ON THE SPECTRUM OF SCHRÖDINGER OPERATORS WITH QUASI-PERIODIC ALGEBRO-GEOMETRIC KDV POTENTIALS

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Dedicated with great pleasure to Vladimir A. Marchenko on the occasion of his 80th birthday.

Abstract. We characterize the spectrum of one-dimensional Schrödinger operators $H = -d^2/dx^2 + V$ in $L^2(\mathbb{R}; dx)$ with quasi-periodic complex-valued algebro-geometric potentials V (i.e., potentials V which satisfy one (and hence infinitely many) equation(s) of the stationary Korteweg–de Vries (KdV) hierarchy) associated with nonsingular hyperelliptic curves. The corresponding problem appears to have been open since the mid-seventies. The spectrum of H coincides with the conditional stability set of H and can explicitly be described in terms of the mean value of the inverse of the diagonal Green's function of H.

As a result, the spectrum of H consists of finitely many simple analytic arcs and one semi-infinite simple analytic arc in the complex plane. Crossings as well as confluences of spectral arcs are possible and discussed as well. These results extend to the $L^p(\mathbb{R}; dx)$-setting for $p \in [1, \infty)$.

1. Introduction

It is well-known since the work of Novikov [51], Marchenko [44], [45], Dubrovin [16], Dubrovin, Matveev, and Novikov [17], Flaschka [23], Its and Matveev [33], Lax [41], McKean and van Moerbeke [48] (see also [7, Sects. 3.4, 3.5], [26, p. 111–112, App. J], [46, Sect. 4.4], [52, Sects. II.6–II.10] and the references therein) that the self-adjoint Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V, \quad \operatorname{dom}(H) = H^{2,2}(\mathbb{R}) \tag{1.1}$$

in $L^2(\mathbb{R}; dx)$ with a real-valued periodic, or more generally, *quasi-periodic* and *real-valued* potential V, that satisfies one (and hence infinitely many) equation(s) of the stationary Korteweg–de Vries (KdV) equations, leads to a finite-gap, or perhaps more appropriately, to a finite-band spectrum $\sigma(H)$ of the form

$$\sigma(H) = \bigcup_{m=0}^{n-1} [E_{2m}, E_{2m+1}] \cup [E_{2n}, \infty). \tag{1.2}$$

It is also well-known, due to work of Serov [57] and Rofe-Beketov [55] in 1960 and 1963, respectively (see also [60]), that if V is *periodic* and *complex-valued* then the spectrum of the non-self-adjoint Schrödinger operator H defined as in (1.1) consists

Date: December 9, 2003.

1991 *Mathematics Subject Classification.* Primary 34L05, 35Q53, 58F07; Secondary 34L40, 35Q51.

Key words and phrases. KdV hierarchy, quasi-periodic algebro-geometric potentials, spectral theory.

either of infinitely many simple analytic arcs, or else, of a finite number of simple analytic arcs and one semi-infinite simple analytic arc tending to infinity. It seems plausible that the latter case is again connected with (complex-valued) stationary solutions of equations of the KdV hierarchy, but to the best of our knowledge, this has not been studied in the literature. In particular, the next scenario in line, the determination of the spectrum of H in the case of *quasi-periodic* and *complex-valued* solutions of the stationary KdV equation apparently has never been clarified. The latter problem is open since the mid-seventies and it is the purpose of this paper to provide a comprehensive solution of it.

To describe our results, a bit of preparation is needed. Let

$$G(z,x,x') = (H-z)^{-1}(x,x'), \quad z \in \mathbb{C}\backslash\sigma(H),\ x,x' \in \mathbb{R}, \tag{1.3}$$

be the Green's function of H (here $\sigma(H)$ denotes the spectrum of H) and denote by $g(z,x)$ the corresponding diagonal Green's function of H defined by

$$g(z,x) = G(z,x,x) = \frac{i\prod_{j=1}^n [z-\mu_j(x)]}{2R_{2n+1}(z)^{1/2}}, \tag{1.4}$$

$$R_{2n+1}(z) = \prod_{m=0}^{2n} (z-E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}, \tag{1.5}$$

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m,m' = 0,1,\dots,2n. \tag{1.6}$$

For any quasi-periodic (in fact, Bohr (uniformly) almost periodic) function f the mean value $\langle f\rangle$ of f is defined by

$$\langle f\rangle = \lim_{R\to\infty} \frac{1}{2R}\int_{-R}^{R} dx\, f(x). \tag{1.7}$$

Moreover, we introduce the set Σ by

$$\Sigma = \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}\big(\langle g(\lambda,\cdot)^{-1}\rangle\big) = 0\big\} \tag{1.8}$$

and note that

$$\langle g(z,\cdot)\rangle = \frac{i\prod_{j=1}^n \big(z-\widetilde\lambda_j\big)}{2R_{2n+1}(z)^{1/2}} \tag{1.9}$$

for some constants $\{\widetilde\lambda_j\}_{j=1}^n \subset \mathbb{C}$.

Finally, we denote by $\sigma_{\rm p}(T)$, $\sigma_{\rm r}(T)$, $\sigma_{\rm c}(T)$, $\sigma_{\rm e}(T)$, and $\sigma_{\rm ap}(T)$, the point spectrum (i.e., the set of eigenvalues), the residual spectrum, the continuous spectrum, the essential spectrum (cf. (4.15)), and the approximate point spectrum of a densely defined closed operator T in a complex Hilbert space, respectively.

Our principal new results, to be proved in Section 4, then read as follows:

Theorem 1.1. *Assume that V is a quasi-periodic* (*complex-valued*) *solution of the nth stationary KdV equation associated with the hyperelliptic curve $y^2 = R_{2n+1}(z)$ subject to* (1.5) *and* (1.6). *Then the following assertions hold:*
(*i*) *The point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\rm p}(H) = \sigma_{\rm r}(H) = \emptyset, \tag{1.10}$$

$$\sigma(H) = \sigma_{\rm c}(H) = \sigma_{\rm e}(H) = \sigma_{\rm ap}(H). \tag{1.11}$$

(ii) The spectrum of H coincides with Σ and equals the conditional stability set of H,

$$\begin{aligned} \sigma(H) &= \big\{\lambda\in\mathbb{C}\,\big|\,\mathrm{Re}\big(\langle g(\lambda,\cdot)^{-1}\rangle\big)=0\big\} && (1.12)\\ &= \{\lambda\in\mathbb{C}\,|\,\text{there exists at least one bounded distributional solution}\\ &\qquad\qquad 0\neq\psi\in L^\infty(\mathbb{R};dx)\ \text{ of } H\psi=\lambda\psi\}. && (1.13)\end{aligned}$$

(iii) $\sigma(H)$ is contained in the semi-strip

$$\sigma(H)\subset\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)\in[M_1,M_2],\ \mathrm{Re}(z)\geq M_3\}, \tag{1.14}$$

where

$$M_1=\inf_{x\in\mathbb{R}}[\mathrm{Im}(V(x))],\quad M_2=\sup_{x\in\mathbb{R}}[\mathrm{Im}(V(x))],\quad M_3=\inf_{x\in\mathbb{R}}[\mathrm{Re}(V(x))]. \tag{1.15}$$

(iv) $\sigma(H)$ consists of finitely many simple analytic arcs and one simple semi-infinite arc. These analytic arcs may only end at the points $\widetilde\lambda_1,\dots,\widetilde\lambda_n$, $E_0,\dots,E_{2n}$, and at infinity. The semi-infinite arc, σ_∞, asymptotically approaches the half-line $L_{\langle V\rangle}=\{z\in\mathbb{C}\,|\,z=\langle V\rangle+x,\ x\geq 0\}$ in the following sense: asymptotically, σ_∞ can be parameterized by

$$\sigma_\infty=\big\{z\in\mathbb{C}\,\big|\,z=R+i\,\mathrm{Im}(\langle V\rangle)+O\big(R^{-1/2}\big)\ \text{as } R\uparrow\infty\big\}. \tag{1.16}$$

(v) Each E_m, $m=0,\dots,2n$, is met by at least one of these arcs. More precisely, a particular E_{m_0} is hit by precisely $2N_0+1$ analytic arcs, where $N_0\in\{0,\dots,n\}$ denotes the number of $\widetilde\lambda_j$ that coincide with E_{m_0}. Adjacent arcs meet at an angle $2\pi/(2N_0+1)$ at E_{m_0}. (Thus, generically, $N_0=0$ and precisely one arc hits E_{m_0}.)
(vi) Crossings of spectral arcs are permitted and take place precisely when

$$\mathrm{Re}\big(\langle g(\widetilde\lambda_{j_0},\cdot)^{-1}\rangle\big)=0\ \text{ for some } j_0\in\{1,\dots,n\}\text{ with } \widetilde\lambda_{j_0}\notin\{E_m\}_{m=0}^{2n}. \tag{1.17}$$

In this case $2M_0+2$ analytic arcs are converging toward $\widetilde\lambda_{j_0}$, where $M_0\in\{1,\dots,n\}$ denotes the number of $\widetilde\lambda_j$ that coincide with $\widetilde\lambda_{j_0}$. Adjacent arcs meet at an angle $\pi/(M_0+1)$ at $\widetilde\lambda_{j_0}$. (Thus, generically, $M_0=1$ and two arcs cross at a right angle.)
(vii) The resolvent set $\mathbb{C}\backslash\sigma(H)$ of H is path-connected.

Naturally, Theorem 1.1 applies to the special case where V is a periodic (complex-valued) solution of the nth stationary KdV equation associated with a nonsingular hyperelliptic curve. Even in this special case, items (v) and (vi) of Theorem 1.1 provide additional new details on the nature of the spectrum of H.

As described in Remark 4.10, these results extend to the $L^p(\mathbb{R};dx)$-setting for $p\in[1,\infty)$.

Theorem 1.1 focuses on stationary quasi-periodic solutions of the KdV hierarchy for the following reasons. First of all, the class of algebro-geometric solutions of the (time-dependent) KdV hierarchy is defined as the class of all solutions of some (and hence infinitely many) equations of the stationary KdV hierarchy. Secondly, time-dependent algebro-geometric solutions of a particular equation of the (time-dependent) KdV hierarchy just represent isospectral deformations (the deformation parameter being the time variable) of a fixed stationary algebro-geometric KdV solution (the latter can be viewed as the initial condition at a fixed time t_0). In the present case of quasi-periodic algebro-geometric solutions of the nth KdV equation, the isospectral manifold of such a given solution is a complex n-dimensional torus, and time-dependent solutions trace out a path in that isospectral torus (cf. the discussion in [26, p. 12]).

