Analyticity of the SRB measure of a lattice of coupled Anosov diffeomorphisms of the torus

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Abstract

We consider the "thermodynamic limit" of a d-dimensional lattice of hyperbolic dynamical systems on the 2-torus, interacting via weak and nearest neighbor coupling. We prove that the SRB measure is analytic in the strength of the coupling. The proof is based on symbolic dynamics techniques that allow us to map the SRB measure into a Gibbs measure for a spin system on a $(d+1)$-dimensional lattice. This Gibbs measure can be studied by an extension (decimation) of the usual "cluster expansion" techniques.

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1 Introduction and Main Results

In recent years a lot of attention has been devoted to the relation between non equilibrium statistical mechanics and dynamical systems theory. According to the point of view of Ruelle, Cohen and Gallavotti [Ru78] [GC95] a mechanical system evolving in a steady state can be described by a hyperbolic dynamical system and its properties can be deduced from the "natural" or SRB distribution (see below for a precise definition) associated with this dynamical system. This line of investigation has already produced several interesting results both analytical, like the "Fluctuation Theorem", see [GC95], or numerical, like the works of Evans and Morris, see [EM90], and Moran and Hoover, see [MH87]. Nonetheless, most of the work has been devoted to low dimensional dynamical system, due to their accessibility both to analytical and to numerical study. In this paper we want to study the properties of the SRB distribution for a class of simple systems in very high dimension. For more references on this kind of systems see [JP98]. The precise model model we study here is taken taken from [BKL03].

We start considering a linear hyperbolic automorphism of the 2-torus $\mathbb{T}^2$. To be definite, we will always consider the so called *Arnold cat map* $s_0 : \mathbb{T}^2 \to \mathbb{T}^2$ defined by the action modulus 2π of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \tag{1}$$

Note that the matrix A admits two orthogonal eigenvectors $v_\pm$ whose respective eigenvalues $\lambda_\pm$ are such that $\lambda_+ > 1 > \lambda_-$ and $\lambda_+\lambda_- = 1$. For this reason the dynamical system s_0 is uniformly hyperbolic and the stable and unstable manifolds at any point $\phi \in \mathbb{T}^2$ are given by $W^\pm_\phi(t) = \phi + v_\pm t \text{ mod} 2\pi$.

¿From s_0 we can construct the *uncoupled* lattice dynamics by considering as phase space the Cartesian product $\mathcal{T} = (\mathbb{T}^2)^{\mathbb{Z}^d}$ (namely any point $\psi \in \mathcal{T}$ has Cartesian components $\{\psi_\xi\}_{\xi\in\mathbb{Z}^d}$), equipped with the metric $d(\psi,\psi') = \sum_\xi 2^{-|\xi|}\hat{d}(\psi_\xi,\psi'_\xi)$ where $\hat{d}(\psi_\xi,\psi'_\xi)$ is the usual metric on $\mathbb{T}^2$ and $|\xi| = \sum_{i=1}^d |\xi_i|$ for $\xi \in \mathbb{Z}^d$. On $\mathcal{T}$ the map S_0 acts simply as

$$S_0^\xi(\psi) = \big(S_0(\psi)\big)_\xi = s_0(\psi_\xi). \tag{2}$$

Note that the stable and unstable manifold of S_0 at a point ψ are the Cartesian product of the stable and unstable manifold of s_0 for the points $\psi_\xi \in \mathbb{T}^2$, *i.e.* $W^\pm_{0,\psi}(\zeta) = \psi + \sum_\xi w^{(\xi)}_{0,\pm}\zeta_\xi$ mod 2π, where $w^{(\xi)}_{0,\pm}$ is the tangent vector to $\mathcal{T}$ that has null component on the tangent space to every $\mathbb{T}^2_\eta$ but for $\mathbb{T}^2_\xi$ where it coincides with $v_\pm$. The action of S_0 on $W^\pm(\psi)$ is naturally given by a diagonal linear transformation.

We observe that the special choice of the matrix A plays no role in the following. Indeed we will show in Appendix A that our results stay true if we replace s_0 with any uniformly hyperbolic analytic automorphism of $\mathbb{T}^2$, not necessarily linear.

To add a coupling to this system we consider an analytic function $g : \mathcal{T} \to \mathbb{T}^2$ and define

$$S^\xi_\varepsilon(\psi) \stackrel{def}{=} s_0(\psi_\xi) + \varepsilon g(\rho^\xi\psi) \stackrel{def}{=} s_0(\psi_\xi) + \varepsilon f^\xi(\psi) \tag{3}$$

where $(\rho^\xi\psi)_\eta = \psi_{\eta+\xi}$, *i.e.* ρ is the group of the translations on $\mathbb{Z}^d$. This means that the function $f : \mathcal{T} \to \mathcal{T}$, whose ξ component is $f^\xi = g \circ \rho^\xi$, is translation invariant. We want f to be short ranged: let the nearest neighbor sites of the site ξ be $nn(\xi) = \{\eta \;:\; |\xi - \eta| \le 1\}$, we will assume that g depended only on $\psi_{nn(0)}$, where we have used the notation $\psi_V = \{\psi_\xi | \xi \in V\}$ for $V \subset \mathbb{Z}^d$. This implies that $S^\xi_\varepsilon(\psi)$ depends only on $\psi_{nn(\xi)}$. More generally we could have assumed that g depends only on ψ_V where V is any finite subset of $\mathbb{Z}^d$ containing 0 but this would not have changed the substance of the following arguments. Moreover we will take g analytic in all its arguments.

The dynamical system S_ε admits many invariant measures. Among them is the "natural" or SRB measure defined as the weak limit of the volume measure on $\mathcal{T}$ under the evolution defined by S_ε, when such a limit exists and is ergodic. Being $\mathcal{T}$ infinite dimensional, to properly define this concept we will consider finite dimensional approximations. Let $\mathcal{T}_N = (\mathbb{T}^2)^{V_N}$ where V_N is the cube of side $2N+1$ in $\mathbb{Z}^d$ centered at the origin. To define the restriction of S_ε to $\mathcal{T}_N$ we have to fix the boundary conditions: we choose periodic ones. To this extent note that $\mathcal{T}_N$ can be naturally identified with the submanifold of $\mathcal{T}$ formed by the points periodic of period $2N+1$. Moreover S_ε leaves such a manifold invariant so that we can define $S_{\varepsilon,N} \stackrel{def}{=} S_\varepsilon\big|_{\mathcal{T}_N}$. If no confusion can arise we will suppress the index N.

We can now define the SRB measure for $S_{\varepsilon,N}$ as $\mu_N^{SRB} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1}(S^*_{\varepsilon,N})^t \mu_N^0$ where the limit must be understood as a weak limit and $\mu_N^0 = \prod_{\xi\in V_N} d\psi_\xi/(2\pi)^2$ is the Lebesgue measure on $\mathcal{T}_N$. The existence of such a measure follows from rather general theorem on hyperbolic dynamical systems, if ε is sufficiently small (see for example [GBG03] and references there). Moreover μ_N^{SRB} is ergodic, always for small ε, and we have that $\mu_N^{SRB}(\mathcal{O}) = \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}\mathcal{O}(S_\varepsilon^t(\psi))$ for μ_N^0 almost every ψ, where $\mathcal{O}$ is an *observable*, *i.e.* a Holdër continuous function from $\mathcal{T}_N$ to $\mathbb{R}$. This means that μ_N^{SRB} is the *statistic* of S_ε. It is well known that the SRB measure is still well defined in the limit $N\to\infty$, for ε small enough. This was first proved by Bunimovich and Sinai in [BS88]. Starting from this work, the model eq.(1.3) (or similar models of coupled expanding automorphisms of the circle) has been widely studied in the literature, see for instance [PS91],[BK95],[BK96],[BK97] [JM95], [JP98], [BEIJK98], [JdL00]. Many properties of such systems are well known, mainly uniqueness of the SRB state in the thermodynamic limit and exponential decay of correlations (see [BK95],[BK96],[BK97] for a proof of these properties).

We further investigate the regularity properties of the limiting measure. We show that μ_N^{SRB} depends analytically on ε. This means that if we consider an analytic observable $\mathcal{O}$, *i.e.* an analytic function from $\mathcal{T}_N$ to $\mathbb{R}$, we have that $\mu_N^{SRB}(\mathcal{O})$ is an analytic function in a domain that depends on the analyticity properties of $\mathcal{O}$.

The main point of this work is to show that such a property remains true when $N\to\infty$, *i.e.* we want to show that the domain of analyticity of μ_N^{SRB} does not shrink to 0 when $N\to\infty$. More precisely we say that $\mathcal{O}:\mathcal{T}\to\mathbb{R}$ is a *local observable* if it depends only on ψ_V for some finite $V\subset\mathbb{Z}^d$. We can summarize our main results as follows

Theorem: *Given S_ε as above and a local observable $\mathcal{O}$ we have*

(1) $\mu^{SRB}(\mathcal{O}) = \lim_{N\to\infty}\mu_N^{SRB}(\mathcal{O})$ exists uniformly in ε for $\varepsilon<\varepsilon_0$ independent on $\mathcal{O}$.

(2) if $\mathcal{O}$ is local and analytic then $\mu^{SRB}(\mathcal{O})$ is analytic in ε for $\varepsilon<\varepsilon_0(\mathcal{O})$.

The proof is mainly based on the possibility of mapping the SRB distribution into the Gibbs state of a suitable spin system on $\mathbb{Z}^{d+1}$ and on the extension of classical techniques used to study such Gibbs states (*i.e.* cluster expansion) to the particular ones that occur in our system. The key point in order to get analyticity of the measure is proving that the SRB potentials (*i.e.* the potentials of the Gibbs state the SRB measure is mapped into) are rapidly decaying. Once this decay is proved, analyticity follows via standard techniques. Analyticity of the measure and convergence of cluster expansion imply in particular uniqueness of the Gibbs measure and exponential decay in space and time of the correlations of Hölder continuous observables, see for instance [GBG03]. Our proof can also be adapted to the case of coupled analytic expanding circle map: in fact also these models can be mapped into spin systems, and proceeding as below one can prove that the SRB potentials satisfy the same decaying properties.

The rest of the paper paper is organized as follows. In section 2 we give a brief review of the main properties of smooth uniformly hyperbolic systems and we briefly describe the construction that allows the above quoted mapping. The detailed proofs of this properties are postponed to sections 3, 4 and 5. Finally in section 6 we complete the proof of the main theorem. Appendix A contain a direct extension of our results to the case in which the uncoupled dynamics is not linear. Appendix B deals with an application. In the contest of the physical application of dynamical systems (see the beginning of this introduction) a special status has been given to a particular observable, the *phase space contraction rate* defined as $\eta_+(\psi) = \log|\det(DS_\varepsilon(\psi))|$ where DS_ε is the differential of S_ε. Being our system infinite it is more interesting to study the *local* phase space contraction rate $\eta_V(\psi)$ defined by taking the determinant of a (large) minor of DS_ε. We show, for a large class of couplings f, that η_V has a positive average and that it obeys a large deviation principle, *i.e.* its large deviation are asymptotically described by a free energy functional.

2 Anosov Systems

2.1 Geometric Properties

A dynamical system on a smooth compact manifold, whose dynamics is given by a uniformly hyperbolic invertible map is called an *Anosov system.* ¿From the general theory we know that Anosov systems are structurally stable, namely, given two Anosov diffeomorphisms S, S' on a manifold Ω that are sufficiently close in the C^2 topology, there exist a *conjugation* $H : \Omega \longleftrightarrow \Omega$ such that $S \circ H = H \circ S'$.

In our situation this implies the existence of a map $h_\varepsilon : \mathcal{T}_N \longleftrightarrow \mathcal{T}_N$ such that

$$S_\epsilon \circ h_\epsilon = h_\epsilon \circ S_0 \ , \tag{1}$$

at least if ε is small enough (*a priori* not uniformly in N). The first step of our proof consist in showing that h_ε is analytic in ε uniformly in N. More precisely we will construct h_ε directly for $N = \infty$. Its finite N version can be obtained by restricting it to $\mathcal{T}_N$. We note that h_ε is, in general, only Holdër continuous in the variable ψ. By this we mean that there exist constant c and β such that $d(h_\varepsilon(\psi), h_\varepsilon(\psi')) \leq c d(\psi, \psi')^\beta$. For this reason we cannot say that the SRB measure of S_ε is just the image under the map h_ε of the SRB measure for S_0, *i.e.* of the Lebesgue measure on $\mathbb{T}^2$. Notwithstanding h_ε will play a crucial role in the construction on μ_N^{SRB}.

As we saw in the introduction the tangent space $\mathrm{T}_\psi \mathcal{T}$ to $\mathcal{T}$ on a point ψ can be split in two subspaces E_ψ^+ and E_ψ^- such that $\mathrm{T}_\psi \mathcal{T} = E_\psi^+ \oplus E_\psi^-$. Moreover the distributions $E^\pm$ are continuous and invariant under S_0, *i.e.* $\left(DS_0\ E_\psi^\pm\right) = E_{S_0\psi}^+$ and we have

$$\begin{aligned} \|DS_0^n w\| &\leq C\lambda_-^n \|w\| \qquad \text{for } w \in E_\psi^- \\ \|DS_0^{-n} w\| &\leq C\lambda_+^{-n} \|w\| \qquad \text{for } w \in E_\psi^+ \end{aligned} \tag{2}$$

E_ψ^+ and E_ψ^- are called the *stable* and *unstable* subspaces, respectively. In the case of S_0 all these properties are trivially true. In particular we can consider on $E_\psi^\pm$ the basis formed by the vectors $\{w_{0,\pm}^{(\xi)}\}_{\xi \in \mathbb{Z}^d}$.

We will show in section 4 that such a splitting can be constructed also for S_ε, again uniformly in N, *i.e.* we will prove the existence of the stable and unstable subspaces $E_{\varepsilon,\psi}^\pm$ for S_ε. Moreover we will show that $E_{\varepsilon,h_\varepsilon(\psi)}^\pm$ is an analytic function of ε, although is only Holdër continuous in ψ. This will turn out to be the right regularity to study the SRB measure. To do this we will directly construct the vectors of the basis $\{w_{\varepsilon,\pm}^{(\xi)}(\psi)\}_{\xi \in \mathbb{Z}^d}$ that coincide with $\{w_{0,\pm}^{(\xi)}\}_{\xi \in \mathbb{Z}^d}$ for $\varepsilon = 0$.

2.2 Symbolic Dynamics

The main property that allow to study analytically the SRB measure for an Anosov map S acting on a manifold $\mathcal{M}$ is the existence of Markov partitions. We call a collection $\mathcal{Q}_i$, $i = 1, \ldots, n$, of closed subsets of $\mathcal{M}$ a *partition* if $\cup_i \mathcal{Q}_i = \mathcal{M}$ and $\mathcal{Q}_i \cap \mathcal{Q}_j = \partial\mathcal{Q}_i \cap \partial\mathcal{Q}_j$ for every $i \neq j$. For every sequence $\sigma = \{\sigma_t\}_{t \in \mathbb{Z}} \in \{1, \ldots, n\}^\mathbb{Z}$ we can define the set $\mathcal{X}(\sigma) = \cap_{t=-\infty}^\infty S^t(\mathcal{Q}_{\sigma_t})$. Due to the hyperbolicity properties of S if $\mathcal{Q}_i$ are small enough $\mathcal{X}(\sigma)$ contains at most one point. This allow us to construct a *symbolic dynamics i.e.* a map from a subset Σ of $\{1, \ldots, n\}^\mathbb{Z}$ to $\mathcal{M}$. In general the structure of the subset Σ is very complex but for Anosov systems it is possible to construct particular partitions for which the set Σ can be described easily. Given a partition $\mathcal{Q}$ we call *compatibility matrix* the $n \times n$ matrix C given by $C_{ij} = 1$ if $\mathrm{int}(S\mathcal{Q}_i) \cap \mathrm{int}(\mathcal{Q}_j) \neq \emptyset$ and 0 otherwise. We say that $\mathcal{Q}$ is a *Markov partition* if the set Σ is formed by the sequences σ such that $C_{\sigma_i,\sigma_{i+1}} = 1$ for every $i \in \mathbb{Z}$. This means that the sequences that satisfy the above nearest neighbor condition code all the points of $\mathcal{M}$. In such a case we will denote $\Sigma = \{1, \ldots, n\}_C^\mathbb{Z}$.

We now show how to construct a Markov partition for our model. We start with s_0. A Markov partition $Q = \{Q_i, i = 1, \ldots, n\}$ for s_0 acting on $\mathbb{T}^2$ can be easily constructed starting from its stable and unstable manifolds. Such a construction is standard and can be found, *e.g.* , in [GBG03]. Let C be its compatibility matrix and $\hat{c}_0$ the associated symbolic dynamics.

