form

$$(-(2n-1)^2,-(2n-1/2)^2),\quad(-(2n-1/2)^2,-(2n)^2),\quad n\in\mathbb{N}.$$

Thus, the convergence of the trace formula

$$-K=Ke^{2ix_0}\Big|_{x_0=\pi/2}=E_0+\sum_{j=1}^{\infty}\left(E_{2j-1}+E_{2j}-2\mu_j(\pi/2)\right)\tag{6.11}$$

implies that there is exactly one eigenvalue $\mu_j(\pi/2)$ in each interval above, and these are all the $\mu_j(\pi/2)$.

Since $J_{-n}(i\sqrt{K}) = (-1)^n J_n(i\sqrt{K})$, $n \in \mathbb{Z}$, and since by (6.9), $f(n) > 0$, $n \in \mathbb{N}_0$, we infer from Lemma 5.1 (4) that

$$\begin{aligned} f(-n) &= (i\sqrt{K}/2)^n J_{-n}(i\sqrt{K}) \\ &= (-1)^n (i\sqrt{K}/2)^{2n} f(n) > 0, \quad n \in \mathbb{N}_0. \end{aligned} \tag{6.12}$$

Next, we show that $f(-2n+1/2) < 0$, $n \in \mathbb{N}$. For every $0 \leq m \leq 2n-2$, repeated use of the formula $\Gamma(\lambda) = \frac{1}{\lambda}\Gamma(\lambda+1)$, implies

$$\Gamma(m-2n+3/2) = \frac{(-2)^{2n-m-1}\Gamma(1/2)}{\prod_{j=1}^{2n-m-1}(4n-2m-(2j+1))}. \tag{6.13}$$

Thus, from (6.9),

$$\begin{aligned} f(-2n+1/2) &= \sum_{m=0}^{\infty} \frac{\left(\frac{\sqrt{K}}{2}\right)^{2m}}{m!\Gamma(m-2n+1/2+1)} \\ &= \sum_{m=0}^{2n-2} (-1)^{m+1} \frac{\prod_{j=1}^{2n-m-1}(4n-2m-(2j+1))}{2^{2n-1} m!\Gamma(1/2)} \left(\frac{K}{2}\right)^m \\ &\quad + \sum_{m=2n-1}^{\infty} \frac{\left(\frac{\sqrt{K}}{2}\right)^{2m}}{m!\Gamma(m-2n+1/2+1)}. \end{aligned} \tag{6.14}$$

Moreover,

$$\sum_{m=2n-1}^{\infty} \frac{\left(\frac{\sqrt{K}}{2}\right)^{2m}}{m!\Gamma(m-2n+1/2+1)} < \frac{\left(\frac{K}{4}\right)^{2n-1} e^{K/4}}{(2n-1)!\sqrt{\pi}}, \quad n \in \mathbb{N}, \tag{6.15}$$

and hence, if $0 < K \leq 1$,

$$\begin{aligned} f(-2n+1/2) &< -\frac{\left(\frac{K}{2}\right)^{2n-2}}{2^{2n-1}(2n-2)!\sqrt{\pi}} + \frac{\left(\frac{K}{4}\right)^{2n-1} e^{K/4}}{(2n-1)!\sqrt{\pi}} \\ &= -\frac{\left(\frac{K}{2}\right)^{2n-2}}{2^{2n-1}\sqrt{\pi}(2n-2)!} \left[1 - \frac{\left(\frac{K}{2}\right) e^{K/4}}{(2n-1)}\right] < 0. \end{aligned} \tag{6.16}$$

Here we used the fact that the sign of an alternating sum with terms of decreasing magnitudes agrees with the sign of the first term.

In order to prove the first part of the theorem, we note that (6.12) holds for every $K > 0$, while (6.16) may not hold for some $K > 1$. Next, for each $K > 1$ and $\frac{1}{K} \leq \varepsilon \leq 1$ we introduce the family of potentials $V(x) = \varepsilon K e^{2ix}$. Again, we use the notation $\mu_j(\varepsilon K, 0)$ to indicate the

ε-dependence of these eigenvalues. Then, for each $j \in \mathbb{N}$, the function $\varepsilon \mapsto \mu_j(\varepsilon K, 0)$ is continuous since $J_\nu(u)$ is an entire function of ν for each fixed $u \neq 0$, and an analytic function of u on the positive real axis.