Finally, we give a brief discussion of the contents of each section. In Section 2 we provide the necessary background material including a quick construction of the KdV hierarchy of nonlinear evolution equations and its Lax pairs using a polynomial recursion formalism. We also discuss the hyperelliptic Riemann surface underlying the stationary KdV hierarchy, the corresponding Baker–Akhiezer function, and the necessary ingredients to describe the Its–Matveev formula for stationary KdV solutions. Section 3 focuses on the diagonal Green's function of the Schrödinger operator H, a key ingredient in our characterization of the spectrum $\sigma(H)$ of H in Section 4 (cf. (1.12)). Our principal Section 4 is then devoted to a proof of Theorem 1.1. Appendix A provides the necessary summary of tools needed from elementary algebraic geometry (most notably the theory of compact (hyperelliptic) Riemann surfaces) and sets the stage for some of the notation used in Sections 2–4. Appendix B provides additional insight into one ingredient of the Its–Matveev formula; Appendix C illustrates our results in the special periodic non-self-adjoint case and provides a simple yet nontrivial example in the elliptic genus one case.

Our methods extend to the case of algebro-geometric non-self-adjoint second order finite difference (Jacobi) operators associated with the Toda lattice hierarchy. Moreover, they extend to the infinite genus limit $n \to \infty$ (cf. (1.2)–(1.5)) using the approach in [25]. This will be studied elsewhere.

Dedication. It is with great pleasure that we dedicate this paper to Vladimir A. Marchenko on the occasion of his 80th birthday. His strong influence on the subject at hand is universally admired.

2. The KdV hierarchy, hyperelliptic curves, and the Its–Matveev formula

In this section we briefly review the recursive construction of the KdV hierarchy and associated Lax pairs following [27] and especially, [26, Ch. 1]. Moreover, we discuss the class of algebro-geometric solutions of the KdV hierarchy corresponding to the underlying hyperelliptic curve and recall the Its–Matveev formula for such solutions. The material in this preparatory section is known and detailed accounts with proofs can be found, for instance, in [26, Ch. 1]. For the notation employed in connection with elementary concepts in algebraic geometry (more precisely, the theory of compact Riemann surfaces), we refer to Appendix A.

Throughout this section we suppose the hypothesis

$$V \in C^\infty(\mathbb{R}) \tag{2.1}$$

and consider the one-dimensional Schrödinger differential expression

$$L = -\frac{d^2}{dx^2} + V. \tag{2.2}$$

To construct the KdV hierarchy we need a second differential expression P_{2n+1} of order $2n+1$, $n \in \mathbb{N}_0$, defined recursively in the following. We take the quickest route to the construction of P_{2n+1}, and hence to that of the KdV hierarchy, by starting from the recursion relation (2.3) below.

Define $\{f_\ell\}_{\ell \in \mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4) f_{\ell-1,xxx} + V f_{\ell-1,x} + (1/2) V_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \tag{2.3}$$

Explicitly, one finds

$$\begin{aligned}
f_0 &= 1, \\
f_1 &= \tfrac{1}{2}V + c_1, \\
f_2 &= -\tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2 + c_1\tfrac{1}{2}V + c_2, \\
f_3 &= \tfrac{1}{32}V_{xxxx} - \tfrac{5}{16}VV_{xx} - \tfrac{5}{32}V_x^2 + \tfrac{5}{16}V^3 \\
&\quad + c_1\big(-\tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2\big) + c_2\tfrac{1}{2}V + c_3, \quad \text{etc.}
\end{aligned} \tag{2.4}$$

Here $\{c_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants which naturally arise when solving (2.3).

Subsequently, it will be convenient to also introduce the corresponding homogeneous coefficients $\hat f_\ell$, defined by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$,

$$\hat f_0 = f_0 = 1, \quad \hat f_\ell = f_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.5}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat f_k, \quad \ell \in \mathbb{N}_0, \tag{2.6}$$

introducing

$$c_0 = 1. \tag{2.7}$$

One can prove inductively that all homogeneous elements $\hat f_\ell$ (and hence all f_ℓ) are differential polynomials in V, that is, polynomials with respect to V and its x-derivatives up to order $2\ell-2$, $\ell \in \mathbb{N}$.

Next we define differential expressions P_{2n+1} of order $2n+1$ by

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell}\frac{d}{dx} - \frac{1}{2}f_{n-\ell,x}\Big)L^\ell, \quad n \in \mathbb{N}_0. \tag{2.8}$$

Using the recursion (2.3), the commutator of P_{2n+1} and L can be explicitly computed and one obtains

$$[P_{2n+1}, L] = 2f_{n+1,x}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

In particular, (L, P_{2n+1}) represents the celebrated *Lax pair* of the KdV hierarchy. Varying $n \in \mathbb{N}_0$, the stationary KdV hierarchy is then defined in terms of the vanishing of the commutator of P_{2n+1} and L in (2.9) by[1],

$$-[P_{2n+1}, L] = -2f_{n+1,x}(V) = \text{s-KdV}_n(V) = 0, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Explicitly,

$$\begin{aligned}
\text{s-KdV}_0(V) &= -V_x = 0, \\
\text{s-KdV}_1(V) &= \tfrac{1}{4}V_{xxx} - \tfrac{3}{2}VV_x + c_1(-V_x) = 0, \\
\text{s-KdV}_2(V) &= -\tfrac{1}{16}V_{xxxxx} + \tfrac{5}{8}V_{xxx} + \tfrac{5}{4}V_xV_{xx} - \tfrac{15}{8}V^2V_x \\
&\quad + c_1\big(\tfrac{1}{4}V_{xxx} - \tfrac{3}{2}VV_x\big) + c_2(-V_x) = 0, \quad \text{etc.,}
\end{aligned} \tag{2.11}$$

represent the first few equations of the stationary KdV hierarchy. By definition, the set of solutions of (2.10), with n ranging in $\mathbb{N}_0$ and c_k in $\mathbb{C}$, $k \in \mathbb{N}$, represents

[1] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ on V, writing $f_\ell(V)$.

the class of algebro-geometric KdV solutions. At times it will be convenient to abbreviate algebro-geometric stationary KdV solutions V simply as KdV *potentials*.

In the following we will frequently assume that V satisfies the nth stationary KdV equation. By this we mean it satisfies one of the nth stationary KdV equations after a particular choice of integration constants $c_k \in \mathbb{C}$, $k = 1, \dots, n$, $n \in \mathbb{N}$, has been made.

Next, we introduce a polynomial F_n of degree n with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_n(z, x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^\ell. \tag{2.12}$$

Explicitly, one obtains

$$\begin{aligned}
&F_0 = 1,\\
&F_1 = z + \tfrac{1}{2}V + c_1,\\
&F_2 = z^2 + \tfrac{1}{2}Vz - \tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2 + c_1\big(\tfrac{1}{2}V + z\big) + c_2,\\
&F_3 = z^3 + \tfrac{1}{2}Vz^2 + \big(-\tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2\big)z + \tfrac{1}{32}V_{xxxx} - \tfrac{5}{16}VV_{xx} - \tfrac{5}{32}V_x^2\\
&\qquad + \tfrac{5}{16}V^3 + c_1\big(z^2 + \tfrac{1}{2}Vz - \tfrac{1}{8}V_{xx} + \tfrac{3}{8}V^2\big) + c_2\big(z + \tfrac{1}{2}V\big) + c_3, \quad \text{etc.}
\end{aligned} \tag{2.13}$$

The recursion relation (2.3) and equation (2.10) imply that

$$F_{n,xxx} - 4(V - z)F_{n,x} - 2V_x F_n = 0. \tag{2.14}$$

Multiplying (2.14) by F_n, a subsequent integration with respect to x results in

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (V - z)F_n^2 = R_{2n+1}, \tag{2.15}$$

where R_{2n+1} is a monic polynomial of degree $2n + 1$. We denote its roots by $\{E_m\}_{m=0}^{2n}$, and hence write

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \tag{2.16}$$

One can show that equation (2.15) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in

$$\text{s-KdV}_n(V) = -2f_{n+1,x}(V) = 0 \tag{2.17}$$

in terms of the zeros $E_0, \dots, E_{2n}$ of the associated polynomial R_{2n+1} in (2.16). In fact, one can prove

$$c_k = c_k(\underline{E}), \quad k = 1, \dots, n, \tag{2.18}$$

where

$$c_k(\underline{E}) = -\sum_{\substack{j_0,\dots,j_{2n}=0\\ j_0+\cdots+j_{2n}=k}}^{k} \frac{(2j_0)! \cdots (2j_{2n})!}{2^{2k}(j_0!)^2 \cdots (j_{2n}!)^2(2j_0 - 1)\cdots(2j_{2n} - 1)} E_0^{j_0} \cdots E_{2n}^{j_{2n}},$$
$$k = 1, \dots, n. \tag{2.19}$$

Remark 2.1. Suppose $V \in C^{2n+1}(\mathbb{R})$ satisfies the nth stationary KdV equation s-KdV$_n(V) = -2f_{n+1,x}(V) = 0$ for a given set of integration constants c_k, $k = 1, \dots, n$. Introducing F_n as in (2.12) with $f_0, \dots, f_n$ given by (2.6) then yields equation (2.14) and hence (2.15). The latter equation in turn, as shown inductively in [29, Prop. 2.1], yields

$$V \in C^\infty(\mathbb{R}) \text{ and } f_\ell \in C^\infty(\mathbb{R}),\ \ell = 0, \dots, n. \tag{2.20}$$

Thus, without loss of generality, we may assume in the following that solutions of s-KdV$_n(V) = 0$ satisfy $V \in C^\infty(\mathbb{R})$.

Next, we study the restriction of the differential expression P_{2n+1} to the two-dimensional kernel (i.e., the formal null space in an algebraic sense as opposed to the functional analytic one) of $(L - z)$. More precisely, let

$$\ker(L - z) = \{\psi\colon \mathbb{R} \to \mathbb{C}_\infty \text{ meromorphic} \mid (L - z)\psi = 0\}, \quad z \in \mathbb{C}. \tag{2.21}$$

Then (2.8) implies

$$P_{2n+1}\big|_{\ker(L-z)} = \Big(F_n(z)\frac{d}{dx} - \frac{1}{2}F_{n,x}(z)\Big)\Big|_{\ker(L-z)}. \tag{2.22}$$

We emphasize that the result (2.22) is valid independently of whether or not P_{2n+1} and L commute. However, if one makes the additional assumption that P_{2n+1} and L commute, one can prove that this implies an algebraic relationship between P_{2n+1} and L.