It is important to note that $\hat{c}_0$ is Holdër continuous in the sense that there exist constants c and β such that, for any two sequences $\sigma, \sigma' \in \{1, \ldots, n\}_C^{\mathbb{Z}}$, $d(\hat{c}_0(\sigma), \hat{c}_0(\sigma')) \leq c\tilde{d}(\sigma, \sigma')^\beta$, with $\tilde{d}(\sigma, \sigma') = e^{-\#(\sigma,\sigma')}$ where $\#(\sigma, \sigma')$ the biggest integer such that $\sigma_j = \sigma'_j$, $\forall |j| \leq \#(\sigma, \sigma')$. In this case we can take $\beta = \ln(\lambda_+)$. Another key property is that C is a mixing matrix, this means that there exists a *decorrelation time* $a \in \mathbb{N}$ such that C^a has all entries strictly positive. This means that we can connect any two element of the Markov partition in a time steps.

For every point $s = \{s_\xi\}_{\xi\in\mathbb{Z}^d} \in \{1, \ldots, n\}^{\mathbb{Z}^d}$ we can consider the Cartesian product $\mathcal{Q}_s = \times_{\xi\in\mathbb{Z}^d} Q_{s_\xi} \subset \mathcal{T}_N$. It is clear that the collection of $\mathcal{Q}_s$ with $s \in \{1, \ldots, n\}^{\mathbb{Z}^d}$ forms a Markov partition for S_0. Note that it is natural to index the element of this partition with the element of $\{1, \ldots, n\}^{\mathbb{Z}^d}$ so that we can associate to this partition the symbolic dynamics $c_0 : \mathbb{Z}^d \times \mathbb{Z} = \mathbb{Z}^{d+1} \to \mathcal{T}_N$ naturally defined from $\hat{c}_0$. We can still call C the compatibility matrix and $\{1, \ldots, n\}_C^{\mathbb{Z}^{d+1}}$ the set of possible sequences (namely $\sigma = \{\sigma_{\xi,i}\}_{\xi\in\mathbb{Z}^d, i\in\mathbb{Z}}$ is in $\{1, \ldots, n\}_C^{\mathbb{Z}^{d+1}}$ if and only if $C_{\sigma_{\xi,i}\sigma_{\xi,i+1}} = 1$ for every $\xi \in \mathbb{Z}^d$ and $i \in \mathbb{Z}$). Given any point $(\xi, i) \in \mathbb{Z}^{d+1}$ we will call ξ its *space component* and i its *time component*.

The key observation is now that the sets $h_\varepsilon(\mathcal{Q}_s)$ form a Markov partition for S_ε. This implies that the space of symbolic sequences for S_ε is the same as that for S_0 and that the symbolic dynamics c_ε for S_ε is given by $c_\varepsilon(\sigma) = h_\varepsilon(c_0(\sigma))$. Clearly c_ε is still Hölder continuous. This completes the construction of the Markov partition for S_ε. We thus obtained that the manifold $\mathcal{T}$ can be mapped to $\{1, \ldots, n\}_C^{\mathbb{Z}^{d+1}}$ where d directions of the lattice $\mathbb{Z}^{d+1}$ represent the d directions of $\mathcal{T} = (\mathbb{T}^2)^{\mathbb{Z}^d}$ and the last represents the time evolution. Indeed the map S_ε on the space $\{1, \ldots, n\}_C^{\mathbb{Z}^{d+1}}$ becomes the shift on the time direction, to be called τ.

2.3 SRB measure

Let now consider the SRB measure μ_N^{SRB} as defined in section 1. In this case we need to keep N finite because it is not easy to give a meaning or construct directly the SRB measure for $N = \infty$.

Let $m_N^{\mathtt{SRB}}$ be the measure on $\{1, \ldots, n\}_C^{V_N\times\mathbb{Z}}$ defined as $m_N^{\mathtt{SRB}}(A) = \mu_N^{\mathtt{SRB}}(c_\varepsilon^{-1}(A))$, *i.e.* $m_N^{\mathtt{SRB}}$ is the image of $\mu_N^{\mathtt{SRB}}$ via symbolic dynamics c_ε. The measure $m_N^{\mathtt{SRB}}$ can be described efficiently through its restrictions to finite subsets of $V_N \times \mathbb{Z}$.

Given $\Lambda \subset V_N \times \mathbb{Z}$, $m_N^{\mathtt{SRB}}(\sigma_\Lambda|\sigma_{\Lambda^c})$ will denote the probability of the event $\{\sigma'|\sigma'_\Lambda = \sigma_\Lambda\}$ conditional to the event $\{\sigma'|\sigma'_{\Lambda^c} = \sigma_{\Lambda^c}\}$ w.r.t. to the probability measure $m_N^{\mathtt{SRB}}$, where $\Lambda^c = (V_N \times \mathbb{Z}) \setminus \Lambda$ and σ_Λ is the collection of the $\sigma_{\xi,i}$ for $(\xi, i) \in \Lambda$.

¿From the theory of SRB measures, see [Si], [GBG03], it follows that $m_N^{\mathtt{SRB}}$ is a Gibbs measure and its conditional probabilities satisfy:

$$\frac{m_N^{\mathtt{SRB}}(\sigma'_\Lambda|\sigma_{\Lambda^c})}{m_N^{\mathtt{SRB}}(\sigma''_\Lambda|\sigma_{\Lambda^c})} = \lim_{K\to\infty} \left[\frac{\mathcal{D}_\varepsilon^{u(2K)}\big(c_\varepsilon(\tau^{-K}\sigma')\big)}{\mathcal{D}_\varepsilon^{u(2K)}\big(c_\varepsilon(\tau^{-K}\sigma'')\big)} \right]^{-1} , \tag{3}$$

where σ' (resp. σ'') is the configuration coinciding with σ'_Λ (resp. σ''_Λ) on Λ and with σ_{Λ^c} on Λ^c; τ is the image of S_ε through c_ε (*i.e.* it is the one step shift in time direction); $\mathcal{D}_\varepsilon^{u(n)}(\psi)$ measures the expansion of the volume on the unstable manifold at the point ψ. To be more precise let $\{w_{\varepsilon,+}^{(\xi)}(\psi)\}_{\xi\in V_N}$ be a basis on $E_{\varepsilon,\psi}^+$. We will construct one such a basis in section 4. Then we have

$$\mathcal{D}_\varepsilon^{u(n)}(\psi) \stackrel{def}{=} \sqrt{\frac{\det_{\xi\eta}\left[(DS_\varepsilon^n\ w_{\varepsilon,+}^{(\xi)}) \cdot (DS_\varepsilon^n\ w_{\varepsilon,+}^{(\eta)})\right]}{\det_{\xi\eta}\left[w_{\varepsilon,+}^{(\xi)} \cdot w_{\varepsilon,+}^{(\eta)}\right]}}(\psi) , \tag{4}$$

where $u \cdot v$ represent the usual scalar product in $\mathbb{R}^{V_N}$ and $\det_{\xi\eta}$ is the determinant of the expression in square brackets thought as a matrix indexed by ξ and η.

Using the invariance of E_ε^+ under S_ε and introducing the *unstable Lyapunov matrix* $\mathcal{L}(\psi)$ satisfying the equation

$$DS_\varepsilon(\psi)w_{\varepsilon,+}^{(\xi)}(\psi) = \sum_\eta w_{\varepsilon,+}^{(\eta)}\big(S_\varepsilon(\psi)\big)\mathcal{L}^{\eta\xi}(\psi),$$

we can rewrite the above expression as

$$\mathcal{D}^{u(2K)}_{\varepsilon}(S_{\varepsilon}^{-K}(\psi)) = \frac{\sqrt{\det_{\xi\eta}\left(w^{(\xi)}_{\varepsilon,+}\cdot w^{(\eta)}_{\varepsilon,+}\right)(S_{\varepsilon}^{K}(\psi))}}{\sqrt{\det_{\xi\eta}\left(w^{(\xi)}_{\varepsilon,+}\cdot w^{(\eta)}_{\varepsilon,+}\right)(S_{\varepsilon}^{-K}(\psi))}} \prod_{j=-K}^{K-1} \left|\det_{\xi\eta}\left[\mathcal{L}^{\xi\eta}(S_{\varepsilon}^{j}\psi)\right]\right| . \tag{5}$$

Now the first ratio in eq.(2.5), when inserted in eq.(2.4), is vanishing ; indeed the uniform Hölder continuity of $w^{(\xi)}_{\varepsilon,+}(h_{\varepsilon}(\psi))$ and the fact that σ' and σ'' are asymptotically identical in the past and in the future implies that:

$$\lim_{K\to\pm\infty}\left(\ln\sqrt{\det_{\xi\eta}(w^{(\xi)}_{\varepsilon,+}\cdot w^{(\eta)}_{\varepsilon,+})}\right)(c_{\varepsilon}(\tau^{K}\sigma')) - \left(\ln\sqrt{\det_{\xi\eta}(w^{(\xi)}_{\varepsilon,+}\cdot w^{(\eta)}_{\varepsilon,+})}\right)(c_{\varepsilon}(\tau^{K}\sigma'')) = 0 \; ; \tag{6}$$

thus the choice of the basis in E^{+} does not change the result, namely the SRB measure does not depend on the choice of the metric as is to be expected from its definition. Calling $\Lambda^{\xi}(\psi)\overset{def}{=}\Big(\ln\mathcal{L}(h_{\varepsilon}(\psi))\Big)^{\xi\xi}$ we finally get:

$$\frac{m_N^{\mathbf{SRB}}(\sigma'_{\Lambda}|\sigma_{\Lambda^c})}{m_N^{\mathbf{SRB}}(\sigma''_{\Lambda}|\sigma_{\Lambda^c})} = \exp\left\{-\sum_{j=-\infty}^{+\infty}\sum_{\xi\in V_N}\left[\Lambda^{\xi}\big(c_0(\tau^{j}\sigma')\big) - \Lambda^{\xi}\big(c_0(\tau^{j}\sigma'')\big)\right]\right\} . \tag{7}$$

Here we used the the fact that $c_{\varepsilon} = h_{\varepsilon}\circ c_0$. Furthermore the Hölder continuity of $\Lambda^{\xi}\big(c_0(\sigma')\big)$ implies absolute convergence of the sum in eq. (2.7) because only points asymptotically equal both in the past and in the future are compared.

The crucial point of this construction is that the matrix $L(\psi)\overset{def}{=}\mathcal{L}\big(h_{\varepsilon}(\psi)\big)$ is analytic in ε due to the fact that depends only on $w^{(\xi)}_{\varepsilon,+}(h_{\varepsilon}(\psi))$. As we already noted $w^{(\xi)}_{\varepsilon,+}(h_{\varepsilon}(\psi))$ are analytic in ε. We will prove this fact in section 4.

In section 6 we will apply to eq. (2.7) the standard methods developed in the study of Gibbs measure in Statistical Mechanics. To do this we will need to decompose the "interaction" $\Lambda^{\xi}\big(c_0(\sigma)\big)$ as the sum of potentials depending only on $\sigma_X\overset{def}{=}\{\sigma_j\}_{j\in X}$ where X is a finite subset of $\mathbb{Z}^{d+1}$. More precisely we will decompose:

$$\sum_{(\xi,i)\in V_N\times\mathbb{Z}}\Lambda^{\xi}\big(c_0(\tau^{i}\sigma)\big) = \sum_{X\subset V_N\times\mathbb{Z}}\phi_X(\sigma_X); \tag{8}$$

(these two series are not convergent: they represent the formal expression for the "Hamiltonian" of a Gibbs measure. See section (5.2) for a more precise statement). We shall show that we can choose ϕ_X analytic in ε, translationally invariant in space and time directions and decaying exponentially in the *tree distance* of the set X, namely the length of the shortest tree connecting all the lattice points in X. In this way (2.7) can be written as

$$\frac{m_N^{\mathbf{SRB}}(\sigma'_{\Lambda}|\sigma_{\Lambda^c})}{m_N^{\mathbf{SRB}}(\sigma''_{\Lambda}|\sigma_{\Lambda^c})} = \exp\Big\{-\sum_{X\cap\Lambda\neq\emptyset}[\phi_X(\sigma'_X)-\phi_X(\sigma''_X)]\Big\} , \tag{9}$$

so that one can finally write:

$$m_N^{\mathbf{SRB}}(\sigma_{\Lambda}|\sigma_{\Lambda^c}) = \frac{\exp\big\{-\sum_{X\cap\Lambda\neq\emptyset}\phi_X(\sigma_X)\big\}}{\sum_{\sigma_\Lambda}\exp\big\{-\sum_{X\cap\Lambda\neq\emptyset}\phi_X(\sigma_X)\big\}} . \tag{10}$$

This will allow us to show our analyticity claim uniformly in N.

3 Perturbative Construction of the SRB measure

In this section we construct the conjugation h_{ε} and prove that it is analytic in ε. The technique we use consists in expanding h_{ε} as a power series in ε and writing a recursive relation linking the n–th order coefficient to the coefficients of order i with $i<n$. This naturally leads to a tree expansion of the form usual in perturbation theory for Quantum Field Theory, *i.e.* the tree we will introduce are the "Feynmann graphs" of our theory. See also [GBG03] and reference therein for similar application to KAM theory.

3.1 The Conjugation

¿From now on we will identify functions from $\mathcal{T}_N$ to $\mathcal{T}_N$ with their lifts as functions from $\mathbb{R}^{2V_N}$ to $\mathbb{R}^{2V_N}$. Using the definition (2.1) and looking for h_ϵ of the form $h_\epsilon(\psi) = \psi + \delta h_\epsilon(\psi)$, we find:

$$\delta h_\varepsilon \circ S_0 - S_0 \circ \delta h_\varepsilon = \varepsilon f \circ (\mathrm{Id} + \delta h_\varepsilon) \, . \tag{1}$$

where Id is the identity map.

Setting $\lambda \stackrel{def}{=} \lambda_- = \lambda_+^{-1}$ and writing $f(\psi) \stackrel{def}{=} \sum_{\xi,\alpha} f^{\xi^\alpha}(\psi) w_{0,\alpha}^{(\xi)}$ and similarly for $\delta h_\varepsilon^{\xi^\pm}$, we get:

$$\begin{cases} \delta h_\varepsilon^{\xi^+}(S_0\psi) - \lambda^{-1}\delta h_\varepsilon^{\xi^+}(\psi) & = \varepsilon f^{\xi^+}\Big(\psi + \delta h_\varepsilon(\psi)\Big) \\ \delta h_\varepsilon^{\xi^-}(S_0\psi) - \lambda\delta h_\varepsilon^{\xi^-}(\psi) & = \varepsilon f^{\xi^-}\Big(\psi + \delta h_\varepsilon(\psi)\Big) \end{cases} \, . \tag{2}$$

Both equations can be implicitly solved by iteration:

$$\delta h_\varepsilon^{\xi^\alpha}(\psi) = -\alpha\varepsilon \sum_{p\geq 0} \lambda^{p+\rho_\alpha} f^{\xi^\alpha}\Big(S_0^{\alpha(p+1-\rho_\alpha)}\psi + \delta h_\varepsilon(S_0^{\alpha(p+1-\rho_\alpha)}\psi)\Big) \, , \tag{3}$$

where $\rho_\alpha = (1+\alpha)/2$.

It is easy to see that the series in eq.(3.3) is absolutely convergent, since $\lambda < 1$ and f is bounded. Expanding $f^{\xi^\alpha}\big(\psi + \delta h_\varepsilon(\psi)\big)$ in power of its argument we find:

$$f^{\mathbf{x}}\big(\psi + \delta h_\varepsilon(\psi)\big) = f^{\mathbf{x}}(\psi) + \sum_{k\geq 1} \varepsilon^k \sum_{s=1}^{k} \sum_{\substack{k_1,\ldots,k_s \\ k_j\geq 1}}^{k_1+\ldots+k_s=k} \Big(\frac{f^{\mathbf{x},\mathbf{x}_1\ldots\mathbf{x}_s}}{s!}\delta h_{(k_1)}^{\mathbf{x}_1}\cdots\delta h_{(k_s)}^{\mathbf{x}_s}\Big)(\psi) \, , \tag{4}$$

where we have introduced the index $\mathbf{x} = (\xi,\alpha)$, with $\alpha = \pm$, and $f^{\mathbf{x},\mathbf{x}_1\ldots\mathbf{x}_s} = \partial_{\mathbf{x}_1}\cdots\partial_{\mathbf{x}_s} f^{\mathbf{x}}$ with $\partial_{(\xi^\alpha)}$ the partial derivative in the direction of $w_{0,\alpha}^{(\xi)}$. Moreover we use the convention of summing on twice repeated indexes. The first order coefficient of the expansion of the conjugation is then:

$$\delta h_{(1)}^{\mathbf{x}}(\psi) = (-\alpha)\sum_{p\geq 0} \lambda^{p+\rho_\alpha} f^{\mathbf{x}}(S_0^{\alpha(p+1-\rho_\alpha)}\psi) \, , \tag{5}$$

while the k–th, $k > 1$, coefficient turns out to be:

$$\delta h_{(k)}^{\mathbf{x}}(\psi) = \sum_{s=1}^{k} \sum_{\substack{k_1,\ldots,k_s \\ k_j\geq 1}}^{k_1+\ldots k_s=k-1} (-\alpha)\sum_{p\geq 0}\lambda^{p+\rho_\alpha}\Big(\frac{f^{\mathbf{x},\mathbf{x}_1\ldots\mathbf{x}_s}}{s!}\delta h_{(k_1)}^{\mathbf{x}_1}\cdots\delta h_{(k_s)}^{\mathbf{x}_s}\Big)(S_0^{\alpha(p+1-\rho_\alpha)}\psi) \, . \tag{6}$$

¿From eq.(3.5) we see that δh_ε is in general non differentiable with respect to ψ. Indeed already differentiating $\delta h_{(1)}^{\xi^+}(\psi)$ with respect to ψ we find a non converging series. On the contrary it is clear that eq.(3.5) is Hölder continuous in ψ for every exponent $\beta < 1$.