When $\varepsilon = \frac{1}{K}$, that is, when $V(x) = e^{2ix}$, we proved above that

$$(2j-1)^2 < \mu_{2j-1}(1, \pi/2) \leq \mu_{2j}(1, \pi/2) < (2j)^2, \quad j \in \mathbb{N}.$$

So as ε increases to 1, the real numbers $\mu_j(\varepsilon K, \pi/2)$ cannot become $(2j-1)^2$ or $(2j)^2$ due to (6.12) which holds for all $K > 0$. This completes the proof. $\square$

Next, we investigate $\nu_k(\pi/2)$ for $k \in \mathbb{N}_0$.

Theorem 6.5. *Suppose that $V(x) = Ke^{2ix}$ and $K > 0$. Then*

$$(2k)^2 < \nu_{2k}(\pi/2) \leq \nu_{2k+1}(\pi/2) < (2k+1)^2, \quad k \in \mathbb{N}_0,$$

and hence $\nu_k(\pi/2) \neq E_m$, $k \in \mathbb{N}_0$.

Moreover, if $0 < K \leq \frac{1}{2}$, then

$$(2k)^2 < \nu_{2k}(\pi/2) < (2k + \frac{1}{2})^2 < \mu_{2k+1}(\pi/2) < (2k+1)^2, \quad k \in \mathbb{N}_0.$$

Proof. One can follow the arguments in the proof of Theorem 6.4. By (5.7) and (6.3), $\nu_k(\pi/2)$ are values of λ for which

$$c'(\lambda, \pi/2, 3\pi/2) = i\pi J'_{-\sqrt{\lambda}}(i\sqrt{K}) J'_{\sqrt{\lambda}}(i\sqrt{K}) = 0.$$

If $\lambda \geq 0$, then $J'_{\sqrt{\lambda}}(i\sqrt{K}) \neq 0$ and hence the zeros of $c'(\lambda, \pi/2, 3\pi/2)$ agree with those of $J'_{-\sqrt{\lambda}}(i\sqrt{K})$ and hence they agree with those of $\left(\frac{i\sqrt{K}}{2}\right)^{\sqrt{\lambda}+1} J'_{-\sqrt{\lambda}}(i\sqrt{K})$. Next, define

$$g(\alpha) = \left(\frac{i\sqrt{K}}{2}\right)^{-\alpha+1} J'_\alpha(i\sqrt{K}), \quad \alpha \in \mathbb{R}.$$

Then one can show that for each $K > 0$,

$$g(n) > 0, \quad n \in \mathbb{Z}. \tag{6.17}$$

Here the identity

$$J'_\nu(u) = \frac{\nu}{u} J_\nu(u) - J_{\nu+1}(u)$$

(see [23, pp. 45]) turns out to be useful.

Moreover, one can show that if $0 < K \leq \frac{1}{2}$, then

$$g(-2n - 1/2) < 0, \quad n \in \mathbb{N}_0.$$

Thus, if $0 < K \leq \frac{1}{2}$, then $g(\alpha)$ has at least one zero in each interval of the form $(-2n-1, -2n-1/2)$ and $(-2n-1/2, -2n)$, $n \in \mathbb{N}_0$. Using the trace formula (3.2) one can show that there are no other zeros and hence this proves the second part of the theorem.

The proof of the first part of the theorem is analogous to that of Theorem 6.4. $\square$

Theorem 6.6. *Suppose that $V(x) = Ke^{2ix}$ and $K > 0$. Then for every $x \in (0, \pi)$, $\mu_j(x)$, $\nu_k(x) \neq E_m$.*

Proof. From (6.3) one knows that $\mu_j(x)$ are zeros of

$$s(\lambda, x, x+\pi) = \pi J_{\sqrt{\lambda}}(e^{ix}\sqrt{K}) J_{-\sqrt{\lambda}}(e^{ix}\sqrt{K}), \tag{6.18}$$

and that $\nu_k(x)$ are zeros of

$$c'(\lambda, x, x+\pi) = \pi\sqrt{K}e^{ix} J'_{\sqrt{\lambda}}(e^{ix}\sqrt{K}) J'_{-\sqrt{\lambda}}(e^{ix}\sqrt{K}). \tag{6.19}$$

Since all zeros of $J_n(u)$ and $J'_n(u)$ are real for $n \geq 0$, we see that if $0 < x < \pi$, then $\mu_j(x)$, $\nu_k(x) \neq n^2$, $n \in \mathbb{N}_0$, because e^{ix} is non-real. $\square$

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