Theorem 2.2. *Fix $n \in \mathbb{N}_0$ and assume that P_{2n+1} and L commute, $[P_{2n+1}, L] = 0$, or equivalently, suppose* s-KdV$_n(V) = -2f_{n+1,x}(V) = 0$. *Then L and P_{2n+1} satisfy an algebraic relationship of the type* (*cf.* (2.16))

$$\begin{aligned} &\mathcal{F}_n(L, -iP_{2n+1}) = -P_{2n+1}^2 - R_{2n+1}(L) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{2.23}$$

The expression $\mathcal{F}_n(L, -iP_{2n+1})$ is called the Burchnall–Chaundy polynomial of the pair (L, P_{2n+1}). Equation (2.23) naturally leads to the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part), where

$$\begin{aligned} &\mathcal{K}_n\colon \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \end{aligned} \tag{2.24}$$

The curve $\mathcal{K}_n$ is compactified by joining the point P_∞ but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n\backslash\{P_\infty\}$ are represented as pairs $P = (z, y)$, where $y(\cdot)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way, see Appendix A. Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0, \dots, E_{2n}$), the integration constants $c_1, \dots, c_n$ in $f_{n+1,x}$ (and hence in the corresponding stationary KdV$_n$ equation) are uniquely determined as is clear from (2.18) and (2.19), which establish the integration constants c_k as symmetric functions of $E_0, \dots, E_{2n}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. The trivial case $n = 0$ which leads to $V(x) = E_0$ is of no interest to us in this paper.

In the following, the zeros[2] of the polynomial $F_n(\cdot,x)$ (cf. (2.12)) will play a special role. We denote them by $\{\mu_j(x)\}_{j=1}^n$ and hence write

$$F_n(z,x)=\prod_{j=1}^n [z-\mu_j(x)]. \tag{2.25}$$

From (2.15) we see that

$$R_{2n+1}+(1/4)F_{n,x}^2=F_nH_{n+1}, \tag{2.26}$$

where

$$H_{n+1}(z,x)=(1/2)F_{n,xx}(z,x)+(z-V(x))F_n(z,x) \tag{2.27}$$

is a monic polynomial of degree $n+1$. We introduce the corresponding roots[3] $\{\nu_\ell(x)\}_{\ell=0}^n$ of $H_{n+1}(\cdot,x)$ by

$$H_{n+1}(z,x)=\prod_{\ell=0}^n [z-\nu_\ell(x)]. \tag{2.28}$$

Explicitly, one computes from (2.4) and (2.12),

$$\begin{aligned}
&H_1=z-V,\\
&H_2=z^2-\tfrac12 Vz+\tfrac14 V_{xx}-\tfrac12 V^2+c_1(z-V),\\
&H_3=z^3-\tfrac12 Vz^2+\tfrac18\big(V_{xx}-V^2\big)z-\tfrac1{16}V_{xxxx}+\tfrac38 V_x^2+\tfrac12 VV_{xx}\\
&\quad\;\; -\tfrac38 V^3+c_1\big(z^2-\tfrac12 Vz+\tfrac14 V_{xx}-\tfrac12 V^2\big)+c_2(z-V), \quad \text{etc.}
\end{aligned} \tag{2.29}$$

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_ℓ of F_n and H_{n+1} from $\mathbb{C}$ to the curve $\mathcal{K}_n$. From (2.26) one infers

$$R_{2n+1}(z)+(1/4)F_{n,x}(z)^2=0, \quad z\in\{\mu_j,\nu_\ell\}_{j=1,\dots,n,\ell=0,\dots,n}. \tag{2.30}$$

We now introduce $\{\hat\mu_j(x)\}_{j=1,\dots,n}\subset\mathcal{K}_n$ and $\{\hat\nu_\ell(x)\}_{\ell=0,\dots,n}\subset\mathcal{K}_n$ by

$$\hat\mu_j(x)=(\mu_j(x),-(i/2)F_{n,x}(\mu_j(x),x)), \quad j=1,\dots,n,\ x\in\mathbb{R} \tag{2.31}$$

and

$$\hat\nu_\ell(x)=(\nu_\ell(x),(i/2)F_{n,x}(\nu_\ell(x),x)), \quad \ell=0,\dots,n,\ x\in\mathbb{R}. \tag{2.32}$$

Due to the $C^\infty(\mathbb{R})$ assumption (2.1) on V, $F_n(z,\cdot)\in C^\infty(\mathbb{R})$ by (2.3) and (2.12), and hence also $H_{n+1}(z,\cdot)\in C^\infty(\mathbb{R})$ by (2.27). Thus, one concludes

$$\mu_j,\nu_\ell\in C(\mathbb{R}),\ j=1,\dots,n,\ \ell=0,\dots,n, \tag{2.33}$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_{n+1} into account. (Away from collisions of zeros, μ_j and ν_ℓ are of course C^∞.)

Next, we define the fundamental meromorphic function $\phi(\cdot,x)$ on $\mathcal{K}_n$,

$$\begin{aligned}
\phi(P,x)&=\frac{iy+(1/2)F_{n,x}(z,x)}{F_n(z,x)} &(2.34)\\
&=\frac{-H_{n+1}(z,x)}{iy-(1/2)F_{n,x}(z,x)}, &(2.35)\\
&\quad P=(z,y)\in\mathcal{K}_n,\ x\in\mathbb{R}
\end{aligned}$$

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[2]If $V\in L^\infty(\mathbb{R};dx)$, these zeros (generically) are the Dirichlet eigenvalues of a closed operator in $L^2(\mathbb{R})$ associated with the differential expression L and a Dirichlet boundary condition at $x\in\mathbb{R}$.

[3]If $V\in L^\infty(\mathbb{R};dx)$, these roots (generically) are the Neumann eigenvalues of a closed operator in $L^2(\mathbb{R})$ associated with L and a Neumann boundary condition at $x\in\mathbb{R}$.

with divisor $(\phi(\cdot,x))$ of $\phi(\cdot,x)$ given by

$$(\phi(\cdot,x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}, \tag{2.36}$$

using (2.25), (2.28), and (2.33). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n) \tag{2.37}$$

(cf. the notation introduced in Appendix A). The stationary Baker–Akhiezer function $\psi(\cdot,x,x_0)$ on $\mathcal{K}_n\backslash\{P_\infty\}$ is then defined in terms of $\phi(\cdot,x)$ by

$$\psi(P,x,x_0) = \exp\bigg(\int_{x_0}^x dx'\,\phi(P,x')\bigg), \quad P\in\mathcal{K}_n\backslash\{P_\infty\},\ (x,x_0)\in\mathbb{R}^2. \tag{2.38}$$

Basic properties of ϕ and ψ are summarized in the following result (where $W(f,g) = fg' - f'g$ denotes the Wronskian of f and g, and P^* abbreviates $P^* = (z,-y)$ for $P=(z,y)$).

Lemma 2.3. *Assume $V\in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (2.10)*. Moreover, let $P=(z,y)\in\mathcal{K}_n\backslash\{P_\infty\}$ and $(x,x_0)\in\mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P) + \phi(P)^2 = V - z, \tag{2.39}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{n+1}(z)}{F_n(z)}, \tag{2.40}$$

$$\phi(P)+\phi(P^*) = \frac{F_{n,x}(z)}{F_n(z)}, \tag{2.41}$$

$$\phi(P)-\phi(P^*) = \frac{2iy}{F_n(z)}. \tag{2.42}$$

Moreover, ψ satisfies

$$(L - z(P))\psi(P) = 0, \quad (P_{2n+1} - iy(P))\psi(P) = 0, \tag{2.43}$$

$$\psi(P,x,x_0) = \bigg(\frac{F_n(z,x)}{F_n(z,x_0)}\bigg)^{1/2} \exp\bigg(iy\int_{x_0}^x dx'\,F_n(z,x')^{-1}\bigg), \tag{2.44}$$

$$\psi(P,x,x_0)\psi(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)}, \tag{2.45}$$

$$\psi_x(P,x,x_0)\psi_x(P^*,x,x_0) = \frac{H_{n+1}(z,x)}{F_n(z,x_0)}, \tag{2.46}$$

$$\psi(P,x,x_0)\psi_x(P^*,x,x_0) + \psi(P^*,x,x_0)\psi_x(P,x,x_0) = \frac{F_{n,x}(z,x)}{F_n(z,x_0)}, \tag{2.47}$$

$$W(\psi(P,\cdot,x_0),\psi(P^*,\cdot,x_0)) = -\frac{2iy}{F_n(z,x_0)}. \tag{2.48}$$

In addition, as long as the zeros of $F_n(\cdot,x)$ are all simple for $x\in\Omega$, $\Omega\subseteq\mathbb{R}$ an open interval, $\psi(\cdot,x,x_0)$ is meromorphic on $\mathcal{K}_n\backslash\{P_\infty\}$ for $x,x_0\in\Omega$.

Combining the polynomial recursion approach with (2.25) readily yields trace formulas for the KdV invariants, that is, expressions of f_ℓ in terms of symmetric functions of the zeros μ_j of F_n.

Lemma 2.4. *Assume $V \in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (2.10). *Then,*

$$V = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j, \tag{2.49}$$

$$V^2 - (1/2)V_{xx} = \sum_{m=0}^{2n} E_m^2 - 2\sum_{j=1}^{n} \mu_j^2, \ \textit{etc.} \tag{2.50}$$

Equation (2.49) represents the trace formula for the algebro-geometric potential V. In addition, (2.50) indicates that higher-order trace formulas associated with the KdV hierarchy can be obtained from (2.25) comparing powers of z. We omit further details and refer to [26, Ch. 1] and [27].

From this point on we assume that the affine part of $\mathcal{K}_n$ is nonsingular, that is,

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2n. \tag{2.51}$$

Since nonspecial divisors play a fundamental role in this context we also recall the following fact.

Lemma 2.5. *Suppose that the affine part of $\mathcal{K}_n$ is nonsingular and assume that $V \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx)$ satisfies the nth stationary KdV equation* (2.10). *Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_n)$ be the Dirichlet divisor of degree n associated with V defined according to* (2.31)*, that is,*

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{2.52}$$

Then $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is nonspecial for all $x \in \mathbb{R}$. Moreover, there exists a constant $C > 0$ such that

$$|\mu_j(x)| \leq C, \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{2.53}$$

Remark 2.6. Assume that $V \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx)$ satisfies the nth stationary KdV equation (2.10). We recall that $f_\ell \in C^\infty(\mathbb{R})$, $\ell \in \mathbb{N}_0$, by (2.20) since f_ℓ are differential polynomials in V. Moreover, we note that (2.53) implies that $f_\ell \in L^\infty(\mathbb{R}; dx)$, $\ell = 0, \dots, n$, employing the fact that f_ℓ, $\ell = 0, \dots, n$, are elementary symmetric functions of $\mu_1, \dots, \mu_n$ (cf. (2.12) and (2.25)). Since $f_{n+1,x} = 0$, one can use the recursion relation (2.3) to reduce f_k for $k \geq n+2$ to a linear combination of $f_1, \dots, f_n$. Thus,

$$f_\ell \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}; dx), \quad \ell \in \mathbb{N}_0. \tag{2.54}$$

Using the fact that for fixed $1 \leq p \leq \infty$,

$$h, h^{(k)} \in L^p(\mathbb{R}; dx) \ \text{ imply } \ h^{(\ell)} \in L^p(\mathbb{R}; dx), \ \ \ell = 1, \dots, k-1 \tag{2.55}$$

(cf., e.g., [6, p. 168–170]), one then infers

$$V^{(\ell)} \in L^\infty(\mathbb{R}; dx), \quad \ell \in \mathbb{N}_0, \tag{2.56}$$

applying (2.55) with $p = \infty$.