We can interpret eq.(3.6) graphically as shown in Fig. 1.

The l.h.s. of the graphical equation in Figure 1 represents $\delta h_{(k)}^{\mathbf{x}}(\psi)$ while the r.h.s., representing the sum in eq. (3.6), is a "simple tree" consisting of a "root" r, a "root branch" $\lambda_v \equiv (r,v)$ coming from the "node" (or "vertex") v and s_v branches "entering v", to be called $\lambda_{v_i} \equiv (v,v_i)$, $i = 1,\ldots,s_v$.

Even if the drawing in the figure does not carry them explicitly, we imagine that some labels are affixed to the node v: more precisely $\mathbf{x}(v) = (\xi(v),\alpha(v)) \in V_N \times \{\pm\}$ and $p_v \in \mathbb{Z}_+$. Furthermore a label $\mathbf{x}_\lambda = (\xi_\lambda,\alpha_\lambda) \in V_N \times \{\pm\}$ is associated to each branch λ. In the figure above $\mathbf{x}_{\lambda_v} \equiv \mathbf{x}$ and $\mathbf{x}_{\lambda_{v_i}} \equiv \mathbf{x}_i$, $i = 1,\ldots,s$.

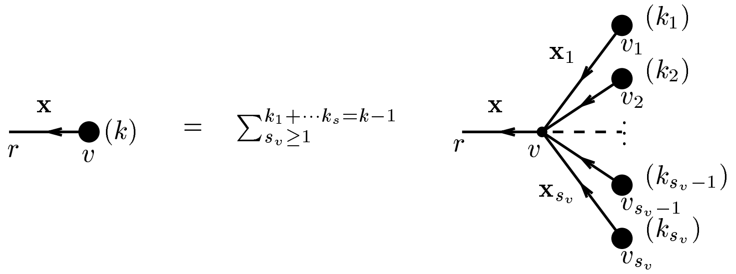

Figure 1: Graphical interpretation of (3.6).

The node v symbolizes the tensor with entries

$$N_{v;\mathbf{x},\mathbf{x}_1,\ldots,\mathbf{x}_s} \stackrel{def}{=} (-\alpha(v))\lambda^{p_v+\rho_{\alpha(v)}} \frac{f^{\mathbf{x},\mathbf{x}_1\ldots\mathbf{x}_{s_v}}}{s_v!}(S_0^{p(v)}\psi) \ , \tag{7}$$

where $p(v) = \alpha(v)(p_v + 1 - \rho_{\alpha(v)})$. Observe that, in order for eq. (3.7) and (3.6) to be non zero, we must have $|\xi_{\lambda_{v_i}} - \xi(v)| \leq 1$, due to our definition of the coupling f.

The line λ_v exiting vertex v symbolizes the *propagator*, that is simply $\delta_{\mathbf{x}_{\lambda_v},\mathbf{x}(v)}$.

The line with label $\mathbf{x}$ exiting from the bullet of the l.h.s. with label (k) represents $\delta h^{\mathbf{x}}_{(k)}(\psi)$; the branches with labels $\mathbf{x}_i$ exiting from the bullets of the r.h.s. with label (k_i) represent $\delta h^{\mathbf{x}_i}_{(k_i)}(S_0^{p(v)}\psi)$.

Even if it is not explicitly written in the figure above, a summation over the free indices $\mathbf{x}(v), \mathbf{x}_{\lambda_{v_i}}$ has to be performed (note that the summation over $\mathbf{x}(v)$ simply fixes $\mathbf{x}(v) = \mathbf{x}$, because of the presence of the propagator $\delta_{\mathbf{x}(v),\mathbf{x}}$).

Since eq. (3.6) is multilinear in $\delta h^{\mathbf{x}_i}_{k_i}$, we can just replace each of the branches exiting from a bullet with the same graphical expression in the r.h.s. of the above figure. And so on, until the labels (k_i) on the bullets (*top nodes*) become equal to 1. In this case the end–points represent $\delta h_{(1)}$, that is a known expression, see eq. (3.5), and we will draw these known end–points as small dots.

Thus we have represented our $\delta h^{\mathbf{x}}_{(k)}$ as a "sum over trees" with k branches and k nodes (we shall not regard the root as a node) of suitable *tree values*. In Fig. 2 we draw a typical tree θ we get via such procedure.

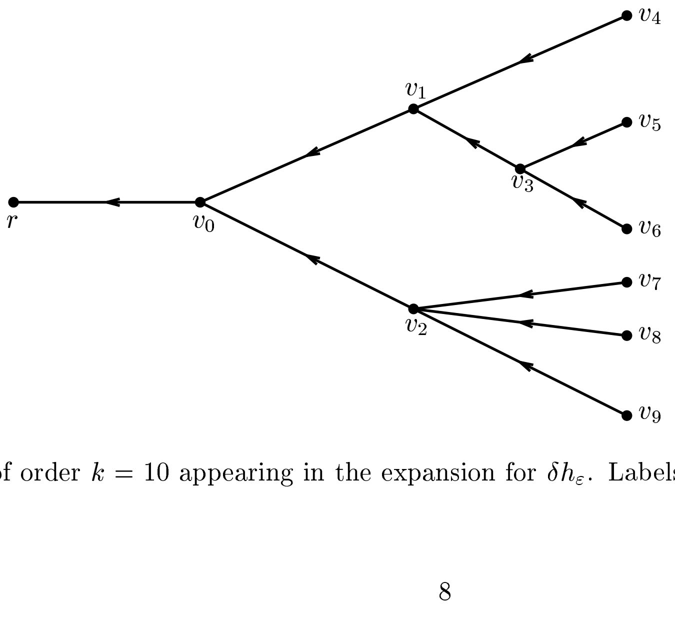

Figure 2: A tree θ of order $k = 10$ appearing in the expansion for δh_ε. Labels $\xi(v_i)$, $\alpha(v_i)$ and p_{v_i} are associated to all vertices v_i.

Note that a tree induces a partial ordering among its nodes: a node w precedes v (and it will be written $w < v$) if there is a path of branches connecting w and v with the arrows pointing from w to v.

Let us now summarize the discussion above. Let $\widehat{\mathcal{T}}_k(\mathbf{x})$ be the set of rooted trees with k branches and k nodes, with labels $\mathbf{x}(v), p_v$ attached to their vertices and $\mathbf{x}(v_0) \equiv \mathbf{x}$, where v_0 is the last vertex preceding the root. Given $\theta \in \widehat{\mathcal{T}}_k(\mathbf{x})$, let the value of θ be defined as

$$\widehat{\mathrm{Val}}(\theta, \psi) = \prod_{v\in\theta} \Big(- \alpha(v)\Big) \lambda^{p_v + \rho_{\alpha(v)}} \frac{f^{\mathbf{x}(v),\, \mathbf{x}(v_1)\ldots\, \mathbf{x}(v_{s_v})}}{s_v!} (S_0^{p(v)}\psi) \; , \tag{8}$$

where $v_1, \ldots, v_{s_v}$ are the nodes immediately preceding v and $p(v) = \sum_{w\geq v} \alpha(v)(p_v + 1 - \rho_{\alpha(v)})$. With these definitions $\delta h^{\mathbf{x}}_{(k)}(\psi)$ can be calculated as $\delta h^{\mathbf{x}}_{(k)}(\psi) = \sum_{\theta\in\widehat{\mathcal{T}}_k(\mathbf{x})} \widehat{\mathrm{Val}}(\theta, \psi)$.

3.2 Convergence and Regularity of the Perturbative Expansion of the Conjugation

By definition $g(\psi)$ depends only on $\psi_{nn(0)}$ so that it is analytic in $\mathcal{D} \stackrel{def}{=} \{\psi^i_\xi \in \mathbb{C} |\; |\mathrm{Im}\, \psi^i_\xi| \leq r_0,\; i = 1, 2,\; \xi \in nn(0)\}$ for some $r_0 > 0$. Calling G the maximum of g on $\mathcal{D}$, from Cauchy's formula we get

$$|f^{\mathbf{x},\, \mathbf{x}_1 \ldots \mathbf{x}_s}(\psi)| \leq G \frac{m_1! \cdots m_D!}{r_0^s} \leq G\frac{s!}{r_0^s}. \tag{9}$$

where $m_1, \ldots m_D$ are the multiplicities of the partial derivatives with respect to the $D \stackrel{def}{=} 2(2d+1) = 2|nn(0)|$ possible variables, (thus $m_1 + \ldots + m_D = s$).

In the same way, if ψ and ψ' are identical on each site but $\xi' \in nn(\xi)$ and if $0 < \beta \leq 1$ we get

$$|f^{\mathbf{x},\mathbf{x}_1 \ldots\, \mathbf{x}_s}(S_0^p\psi) - f^{\mathbf{x},\mathbf{x}_1\ldots\mathbf{x}_s}(S_0^p\psi')| \leq G\frac{(s+1)!}{r_0^{s+1}} (2\pi^2)^{\frac{1-\beta}{2}} \lambda^{-\beta p} |\psi_{\xi'} - \psi'_{\xi'}|^\beta \tag{10}$$

where we have used the periodicity of f. Next we bound the value of a tree $\theta \in \widehat{\mathcal{T}}_k(\mathbf{x})$. Using eq. (3.9), for $\theta \in \widehat{\mathcal{T}}_k(\mathbf{x})$, we find:

$$||\widehat{\mathrm{Val}}(\theta, \cdot)||_\infty \leq \prod_{v\in\theta} \lambda^{p_v} \frac{G}{r_0^{s_v}} = \frac{G^k}{r_0^{k-1}} \prod_{v\in\theta} \lambda^{p_v} \; , \tag{11}$$

where we used that, if $\theta \in \widehat{\mathcal{T}}_k(\mathbf{x})$, $\sum_{v\in\theta} s_v = k - 1$.

The sum over the trees can be interpreted as a sum over the topological trees and a sum over the labels attached to the trees. If Θ_k is the set of topological trees of order k, we get:

$$\begin{aligned} ||\delta h^{\mathbf{x}}_{(k)}||_\infty &\leq \sum_{\theta\in\Theta_k} \sum_{\substack{\mathbf{x}(v),\alpha(v)\\ v\in\theta}} \frac{G^k}{r_0^{k-1}} \sum_{\substack{p_v\\ v\in\theta}} \prod_{v\in\theta} \lambda^{p_v} = \sum_{\theta\in\Theta_k} \sum_{\substack{\xi(v),\alpha(v)\\ v\in\theta}} \frac{G^k}{r_0^{k-1}} (\frac{1}{1-\lambda})^k \leq \\ &\leq \sum_{\theta\in\Theta_k} 2^k (2d+1)^k \frac{G^k}{r_0^{k-1}} (\frac{1}{1-\lambda})^k \leq 2^{2k} 2^k (2d+1)^k \frac{G^k}{r_0^{k-1}} (\frac{1}{1-\lambda})^k \; , \end{aligned} \tag{12}$$

where we used that:

1. 2^k is the number of terms in the sum over the $\alpha(v)$ indices;

2. $(2d+1)^k$ is a bound on the number of terms in the sum over the values of the $\xi(v)$ indices not making $\widehat{\mathrm{Val}}(\theta, \psi)$ vanish (observe that, given a tree θ, its value $\widehat{\mathrm{Val}}(\theta, \psi)$ is vanishing unless $|\xi(v') - \xi(v)| \leq 1$, where v' is the node immediately preceding v);

3. 2^{2k} is a bound on the number of unlabeled rooted trees with k nodes.

In the same way we find that, if ψ and ψ' are identical on each site but ξ' and if $0<\beta<1$:

$$\frac{|\delta h^{\mathbf{x}}_{(k)}(\psi)-\delta h^{\mathbf{x}}_{(k)}(\psi')|}{|\psi_{\xi'}-\psi'_{\xi'}|^\beta} \le \sum_{\theta\in\Theta}\left(\frac{1}{1-\lambda^{1-\beta}}\right)^k (2\pi^2)^{\frac{1-\beta}{2}}\frac{G^k}{r_0^k}2^k(2d+1)^k\sum_{v\in\theta}(s_v+1)\le \tag{13}$$

$$\le 2^{2k}\left(\frac{1}{1-\lambda^{1-\beta}}\right)^k (2\pi^2)^{\frac{1-\beta}{2}}\frac{G^k}{r_0^k}2^k(2d+1)^k(2k-1)\ .$$

So the map $h_\varepsilon:\mathcal{T}_N\to\mathcal{T}_N$ exists, it is hölder continuous w.r.t. ψ and analytic w.r.t. ε in the complex disc $|\varepsilon|\le\varepsilon_\beta$, with

$$\varepsilon_\beta=\left[\frac{1}{1-\lambda^{1-\beta}}2^3\frac{G}{r_0}(2d+1)\right]^{-1}\ . \tag{14}$$

In order to prove that $h_\varepsilon(\psi)$ is an *homeomorphism*, we have to show that it is invertible. The proof is easy and standard. Regarding injectivity, note that two *distinct* points ψ_1,ψ_2, are necessarily far order one in the "future" or in the "past", namely there exists an integer $n\in\mathbb{Z}$ such that $|S_0^n\psi_1-S_0^n\psi_2|=O(1)$; then $S_\varepsilon^n\Big(h_\varepsilon(\psi_1)\Big)-S_\varepsilon^n\Big(h_\varepsilon(\psi_2)\Big)=S_0^n(\psi_1-\psi_2)+\delta h_\varepsilon(S_0^n\psi_1)-\delta h_\varepsilon(S_0^n\psi_2)$, cannot vanish as the first term is order one, the other two of order ε; thus it cannot be but $h_\varepsilon(\psi_1)\ne h_\varepsilon(\psi_2)$. Regarding surjectivity, since f is a continuous injective mapping on a torus, f is necessarily surjective (the proof is trivial on $\mathbb{T}^1$ and it can be easily extended by induction to $\mathcal{T}_N$).

4 The Unstable Direction

In order to explicitly compute the SRB measure, we have to construct a basis for the unstable subspace E^+_ψ, and the expansion coefficient $\mathcal{D}^{(n)}_\varepsilon$ associated to it, as explained in section 2.2 above. Note that we cannot use h_ε to find a basis for E^+_ψ because it is only Hölder continuous.

To find the unstable base $\{w^{(\xi)}_{\varepsilon,+}(\psi)\}_{\xi\in V}$ and the *Lyapunov matrix* $\mathcal{L}(\psi)$ we have to solve the following equation

$$\Big(DS_\varepsilon\ w^{(\eta)}_{\varepsilon,+}\Big)(\psi)=w^{(\xi)}_{\varepsilon,+}\Big(S_\varepsilon(\psi)\Big)\mathcal{L}^{\xi\eta}(\psi)\ . \tag{1}$$

In general this equation cannot have solutions analytic in ε. In fact, from the general theory we know that the unstable vectors $\{w^{(\xi)}_{\varepsilon,+}(\psi)\}_{\xi\in V}$ are not differentiable with respect to ψ. But, as we previously pointed out, to compute the SRB measure we need only to know the expansion coefficient at the point $h_\varepsilon(\psi)$, *i.e.* $\mathcal{D}^{(n)}_\varepsilon(h_\varepsilon(\psi))$. Let us define $w^{(\xi)}_{\varepsilon,+}(h_\varepsilon(\psi))\stackrel{def}{=}v^{(\xi)}_\varepsilon(\psi)$ for $\xi\in V$ and note that $v^{(\xi)}_\varepsilon(\psi)$ satisfies the equation

$$\big(DS_\varepsilon\big)\big(h_\varepsilon(\psi)\big)v^{(\eta)}_\varepsilon(\psi)=v^{(\xi)}_\varepsilon\,(S_0\psi)\,L^{\xi\eta}(\psi)\quad,\quad L(\psi)=\mathcal{L}(h_\varepsilon(\psi))\ . \tag{2}$$

We will show that this equation admit a solution analytic in ε. Moreover the determinant of $L(\psi)$ is all what we need to compute the SRB measure.

At this point, it is convenient to write eq.(4.2) in components. Denoting by $\mathbf{y}$ the double index η^β (again $\mathbf{x}\stackrel{def}{=}\xi^\alpha$), defining $v^{(\xi)}_\varepsilon(\psi)\stackrel{def}{=}\sum_\eta V^{(\xi)}_{\varepsilon,\mathbf{y}}(\psi)w^{(\eta)}_{0,\beta}$ and $\big(DS_\varepsilon\ w^{(\eta)}_{0,\beta}\big)(\psi)\stackrel{def}{=}\sum_{\mathbf{x}}S^{\mathbf{x},\mathbf{y}}_\varepsilon(\psi)w^{(\xi)}_{0,\alpha}$, we get

$$S^{\mathbf{x},\mathbf{y}}_\varepsilon\Big(h_\varepsilon(\psi)\Big)V^{(\rho)}_{\varepsilon,\mathbf{y}}(\psi)=V^{(\zeta)}_{\varepsilon,\mathbf{x}}(S_0\psi)L^{\zeta\rho}(\psi)\ . \tag{3}$$

Now, defining the corrections δL and δV as follows:

$$L^{\xi\eta}(\psi)\stackrel{def}{=}\lambda^{-1}\delta_{\xi\eta}+\delta L^{\xi\eta}(\psi)\quad,\quad V^{(\xi)}_{\varepsilon,\mathbf{x}}(\psi)\stackrel{def}{=}V^{(\xi)}_{0,\mathbf{x}}+\delta V^{(\xi)}_{\mathbf{x}}(\psi)\quad\text{with}\quad V^{(\xi)}_{0,\eta^+}=\delta_{\xi,\eta}\quad,\quad V^{(\xi)}_{0,\eta^-}=0\ , \tag{4}$$

we find that (4.3) is equivalent to:

$$\left\{\begin{array}{l} \delta L^{\xi\rho}(\psi) = \lambda^{-1}\left[\delta V^{(\rho)}_{\xi^+}(\psi) - \delta V^{(\rho)}_{\xi^+}(S_0\psi)\right] + \varepsilon f^{\xi^+,\rho^+}(h_\varepsilon(\psi)) + \\ \qquad + \varepsilon f^{\xi^+,\mathbf{y}}(h_\varepsilon(\psi))\delta V^{(\rho)}_{\mathbf{y}}(\psi) - \delta V^{(\zeta)}_{\xi^+}(S_0\psi)\delta L^{\zeta\rho}(\psi) \\ \\ \lambda\delta V^{(\rho)}_{\xi^-}(\psi) - \lambda^{-1}\delta V^{(\rho)}_{\xi^-}(S_0\psi) = -\varepsilon f^{\xi^-,\rho^+}(h_\varepsilon(\psi)) - \varepsilon f^{\xi^-,\mathbf{y}}(h_\varepsilon(\psi))\delta V^{(\rho)}_{\mathbf{y}}(\psi) + \\ \qquad + \delta V^{(\zeta)}_{\xi^-}(S_0\psi)\delta L^{\zeta\rho}(\psi)\ . \end{array}\right. \tag{5}$$

Of course the above equations cannot determine completely the basis and its associated matrix: indeed given a solution $\{V^{(\rho)}_{\mathbf{y}}(\psi)\}$, $\{L^{\zeta\rho}(\psi)\}$ of eq.(4.2) and a generic invertible Hölder continuous matrix $R^{\gamma\rho}(\psi)$, also $\{V^{(\gamma)}_{\mathbf{y}}(\psi)R^{\gamma\rho}(\psi)\}$, $\{R^{-1,\zeta\delta}(S_0\psi)L^{\delta\gamma}(\psi)R^{\gamma\rho}(\psi)\}$ solve (4.2). Thus it is possible to add a constrain to $\delta V^{(\rho)}_{\mathbf{y}}(\psi)$: a possible choice, which simplifies a lot the expressions above, consists in taking $\delta V^{(\xi)}_{\rho^+}(\psi) = 0$, so that (4.5) becomes:

$$\left\{\begin{array}{l} \delta L^{\xi\rho}(\psi) = \varepsilon f^{\xi^+,\rho^+}(h_\varepsilon(\psi)) + \varepsilon f^{\xi^+,\eta^-}(h_\varepsilon(\psi))\delta V^{(\rho)}_{\eta^-}(\psi) \\ \\ \lambda\delta V^{(\rho)}_{\xi^-}(\psi) - \lambda^{-1}\delta V^{(\rho)}_{\xi^-}(S_0\psi) = -\varepsilon f^{\xi^-,\rho^+}(h_\varepsilon(\psi)) - \varepsilon f^{\xi^-,\eta^-}(h_\varepsilon(\psi))\delta V^{(\rho)}_{\eta^-}(\psi) + \\ \qquad + \delta V^{(\zeta)}_{\xi^-}(S_0\psi)\delta L^{\zeta\rho}(\psi)\ . \end{array}\right. \tag{6}$$

An implicit solution of (4.6) (to be inverted iteratively by a new tree expansion, see below) is:

$$\left\{\begin{array}{ll} \delta L^{\xi\rho}(\psi) & = \varepsilon f^{\xi^+,\rho^+}(h_\varepsilon(\psi)) + \varepsilon f^{\xi^+,\eta^-}(h_\varepsilon(\psi))\delta V^{(\rho)}_{\eta^-}(\psi) \\ \\ \delta V^{(\rho)}_{\xi^-}(\psi) & = \sum_{j\ge 0}\lambda^{2j+1}\Big[\varepsilon f^{\xi^-,\rho^+}(h_\varepsilon(S_0^{-j}\psi)) + \varepsilon f^{\xi^-,\eta^-}(h_\varepsilon(S_0^{-j}\psi))\delta V^{(\rho)}_{\eta^-}(S_0^{-j}\psi) + \\ & \qquad - \delta V^{(\zeta)}_{\xi^-}(S_0^{-j+1}\psi)\delta L^{\zeta\rho}(S_0^{-j}\psi)\Big] \end{array}\right. \tag{7}$$

As for the construction of the conjugation, we can expand in power series of ε both sides of eq.(4.7) and equate the coefficients of the same order, thus finding an iterative solution of $\delta L_{(k)}$ and $\delta V_{(k)}$. The first order coefficients are given by:

$$\begin{array}{l} \delta L^{\xi\rho}_{(1)}(\psi) = f^{\xi^+,\rho^+}(\psi) \\ \delta V^{(\rho)}_{\xi^-(1)}(\psi) = \displaystyle\sum_{j\ge 0}\lambda^{2j+1} f^{\xi^-,\rho^+}(S_0^{-j}\psi)\ , \end{array} \tag{8}$$

while, for $k+1\ge 2$:

$$\begin{array}{rl} \delta L^{\xi\rho}_{(k+1)}(\psi) = & \displaystyle\sum_{s\ge 1,\ k_i\ge 1}^{k_1+\cdots+k_s=k}\Big(\frac{f^{\xi^+,\rho^+\mathbf{x}_1\ldots\mathbf{x}_s}}{s!}\delta h^{\mathbf{x}_1}_{(k_1)}\cdots\delta h^{\mathbf{x}_s}_{(k_s)}\Big)(\psi) + \\ + & \displaystyle\sum_{s\ge 1,\ k_i\ge 1}^{k_1+\cdots+k_s=k}\Big(\frac{f^{\xi^+,\eta^-\mathbf{x}_2\ldots\mathbf{x}_s}}{(s-1)!}\delta V^{(\rho)}_{\eta^-(k_1)}\delta h^{\mathbf{x}_2}_{(k_2)}\cdots\delta h^{\mathbf{x}_s}_{(k_s)}\Big)(\psi) \end{array} \tag{9}$$

and

$$\begin{array}{rl} \delta V^{(\rho)}_{\xi^-(k+1)}(\psi) = & \displaystyle\sum_{s\ge 1,\ k_i\ge 1}^{k_1+\cdots+k_s=k}\sum_{j\ge 0}\lambda^{2j+1}\Big(\frac{f^{\xi^-,\rho^+\mathbf{x}_1\cdots\mathbf{x}_s}}{s!}\delta h^{\mathbf{x}_1}_{(k_1)}\cdots\delta h^{\mathbf{x}_s}_{(k_s)}\Big)(S_0^{-j}\psi) + \\ + & \displaystyle\sum_{s\ge 1,\ k_i\ge 1}^{k_1+\cdots+k_s=k}\sum_{j\ge 0}\lambda^{2j+1}\Big(\frac{f^{\xi^-,\eta^-\mathbf{x}_2\cdots\mathbf{x}_s}}{(s-1)!}\delta V^{(\rho)}_{\eta^-(k_1)}\delta h^{\mathbf{x}_2}_{(k_2)}\cdots\delta h^{\mathbf{x}_s}_{(k_s)}\Big)(S_0^{-j}\psi) + \end{array} \tag{10}$$

$$- \sum_{s\geq 1,\ k_i\geq 1}^{k_1+\cdots+k_s=k} \sum_{j\geq 0} \lambda^{2j+1} \Big(\frac{f^{\zeta^+,\rho^+ \mathbf{x}_2\ldots\mathbf{x}_s}}{(s-1)!} (\delta V^{(\zeta)}_{\xi^-(k_1)} \circ S_0)\, \delta h^{\mathbf{x}_2}_{(k_2)} \cdots \delta h^{\mathbf{x}_s}_{(k_s)} \Big) (S_0^{-j}\psi) +$$

$$- \sum_{s\geq 2,\ k_i\geq 1}^{k_1+\cdots+k_s=k} \sum_{j\geq 0} \lambda^{2j+1} \Big(\frac{f^{\zeta^+,\eta^- \mathbf{x}_3\ldots\mathbf{x}_s}}{(s-2)!} (\delta V^{(\zeta)}_{\xi^-(k_1)} \circ S_0)\, \delta V^{(\rho)}_{\eta^-(k_2)} \delta h^{\mathbf{x}_3}_{(k_3)} \cdots \delta h^{\mathbf{x}_s}_{(k_s)} \Big) (S_0^{-j}\psi) \; .$$

This two relations, together with (3.6), allow a recursive construction of δL and δV. Obviously, repeating the discussion of §3.1, one finds that δL and δV can be expressed as sums over trees, obtained by suitably modifying the construction of previous section. It can be easily realized that the estimates for the tree values are qualitatively the same as before, see eq.(3.11),(3.12),(3.13). We point out the differences appearing in the tree expansion for δV:

1. the nodes can be of 4 different types (corresponding to the 4 lines in eq. (4.10)); so that the number of possible labels for a tree of order k is larger of a factor 4^k;

2. the number D_v of derivatives acting on a node function can be either s_v or s_v+1, see eq.(4.9),(4.10), so that $D_v!$ differs from the combinatorial factor $s_v!$ by at most s_v+1; then the final estimate contains a factor that can be bounded by $(1/r_0)\prod_v (s_v+1) \leq e^k/r_0$.

A similar discussion can be made for the tree expansion of δL.

The result is that L and V are analytic in ε and Hölder continuous in ψ with exponent $0<\beta<1$ in a disc $|\varepsilon| \leq \varepsilon'_\beta$, with ε'_β smaller than the convergence radius ε_β of h_ε, see eq.(3.14). Note that also in this case ε'_β is independent of N.

As already explained (see section 2.2 and in particular eq.(2.7)), in order to compute the SRB measure we need $\Lambda^\xi = (\log L)^{\xi\xi} \stackrel{def}{=} -\log\lambda + \delta\Lambda^\xi$, where:

$$\delta\Lambda^\xi(\psi) = \Big[\log(I+\lambda\delta L)\Big]^{\xi\xi} = \sum_{s\geq 1} \frac{(-1)^{s+1}}{s} \lambda^s \delta L^{\xi\eta_1}(\psi)\cdots\delta L^{\eta_{s-1}\xi}(\psi) \; . \tag{11}$$

(no summation on ξ is intended). Expanding eq.(4.11) in series of ε, we get:

$$\delta\Lambda^\xi_{(k)}(\psi) = \sum_{s\geq 1,\ k_i\geq 1}^{k_1+\cdots k_s=k} \frac{(-1)^{s+1}}{s} \lambda^s \delta L^{\xi\eta_1}_{(k_1)}(\psi)\cdots\delta L^{\eta_{s-1}\xi}_{(k_s)}(\psi) \; . \tag{12}$$

Again, the last equation, together with (4.10) and (3.6), allows a recursive construction of the coefficients $\delta\Lambda^\xi_{(k)}$ and the result is that Λ^ξ is a sum over (suitably modified) trees. The bounds are still qualitatively the same, so that Λ^ξ is analytic w.r.t. ε in a suitably small complex disc (independent of N) and Hölder continuous w.r.t. ψ.

5 SRB Potentials

The next step towards the construction of the SRB measure and the proof of its analyticity consists in the expansion of Λ^ξ in potentials ϕ_X. From the analysis of previous sections follows that Λ^ξ, as well as $h^{\mathbf{x}}_\varepsilon$, $V^{(\rho)}_{\xi^-}$ and $L^{\xi\eta}$, can be expanded in convergent sums over tree values. We will discuss here how to expand h in potentials, since the analogous expansion for V, L and Λ is conceptually similar, just more involved due to the more complex structure of the trees.

We will proceed as follows. We first write the values of the trees in terms of the symbolic variables σ. We then decompose each of these values as a sum of terms only depending on the σ's on finite but arbitrary large sets. Finally we define the associated potentials by collecting together the contributions which depend on the same σ's. Our goal is to obtain potentials defined over sets with rather arbitrary shape but decaying exponentially with the *tree distance* (see after eq.(2.8) for a precise definition) of their support.

To begin with we expand the derivatives of the perturbation function f via a telescopic sum. Given the digits s and $s' \in \{1,\dots,n\}$ we can always find a sequence of digits $\Sigma(s,s') = s_1 s_2 \dots s_{a-1}$ such that the sequence $s\Sigma(s,s')s'$ compatible, *i.e.* such that $C_{s_i,s_{i+1}} = 1$ for $i = 0,\dots a-1$, where $s_0 = s$ and $s_a = s'$. Chosen a sequence $\hat\sigma \in \{1,\dots,n\}^{\mathbb{Z}}_C$ once and for all, given $\sigma \in \{1,\dots,n\}^{\mathbb{Z}}_C$ we can define its restriction to time j, σ^j as follows: $\sigma^j_{\xi,t} = \sigma_{\xi,t}$ if $|t| \le j$, $\sigma^j_{\xi,t} = \hat\sigma_{\xi,t}$ if $|t| > j + a$ and the gap is filled with the sequence constructed above for $s = \sigma_{\xi,\pm j}$ and $s' = \hat\sigma_{\xi,\pm(j+a)}$. We can now define:

$$
\begin{aligned}
f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}(c_0(\sigma)) &= f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}(c_0(\sigma^0)) + \sum_{j\ge1}\left[f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}(c_0(\sigma^j)) - f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}(c_0(\sigma^{j-1}))\right] \stackrel{def}{=} \\
&\stackrel{def}{=} \sum_{j\ge0} f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}_{(j)}(\sigma_{nn^{(j)}(\xi)})
\end{aligned} \tag{1}
$$

where ξ is the spatial coordinate associated to $\mathbf{x}$ and $nn^{(j)}(\xi) = nn(\xi)\times I_j$, $I_j = [-j,j]\cap\mathbb{Z}$. Since $|c_0(\sigma^j) - c_0(\sigma^{j-1})| \le c\lambda^j$ for some $c > 0$, $f^{\mathbf{x},\mathbf{x}_1\cdots\mathbf{x}_s}_{(j)}$ is bounded by:

$$
||f^{\mathbf{x},\,\mathbf{x}_1\dots\mathbf{x}_s}_{(j)}||_\infty \le G\frac{(s+1)!}{r_0^{s+1}}c\lambda^j . \tag{2}
$$

5.1 Decay of the Potentials for the Conjugation

Inserting expansion (5.1) in the definition of the value of a tree eq. (3.8), we find:

$$
\widehat{\mathrm{Val}}(\theta, c_0(\sigma)) = \prod_{v\in\theta}\sum_{j_v\ge0}\big(-\alpha(v)\big)\lambda^{p_v+\rho_{\alpha(v)}}\frac{f^{\mathbf{x}(v),\,\mathbf{x}(v_1)\dots\,\mathbf{x}(v_{s_v})}_{(j_v)}}{s_v!}\big(\tau^{p(v)}\sigma_{nn^{(j_v)}(\xi(v))}\big) , \tag{3}
$$

where we recall that $\rho_{\alpha(v)} = (1+\alpha(v))/2$, $p(v) = \sum_{w\ge v}\alpha(v)(p_v + 1 - \rho_{\alpha(v)})$ and $nn^{(j)}(\xi) = nn(\xi)\times I_j$, $I_j = [-j,j]\cap\mathbb{Z}$. The above expression can be seen as a sum over the values of a new kind of trees, identical to the ones described in section. 3.1, but with a new label $j_v \in \mathbb{N}$ attached to each node. Let $\mathcal{T}_k(\mathbf{x})$ be the set of these new trees of order k contributing to $\delta h^{\mathbf{x}}_{(k)}$, *i.e.* $\theta \in \mathcal{T}_k(\mathbf{x})$ is a tree with k branches and k nodes (the root is not a node) with the following labels attached to the nodes $v \in \theta$: $\xi(v) \in V$, $p_v \in \mathbb{N}$, $j_v \in \mathbb{N}$ and $\alpha(v) \in \{-1,+1\}$.