We continue with the theta function representation for ψ and V. For general background information and the notation employed we refer to Appendix A.

Let θ denote the Riemann theta function associated with $\mathcal{K}_n$ (whose affine part is assumed to be nonsingular) and a fixed homology basis $\{a_j, b_j\}_{j=1}^n$ on $\mathcal{K}_n$. Next, choosing a base point $Q_0 \in \mathcal{K}_n \backslash P_\infty$, the Abel maps $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ are defined by (A.41) and (A.42), and the Riemann vector $\underline{\Xi}_{Q_0}$ is given by (A.54).

Next, let $\omega^{(2)}_{P_\infty,0}$ denote the normalized differential of the second kind defined by

$$\omega^{(2)}_{P_\infty,0} = -\frac{1}{2y}\prod_{j=1}^n (z-\lambda_j)dz \underset{\zeta\to 0}{=} \big(\zeta^{-2}+O(1)\big)d\zeta \text{ as } P\to P_\infty, \qquad \zeta=\sigma/z^{1/2},\ \sigma\in\{1,-1\}, \tag{2.57}$$

where the constants $\lambda_j\in\mathbb{C}$, $j=1,\dots,n$, are determined by employing the normalization

$$\int_{a_j}\omega^{(2)}_{P_\infty,0}=0, \quad j=1,\dots,n. \tag{2.58}$$

One then infers

$$\int_{Q_0}^P \omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} -\zeta^{-1}+e^{(2)}_0(Q_0)+O(\zeta) \text{ as } P\to P_\infty \tag{2.59}$$

for some constant $e^{(2)}_0(Q_0)\in\mathbb{C}$. The vector of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ is denoted by

$$\underline{U}^{(2)}_0=(U^{(2)}_{0,1},\dots,U^{(2)}_{0,n}), \quad U^{(2)}_{0,j}=\frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_\infty,0},\ j=1,\dots,n. \tag{2.60}$$

By (A.26) one concludes

$$U^{(2)}_{0,j}=-2c_j(n), \quad j=1,\dots,n. \tag{2.61}$$

In the following it will be convenient to introduce the abbreviation

$$\underline{z}(P,\underline{Q})=\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),\ P\in\mathcal{K}_n,\ \underline{Q}=\{Q_1,\dots,Q_n\}\in\operatorname{Sym}^n(\mathcal{K}_n). \tag{2.62}$$

We note that $\underline{z}(\cdot,\underline{Q})$ is independent of the choice of base point Q_0.

Theorem 2.7. *Suppose that $V\in C^\infty(\mathbb{R})\cap L^\infty(\mathbb{R};dx)$ satisfies the nth stationary KdV equation (2.10) on $\mathbb{R}$. In addition, assume the affine part of $\mathcal{K}_n$ to be nonsingular and let $P\in\mathcal{K}_n\backslash\{P_\infty\}$ and $x,x_0\in\mathbb{R}$. Then $\mathcal{D}_{\underline{\hat\mu}(x)}$ and $\mathcal{D}_{\underline{\hat\nu}(x)}$ are nonspecial for $x\in\mathbb{R}$. Moreover,*[4]

$$\begin{aligned}\psi(P,x,x_0)&=\frac{\theta(\underline{z}(P_\infty,\underline{\hat\mu}(x_0)))\theta(\underline{z}(P,\underline{\hat\mu}(x)))}{\theta(\underline{z}(P_\infty,\underline{\hat\mu}(x)))\theta(\underline{z}(P,\underline{\hat\mu}(x_0)))}\\&\quad\times\exp\bigg[-i(x-x_0)\bigg(\int_{Q_0}^P\omega^{(2)}_{P_\infty,0}-e^{(2)}_0(Q_0)\bigg)\bigg],\end{aligned} \tag{2.63}$$

with the linearizing property of the Abel map,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)})=\Big(\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0)})+i\underline{U}^{(2)}_0(x-x_0)\Big) \pmod{L_n}. \tag{2.64}$$

The Its–Matveev formula for V reads

$$V(x)=E_0+\sum_{j=1}^n(E_{2j-1}+E_{2j}-2\lambda_j)-2\partial_x^2\ln\big(\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)}))\big). \tag{2.65}$$

[4]To avoid multi-valued expressions in formulas such as (2.63), etc., we agree to always choose the same path of integration connecting Q_0 and P and refer to Remark A.4 for additional tacitly assumed conventions.

Combining (2.64) and (2.65) shows the remarkable linearity of the theta function with respect to x in the Its–Matveev formula for V. In fact, one can rewrite (2.65) as

$$V(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)), \tag{2.66}$$

where

$$\underline{A} = \Xi_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i\underline{U}_0^{(2)} x_0 + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}), \tag{2.67}$$

$$\underline{B} = i\underline{U}_0^{(2)}, \tag{2.68}$$

$$\Lambda_0 = E_0 + \sum_{j=1}^n (E_{2j-1} + E_{2j} - 2\lambda_j). \tag{2.69}$$

Hence the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point x_0.

Remark 2.8. If one assumes V in (2.65) (or (2.66)) to be quasi-periodic (cf. (3.16) and (3.17)), then there exists a homology basis $\{\tilde{a}_j, \tilde{b}_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that $\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)}$ satisfies the constraint

$$\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)} \in \mathbb{R}^n. \tag{2.70}$$

This is studied in detail in Appendix B.

An example illustrating some of the general results of this section is provided in Appendix C.

3. The diagonal Green's function of H

In this section we focus on the diagonal Green's function of H and derive a variety of results to be used in our principal Section 4.

We start with some preparations. We denote by

$$W(f,g)(x) = f(x)g_x(x) - f_x(x)g(x) \text{ for a.e. } x \in \mathbb{R} \tag{3.1}$$

the Wronskian of $f, g \in AC_{\mathrm{loc}}(\mathbb{R})$ (with $AC_{\mathrm{loc}}(\mathbb{R})$ the set of locally absolutely continuous functions on $\mathbb{R}$).

Lemma 3.1. *Assume*[5] $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, *define* $\tau = -d^2/dx^2 + q$, *and let* $u_j(z)$, $j = 1, 2$ *be two (not necessarily distinct) distributional solutions*[6] *of* $\tau u = zu$ *for some* $z \in \mathbb{C}$. *Define* $U(z,x) = u_1(z,x)u_2(z,x)$, $(z,x) \in \mathbb{C} \times \mathbb{R}$. *Then,*

$$2U_{xx}U - U_x^2 - 4(q-z)U^2 = -W(u_1,u_2)^2. \tag{3.2}$$

If in addition $q_x \in L^1_{\mathrm{loc}}(\mathbb{R})$, *then*

$$U_{xxx} - 4(q-z)U_x - 2q_xU = 0. \tag{3.3}$$

Proof. Equation (3.3) is a well-known fact going back to at least Appell [2]. Equation (3.2) either follows upon integration using the integrating factor U, or alternatively, can be verified directly from the definition of U. We omit the straightforward computations. □

[5] One could admit more severe local singularities; in particular, one could assume q to be meromorphic, but we will not need this in this paper.

[6] That is, $u, u_x \in AC_{\mathrm{loc}}(\mathbb{R})$.

Introducing

$$\mathfrak{g}(z,x) = u_1(z,x)u_2(z,x)/W(u_1(z),u_2(z)), \quad z\in\mathbb{C},\ x\in\mathbb{R}, \tag{3.4}$$

Lemma 3.1 implies the following result.

Lemma 3.2. *Assume that $q\in L^1_{\text{loc}}(\mathbb{R})$ and $(z,x)\in\mathbb{C}\times\mathbb{R}$. Then,*

$$2\mathfrak{g}_{xx}\mathfrak{g} - \mathfrak{g}_x^2 - 4(q-z)\mathfrak{g}^2 = -1, \tag{3.5}$$

$$-\big(\mathfrak{g}^{-1}\big)_z = 2\mathfrak{g} + \big\{\mathfrak{g}\big[u_1^{-2}W(u_1,u_{1,z}) + u_2^{-2}W(u_2,u_{2,z})\big]\big\}_x, \tag{3.6}$$

$$-\big(\mathfrak{g}^{-1}\big)_z = 2\mathfrak{g} - \mathfrak{g}_{xxz} + \big[\mathfrak{g}^{-1}\mathfrak{g}_x\mathfrak{g}_z\big]_x \tag{3.7}$$

$$= 2\mathfrak{g} - \Big\{\Big[\big(\mathfrak{g}^{-1}\big)\big(\mathfrak{g}^{-1}\big)_{zx} - \big(\mathfrak{g}^{-1}\big)_x\big(\mathfrak{g}^{-1}\big)_z\Big]\Big/\big(\mathfrak{g}^{-3}\big)\Big\}_x. \tag{3.8}$$

If in addition $q_x\in L^1_{\text{loc}}(\mathbb{R})$, then

$$\mathfrak{g}_{xxx} - 4(q-z)\mathfrak{g}_x - 2q_x\mathfrak{g} = 0. \tag{3.9}$$

Proof. Equations (3.9) and (3.5) are clear from (3.3) and (3.2). Equation (3.6) follows from

$$\big(\mathfrak{g}^{-1}\big)_z = u_2^{-2}W(u_2,u_{2,z}) - u_1^{-2}W(u_1,u_{1,z}) \tag{3.10}$$

and

$$W(u_j,u_{j,z})_x = -u_j^2, \quad j=1,2. \tag{3.11}$$

Finally, (3.8) (and hence (3.7)) follows from (3.4), (3.5), and (3.6) by a straightforward, though tedious, computation. □

Equation (3.7) is known and can be found, for instance, in [24]. Similarly, (3.6) can be inferred, for example, from the results in [12, p. 369].

Next, we turn to the analog of $\mathfrak{g}$ in connection with the algebro-geometric potential V in (2.65). Introducing

$$g(P,x) = \frac{\psi(P,x,x_0)\psi(P^*,x,x_0)}{W(\psi(P,\cdot,x_0),\psi(P^*,\cdot,x_0))}, \quad P\in\mathcal{K}_n\backslash\{P_\infty\},\ x,x_0\in\mathbb{R}, \tag{3.12}$$

equations (2.45) and (2.48) imply

$$g(P,x) = \frac{iF_n(z,x)}{2y}, \quad P=(z,y)\in\mathcal{K}_n\backslash\{P_\infty\},\ x\in\mathbb{R}. \tag{3.13}$$

Together with $g(P,x)$ we also introduce its two branches $g_\pm(z,x)$ defined on the upper and lower sheets $\Pi_\pm$ of $\mathcal{K}_n$ (cf. (A.3), (A.4), and (A.14))

$$g_\pm(z,x) = \pm\frac{iF_n(z,x)}{2R_{2n+1}(z)^{1/2}}, \quad z\in\Pi,\ x\in\mathbb{R} \tag{3.14}$$

with $\Pi=\mathbb{C}\backslash\mathcal{C}$ the cut plane introduced in (A.4). A comparison of (3.4), (3.12)–(3.14), then shows that $g_\pm(z,\cdot)$ satisfy (3.5)–(3.9).