Given $\theta \in \mathcal{T}_k(\mathbf{x})$, its value is given by:

$$
\mathrm{Val}\ \theta(\sigma) = \prod_{v\in\theta}\big(-\alpha(v)\big)\lambda^{p_v+\rho_{\alpha(v)}}\frac{f^{\mathbf{x}(v),\,\mathbf{x}(v_1)\dots\,\mathbf{x}(v_{s_v})}_{(j_v)}}{s_v!}\big(\tau^{p(v)}\sigma_{nn^{(j_v)}(\xi(v))}\big) . \tag{4}
$$

so that we have

$$
\delta h^{\mathbf{x}}_{(k)}(c_0(\sigma)) = \sum_{\theta\in\mathcal{T}_k(\mathbf{x})}\mathrm{Val}\ \theta(\sigma) \qquad \text{with} \quad ||\mathrm{Val}\ \theta||_\infty \le \left(\frac{ceG}{\lambda r_0^2}\right)^k\prod_{v\in\theta}\lambda^{j_v+p_v+1} . \tag{5}
$$

where we have used eq.(5.2). We can now define the support $X(\theta) \subset V\times\mathbb{Z}$ of a tree $\theta \in \mathcal{T}_k(\mathbf{x})$, as the support of the spin variables on which $\mathrm{Val}\,\theta$ depends in a non trivial way, plus a *center* $(\xi,0)$. More precisely:

$$
X(\theta) \stackrel{def}{=} \Big\{(\xi,0)\Big\} \cup \bigcup_{v\in\theta}\mathcal{C}\Big(\xi(v),p(v),j_v\Big) , \qquad \text{where} \qquad \mathcal{C}\Big(\xi(v),p(v),j_v\Big) \stackrel{def}{=} \bigcup_{\eta\in nn(\xi(v))}\ \bigcup_{|i|\le j_v}(\eta, p(v)+i) . \tag{6}
$$

namely $\mathcal{C}(\xi,p,j)$ is a cylinder centered in (ξ,p), with the spatial base equal to the set of nearest neighbors of ξ and with height equal to $2j$. Then $X(\theta)$ is the union of $(\xi,0)$ and of cylinders of this kind, one for each node v of the tree. The point $(\xi,0)$ has the role of center of $X(\theta)$ and is added to $X(\theta)$ for later convenience (note in fact that $\mathrm{Val}\ \theta(\sigma)$ could not depend on $\sigma_{(\xi,0)}$).

Given a set $X \subset \mathbb{Z}^{d+1}$ we can partition it in a natural way as a union of time-like segments. More precisely given $\xi \in \mathbb{Z}^d$ let $T_\xi = \{(\xi, i) \in \mathbb{Z}^{d+1} \,|\, i \in \mathbb{Z}\}$. The intersection between T_ξ and X can be uniquely partitioned as a union of n_ξ maximal connected segments. The collection of all this segments forms a partition of X in n_X time-like segments $\{R_i(X)\}_{i=1,\dots,n_X}$. Let now r_i the center of $R_i(X)$. If Y is a subset of $\mathbb{Z}^{d+1}$ we call tree distance of Y, $d_t(Y)$, the length of the minimal tree connection all the points of Y. Finally let $d_c(X)$ the tree distance of the set $\{r_i\}_{i=1,\dots,n_X}$.

¿From the previous bound on the value of a tree $\theta \in \mathcal{T}_k(\mathbf{x})$ eq. (5.5) can be interpreted as the tree distance decay of the contribution of order k to δh. Indeed:

$$|\varepsilon|^k ||\mathrm{Val}\,\theta||_\infty \le \Big(\frac{ce\,G|\varepsilon|^{1/2}}{\lambda r_0^2}\Big)^k \left[\lambda^{d_c(X(\theta))}|\varepsilon|^{n_{X(\theta)}/2}\prod_{i=1}^{n_{X(\theta)}}\lambda^{|R_i(X(\theta))|}\right]^{\frac{1}{2d+1}} , \tag{7}$$

where:

1. the factor $\lambda^{d_c(X(\theta))}$ comes from $\prod_{v\in\theta}\lambda^{1+p_v}$; in fact p_v is the displacement in the time direction of the cylinder associated to the node v w.r.t. the one associated to the node v' immediately following v, and 1 is their maximum displacement in spatial direction, so that $\sum_v(1+p_v) \ge d_c(X(\theta))$;
2. we used that $n_X \le (2d+1)k$ in order to bound $|\varepsilon|^{k/2}$ with $|\varepsilon|^{\frac{n_X}{2(2d+1)}}$;
3. the factor $\prod_{i=1}^{n_{X(\theta)}}\lambda^{|R_i(X(\theta))|}$ comes from $\prod_{v\in\theta}\lambda^{j_v}$;
4. the global power $1/(2d+1)$ in (5.7) comes from the size of the base of each cylinder, namely we used the fact that the number n_X of segments is less than $2d+1$ times the number of cylinders in $X(\theta)$.

Collecting together all the trees θ which have support $X(\theta) = X$ for a given X, we get:

$$\delta h_\varepsilon^{\mathbf{x}}(c_0(\sigma)) = \sum_{X\ni(\xi,0)}\delta h_X^{\mathbf{x}}(\sigma_X) \quad \text{with} \quad \delta h_X^{\mathbf{x}}(\sigma_X) \stackrel{def}{=} \sum_{k\ge1}\varepsilon^k \sum_{\theta\in\mathcal{T}_k(\mathbf{x})}^{X(\theta)\equiv X} \mathrm{Val}\;\theta(\sigma) \; . \tag{8}$$

So, using the bound (5.7) for $|\varepsilon|$ small enough, $\gamma_0 = \frac{1}{2(2d+1)}$, $\kappa_0 = -2\gamma_0\log\lambda$, $\nu_0 = |\varepsilon|^{\gamma_0}$ and a suitable $c > 0$ we get

$$||\delta h_X^{\mathbf{x}}||_\infty \le ce^{-\kappa_0 d_c(X)}\nu_0^{n_X}\prod_{i=1}^{n_X} e^{-\kappa_0|R_i(X)|} \; , \tag{9}$$

namely $\delta h_X^{\mathbf{x}}$ decays exponentially with the tree distance of X.

5.2 SRB Potentials and Their Decay

Proceeding as above for the function $\Lambda^\xi(c_0(\sigma))$ we obtain that we can write it as:

$$\Lambda^\xi(c_0(\sigma)) \stackrel{def}{=} \sum_{X\subset(V_N\times\mathbb{Z})}\phi_X^{(\xi,0)}(\sigma_X)$$

where by construction $\phi_X^{(\xi,0)}$ is different from 0 only if $(\xi,0) \in X$. The function $\phi_X^{(\xi,0)}$ is again given by a tree expansion analogous to that in eq. (5.8). Moreover we will set

$$\phi_X^{(\xi,j)}(\sigma_X) \stackrel{def}{=} \phi_{\tau^{-j}X}^{(\xi,0)}(\sigma_X).$$

We can define:

$$\phi_X(\sigma_X) \stackrel{def}{=} \sum_{(\xi,j)\in X}\phi_X^{(\xi,j)}(\sigma_X),$$

so that we formally obtain eq.(2.8), namely, given $I_T = [-T/2, T/2] \cap \mathbb{Z}$ (T even) and calling $\Lambda = V_N \times I_T$,

$$\sum_{(\xi,i)\in\Lambda} \Lambda^\xi\big(c_0(\tau^i\sigma)\big) - \sum_{X\cap\Lambda\neq\emptyset} \phi_X(\sigma_X) = O(\partial\Lambda) ,$$

where $\partial\Lambda$ is the boundary of Λ and the correction can be exactly computed from the definitions above.

Note the potential $\phi_X(\sigma_X)$ is invariant under time and space translations (respectively for the definition of $\phi_X^{(\xi,i)}$ and for the periodic boundary conditions), namely:

$$\phi_X(\sigma_X) = \phi_{\rho^\xi\tau^j X}(\sigma_X) \quad \text{for any} \quad (\xi, j) \in V_N \times \mathbb{Z}. \tag{10}$$

Moreover it can be bounded by:

$$||\phi_X||_\infty \leq c e^{-\kappa_1 d_c(X)} \nu_1^{n_X} \prod_{i=1}^{n_X} e^{-\kappa_1 |R_i(X)|} , \tag{11}$$

for suitable $c, \gamma_1, \kappa_1 > 0$ and $\nu_1 = |\varepsilon|^{\gamma_1}$.

6 Analyticity of SRB measure

In the previous sections, we wrote the SRB measure as a Gibbs measure with translationally invariant potentials ϕ_X, decaying as in (5.11), and with hard core interaction in time direction. Moreover the potential ϕ_X is analytic in ε in a small disc in $\mathbb{C}$ around the origin (independent of N). A well known technique to show analyticity of the Gibbs measure w.r.t. ε is the so called *cluster expansion.*

If $\Lambda = V_N \times I_T$, with $I_T \overset{def}{=} [-T/2, T/2] \cap \mathbb{Z}$ for some even $T \in \mathbb{N}$, we call $\Lambda_a = V_N \times I_{T+2a}$. Given a *boundary condition* $\bar\sigma \in \{1, \ldots, n\}_C^{\mathbb{Z}^{d+1}}$, we define the *pressure* P_Λ as

$$P_\Lambda \overset{def}{=} |\Lambda|^{-1} \log \sum_\sigma e^{-\sum_{X\cap\Lambda\neq\emptyset} \phi_X(\sigma_X)} . \tag{1}$$

where the sum is over all the σ that coincide with σ_Λ on Λ, to $\bar\sigma$ on Λ_a^c and with $\Sigma(\sigma_{\xi,T/2}, \bar\sigma_{\xi,T/2+a})$ in the space remaining. It is well known that the pressure P_Λ can be considered as the generating functional for the Gibbs states. From its analyticity our main theorem will follow easily, as we will see in section 6.4.

6.1 Decimation

In presence of hard cores we cannot proceed in the standard way (Mayer's expansion), since the standard proof, see [GMM73], requires weakness of the original interactions. We can overcome this obstacle by a *decimation*, see [CO81], namely considering the statistical system on scales larger than the length of decorrelation of the hard core.

6.1.1 Decimated Lattice Λ_D

For each $\xi \in V_N$, we divide the time interval $I_T^\xi \overset{def}{=} \{\xi\} \times I_T$ into an alternating sequence of blocks, called of "B-type" and "H-type", $B_\xi^{(0)}, H_\xi^{(0)}, B_\xi^{(1)}, H_\xi^{(1)}, \ldots, B_\xi^{(\ell-1)}, H_\xi^{(\ell-1)}, B_\xi^{(\ell)}$, containing a number of spins respectively equal to $b = 1$ and $h = h_0 a - 1$, with $h_0 \in \mathbb{N}$ to be chosen later. For this reason we choose the number of points in I_T^ξ to be $|I_T^\xi| = \ell h_0 a + 1$, namely $T = \ell h_0 a$.

Remark The choice $b = 1$ is special for the present case, in which the unperturbed potential is vanishing. In general one could treat with the same technique the case in which the unperturbed potential is order one, with a sufficiently fast decay of the tails, and in that case b should be chosen suitably large, see [CO81]. Such a case arises, for instance, when the unperturbed system is the product of non linear Anosov maps on $\mathbb{T}^2$, namely in the case treated in Appendix A. The present discussion could be easily adapted to cover that case.

Let $\beta_\xi^{(i)} \stackrel{def}{=} \sigma_{(\xi,-T/2+ih_0 a)}$, $\xi \in V_N$, $i = 0,\ldots,\ell$, be the spin in the block $B_\xi^{(i)}$ and $\eta_\xi^{(i)} \stackrel{def}{=} \{\sigma_{(\xi,p)}\}_{(\xi,p)\in H_\xi^{(i)}}$, $\xi \in V_N$, $i = 0,\ldots,\ell-1$, be the collection of spins belonging to the block $H_\xi^{(i)}$; it will be regarded as a sequence of h β spins: $\eta_\xi^{(i)} = \big(\beta_1(\eta_\xi^{(i)}),\ldots,\beta_h(\eta_\xi^{(i)})\big)$. The lattice obtained considering the H and B blocks as points:

$$\Lambda_D \stackrel{def}{=} \{B_\xi^{(p)}, H_\xi^{(q)}\}_{\xi\in V_N,\, p=0,\ldots,\ell}^{q=0,\ldots,\ell-1} \tag{2}$$

will be called the *decimated lattice*; on Λ_D the distances will be computed by thinking of it as having its sites spaced by 1 also in the time direction.

If $X \subset \Lambda$, $Y(X)$ will denote the corresponding subset in Λ_D, namely the smaller subset $Y \subset \Lambda_D$ such that the union of the B and H–blocks in Y contains the set X. Defining $\Phi_Y(\beta_Y,\eta_Y) \stackrel{def}{=} \sum_{X:Y(X)=Y} \phi_X(\sigma_X)$, eq.(6.1) can be rewritten as

$$P_\Lambda = \frac{1}{|\Lambda|} \log \sum_{\beta_\Lambda} \sum_{\eta_\Lambda} e^{-\sum_{Y\subset\Lambda_D} \Phi_Y(\beta_Y,\eta_Y)} \prod_{\xi\in V} \prod_{i=0}^{\ell-1} Z(\beta_\xi^{(i)},\eta_\xi^{(i)},\beta_\xi^{(i+1)}) \,,$$

$$\text{where} \qquad Z(\beta,\eta,\beta') = C_{\beta\beta_1(\eta)} C_{\beta_1(\eta)\beta_2(\eta)} \cdots C_{\beta_{h-1}(\eta)\beta_h(\eta)} C_{\beta_h(\eta)\beta'} \,. \tag{3}$$

Observe that, from eq. (5.11), if Y do not coincide with a single H–block, Φ_Y satisfies a qualitatively equivalent bound:

$$||\Phi_Y||_\infty \le c\, e^{-\tilde\kappa d_c(Y)} \tilde\nu^{n_Y} \prod_{i=1}^{n_Y} e^{-\tilde\kappa |R_i(Y)|} \quad , \quad Y \ne H_\xi^{(i)} \,, \tag{4}$$

for some $c, \tilde\kappa, \tilde\gamma > 0$ and $\tilde\nu = |\varepsilon|^{\tilde\gamma}$. Whereas if $Y = H_\xi^{(i)}$ for some $\xi \in V$ and some $i = 0,\ldots,\ell-1$, we have $||\Phi_Y||_\infty \le h\tilde\nu$.

6.1.2 Averaging over many degree of freedom: the Perron-Frobenius's theorem

Decimation is a *renormalization group* technique, consisting in summing first on the H-type spins, thus getting an effective statistical system for the B-blocks: the idea is that if the B-blocks are sufficiently far apart, after the averaging of the η's, the β's should be *almost independent*, as if there were only small interactions among them. The technical tool we shall use to prove rigorously that the effective interactions between the β's are small is Perron-Frobenius theorem.

Let $Z(\beta,\beta')$ be defined, with a little abuse of notation, as:

$$Z(\beta,\beta') \stackrel{def}{=} \sum_\eta Z(\beta,\eta,\beta') = C^{ah_0}_{\beta\beta'} \tag{5}$$

Observe that $1 \le C^a_{\sigma\sigma'} \le q^a$. Since C^a has strictly positive entries, we can apply the Perron-Frobenius theorem and obtain that C^a and its transpose $C^{a,T}$ admit a non degenerate eigenvalue $l > 0$ with eigenvectors π and π^* respectively such that $\pi_\sigma,\ \pi^*_\sigma > 0$ for any $\sigma = 1,\ldots q$, and $\sum_\sigma \pi^*_\sigma \pi_\sigma = 1$. The eigenvalue l is maximal in the spectrum of C^a; namely, if we define P as the projection matrix $P_{\sigma\sigma'} = \delta_{\sigma\sigma'} - \pi_\sigma \pi^*_{\sigma'}$, we have:

$$||(l^{-1}C^a)^k P\omega||_\infty \le c_\alpha e^{-\alpha k} ||\omega||_\infty \,, \tag{6}$$

for any $\omega \in \mathbb{R}^q$ and with

$$\alpha \stackrel{def}{=} -\log(1 - [\min(C^a_{\sigma\sigma''}/C^a_{\sigma\sigma'})]^2) \ge q^{-2a} \,. \tag{7}$$

As a consequence,

$$Z(\beta,\beta') \qquad = C^{ah_0}_{\beta\beta'} = \sum_\sigma C^{ah_0}_{\beta\sigma}(\pi_\sigma\pi^*_{\beta'} + P_{\sigma\beta'}) =$$

$$= l^{h_0}\pi_\beta\pi^*_{\beta'}\Big[1 + \frac{(l^{-h_0}C^{ah_0}P)_{\beta\beta'}}{\pi_\beta\pi^*_{\beta'}}\Big] \stackrel{def}{=} l^{h_0}\pi_\beta\pi^*_{\beta'}e^{-I(\beta,\beta')} \ , \tag{8}$$

with $I(\beta,\beta') = O(e^{-h_0 q^{-2a}})$. It is now clear that taking h_0 big enough we can make the two body potential $I(\beta,\beta')$ as small as needed.