For convenience we will subsequently focus on g_+ whenever possible and then use the simplified notation

$$g(z,x) = g_+(z,x), \quad z\in\Pi,\ x\in\mathbb{R}. \tag{3.15}$$

Next, we assume that V is quasi-periodic and compute the mean value of $g(z,\cdot)^{-1}$ using (3.7). Before embarking on this task we briefly review a few properties of quasi-periodic functions.

We denote by $CP(\mathbb{R})$ and $QP(\mathbb{R})$, the sets of continuous periodic and quasi-periodic functions on $\mathbb{R}$, respectively. In particular, f is called quasi-periodic with fundamental periods $(\Omega_1, \dots, \Omega_N) \in (0,\infty)^N$ if the frequencies $2\pi/\Omega_1, \dots, 2\pi/\Omega_N$ are linearly independent over $\mathbb{Q}$ and if there exists a continuous function $F \in C(\mathbb{R}^N)$, periodic of period 1 in each of its arguments

$$F(x_1, \dots, x_j + 1, \dots, x_N) = F(x_1, \dots, x_N), \quad x_j \in \mathbb{R},\ j = 1, \dots, N, \tag{3.16}$$

such that

$$f(x) = F(\Omega_1^{-1}x, \dots, \Omega_N^{-1}x), \quad x \in \mathbb{R}. \tag{3.17}$$

The frequency module $\mathrm{Mod}\,(f)$ of f is then of the type

$$\mathrm{Mod}\,(f) = \{2\pi m_1/\Omega_1 + \cdots + 2\pi m_N/\Omega_N \,|\, m_j \in \mathbb{Z},\ j = 1, \dots, N\}. \tag{3.18}$$

We note that $f \in CP(\mathbb{R})$ if and only if there are $r_j \in \mathbb{Q}\backslash\{0\}$ such that $\Omega_j = r_j\widehat{\Omega}$ for some $\widehat{\Omega} > 0$, or equivalently, if and only if $\Omega_j = m_j\widetilde{\Omega}$, $m_j \in \mathbb{Z}\backslash\{0\}$ for some $\widetilde{\Omega} > 0$. f has the fundamental period $\Omega > 0$ if every period of f is an integer multiple of Ω.

For any quasi-periodic (in fact, Bohr (uniformly) almost periodic) function f, the mean value $\langle f\rangle$ of f, defined by

$$\langle f\rangle = \lim_{R\to\infty} \frac{1}{2R} \int_{x_0-R}^{x_0+R} dx\, f(x), \tag{3.19}$$

exists and is independent of $x_0 \in \mathbb{R}$. Moreover, we recall the following facts (also valid for Bohr (uniformly) almost periodic functions on $\mathbb{R}$), see, for instance, [8, Ch. I], [11, Sects. 39–92], [15, Ch. I], [22, Chs. 1,3,6], [34], [43, Chs. 1,2,6], and [56].

Theorem 3.3. *Assume $f, g \in QP(\mathbb{R})$ and $x_0, x \in \mathbb{R}$. Then the following assertions hold:*
(i) f is uniformly continuous on $\mathbb{R}$ and $f \in L^\infty(\mathbb{R}; dx)$.
(ii) $\overline{f}$, $d\,f$, $d \in \mathbb{C}$, $f(\cdot + c)$, $f(c\cdot)$, $c \in \mathbb{R}$, $|f|^\alpha$, $\alpha \geq 0$ are all in $QP(\mathbb{R})$.
(iii) $f + g, fg \in QP(\mathbb{R})$.
(iv) $f/g \in QP(\mathbb{R})$ if and only if $\inf_{s\in\mathbb{R}}[|g(s)|] > 0$.
(v) Let G be uniformly continuous on $\mathcal{M} \subseteq \mathbb{R}$ and $f(s) \in \mathcal{M}$ for all $s \in \mathbb{R}$. Then $G(f) \in QP(\mathbb{R})$.
(vi) $f' \in QP(\mathbb{R})$ if and only if f' is uniformly continuous on $\mathbb{R}$.
(vii) Let $\langle f\rangle = 0$, then $\int_{x_0}^x dx'\, f(x') \underset{|x|\to\infty}{=} o(|x|)$.
(viii) Let $F(x) = \int_{x_0}^x dx'\, f(x')$. Then $F \in QP(\mathbb{R})$ if and only if $F \in L^\infty(\mathbb{R}; dx)$.
(ix) If $0 \leq f \in QP(\mathbb{R})$, $f \not\equiv 0$, then $\langle f\rangle > 0$.
(x) If $f = |f|\exp(i\varphi)$, then $|f| \in QP(\mathbb{R})$ and φ is of the type $\varphi(x) = cx + \psi(x)$, where $c \in \mathbb{R}$ and $\psi \in QP(\mathbb{R})$ (and real-valued).
(xi) If $F(x) = \exp\Big(\int_{x_0}^x dx'\, f(x')\Big)$, then $F \in QP(\mathbb{R})$ if and only if $f(x) = i\beta + \psi(x)$, where $\beta \in \mathbb{R}$, $\psi \in QP(\mathbb{R})$, and $\Psi \in L^\infty(\mathbb{R}; dx)$, where $\Psi(x) = \int_{x_0}^x dx'\, \psi(x')$.

For the rest of this section and the next it will be convenient to introduce the following hypothesis:

Hypothesis 3.4. Assume the affine part of $\mathcal{K}_n$ to be nonsingular. Moreover, suppose that $V \in C^\infty(\mathbb{R}) \cap QP(\mathbb{R})$ satisfies the nth stationary KdV equation (2.10) on $\mathbb{R}$.

Next, we note the following result.

Lemma 3.5. *Assume Hypothesis* 3.4. *Then* $V^{(k)}$, $k \in \mathbb{N}$, *and* f_ℓ, $\ell \in \mathbb{N}$, *and hence all* x *and* z*-derivatives of* $F_n(z,\cdot)$, $z \in \mathbb{C}$, *and* $g(z,\cdot)$, $z \in \Pi$, *are quasi-periodic. Moreover, taking limits to points on* $\mathcal{C}$, *the last result extends to either side of the cuts in the set* $\mathcal{C}\backslash\{E_m\}_{m=0}^{2n}$ *(cf.* (A.3)*) by continuity with respect to* z.

Proof. Since by hypothesis $V \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R};dx)$, s-KdV$_n(V) = 0$ implies $V^{(k)} \in L^\infty(\mathbb{R};dx)$, $k \in \mathbb{N}$ and $f_\ell \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R};dx)$, $\ell \in \mathbb{N}_0$, applying Remark 2.6. In particular $V^{(k)}$ is uniformly continuous on $\mathbb{R}$ and hence quasi-periodic for all $k \in \mathbb{N}$. Since the f_ℓ are differential polynomials with respect to V, also f_ℓ, $\ell \in \mathbb{N}$ are quasi-periodic. The corresponding assertion for $F_n(z,\cdot)$ then follows from (2.12) and that for $g(z,\cdot)$ follows from (3.14). □

For future purposes we introduce the set

$$\begin{aligned}\Pi_C = \Pi\Big\backslash\Big\{&\{z \in \mathbb{C} \,|\, |z| \le C+1\} \cup \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \ge \min_{m=0,\dots,2n}[\mathrm{Re}(E_m)] - 1, \\ &\min_{m=0,\dots,2n}[\mathrm{Im}(E_m)] - 1 \le \mathrm{Im}(z) \le \max_{m=0,\dots,2n}[\mathrm{Im}(E_m)] + 1\}\Big\}, \end{aligned} \tag{3.20}$$

where $C > 0$ is the constant in (2.53). Moreover, without loss of generality, we may assume Π_C contains no cuts, that is,

$$\Pi_C \cap \mathcal{C} = \emptyset. \tag{3.21}$$

Lemma 3.6. *Assume Hypothesis* 3.4 *and let* $z, z_0 \in \Pi$. *Then*

$$\langle g(z,\cdot)^{-1}\rangle = -2\int_{z_0}^{z} dz' \, \langle g(z',\cdot)\rangle + \langle g(z_0,\cdot)^{-1}\rangle, \tag{3.22}$$

where the path connecting z_0 *and* z *is assumed to lie in the cut plane* Π. *Moreover, by taking limits to points on* $\mathcal{C}$ *in* (3.22), *the result* (3.22) *extends to either side of the cuts in the set* $\mathcal{C}$ *by continuity with respect to* z.

Proof. Let $z, z_0 \in \Pi_C$. Integrating equation (3.7) from z_0 to z along a smooth path in Π_C yields

$$\begin{aligned} g(z,x)^{-1} - g(z_0,x)^{-1} &= -2\int_{z_0}^{z} dz' \, g(z',x) + [g_{xx}(z,x) - g_{xx}(z_0,x)] \\ &\quad - \int_{z_0}^{z} dz' \left[g(z',x)^{-1} g_x(z',x) g_z(z',x)\right]_x \\ &= -2\int_{z_0}^{z} dz' \, g(z',x) + g_{xx}(z,x) - g_{xx}(z_0,x) \\ &\quad - \left[\int_{z_0}^{z} dz' \, g(z',x)^{-1} g_x(z',x) g_z(z',x)\right]_x. \end{aligned} \tag{3.23}$$

By Lemma 3.5 $g(z,\cdot)$ and all its x-derivatives are quasi-periodic,

$$\langle g_{xx}(z,\cdot)\rangle = 0, \quad z \in \Pi. \tag{3.24}$$

Since we actually assumed $z \in \Pi_C$, also $g(z,\cdot)^{-1}$ is quasi-periodic. Consequently, also

$$\int_{z_0}^{z} dz' \, g(z',\cdot)^{-1} g_x(z',\cdot) g_z(z',\cdot), \quad z \in \Pi_C, \tag{3.25}$$

is a family of uniformly almost periodic functions for z varying in compact subsets of Π_C as discussed in [22, Sect. 2.7] and one obtains

$$\left\langle \left[\int_{z_0}^{z} dz'\, g(z',\cdot)^{-1} g_x(z',\cdot) g_z(z',\cdot) \right]_x \right\rangle = 0. \tag{3.26}$$

Hence, taking mean values in (3.23) (taking into account (3.24) and (3.26)), proves (3.22) for $z \in \Pi_C$. Since f_ℓ, $\ell \in \mathbb{N}_0$, are quasi-periodic by Lemma 3.5 (we recall that $f_0 = 1$), (2.12) and (3.13) yield

$$\int_{z_0}^{z} dz'\, \langle g(z',\cdot)\rangle = \frac{i}{2} \sum_{\ell=0}^{n} \langle f_{n-\ell}\rangle \int_{z_0}^{z} dz'\, \frac{(z')^\ell}{R_{2n+1}(z')^{1/2}}. \tag{3.27}$$

Thus, $\int_{z_0}^{z} dz'\, \langle g(z',\cdot)\rangle$ has an analytic continuation with respect to z to all of Π and consequently, (3.22) for $z \in \Pi_C$ extends by analytic continuation to $z \in \Pi$. By continuity this extends to either side of the cuts in $\mathcal{C}$. Interchanging the role of z and z_0, analytic continuation with respect to z_0 then yields (3.22) for $z, z_0 \in \Pi$. $\square$

Remark 3.7. For $z \in \Pi_C$, $g(z,\cdot)^{-1}$ is quasi-periodic and hence $\langle g(z,\cdot)^{-1}\rangle$ is well-defined. If one analytically continues $g(z,x)$ with respect to z, $g(z,x)$ will acquire zeros for some $x \in \mathbb{R}$ and hence $g(z,\cdot)^{-1} \notin QP(\mathbb{R})$. Nevertheless, as shown by the right-hand side of (3.22), $\langle g(z,\cdot)^{-1}\rangle$ admits an analytic continuation in z from Π_C to all of Π, and from now on, $\langle g(z,\cdot)^{-1}\rangle$, $z \in \Pi$, always denotes that analytic continuation (cf. also (3.29)).

Next, we will invoke the Baker–Akhiezer function $\psi(P,x,x_0)$ and analyze the expression $\langle g(z,\cdot)^{-1}\rangle$ in more detail.

Theorem 3.8. *Assume Hypothesis* 3.4*, let $P = (z,y) \in \Pi_\pm$, and $x, x_0 \in \mathbb{R}$. Moreover, select a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that $\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)}$, with $\underline{\widetilde U}_0^{(2)}$ the vector of $\tilde b$-periods of the normalized differential of the second kind, $\widetilde\omega_{P_\infty,0}^{(2)}$, satisfies the constraint*

$$\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)} \in \mathbb{R}^n \tag{3.28}$$

(*cf. Appendix* B). *Then,*

$$\mathrm{Re}\big(\langle g(P,\cdot)^{-1}\rangle\big) = -2\mathrm{Im}\big(y\langle F_n(z,\cdot)^{-1}\rangle\big) = 2\mathrm{Im}\bigg(\int_{Q_0}^{P} \widetilde\omega_{P_\infty,0}^{(2)} - \tilde e_0^{(2)}(Q_0)\bigg). \tag{3.29}$$

Proof. Using (2.44), one obtains for $z \in \Pi_C$,

$$\begin{aligned} \psi(P,x,x_0) &= \bigg(\frac{F_n(z,x)}{F_n(z,x_0)}\bigg)^{1/2} \exp\bigg(iy \int_{x_0}^{x} dx'\, F_n(z,x')^{-1}\bigg) \\ &= \bigg(\frac{F_n(z,x)}{F_n(z,x_0)}\bigg)^{1/2} \exp\bigg(iy \int_{x_0}^{x} dx'\, \big[F_n(z,x')^{-1} - \langle F_n(z,\cdot)^{-1}\rangle\big]\bigg) \\ &\quad \times \exp\big(i(x-x_0)y\langle F_n(z,\cdot)^{-1}\rangle\big), \\ &\quad P = (z,y) \in \Pi_\pm,\ z \in \Pi_C,\ x, x_0 \in \mathbb{R}. \end{aligned} \tag{3.30}$$

Since $\big[F_n(z,x')^{-1} - \langle F_n(z,\cdot)^{-1}\rangle\big]$ has mean zero,

$$\bigg| \int_{x_0}^{x} dx'\, \big[F_n(z,x')^{-1} - \langle F_n(z,\cdot)^{-1}\rangle\big] \bigg| \underset{|x|\to\infty}{=} o(|x|), \quad z \in \Pi_C \tag{3.31}$$

by Theorem 3.3 (vii). In addition, the factor $F_n(z,x)/F_n(z,x_0)$ in (3.30) is quasi-periodic and hence bounded on $\mathbb{R}$.

On the other hand, (2.63) yields

$$\begin{aligned}\psi(P,x,x_0) &= \frac{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0)))} \\ &\quad\times \exp\bigg[-i(x-x_0)\bigg(\int_{Q_0}^P \widetilde{\omega}^{(2)}_{P_\infty,0} - \tilde{e}^{(0)}_0(Q_0)\bigg)\bigg] \\ &= \Theta(P,x,x_0)\exp\bigg[-i(x-x_0)\bigg(\int_{Q_0}^P \widetilde{\omega}^{(2)}_{P_\infty,0} - \tilde{e}^{(2)}_0(Q_0)\bigg)\bigg], \\ &\qquad P\in\mathcal{K}_n\backslash\big\{\{P_\infty\}\cup\{\hat{\mu}_j(x_0)\}_{j=1}^n\big\}.\end{aligned} \tag{3.32}$$

Taking into account (2.62), (2.64), (2.70), (A.30), and the fact that by (2.53) no $\hat{\mu}_j(x)$ can reach P_∞ as x varies in $\mathbb{R}$, one concludes that

$$\Theta(P,\cdot,x_0)\in L^\infty(\mathbb{R};dx), \quad P\in\mathcal{K}_n\backslash\{\hat{\mu}_j(x_0)\}_{j=1}^n. \tag{3.33}$$

A comparison of (3.30) and (3.32) then shows that the $o(|x|)$-term in (3.31) must actually be bounded on $\mathbb{R}$ and hence the left-hand side of (3.31) is quasi-periodic. In addition, the term

$$\exp\bigg(iR_{2n+1}(z)^{1/2}\int_{x_0}^x dx'\,\big[F_n(z,x')^{-1} - \big\langle F_n(z,\cdot)^{-1}\big\rangle\big]\bigg), \quad z\in\Pi_C, \tag{3.34}$$

is then quasi-periodic by Theorem 3.3 (xi). A further comparison of (3.30) and (3.32) then yields (3.29) for $z\in\Pi_C$. Analytic continuation with respect to z then yields (3.29) for $z\in\Pi$. By continuity with respect to z, taking boundary values to either side of the cuts in the set $\mathcal{C}$, this then extends to $z\in\mathcal{C}$ (cf. (A.3), (A.4)) and hence proves (3.29) for $P=(z,y)\in\mathcal{K}_n\backslash\{P_\infty\}$. □

4. Spectra of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials

In this section we establish the connection between the algebro-geometric formalism of Section 2 and the spectral theoretic description of Schrödinger operators H in $L^2(\mathbb{R};dx)$ with quasi-periodic algebro-geometric KdV potentials. In particular, we introduce the conditional stability set of H and prove our principal result, the characterization of the spectrum of H. Finally, we provide a qualitative description of the spectrum of H in terms of analytic spectral arcs.

Suppose that $V\in C^\infty(\mathbb{R})\cap QP(\mathbb{R})$ satisfies the nth stationary KdV equation (2.10) on $\mathbb{R}$. The corresponding Schrödinger operator H in $L^2(\mathbb{R};dx)$ is then introduced by

$$H = -\frac{d^2}{dx^2}+V, \quad \operatorname{dom}(H) = H^{2,2}(\mathbb{R}). \tag{4.1}$$

Thus, H is a densely defined closed operator in $L^2(\mathbb{R};dx)$ (it is self-adjoint if and only if V is real-valued).

Before we turn to the spectrum of H in the general non-self-adjoint case, we briefly mention the following result on the spectrum of H in the self-adjoint case with a quasi-periodic (or almost periodic) real-valued potential q. We denote by $\sigma(A)$, $\sigma_{\mathrm{e}}(A)$, and $\sigma_{\mathrm{d}}(A)$ the spectrum, essential spectrum, and discrete spectrum of a self-adjoint operator A in a complex Hilbert space, respectively.

Theorem 4.1 (See, e.g., [58]). *Let $V \in QP(\mathbb{R})$ and q be real-valued. Define the self-adjoint Schrödinger operator H in $L^2(\mathbb{R}; dx)$ as in* (4.1)*. Then,*

$$\sigma(H) = \sigma_{\rm e}(H) \subseteq \big[\min_{x\in\mathbb{R}}(V(x)), \infty\big), \quad \sigma_{\rm d}(H) = \emptyset. \tag{4.2}$$

Moreover, $\sigma(H)$ contains no isolated points, that is, $\sigma(H)$ is a perfect set.

In the special periodic case where $V \in CP(\mathbb{R})$ is real-valued, the spectrum of H is purely absolutely continuous and either a finite union of some compact intervals and a half-line or an infinite union of compact intervals (see, e.g., [19, Sect. 5.3], [54, Sect. XIII.16]). If $V \in CP(\mathbb{R})$ and V is complex-valued, then the spectrum of H is purely continuous and it consists of either a finite union of simple analytic arcs and one simple semi-infinite analytic arc tending to infinity or an infinite union of simple analytic arcs (cf. [55], [57], and [60])[7].

Remark 4.2. Here $\sigma \subset \mathbb{C}$ is called an *arc* if there exists a parameterization $\gamma \in C([0,1])$ such that $\sigma = \{\gamma(t) \,|\, t \in [0,1]\}$. The arc σ is called *simple* if there exists a parameterization γ such that $\gamma\colon [0,1] \to \mathbb{C}$ is injective. The arc σ is called *analytic* if there is a parameterization γ that is analytic at each $t \in [0,1]$. Finally, σ_∞ is called a *semi-infinite* arc if there exists a parameterization $\gamma \in C([0,\infty))$ such that $\sigma_\infty = \{\gamma(t) \,|\, t \in [0,\infty)\}$ and σ_∞ is an unbounded subset of $\mathbb{C}$. Analytic semi-infinite arcs are defined analogously and by a simple semi-infinite arc we mean one that is without self-intersection (i.e., corresponds to a injective parameterization) with the additional restriction that the unbounded part of σ_∞ consists of precisely one branch tending to infinity.

Now we turn to the analyis of the generally non-self-adjoint operator H in (4.1). Assuming Hypothesis 3.4 we now introduce the set $\Sigma \subset \mathbb{C}$ by

$$\Sigma = \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}\big(\big\langle g(\lambda,\cdot)^{-1}\big\rangle\big) = 0\big\}. \tag{4.3}$$

Below we will show that Σ plays the role of the conditional stability set of H, familiar from the spectral theory of one-dimensional periodic Schrödinger operators (cf. [19, Sect. 5.3], [55], [64], [65]).