Using eq. (6.8), introducing a new effective potential W including the contributions from Φ and I, defining

$$\prod_{i=0}^{\ell} e^{-U^{(i)}(\beta^{(i)}_\xi)} \stackrel{def}{=} \prod_{i=0}^{\ell-1} \pi_{\beta^{(i)}_\xi}\pi^*_{\beta^{(i+1)}_\xi} \tag{9}$$

and using $\lim_{\Lambda\to\infty}|\Lambda|^{-1}\log\prod_{\xi,i}\sum_{\beta^{(i)}_\xi} e^{-U^{(i)}(\beta^{(i)}_\xi)} = 0$ (as it follows from the normalization condition $\sum_\sigma \pi^*_\sigma\pi_\sigma = 1$), we can rewrite P_Λ as

$$P_\Lambda = \frac{1}{a}\log l + \frac{1}{|\Lambda|}\log\sum_{\beta_\Lambda}\sum_{\eta_\Lambda} m(\beta_\Lambda,\eta_\Lambda)\prod_{Y\subset\overline{\Lambda}} e^{-W_Y(\beta_Y,\eta_Y)}$$

$$\text{with} \quad m(\beta_\Lambda,\eta_\Lambda) \stackrel{def}{=} \prod_{\xi\in V}\prod_{i=0}^{\ell}\frac{e^{-U^{(i)}(\beta^{(i)}_\xi)}}{\sum_{\beta_\xi} e^{-U^{(i)}(\beta_\xi)}}\prod_{i=0}^{\ell-1}\frac{Z(\beta^{(i)}_\xi,\eta^{(i)}_\xi,\beta^{(i+1)}_\xi)}{Z(\beta^{(i)}_\xi,\beta^{(i+1)}_\xi)} \tag{10}$$

where $m(\beta_\Lambda,\eta_\Lambda)$ is a probability density. Observe that, if one chooses $h_0 \simeq -\log\widetilde{\nu}$ (so that both $h\widetilde{\nu}$ and $I(\beta,\beta')$ are small), the new interaction W satisfies a bound similar the one of Φ:

$$||W_Y||_\infty \le ce^{-\overline{\kappa}d_c(Y)}\overline{\nu}^{n_Y}\prod_{i=1}^{n_Y} e^{-\bar{\kappa}|R_i(Y)|} \quad , \quad \forall Y \subset \Lambda_D \ , \tag{11}$$

for some $c, \overline{\kappa}, \overline{\gamma} > 0$, $\overline{\nu} = |\varepsilon|^{\overline{\gamma}}$.

6.2 Mayer's Expansion and Polymer Lattice Gas

We shall now expand the small potential appearing in the expression for P_Λ, via a *Mayer's expansion*, obtaining the pressure for $\varepsilon = 0$ plus a correction.

It will be convenient to collect together the contributions of the potentials whose supports have the same *closure*, in the following sense: for a set formed by a unique point $H^{(i)}_\xi \in \Lambda_D$ we define its *closure* as $\overline{(H^{(i)}_\xi)} \stackrel{def}{=} (B^{(i)}_\xi, H^{(i)}_\xi, B^{(i+1)}_\xi)$ while for a set formed by a unique point $B^{(i)}_\xi \in \Lambda_D$ we define $\overline{(B^{(i)}_\xi)} \stackrel{def}{=} (B^{(i)}_\xi)$; finally for $Y \subset \Lambda_D$ we define its closure as $\overline{Y} \stackrel{def}{=} \cup_{G\in Y}\overline{(G)}$.

We say that a collection $\mathcal{C} = \{Y_m\}_{m=1}^n$ of sets $Y_i \subset \mathbb{Z}^{d+1}$ (think to them as *molecules*) is *connected* if, given a couple $(Y_{\rm in}, Y_{\rm fin}) \in \mathcal{C}\times\mathcal{C}$, it is possible to find $\{Y_{m_j}\}_{j=1}^p$, such that $\overline{Y}_{\rm in}\cap\overline{Y}_{m_1} \ne \emptyset$, $\overline{Y}_{m_i}\cap\overline{Y}_{m_{i+1}} \ne \emptyset$ and $\overline{Y}_{m_p}\cap\overline{Y}_{\rm fin} \ne \emptyset$.

Writing $e^{-W_Y(\beta_Y,\eta_Y)}$ as the value for $\varepsilon = 0$ plus the correction, namely $1 + \Big(e^{-W_Y(\beta_Y,\eta_Y)} - 1\Big)$, expanding the product over $Y \subset \Lambda_D$ and collecting together the connected components, we can rewrite eq.(6.10) as:

$$P_\Lambda - \frac{1}{a}\log l = \frac{1}{|\Lambda|}\log\sum_{\beta_\Lambda}\sum_{\eta_\Lambda} m(\beta_\Lambda,\eta_\Lambda)\sum_{\Gamma\subset\Lambda_D}\Upsilon(\Gamma)\prod_{\gamma\in\Gamma}\rho(\gamma|\beta_\gamma,\eta_\gamma) \ , \tag{12}$$

where:

1. γ is a subset of Λ_D, to be called in the following *polymer* (they are, indeed, the union of a connected collection of molecules);
2. Γ is a collection of polymers: $\Gamma = (\gamma_1, \ldots, \gamma_n)$, $n \geq 1$ and $\Gamma \subset \Lambda_D$ means that $\gamma \subset \Lambda_D$, $\forall \gamma \in \Gamma$;
3. $\Upsilon(\Gamma)$ is the function equal to 1 if $\gamma \cap \gamma' = \emptyset$ for every $\gamma, \gamma' \in X$ with $\gamma \neq \gamma'$ and 0 otherwise;
4. $\rho(\gamma|\beta_\gamma, \eta_\gamma)$, $\gamma \subset \Lambda_D$ is defined as

$$\rho(\gamma|\beta_\gamma, \eta_\gamma) \stackrel{def}{=} \sum_{q \geq 1} \frac{1}{q!} \sum_{\substack{Y_1, \ldots, Y_q \\ \cup_i \overline{Y}_i = \gamma}}^{*} \prod_{i=1}^{q} \left(e^{-W_{Y_i}(\beta_{Y_i}, \eta_{Y_i})} - 1 \right) \tag{13}$$

   where the $*$ on the sum means that $Y_1, \ldots, Y_q$ is a connected collection of subsets of Λ_D;
5. the term corresponding to $\Gamma = \emptyset$ must be interpreted as equal to 1.

The key observation is that, thanks to the above definition of closure, in (6.12) we can sum over η spins before summing over the β spins. After doing this the measure $m(\beta_\Lambda, \eta_\Lambda)$ factorizes, *i.e.*

$$\begin{aligned} P_\Lambda - \frac{1}{a} \log l \quad &= \frac{1}{|\Lambda|} \log \sum_{\Gamma \subset \Lambda_D} \Upsilon(\Gamma) \prod_{\gamma \in \Gamma} \left[\sum_{\beta_\gamma} \sum_{\eta_\gamma} m(\beta_\gamma, \eta_\gamma) \rho(\gamma|\beta_\gamma, \eta_\gamma) \right] \stackrel{def}{=} \\ &\stackrel{def}{=} \frac{1}{|\Lambda|} \log \sum_{\Gamma \subset \Lambda_D} \Upsilon(\Gamma) \prod_{\gamma \in \Gamma} \rho(\gamma) \ , \end{aligned} \tag{14}$$

Namely we have rewritten P_Λ as the pressure for $\varepsilon = 0$ plus a correction having the form of the pressure of a "polymer lattice gas", with activities $\rho(\gamma)$ and hard core potentials $\Upsilon(\Gamma)$.

6.3 Cluster Expansion and Its Convergence

A standard argument, exposed for instance in [GMM73], [PS99] or [GBG03], leads to

$$P_\Lambda - \frac{1}{a} \log l = \frac{1}{|\Lambda|} \log \sum_{\Gamma \subset \Lambda_D} \Upsilon(\Gamma) \rho(\Gamma) = \frac{1}{|\Lambda|} \sum_{\Gamma \subset \Lambda_D} \Upsilon^T(\Gamma) \rho(\Gamma) \ , \tag{15}$$

where Υ^T is the Mayer function, defined as

$$\Upsilon^T(\gamma_1, \ldots, \gamma_n) \stackrel{def}{=} \begin{cases} \sum_{g \in \mathcal{G}(n)} \prod_{(i,j) \in g} f(\gamma_i, \gamma_j) & \text{if} \ \ n > 1 \\ 1 & \text{if} \ \ n = 1. \end{cases} \ , \tag{16}$$

where $\mathcal{G}(n)$ is the set of connected graphs which can be drawn on n vertices labeled $1, \ldots, n$ by connecting with links couples of distinct vertices; the function $f(\gamma_i, \gamma_j)$ is equal to 1 if $\gamma_i \cap \gamma_j \neq \emptyset$ and 0 otherwise. By construction, $\Upsilon^T(\Gamma)$ is different from zero only if Γ is a connected collection of polymers. Observe that Γ could contain many copies of the same γ. More precisely here Γ represents a function from the subsets of Λ_D to $\mathbb{N}$ (and we can think $\Gamma(\gamma)$ as representing the number of copies of γ) such that $\sum_{\gamma \subset \Lambda_D} \Gamma(\gamma) \leq \infty$.

A bound for $\rho(\gamma)$ can be obtained as follows:

$$|\rho(\gamma)| \leq ||\rho(\gamma|\cdot, \cdot)||_\infty \leq \sum_{p \geq 1} \frac{1}{p!} \sum_{Y_i, \ \cup_i \overline{Y}_i = \gamma}^{*} \prod_{i=1}^{p} ||W_{Y_i}||_\infty e^{||W_{Y_i}||_\infty} \ , \tag{17}$$

Using the bound (6.11) (and that, if $\cup_{i=1}^{p}\overline{Y}_i=\gamma$, one has $\sum_{i=1}^{p}||W_{Y_i}||\le c\overline{\nu}|\gamma|$) we find:

$$|\rho(\gamma)|\le e^{c\overline{\nu}|\gamma|}\sum_{p\ge1}\frac{1}{p!}\sum_{Y_i,\ \cup_i\overline{Y}_i=\gamma}^{*}\prod_{i=1}^{p}ce^{-\overline{\kappa}d_c(Y_i)}\overline{\nu}^{n_{Y_i}}\prod_{j=1}^{n_{Y_i}}e^{-\overline{\kappa}|R_j(Y_i)|}\ . \tag{18}$$

We can now use the connectedness constraint on the sum in order to extract a factor exponentially small in the size of γ. Indeed, if $\cup_{i=1}^{p}\overline{Y}_i=\gamma$, one has both $\sum_{i=1}^{p}d_c(Y_i)\ge d_c(Y_i)$ and $\sum_{i=1}^{p}n_{Y_i}\ge n_\gamma$. After extracting such a factor we can relax the constraints on the sum, so that:

$$|\rho(\gamma)|\le ce^{c\overline{\nu}|\gamma|}\left[e^{-\frac{\overline{\kappa}}{2}d_c(\gamma)}\overline{\nu}^{n_\gamma/2}\prod_{i=1}^{n_\gamma}e^{-\frac{\overline{\kappa}}{2}|R_i(\gamma)|}\right]\sum_{p\ge1}\frac{1}{p!}\Big(\sum_{\overline{Y}\subset\gamma}e^{-\frac{\overline{\kappa}}{2}d_c(Y)}\overline{\nu}^{n_Y/2}\prod_{j=1}^{n_Y}e^{-\frac{\overline{\kappa}}{2}|R(Y)|}\Big)^p\ . \tag{19}$$

It is easy to see that the last sum is bounded by $c|\gamma|\overline{\nu}^{1/4}$, so that:

$$|\rho(\gamma)|\le ce^{c\bar{\nu}|\gamma|}\left[e^{-\frac{\bar{\kappa}}{2}d_c(\gamma)}\bar{\nu}^{n_\gamma/2}\prod_{i=1}^{n_\gamma}e^{-\frac{\bar{\kappa}}{2}|R_i(\gamma)|}\right]\sum_{p\ge1}\frac{1}{p!}(c|\gamma|\bar{\nu}^{1/4})^p\le ce^{-\kappa' d_c(\gamma)}(\nu')^{n_\gamma}\prod_{i=1}^{n_\gamma}e^{-\kappa'|R_i(\gamma)|}\ , \tag{20}$$

for some $c,\kappa',\gamma'>0$ and $\nu'=|\varepsilon|^{\gamma'}$. Using the preceding bound we can easily prove that:

$$\sup_{x\in\mathbb{Z}^{d+1}}\sum_{\delta(\gamma)\ge r}^{\gamma\ni x}|\rho(\gamma)|\le c(\nu')^{1/2}e^{-(\kappa'/2)r}\ . \tag{21}$$

where $\delta(\gamma)$ is the diameter of the polymer γ. A standard theorem, proved for instance in [GMM73][GBG03], states that, if $\rho(\gamma)$ satisfies (6.21), then

$$\sup_{x\in\Lambda_D}\sum_{\delta(\Gamma)\ge r}^{\Gamma\ni x}\Upsilon^T(\Gamma)|\rho(\Gamma)|\le c(\nu')^{1/4}e^{-(\kappa'/4)r}\ . \tag{22}$$

This implies that, varying Λ, P_Λ is a uniformly convergent sequence of analytic functions in a domain independent from Λ. The limit, still analytic in the same domain (thanks to Vitali's convergence theorem), is independent of the way the thermodynamic limit is performed (*i.e.* one can send the time side of Λ to ∞ either before the spatial side is sent to ∞ or together with it), thanks to the exponentially fast convergence of the sequence, implied by (6.22). For the same reason, the limit is also independent of the choice of boundary conditions and, because of translational invariance, it is equal to:

$$P\stackrel{def}{=}\lim_{|\Lambda|\to\infty}P_\Lambda=\frac{1}{a}\log l+\frac{2}{h_0a}\sum_{\Gamma\subset\mathbb{Z}^{d+1}}^{\Gamma\ni(0,0)}\frac{\Upsilon^T(\Gamma)\rho(\Gamma)}{|\Gamma|}\ , \tag{23}$$

where $|\Gamma|\stackrel{def}{=}|\cup_{\gamma\in\Gamma}\gamma|$ and $2/(h_0a)=\lim_{|\Lambda|\to\infty}|\Lambda_D|/|\Lambda|$.

6.4 Analyticity of the Mean Values

The analyticity for the mean value of an analytic local observable $\mathcal{O}(\psi_V)$ (depending on the variables in the finite set $V\subset\mathbb{Z}^d$) is an easy corollary of the previous result.