Lemma 4.3. *Assume Hypothesis* 3.4*. Then Σ coincides with the conditional stability set of H, that is,*

$$\begin{aligned} \Sigma = \{\lambda \in \mathbb{C} \,|\, &\textit{there exists at least one bounded distributional solution} \\ &0 \neq \psi \in L^\infty(\mathbb{R}; dx) \ \textit{of } H\psi = \lambda\psi.\} \end{aligned} \tag{4.4}$$

Proof. By (3.32) and (3.33),

$$\psi(P,x) = \frac{\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x)))} \exp\bigg[-ix\bigg(\int_{Q_0}^P \widetilde{\omega}^{(2)}_{P_\infty,0} - \tilde{e}^{(0)}_0(Q_0)\bigg)\bigg], \quad P=(z,y)\in\Pi_\pm, \tag{4.5}$$

is a distributional solution of $H\psi = z\psi$ which is bounded on $\mathbb{R}$ if and only if the exponential function in (4.5) is bounded on $\mathbb{R}$. By (3.29), the latter holds if and only if

$$\mathrm{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big) = 0. \tag{4.6}$$

□

[7] in either case the resolvent set is connected.

Remark 4.4. At first sight our *a priori* choice of cuts $\mathcal{C}$ for $R_{2n+1}(\cdot)^{1/2}$, as described in Appendix A, might seem unnatural as they completely ignore the actual spectrum of H. However, the spectrum of H is not known from the outset, and in the case of complex-valued periodic potentials, spectral arcs of H may actually cross each other (cf. [28], [53], and Theorem 4.9 (iv)) which renders them unsuitable for cuts of $R_{2n+1}(\cdot)^{1/2}$.

Before we state our first principal result on the spectrum of H, we find it convenient to recall a number of basic definitions and well-known facts in connection with the spectral theory of non-self-adjoint operators (we refer to [20, Chs. I, III, IX], [31, Sects. 1, 21–23], [35, Sects. IV.5.6, V.3.2], and [54, p. 178–179] for more details). Let S be a densely defined closed operator in a complex separable Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ and by $\ker(T)$ and $\operatorname{ran}(T)$ the kernel (null space) and range of a linear operator T in $\mathcal{H}$. The resolvent set, $\rho(S)$, spectrum, $\sigma(S)$, point spectrum (the set of eigenvalues), $\sigma_{\mathrm{p}}(S)$, continuous spectrum, $\sigma_{\mathrm{c}}(S)$, residual spectrum, $\sigma_{\mathrm{r}}(S)$, field of regularity, $\pi(S)$, approximate point spectrum, $\sigma_{\mathrm{ap}}(S)$, two kinds of essential spectra, $\sigma_{\mathrm{e}}(S)$, and $\widetilde{\sigma}_{\mathrm{e}}(S)$, the numerical range of S, $\Theta(S)$, and the sets $\Delta(S)$ and $\widetilde{\Delta}(S)$ are defined as follows:

$$\rho(S) = \{z \in \mathbb{C} \,|\, (S - zI)^{-1} \in \mathcal{B}(\mathcal{H})\}, \tag{4.7}$$

$$\sigma(S) = \mathbb{C}\backslash\rho(S), \tag{4.8}$$

$$\sigma_{\mathrm{p}}(S) = \{\lambda \in \mathbb{C} \,|\, \ker(S - \lambda I) \neq \{0\}\}, \tag{4.9}$$

$$\begin{aligned}\sigma_{\mathrm{c}}(S) = \{\lambda \in \mathbb{C} \,|\, &\ker(S - \lambda I) = \{0\} \text{ and } \operatorname{ran}(S - \lambda I) \text{ is dense in } \mathcal{H}\\ &\text{but not equal to } \mathcal{H}\},\end{aligned} \tag{4.10}$$

$$\sigma_{\mathrm{r}}(S) = \{\lambda \in \mathbb{C} \,|\, \ker(S - \lambda I) = \{0\} \text{ and } \operatorname{ran}(S - \lambda I) \text{ is not dense in } \mathcal{H}\}, \tag{4.11}$$

$$\begin{aligned}\pi(S) = \{z \in \mathbb{C} \,|\, &\text{there exists } k_z > 0 \text{ s.t. } \|(S - zI)u\|_{\mathcal{H}} \geq k_z\|u\|_{\mathcal{H}}\\ &\text{for all } u \in \operatorname{dom}(S)\},\end{aligned} \tag{4.12}$$

$$\sigma_{\mathrm{ap}}(S) = \mathbb{C}\backslash\pi(S), \tag{4.13}$$

$$\Delta(S) = \{z \in \mathbb{C} \,|\, \dim(\ker(S - zI)) < \infty \text{ and } \operatorname{ran}(S - zI) \text{ is closed}\}, \tag{4.14}$$

$$\sigma_{\mathrm{e}}(S) = \mathbb{C}\backslash\Delta(S), \tag{4.15}$$

$$\widetilde{\Delta}(S) = \{z \in \mathbb{C} \,|\, \dim(\ker(S - zI)) < \infty \text{ or } \dim(\ker(S^* - \overline{z}I)) < \infty\}, \tag{4.16}$$

$$\widetilde{\sigma}_{\mathrm{e}}(S) = \mathbb{C}\backslash\widetilde{\Delta}(S), \tag{4.17}$$

$$\Theta(S) = \{(f, Sf) \in \mathbb{C} \,|\, f \in \operatorname{dom}(S),\, \|f\|_{\mathcal{H}} = 1\}, \tag{4.18}$$

respectively. One then has

$$\begin{aligned}\sigma(S) &= \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{c}}(S) \cup \sigma_{\mathrm{r}}(S) \quad \text{(disjoint union)} &(4.19)\\ &= \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{e}}(S) \cup \sigma_{\mathrm{r}}(S), &(4.20)\end{aligned}$$

$$\sigma_{\mathrm{c}}(S) \subseteq \sigma_{\mathrm{e}}(S)\backslash(\sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{r}}(S)), \tag{4.21}$$

$$\sigma_{\mathrm{r}}(S) = \sigma_{\mathrm{p}}(S^*)^*\backslash\sigma_{\mathrm{p}}(S), \tag{4.22}$$

$$\begin{aligned}\sigma_{\mathrm{ap}}(S) = \{\lambda \in \mathbb{C} \,|\, &\text{there exists a sequence } \{f_n\}_{n\in\mathbb{N}} \subset \operatorname{dom}(S)\\ &\text{with } \|f_n\|_{\mathcal{H}} = 1,\, n \in \mathbb{N}, \text{ and } \lim_{n\to\infty} \|(S - \lambda I)f_n\|_{\mathcal{H}} = 0\},\end{aligned} \tag{4.23}$$

$$\widetilde{\sigma}_{\mathrm{e}}(S) \subseteq \sigma_{\mathrm{e}}(S) \subseteq \sigma_{\mathrm{ap}}(S) \subseteq \sigma(S) \ \text{ (all four sets are closed)}, \tag{4.24}$$

$$\rho(S) \subseteq \pi(S) \subseteq \Delta(S) \subseteq \widetilde{\Delta}(S) \quad \text{(all four sets are open)}, \tag{4.25}$$

$$\widetilde{\sigma}_{\rm e}(S) \subseteq \overline{\Theta(S)}, \quad \Theta(S) \text{ is convex}, \tag{4.26}$$

$$\widetilde{\sigma}_{\rm e}(S) = \sigma_{\rm e}(S) \text{ if } S = S^*. \tag{4.27}$$

Here σ^* in the context of (4.22) denotes the complex conjugate of the set $\sigma \subseteq \mathbb{C}$, that is,

$$\sigma^* = \{\overline{\lambda} \in \mathbb{C} \,|\, \lambda \in \sigma\}. \tag{4.28}$$

We note that there are several other versions of the concept of the essential spectrum in the non-self-adjoint context (cf. [20, Ch. IX]) but we will only use the two in (4.15) and in (4.17) in this paper.

Finally, we recall the following result due to Talenti [59] and Tomaselli [63] (see also Chisholm and Everitt [13], Chisholm, Everitt, and Littlejohn [14], and Muckenhoupt [49]).

Lemma 4.5. *Let $f \in L^2(\mathbb{R}; dx)$, $U \in L^2((-\infty, R]; dx)$, and $V \in L^2([R, \infty); dx)$ for all $R \in \mathbb{R}$. Then the following assertions (i)–(iii) are equivalent:*
(i) There exists a finite constant $C > 0$ such that

$$\int_{\mathbb{R}} dx \left| U(x) \int_x^\infty dx' \, V(x') f(x') \right|^2 \leq C \int_{\mathbb{R}} dx \, |f(x)|^2. \tag{4.29}$$

(ii) There exists a finite constant $D > 0$ such that

$$\int_{\mathbb{R}} dx \left| V(x) \int_{-\infty}^x dx' \, U(x') f(x') \right|^2 \leq D \int_{\mathbb{R}} dx \, |f(x)|^2. \tag{4.30}$$

(iii)

$$\sup_{r \in \mathbb{R}} \left[\left(\int_{-\infty}^r dx \, |U(x)|^2 \right) \left(\int_r^\infty dx \, |V(x)|^2 \right) \right] < \infty. \tag{4.31}$$

We start with the following elementary result.

Lemma 4.6. *Let H be defined as in* (4.1)*. Then,*

$$\sigma_{\rm e}(H) = \widetilde{\sigma}_{\rm e}(H) \subseteq \overline{\Theta(H)}. \tag{4.32}$$

Proof. Since H and H^* are second-order ordinary differential operators on $\mathbb{R}$,

$$\dim(\ker(H - zI)) \leq 2, \quad \dim(\ker(H^* - \overline{z}I)) \leq 2. \tag{4.33}$$

Equations (4.14)–(4.17) and (4.26) then prove (4.32). □

Theorem 4.7. *Assume Hypothesis* 3.4*. Then the point spectrum and residual spectrum of H are empty and hence the spectrum of H is purely continuous,*

$$\sigma_{\rm p}(H) = \sigma_{\rm r}(H) = \emptyset, \tag{4.34}$$

$$\sigma(H) = \sigma_{\rm c}(H) = \sigma_{\rm e}(H) = \sigma_{\rm ap}(H). \tag{4.35}$$

Proof. First we prove the absence of the point spectrum of H. Suppose $z \in \Pi \backslash \{\Sigma \cup \{\mu_j(x_0)\}_{j=1}^n\}$. Then $\psi(P, \cdot, x_0)$ and $\psi(P^*, \cdot, x_0)$ are linearly independent distributional solutions of $H\psi = z\psi$ which are unbounded at $+\infty$ or $-\infty$. This argument extends to all $z \in \Pi \backslash \Sigma$ by multiplying $\psi(P, \cdot, x_0)$ and $\psi(P^*, \cdot, x_0)$ with an appropriate function of z and x_0 (independent of x). It also extends to either side of the cut $\mathcal{C} \backslash \Sigma$ by continuity with respect to z. On the other hand, since