We first observe that $\mu^{SRB}(\mathcal{O})=\lim_{N,T\to\infty}\frac{1}{|V_N||I_T|}\sum_{(\xi,i)\in(V_N\times I_T)}\mu^{SRB}(\mathcal{O}\circ\rho^\xi\circ S_\varepsilon^i)$. This is true thanks to the time and space translation invariance of μ^{SRB}. Moreover it is possible to decompose $\mathcal{O}$ as

$$\mathcal{O}\big(h_\varepsilon(c_0(\sigma))_V\big)=\sum_{X\cap(V\times\{0\})=\emptyset}\mathcal{O}_X^{(0,0)}(\sigma_X)$$

This can be done expanding $\mathcal{O}(h_\varepsilon)$ in power of ψ, using the representation of h_ε given in section 3 and 5 and collecting the terms with the same support. Moreover we will set:

$$\mathcal{O}_X^{(\xi,j)}(\sigma_X) \stackrel{def}{=} \mathcal{O}^{(0,0)}_{\rho^{-\xi}\tau^{-j}X}(\sigma_X)$$

and

$$\mathcal{O}_X(\sigma_X) \stackrel{def}{=} \sum_{\substack{(\xi,j) \\ \rho^\xi V\times\{j\}\cap X\neq\emptyset}} \mathcal{O}_X^{(\xi,j)}(\sigma_X)$$

It is easy to realize that $\mathcal{O}_X$ is invariant under space and time translations, and satisfies

$$||\mathcal{O}_X||_\infty \leq c_V \nu^{n_X} e^{-\kappa d_c(X)} \prod_{i=1}^{n_X} e^{-\kappa|R_i(X)|} \;, \tag{24}$$

for some $\kappa, \gamma > 0$, $\nu = |\varepsilon|^\gamma$ and some constant $c_V > 0$ which depends on the size of V. Setting $\Lambda = V_N \times I_T$, the thermodynamic limit of the mean value of $\mathcal{O}(\psi_V)$ can be written as:

$$\mu^{\mathbf{SRB}}(\mathcal{O}) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \partial_\zeta \log \frac{\sum_{\sigma_\Lambda} e^{-\sum_{X\cap\Lambda\neq\emptyset}[\phi_X(\sigma_X)-\zeta\mathcal{O}_X(\sigma_X)]}}{\sum_{\sigma_\Lambda} e^{-\sum_{X\cap\Lambda\neq\emptyset}\phi_X(\sigma_X)}}\Bigg|_{\zeta=0} \stackrel{def}{=} \partial_\zeta P_{\mathcal{O}}(\zeta) \;. \tag{25}$$

Via a new cluster expansion we find:

$$\mu^{\mathbf{SRB}}(\mathcal{O}) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \partial_\zeta \sum_{\Gamma\cap\Lambda_D\neq\emptyset} \Upsilon^T(\Gamma)\big(\rho^\zeta(\Gamma) - \rho(\Gamma)\big)\Big|_{\zeta=0} \;, \tag{26}$$

where $\rho^\zeta(\gamma)$ are the activities corresponding to the potential $\phi_X - \zeta\mathcal{O}_X$. For $|\zeta|$ small enough, the potential $\phi_X - \zeta\mathcal{O}_X$ satisfies the same bounds of ϕ_X so that $\sum_{\Gamma\cap\Lambda_D\neq\emptyset} \Upsilon^T(\Gamma)\big(\rho^\zeta(\Gamma)-\rho(\Gamma)\big)$ is a uniformly convergent sequence of functions, analytic in ε and ζ in the product of two small discs. This implies that $\mu^{\mathbf{SRB}}(\mathcal{O})$ is analytic in ε and given by:

$$\mu^{\mathbf{SRB}}(\mathcal{O}) = \frac{2}{h_0 a} \sum_{\Gamma\subset\mathbb{Z}^{d+1}}^{\Gamma\ni(0,0)} \frac{\Upsilon^T(\Gamma)}{|\Gamma|} \partial_\zeta \big(\rho^\zeta(\Gamma) - \rho(\Gamma)\big)\Big|_{\zeta=0} \;. \tag{27}$$

A Unperturbed Nonlinear Dynamics

The result about analyticity can be extended to the case in which the unperturbed dynamic is made up of independent *nonlinear* analytic Anosov systems $s_0 : \mathbb{T}^2 \to \mathbb{T}^2$. We suppose that there exist $v_\pm(\psi)$ and $\lambda_\pm(\psi)$ such that:

$$\big(Ds_0\ v_+\big)(\phi) = \lambda_+(\phi) v_+\big(s_0(\phi)\big) \quad , \quad \big(Ds_0\ v_-\big)(\phi) = \lambda_-(\phi) v_-\big(s_0(\phi)\big) \ , \tag{1}$$

with $\phi \in \mathbb{T}^2$, $v_\pm(\phi)$ and $\lambda_\pm(\phi)$ Hölder continuous and $|\lambda_+(\phi)|^{-1}$, $|\lambda_-(\phi)| \leq \lambda < 1$. Then we consider again a perturbation $f(\psi)$ on $\mathcal{T}_N$ analytic in ψ. Observe however that in this case the most naive example of perturbation, $f(\psi) \stackrel{def}{=} f^+(\psi) v_+(\psi)$, with $f^+(\psi)$ analytic, is *no longer* an analytic perturbation.

A.1 Conjugation

The constitutive equation for δh_ε, lifted on $\mathbb{R}^{2V_N}$, is:

$$S_0\big(h_\varepsilon(\psi)\big) + \varepsilon f\big(h_\varepsilon(\psi)\big) = S_0(\psi) + \delta h_\varepsilon\big(S_0(\psi)\big). \tag{2}$$

In order to exploit the hyperbolicity, it is convenient to arrange the terms as follows:

$$\big(DS_0\ \delta h_\varepsilon\big)(\psi) - \delta h_\varepsilon\big(S_0(\psi)\big) = -\varepsilon f\big(\psi + \delta h_\varepsilon(\psi)\big) - \big[S_0\big(h_\varepsilon(\psi)\big) - S_0(\psi) - \big(DS_0\ \delta h_\varepsilon\big)(\psi)\big]. \tag{3}$$

Define

$$f^{\ ,\mathbf{x}_1 \cdots\ \mathbf{x}_s}(\psi) \stackrel{def}{=} \frac{\partial^s}{\partial \zeta_1, \ldots\ \partial \zeta_s} f\Big(\psi + \zeta_1 w_{0,\alpha_1}^{(\xi_1)}(\psi) + \ldots\ \zeta_s w_{0,\alpha_s}^{(\xi_s)}(\psi)\Big)\Big|_{\zeta_1 = \ldots\ \zeta_s = 0}$$

whereas

$$f^{\ ,\mathbf{x}_1 \cdots\ \mathbf{x}_s}(\psi) \stackrel{def}{=} \sum_{\mathbf{x}} f^{\mathbf{x},\mathbf{x}_1 \cdots \mathbf{x}_s}(\psi) w_{0,\alpha}^{(\xi)}\big(S_0(\psi)\big) \qquad \text{and} \qquad S_0^{\ ,\mathbf{x}_1 \cdots \mathbf{x}_s}(\psi) \stackrel{def}{=} \sum_{\mathbf{x}} S_0^{\mathbf{x},\mathbf{x}_1 \cdots \mathbf{x}_s}(\psi) w_{0,\alpha}^{(\xi)}\big(S_0(\psi)\big);$$

Writing $\delta h_\varepsilon(\psi) \stackrel{def}{=} \sum_{\mathbf{x}} \delta h_\varepsilon^{\mathbf{x}}(\psi) w_{0,\alpha}^{(\xi)}(\psi)$, and $\big(DS_0\ w_{0,\alpha}^{(\xi)}\big)(\psi) \stackrel{def}{=} \sum_{\mathbf{y}} S_0^{\mathbf{y},\mathbf{x}}(\psi) w_{0,\beta}^{(\eta)}\big(S_0(\psi)\big)$, with $S_0^{\mathbf{y},\mathbf{x}}(\psi) = \lambda_\alpha(\psi_\xi)\delta_{\mathbf{x},\mathbf{y}}$, we get:

$$\lambda_\alpha \delta h_\varepsilon^{\mathbf{x}}(\psi) - \delta h_\varepsilon^{\mathbf{x}}\big(S_0(\psi)\big) = -\varepsilon \sum_{s\geq 0} \Big(\frac{f^{\mathbf{x},\mathbf{x}_1 \cdots \mathbf{x}_s}}{s!} \delta h_\varepsilon^{\mathbf{x}_1} \cdots \delta h_\varepsilon^{\mathbf{x}_s}\Big)(\psi) - \sum_{s \geq 2} \Big(\frac{S_0^{\mathbf{x},\mathbf{x}_1 \ldots\ \mathbf{x}_s}}{s!} \delta h_\varepsilon^{\mathbf{x}_1} \ldots\ \delta h_\varepsilon^{\mathbf{x}_s}\Big)(\psi), \tag{4}$$

Finally, the recursive equation for the Taylor coefficients of $\delta h_\varepsilon^{\xi^+}(\psi)$ is:

$$\begin{aligned} \delta h_{(k+1)}^{\xi^+}(\psi) = \quad & -\sum_{p\geq 0} \Big(\prod_{m=0}^{p} \lambda_+^{-1}\big(s_0^m(\psi_\xi)\big)\Big) \sum_{s \geq 0} \sum_{\substack{k_1 + \cdots + k_s = k \\ k_i \geq 1}} \Big(\frac{f^{\xi^+,\mathbf{x}_1 \cdots \mathbf{x}_s}}{s!} \delta h_{(k_1)}^{\mathbf{x}_1} \cdots \delta h_{(k_s)}^{\mathbf{x}_s}\Big)\big(S_0^p(\psi)\big) + \\ & + \sum_{p\geq 0} \Big(\prod_{m=0}^{p} \lambda_+^{-1}\big(s_0^m(\psi_\xi)\big)\Big) \sum_{s \geq 2} \sum_{\substack{k_1 + \cdots + k_s = k+1 \\ k_i \geq 1}} \Big(\frac{S_0^{\xi^+,\mathbf{x}_1 \cdots \mathbf{x}_s}}{s!} \delta h_{(k_1)}^{\mathbf{x}_1} \cdots \delta h_{(k_s)}^{\mathbf{x}_s}\Big)\big(S_0^p(\psi)\big)\ . \end{aligned} \tag{5}$$

A similar equation holds for $\mathbf{x} = \xi^-$.

¿From now on, the construction of the conjugation function goes on as in the linear case with similar considerations. We have only to take in account the fact that a tree of order k (w.r.t. ε) does not necessarily have k branches, because of the term on the last line of (1.5) (to be called a vertex of type 0). Since the number of lines entering a vertex of type 0 is ≥ 2, one can easily prove that the number b_k of branches of a tree of order k is bounded by $k \leq b_k \leq 2k - 1$, so that nothing qualitatively changes in the bounds and the proof of analyticity of δh_ε proceed as in sections sections 3 and 6.

A.2 Unstable Direction

The perturbed unstable direction in the point $h_\varepsilon(\psi)$ is given by the equation:

$$\Big(DS_\varepsilon \; w^{(\xi)}_{\varepsilon,+}\Big)\big(h_\varepsilon(\psi)\big) = w^{(\eta)}_{\varepsilon,+}\Big(h_\varepsilon\big(S_0(\psi)\big)\Big)L^{\eta\xi}(\psi). \tag{6}$$

Setting $w^{(\xi)}_{\varepsilon,+}\big(h_\varepsilon(\psi)\big)\stackrel{def}{=}v^{(\xi)}_\varepsilon(\psi)$, it is convenient to rearrange the terms of the equation in the following way:

$$\begin{aligned}\big(DS_0\; v^{(\xi)}_\varepsilon\big)(\psi) - \lambda_+(\psi_\xi)v^{(\xi)}_e\big(S_0(\psi)\big) = \quad & \delta L^{\eta\xi}(\psi)v^{(\eta)}_\varepsilon\big(S_0(\psi)\big)\\ & -\varepsilon\Big(Df\Big)\big(h_\varepsilon(\psi)\big)v^{(\xi)}_\varepsilon(\psi) - \Big[DS_0\big(h_\varepsilon(\psi)\big) - DS_0(\psi)\Big]v^{(\xi)}_\varepsilon(\psi).\end{aligned} \tag{7}$$

Defining $v^{(\eta)}_\varepsilon(\psi) = \sum_{\mathbf{x}} V^{(\eta)}_{\varepsilon,\mathbf{x}}(\psi)w^{(\xi)}_{0,\alpha}(\psi)$, and using again the considerations of section 4, we finally get

$$\begin{aligned}\lambda_\alpha(\psi_\rho)V^{(\rho)}_{\varepsilon,\mathbf{x}}(\psi) - \lambda_+(\psi_\rho)V^{(\rho)}_{\varepsilon,\mathbf{x}}\big(S_0(\psi)\big) = \quad & +\delta L^{\zeta\rho}(\psi)V^{(\zeta)}_{\varepsilon,\mathbf{x}}\big(S_0(\psi)\big)\\ & -\varepsilon\sum_{s\geq 0}\Big(\frac{f^{\mathbf{x},\mathbf{y}\mathbf{x}_1\cdots\mathbf{x}_s}}{s!}V^{(\rho)}_{\varepsilon,\mathbf{y}}\delta h^{\mathbf{x}_1}_\varepsilon\cdots\delta h^{\mathbf{x}_s}_\varepsilon\Big)(\psi)\\ & -\sum_{s\geq 1}\Big(\frac{S_0^{\mathbf{x}\mathbf{y},\mathbf{x}_1\dots\,\mathbf{x}_s}}{s!}V^{(\rho)}_{\varepsilon,\mathbf{y}}\delta h^{\mathbf{x}_1}_\varepsilon\dots\delta h^{\mathbf{x}_s}_\varepsilon\Big)(\psi)\end{aligned}$$

(with $\big(DS_0{}^{,\mathbf{x}_1\dots\,\mathbf{x}_s}w^{(\xi)}_a\big)(\psi)\stackrel{def}{=}S_0^{\mathbf{x}\mathbf{y},\mathbf{x}_1\dots\,\mathbf{x}_s}(\psi)w^{(\eta)}_b(S_0\psi)\big)$. Again, because of the third term in the r.h.s. of eq.(1.7) , the number of branches of a tree appearing in the construction of δV and δL is greater (in general) than the order of the tree itself. This is not a problem, since one can easily realize that, again, the number b_k of branches of a tree of order k is such that $k \leq b_k \leq 2k$.

A.3 SRB Interactions

Following the proof in section 4 and, proceeding as in sections 5 and 6 one proves analyticity of SRB distribution. In fact the only (slight) difference in the construction of SRB potentials is in the telescopic cutting necessary to represent h, L, V and Λ as sums of local functions of spin variables. Notice that now each tree node is associated to the product of a node function $f_v(\psi)$ (*e.g.* in the case of a tree contributing to δh, f_v can be a derivative of f or a derivative of S_0, see (1.5)) times a product of local Lyapunov exponents, like the factor $\prod_{m=0}^{p}\lambda_+^{-1}\big(s_0^m(\psi_\xi)\big)\stackrel{def}{=}\Theta_+(p, s_0^{p(v)}(\psi_\xi))$ in (1.5); the analogous expression appearing in a vertex with $\alpha(v) = -$ will be denoted by $\Theta_-(p, s_0^{p(v)}\psi_\xi)$. So the total node function associated to a vertex v will be now of the form

$$F^v(p(v), S_0^{p(v)}(\psi))\stackrel{def}{=}\Theta_{\alpha(v)}(p(v), s_0^{p(v)}(\psi_{\xi_v}))f_v(S_0^{p(v)}(\psi)) \;, \tag{8}$$

where v' is the vertex immediately following v. The telescopic expansion (5.1) has to be done *separately* for each of the factor in the above equation ($\lambda(\psi)$ is hölder continuous), getting in the end potentials with the same kind of decay rate. The bounds are not qualitatively changed and the subsequent analysis of section 5 follows so that, by suitably modifying the decimation procedure, analyticity of SRB measure can be proved. We point out that a main difference in the proof of convergence of the cluster expansion is that now the unperturbed potentials are not vanishing, but have support only on time like segments $I \subset \mathbb{Z}$, and are exponentially decaying with the diameter of I. For this reason one cannot proceed exactly as in section 6. The standard way to treat this problem, see [CO81], is to choose a length r such that the unperturbed interactions on sets I, $\mathrm{diam}(I) > r$, are small enough for the cluster expansion. Then one fix the size of the B-blocks $b = r$, and the size of the H-blocks, h, such that the Perron-Frobenius theorem is true for the *reduced partition function* $Z_r(\beta^{(p)}_\xi, \eta^{(p)}_\xi, \beta^{(p+1)}_\xi)$, in which only the interaction on sets $I \subset (B^{(p)}_\xi \cup H^{(p)}_\xi \cup B^{(p+1)}_\xi)$, $\mathrm{diam}(I) \leq r$ are taken in account.

B Green-Kubo Formula and Large Deviation

In this section we deal with an application. We introduce the *local phase space contraction rate* on a volume $V_0 \subset V_N$ averaged on a time T_0, given by

$$\eta_{\Lambda_0}(\psi) \stackrel{def}{=} \frac{1}{|\Lambda_0|} \sum_{j \in I_{T_0}} \log \left| \det(DS_\varepsilon)_{V_0}\left(S_\varepsilon^j(\psi)\right) \right| , \tag{1}$$

with $\Lambda_0 = V_0 \times I_0$ and $I_0 \stackrel{def}{=} [-T_0/2, T_0/2] \cap \mathbb{Z}$. We prove a Green-Kubo formula for η_{Λ_0}, from which it will come out that generically its mean value η_+ is strictly negative. Furthermore, we can show the large fluctuations of η_{Λ_0} around η_+ satisfy a large deviation principle, namely they are asymptotically described by a strictly convex *free energy functional* $F(\eta)$: it can be obtained as the Legendre transform of the generating functional $P(\zeta) = P_{\eta_{\Lambda_0}}(\zeta)$, see eq. (6.25).

For the rest of the appendix the SRB interaction will be called $\{\phi_X^+\}_{X \subset \mathbb{Z}^{d+1}}$, to remind that they are derived from the unstable restriction of DS_ε.

Theorem B1: *Given S_ε such that $\eta_+ < 0$,*

1. *$P(\zeta)$ is analytic and strictly convex in ζ, for $|\varepsilon| < \varepsilon_0$, $|\zeta| \le 1$, with ε_0 small enough;*
2. *the Green-Kubo formula is valid:*

$$\partial_\varepsilon^2 P'(0)|_{\varepsilon=0} = -\frac{1}{2}\partial_\varepsilon^2 P''(0)|_{\varepsilon=0}. \tag{2}$$

Theorem B2: *Given S_ε such that $\eta_+ < 0$,*

1. *the free energy $F(\eta)$ is analytic in η, for $|\varepsilon| < \varepsilon_0$, and $\eta \in [P'(-1), P'(1)]$;*
2. *if $[a,b] \subset [P'(-1), P'(1)]$, then*

$$\lim_{|\Lambda_0| \to \infty} \frac{1}{|\Lambda_0|} \log \mu^{SRB}\big(\eta_{\Lambda_0} \in [a,b]\big) = \max_{\eta \in [a,b]} -\Delta F(\eta, \eta_+) , \tag{3}$$

with $\Delta F(\eta, \eta_+) \stackrel{def}{=} F(\eta) - F(\eta_+)$.

B.1 Local Phase Space Contraction Rate

Repeating the construction of SRB potentials leading to (2.8), we set:

$$\eta_{\Lambda_0}(h_\varepsilon(c_0(\sigma))) \stackrel{def}{=} \frac{1}{|\Lambda_0|} \sum_{X \subset \mathbb{Z}^{d+1}}^{X \cap \Lambda_0 \neq 0} \phi_X(\sigma_X) , \tag{4}$$

for a suitable potential ϕ_X, satisfying

$$||\phi_X||_\infty \le c e^{-\kappa d_c(X)} \nu^{n_X} \prod_{i=1}^{n_X} e^{-\kappa |R_i(X)|} , \tag{5}$$

for some $c, \kappa, \gamma > 0$ and $\nu = |\varepsilon|^\gamma$. ¿From the invariance under time translations of the SRB measure, we have

$$\eta_+ \stackrel{def}{=} \lim_{|V_0| \to \infty} \frac{1}{|V_0|} \mu^{SRB}\big(\log \left| \det(DS_\varepsilon)_{V_0} \right|\big) = \lim_{|\Lambda_0| \to \infty} \mu^{SRB}(\eta_{\Lambda_0}) =$$

$$= \lim_{|\Lambda_0|\to\infty} \frac{1}{|\Lambda_0|} \sum\nolimits_{X\cap\Lambda_0\neq\emptyset} \mu^{\mathbf{SRB}}(\phi_X) = \tag{6}$$

$$= \lim_{|\Lambda_0|\to\infty} \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda_0|} \partial_\zeta \log \left. \frac{\sum_{\sigma_\Lambda} e^{-\sum_{X\cap\Lambda\neq\emptyset} \phi_X^+(\sigma_X) + \zeta \sum_{X\cap\Lambda_0\neq\emptyset} \phi_X(\sigma_X)}}{\sum_{\sigma_\Lambda} e^{-\sum_{X\cap\Lambda\neq\emptyset} \phi_X^+(\sigma_X)}} \right|_{\zeta=0} .$$

It is easy to show the last expression is equal to the one with the summations over $X \cap \Lambda \neq \emptyset$ and $X \cap \Lambda_0 \neq \emptyset$ replaced by $X \subset \Lambda_0$ and without the limit in Λ (since the correction is only a border effect; or simply using again the cluster expansion developed in section 6.4). In this way, defining the *generating function* $P(\zeta)$, as

$$P(\zeta) \stackrel{def}{=} \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log \frac{\sum_{\sigma_\Lambda} e^{-\sum_{X\subset\Lambda} (\phi_X^+ - \zeta\phi_X)(\sigma_X)}}{\sum_{\sigma_\Lambda} e^{-\sum_{X\subset\Lambda} \phi_X^+(\sigma_X)}} . \tag{7}$$

we finally get:

$$\eta_+ = P'(0) . \tag{8}$$

Analyticity is achieved by cluster expansion (we do not need ζ small, but we can take, say, $|\zeta| \leq 1$, since $\{\phi_X\}_X$ are $O(\varepsilon)$).

B.2 Green-Kubo Formula

Consider the case in which s_0 is the Arnold's cat map defined by (1.1).

Using the definition of pressure (2.7) and the fast convergence properties of the cluster expansion of $P(\zeta)$, we find:

$$P(\zeta) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log \frac{\mu_{N,0}^{\mathbf{SRB}}\left(e^{-\sum_{j\in I_T} \log|\det L\circ S_0^j| + \zeta \sum_{j\in I_T} \log|\det DS_\varepsilon \circ h_\varepsilon \circ S_0^j|}\right)}{\mu_{N,0}^{\mathbf{SRB}}\left(e^{-\sum_{j\in I_T} \log|\det L\circ S_0^j|}\right)} , \tag{9}$$

where:

1. the matrix $L = \mathcal{L} \circ h_\varepsilon$ was introduced in section 2.2 above;
2. $\mu_{N,0}^{\mathbf{SRB}}$ is the unperturbed SRB measure: if $\mathcal{O}(\psi)$ is a local hölder continuous observable, it is defined as

$$\mu_{N,0}^{\mathbf{SRB}}(\mathcal{O}) = \lim_{|\Lambda|\to\infty} \frac{\sum_{\sigma_\Lambda} \mathcal{O}(c_0(\sigma_\Lambda|\hat{\sigma}_{\Lambda^c}))}{\sum_{\sigma_\Lambda} 1} \tag{10}$$

   and, independently of the boundary conditions, it is equal to the Lebesgue measure.

Defining U_ζ as

$$U_\zeta = \log|\det L| - \zeta \log|\det S_0^{-1} \circ DS_\varepsilon \circ h_\varepsilon| , \tag{11}$$

and using that $\mu_{N,0}^{\mathbf{SRB}}$ is the Lebesgue measure on $\mathcal{T}_N$, we find:

$$P(\zeta) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log \frac{\int d\psi\, e^{-\sum_{j\in I_T} U_\zeta(S_0^j\psi)}}{\int d\psi\, e^{-\sum_{j\in I_T} U_0(S_0^j\psi)}} , \tag{12}$$

so that $P'(0)$ is equal to

$$P'(0) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{j\in I_T} \frac{\int d\psi \log|\det S_0^{-1} DS_\varepsilon(h_\varepsilon(S_0^j\psi))|\, e^{-\sum_{j\in I_T} U_0(S_0^j\psi)}}{\int d\psi\, e^{-\sum_{j\in I_T} U_0(S_0^j\psi)}} . \tag{13}$$

Since $P'(0)|_{\varepsilon=0}$ is trivially $=0$, we can try to see if $\partial_\varepsilon P'(0)|_{\varepsilon=0}$ is different from zero (if it were, $P'(0)$ would be different from zero for $\varepsilon \neq 0$ small enough). Recalling that $f(\psi)$ is the perturbing function and $f^\xi(\psi)$ is its projection on the ξ–th site, we get:

$$\begin{aligned}
\partial_\varepsilon P'(0)|_{\varepsilon=0} \qquad &= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{j\in I_T} \int \frac{\mathrm{d}\psi}{(2\pi)^{2|V_N|}} \mathrm{Tr}\,[S_0^{-1} Df(S_0^j\psi)] = \\
&= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{j\in I_T} \sum_{\substack{\alpha=\pm\\ \xi\in V_N}} \int \frac{\mathrm{d}\psi}{(2\pi)^{2|V_N|}} \lambda^\alpha f^{\xi^\alpha,\xi^\alpha}(S_0^j\psi)\ .
\end{aligned} \tag{14}$$

Since f is periodic we have $\partial_\varepsilon P'(0)|_{\varepsilon=0}=0$.

A straightforward calculation shows that

$$\begin{aligned}
&\frac{1}{2}\partial_\varepsilon^2 P'(0)|_{\varepsilon=0} = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{j\in I_T} \int \frac{\mathrm{d}\psi}{(2\pi)^{2|V_N|}} \Big\{ \mathrm{Tr}\,[S_0^{-1} D^2 f(S_0^j\psi)\delta h_{(1)}(S_0^j\psi)] - \\
&-\frac{1}{2}\mathrm{Tr}\,[(S_0^{-1}Df(S_0^j\psi))^2] - \sum_{j'\in I_T} \mathrm{Tr}\,(S_0^{-1}Df(S_0^j\psi))\mathrm{Tr}^{\,(u)}(S_0^{-1}Df(S_0^{j'}\psi)) \Big\}\ ,
\end{aligned} \tag{15}$$

where $\mathrm{Tr}^{\,(u)}$ is the trace restricted to the (unperturbed) unstable manifold. The preceding expression can be rewritten in a more convenient way. Using the explicit expression of $\delta h_{(1)}$ eq.(3.5), and defining $A_0 = \cup_{\xi\in nn(0)} nn(\xi)$, we find that the first term in eq.(2.15) is equal to

$$\begin{aligned}
&\sum_{\substack{\alpha_i=\pm\\ |\xi|\le 1}} \sum_{p\ge 0} \int \frac{\mathrm{d}\psi_{A_0}}{(2\pi)^{2|A_0|}} \lambda^{\alpha_1} f^{0^{\alpha_1},0^{\alpha_1}\xi^{\alpha_2}}(\psi)(-\alpha_2)\lambda^{p+\rho_{\alpha_2}} f^{\xi^{\alpha_2}}(S_0^{\alpha_2(p+1-\rho_{\alpha_2})}\psi) = \\
&= \sum_{\substack{\alpha_i=\pm\\ |\xi|\le 1}} \sum_{p\ge 0} \int \frac{\mathrm{d}\psi_{A_0}}{(2\pi)^{2|A_0|}} \lambda^{\alpha_1} f^{0^{\alpha_1},0^{\alpha_1}}(\psi)\alpha_2\lambda^{\alpha_2} f^{\xi^{\alpha_2},\xi^{\alpha_2}}(S_0^{\alpha_2(p+1-\rho_{\alpha_2})}\psi)\ .
\end{aligned} \tag{16}$$

Integrating by parts, we see that the sum of the second and third term in eq. (2.15) is equal to:

$$-\frac{1}{2} \sum_{\substack{\alpha_i=\pm\\ |\xi|\le 1}} \int \frac{\mathrm{d}\psi_{A_0}}{(2\pi)^{2|A_0|}} \lambda^{\alpha_1} f^{0^{\alpha_1},0^{\alpha_1}}(\psi) \Big[\lambda^{\alpha_2} f^{\xi^{\alpha_2},\xi^{\alpha_2}}(\psi) + \lambda \sum_{p\in\mathbb{Z}} f^{\xi^+,\xi^+}(S_0^p\psi) \Big]\ . \tag{17}$$

Combining the three contributions, we finally find:

$$\begin{aligned}
\partial_\varepsilon^2 P'(0)|_{\varepsilon=0} \qquad &= -\sum_{\substack{\alpha_i=\pm\\ |\xi|\le 1}} \sum_{p\in\mathbb{Z}} \int \frac{\mathrm{d}\psi_{A_0}}{(2\pi)^{2|A_0|}} \lambda^{\alpha_1} f^{0^{\alpha_1},0^{\alpha_1}}(\psi)\lambda^{\alpha_2} f^{\xi^{\alpha_2},\xi^{\alpha_2}}(S_0^p\psi) = \\
&= -\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \int \frac{\mathrm{d}\psi}{(2\pi)^{2|V_N|}} \Big(\sum_{j\in I_T} \mathrm{Tr}\,[S_0^{-1}Df(S_0^j\psi)] \Big)^2 = -\frac{1}{2}\partial_\varepsilon^2 P''(0)|_{\varepsilon=0}\ ,
\end{aligned} \tag{18}$$

that is the expected Green-Kubo relation (see [GR97]).

¿From eq.(2.18) and eq.(2.8), we see that, for ε small enough, η_+ is negative and, generically, strictly negative (the condition for f to be *generic* is just that the first line in eq.(2.18) is different from 0).

Let us now compute eq.(2.18) in a special case, essentially the simpler possible. Let

$$f^{\xi^+}(\psi) = \sum_{\eta\in nn(\xi)} \sin(\psi^1_\xi - \psi^1_\eta) \quad , \quad f^{\xi^-}(\psi) = 0\ . \tag{19}$$

Substituting such choice in eq.(2.18), we find:

$$\partial_\varepsilon^2 P'(0)|_{\varepsilon=0} = -2\sum_{|\xi|=1}\int \frac{\mathrm{d}\psi_0}{(2\pi)^2}\frac{\mathrm{d}\psi_\xi}{(2\pi)^2}\lambda^2\cos^2(\psi_0^1-\psi_\xi^1)(v_+\cdot\hat{e}_1)^2 = -\frac{2d}{1+\lambda^{-2}}\ , \tag{20}$$

where $\hat{e}_1 = (1,0)$ and we used that $v_+ = (\frac{1}{\sqrt{1+\lambda^2}}, -\frac{\lambda}{\sqrt{1+\lambda^2}})$.

So, choosing $\varepsilon \in \mathbb{R}$ small enough and different from zero, $\eta_+ = P'(0) = -\frac{d}{1+\lambda^{-2}}\varepsilon^2 + O(\varepsilon^3) < 0$. Furthermore, if $\zeta \in \mathbb{R}$ has modulus smaller than 1, $P(\zeta)$ is strictly convex (since $\frac{1}{2}P''(0) = -P'(0) > 0$ and $P(\zeta)$ is analytic for $|\zeta| \le 1$ and ε small enough).

B.3 Large deviations

In the present section we shall prove a large deviations property for η_{Λ_0}. We will follow the classical strategy set up in [Si], [GLM02] (in particular we will refer to the formulas in section 5 of the latter). The proof below will hold in the case $\eta_+ < 0$, namely in the generic case or, to be definite, in the case the perturbation is chosen as in eq.(2.19).

Thanks to the convexity of $P(\zeta)$, given $\eta \in [P'(-1), P'(1)]$, there exist a unique point $Z(\eta) \in [-1,1]$ such that $P'\big(Z(\eta)\big) \equiv \eta$; considering such a point η and its neighbor of radius δ, $I_\delta(\eta)$, such that $I_\delta(\eta) \subset [P'(-1), P'(1)]$, from the "large deviation property III", section 5 of [GLM02], we get:

$$\mu^{\mathbf{SRB}}\big(\eta_+ \in I_\delta(\eta)\big) = O(1)e^{O(\delta|\Lambda_0|)}e^{O(|\partial\Lambda_0|)}\exp\Big\{\big[P\big(Z(\eta)\big) - P(0) - Z(\eta)\eta\big]|\Lambda_0|\Big\}\ . \tag{21}$$

In our case $P(0) = 0$. Still for $\eta \in [P'(-1), P'(1)]$, we define the free energy $F(\eta)$ as the Laplace transform of the generating function $P(\zeta)$:

$$F(\eta) \stackrel{def}{=} \max_\zeta\{\zeta\eta - P(\zeta)\} = Z(\eta)\eta - P\big(Z(\eta)\big)\ ; \tag{22}$$

therefore, for $I_\delta(\eta) \subset [P'(-1), P'(1)]$

$$\mu^{\mathbf{SRB}}\Big(\eta_+ \in I_\delta(\eta)\Big) = O(1)e^{O(\delta|\Lambda_0|)}e^{O(|\partial\Lambda_0|)}\exp\Big\{-|\Lambda_0|\Delta F(\eta,\eta_+)\Big\}\ ; \tag{23}$$

where $\Delta F(\eta,\eta_+) \stackrel{def}{=} F(\eta) - F(\eta_+)$ (indeed $F(\eta_+) = -P(0) = 0$).

Finally, if $[a,b] \subset [P'(-1), P'(1)]$, it is suitable to take $\delta_{\Lambda_0} = |\Lambda_0|^{-\beta}$, $0 < \beta < 1$, and divide the interval $[a,b]$ in $|b-a||\Lambda_0|^\beta$ identical disjoint subintervals centered in $\eta_n \stackrel{def}{=} a + (n-1/2)\delta_{\Lambda_0}$, we find:

$$\begin{aligned}\mu^{\mathbf{SRB}}\Big(\eta_{\Lambda_0} \in [a,b]\Big) &= \sum_{n=1}^{|b-a||\Lambda_0|^\beta}\mu^{\mathbf{SRB}}\Big(\eta_+ \in I_{\delta_{\Lambda_0}}(\eta_n)\Big)\\ &= O(1)|\Lambda_0|^\beta e^{O(|\Lambda_0|^{1-\beta})}e^{O(|\partial\Lambda_0|)}\exp\Big\{|\Lambda_0|\max_{\eta\in[a,b]}\big[-\Delta F(\eta,\eta_+)\big]\Big\}\ ;\end{aligned} \tag{24}$$

namely the result in the second theorem.

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