$V^{(k)} \in L^\infty(\mathbb{R}; dx)$ for all $k \in \mathbb{N}_0$, any distributional solution $\psi(z, \cdot) \in L^2(\mathbb{R}; dx)$ of $H\psi = z\psi$, $z \in \mathbb{C}$, is necessarily bounded. In fact,

$$\psi^{(k)}(z, \cdot) \in L^\infty(\mathbb{R}; dx) \cap L^2(\mathbb{R}; dx), \quad k \in \mathbb{N}_0, \tag{4.36}$$

applying $\psi''(z, x) = (V(x) - z)\psi(z, x)$ and (2.55) with $p = 2$ and $p = \infty$ repeatedly. (Indeed, $\psi(z, \cdot) \in L^2(\mathbb{R}; dx)$ implies $\psi''(z, \cdot) \in L^2(\mathbb{R}; dx)$ which in turn implies $\psi'(z, \cdot) \in L^2(\mathbb{R}; dx)$. Integrating $(\psi^2)' = 2\psi\psi'$ then yields $\psi(z, \cdot) \in L^\infty(\mathbb{R}; dx)$. The latter yields $\psi''(z, \cdot) \in L^\infty(\mathbb{R}; dx)$, etc.) Thus,

$$\{\mathbb{C}\backslash\Sigma\} \cap \sigma_{\mathrm{p}}(H) = \emptyset. \tag{4.37}$$

Hence, it remains to rule out eigenvalues located in Σ. We consider a fixed $\lambda \in \Sigma$ and note that by (2.45), there exists at least one distributional solution $\psi_1(\lambda, \cdot) \in L^\infty(\mathbb{R}; dx)$ of $H\psi = \lambda\psi$. Actually, a comparison of (2.44) and (4.3) shows that we may choose $\psi_1(\lambda, \cdot)$ such that $|\psi_1(\lambda, \cdot)| \in QP(\mathbb{R})$ and hence $\psi_1(\lambda, \cdot) \notin L^2(\mathbb{R}; dx)$. As in (4.36) one then infers from repeated use of $\psi''(\lambda) = (V - \lambda)\psi(\lambda)$ and (2.55) with $p = \infty$ that

$$\psi_1^{(k)}(\lambda, \cdot) \in L^\infty(\mathbb{R}; dx), \quad k \in \mathbb{N}_0. \tag{4.38}$$

Next, suppose there exists a second distributional solution $\psi_2(\lambda, \cdot)$ of $H\psi = \lambda\psi$ which is linearly independent of $\psi_1(\lambda, \cdot)$ and which satisfies $\psi_2(\lambda, \cdot) \in L^2(\mathbb{R}; dx)$. Applying (4.36) then yields

$$\psi_2^{(k)}(\lambda, \cdot) \in L^2(\mathbb{R}; dx), \quad k \in \mathbb{N}_0. \tag{4.39}$$

Combining (4.38) and (4.39), one concludes that the Wronskian of $\psi_1(\lambda, \cdot)$ and $\psi_2(\lambda, \cdot)$ lies in $L^2(\mathbb{R}; dx)$,

$$W(\psi_1(\lambda, \cdot), \psi_2(\lambda, \cdot)) \in L^2(\mathbb{R}; dx). \tag{4.40}$$

However, by hypothesis, $W(\psi_1(\lambda, \cdot), \psi_2(\lambda, \cdot)) = c(\lambda) \neq 0$ is a nonzero constant. This contradiction proves that

$$\Sigma \cap \sigma_{\mathrm{p}}(H) = \emptyset \tag{4.41}$$

and hence $\sigma_{\mathrm{p}}(H) = \emptyset$.

Next, we note that the same argument yields that H^* also has no point spectrum,

$$\sigma_{\mathrm{p}}(H^*) = \emptyset. \tag{4.42}$$

Indeed, if $V \in C^\infty(\mathbb{R}) \cap QP(\mathbb{R})$ satisfies the nth stationary KdV equation (2.10) on $\mathbb{R}$, then $\overline{V}$ also satisfies one of the nth stationary KdV equations (2.10) associated with a hyperelliptic curve of genus n with $\{E_m\}_{m=0}^{2n}$ replaced by $\{\overline{E}_m\}_{m=0}^{2n}$, etc. Since by general principles (cf. (4.28)),

$$\sigma_{\mathrm{r}}(B) \subseteq \sigma_{\mathrm{p}}(B^*)^* \tag{4.43}$$

for any densely defined closed linear operator B in some complex separable Hilbert space (see, e.g., [32, p. 71]), one obtains $\sigma_{\mathrm{r}}(H) = \emptyset$ and hence (4.34). This proves that the spectrum of H is purely continuous, $\sigma(H) = \sigma_{\mathrm{c}}(H)$. The remaining equalities in (4.35) then follow from (4.21) and (4.24). □

The following result is a fundamental one:

Theorem 4.8. *Assume Hypothesis* 3.4. *Then the spectrum of H coincides with Σ and hence equals the conditional stability set of H,*

$$\sigma(H) = \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}\big(\langle g(\lambda, \cdot)^{-1}\rangle\big) = 0\big\} \tag{4.44}$$

$$= \{\lambda \in \mathbb{C} \,|\, \textit{there exists at least one bounded distributional solution}$$
$$0 \neq \psi \in L^\infty(\mathbb{R}; dx) \text{ of } H\psi = \lambda\psi\}. \tag{4.45}$$

In particular,

$$\{E_m\}_{m=0}^{2n} \subset \sigma(H), \tag{4.46}$$

and $\sigma(H)$ contains no isolated points.

Proof. First we will prove that

$$\sigma(H) \subseteq \Sigma \tag{4.47}$$

by adapting a method due to Chisholm and Everitt [13]. For this purpose we temporarily choose $z \in \Pi\backslash\{\Sigma \cup \{\mu_j(x_0)\}_{j=1}^n\}$ and construct the resolvent of H as follows. Introducing the two branches $\psi_\pm(P,x,x_0)$ of the Baker–Akhiezer function $\psi(P,x,x_0)$ by

$$\psi_\pm(P,x,x_0) = \psi(P,x,x_0), \quad P = (z,y) \in \Pi_\pm, \ x, x_0 \in \mathbb{R}, \tag{4.48}$$

we define

$$\hat\psi_+(z,x,x_0) = \begin{cases} \psi_+(z,x,x_0) & \text{if } \psi_+(z,\cdot,x_0) \in L^2((x_0,\infty); dx), \\ \psi_-(z,x,x_0) & \text{if } \psi_-(z,\cdot,x_0) \in L^2((x_0,\infty); dx), \end{cases} \tag{4.49}$$

$$\hat\psi_-(z,x,x_0) = \begin{cases} \psi_-(z,x,x_0) & \text{if } \psi_-(z,\cdot,x_0) \in L^2((-\infty,x_0); dx), \\ \psi_+(z,x,x_0) & \text{if } \psi_+(z,\cdot,x_0) \in L^2((-\infty,x_0); dx), \end{cases} \tag{4.50}$$
$$z \in \Pi\backslash\Sigma, \ x, x_0 \in \mathbb{R},$$

and

$$G(z,x,x') = \frac{1}{W(\hat\psi_+(z,x,x_0), \hat\psi_-(z,x,x_0))} \begin{cases} \hat\psi_-(z,x',x_0)\hat\psi_+(z,x,x_0), & x \geq x', \\ \hat\psi_-(z,x,x_0)\hat\psi_+(z,x',x_0), & x \leq x', \end{cases}$$
$$z \in \Pi\backslash\Sigma, \ x, x_0 \in \mathbb{R}. \tag{4.51}$$

Due to the homogeneous nature of G, (4.51) extends to all $z \in \Pi$. Moreover, we extend (4.49)–(4.51) to either side of the cut $\mathcal{C}$ except at possible points in Σ (i.e., to $\mathcal{C}\backslash\Sigma$) by continuity with respect to z, taking limits to $\mathcal{C}\backslash\Sigma$. Next, we introduce the operator $R(z)$ in $L^2(\mathbb{R}; dx)$ defined by

$$(R(z)f)(x) = \int_{\mathbb{R}} dx'\, G(z,x,x') f(x'), \quad f \in C_0^\infty(\mathbb{R}), \ z \in \Pi, \tag{4.52}$$

and extend it to $z \in \mathcal{C}\backslash\Sigma$, as discussed in connection with $G(\cdot,x,x')$. The explicit form of $\hat\psi_\pm(z,x,x_0)$, inferred from (3.32) by restricting P to $\Pi_\pm$, then yields the estimates

$$|\hat\psi_\pm(z,x,x_0)| \leq C_\pm(z,x_0) e^{\mp\kappa(z)x}, \quad z \in \Pi\backslash\Sigma, \ x \in \mathbb{R} \tag{4.53}$$

for some constants $C_\pm(z,x_0) > 0$, $\kappa(z) > 0$, $z \in \Pi\backslash\Sigma$. An application of Lemma 4.5 identifying $U(x) = \exp(-\kappa(z)x)$ and $V(x) = \exp(\kappa(z)x)$ then proves that $R(z)$, $z \in \mathbb{C}\backslash\Sigma$, extends from $C_0^\infty(\mathbb{R})$ to a bounded linear operator defined on all of $L^2(\mathbb{R}; dx)$. (Alternatively, one can follow the second part of the proof of Theorem 5.3.2 in [19] line by line.) A straightforward differentiation then proves

$$(H - zI)R(z)f = f, \quad f \in L^2(\mathbb{R}; dx), \ z \in \mathbb{C}\backslash\Sigma \tag{4.54}$$

and hence also

$$R(z)(H - zI)g = g, \quad g \in \mathrm{dom}(H), \ z \in \mathbb{C}\backslash\Sigma. \tag{4.55}$$

Thus, $R(z) = (H - zI)^{-1}$, $z \in \mathbb{C}\backslash\Sigma$, and hence (4.47) holds.

Next we will prove that

$$\sigma(H) \supseteq \Sigma. \tag{4.56}$$

We will adapt a strategy of proof applied by Eastham in the case of (real-valued) periodic potentials [18] (reproduced in the proof of Theorem 5.3.2 of [19]) to the (complex-valued) quasi-periodic case at hand. Suppose $\lambda \in \Sigma$. By the characterization (4.4) of Σ, there exists a bounded distributional solution $\psi(\lambda,\cdot)$ of $H\psi = \lambda\psi$. A comparison with the Baker-Akhiezer function (2.44) then shows that we can assume, without loss of generality, that

$$|\psi(\lambda,\cdot)| \in QP(\mathbb{R}). \tag{4.57}$$

Moreover, by the same argument as in the proof of Theorem 4.7 (cf. (4.38)), one obtains

$$\psi^{(k)}(\lambda,\cdot) \in L^\infty(\mathbb{R}; dx), \quad k \in \mathbb{N}_0. \tag{4.58}$$

Next, we pick $\Omega > 0$ and consider $g \in C^\infty([0,\Omega])$ satisfying

$$\begin{aligned}
&g(0) = 0, \quad g(\Omega) = 1, \\
&g'(0) = g''(0) = g'(\Omega) = g''(\Omega) = 0, \\
&0 \leq g(x) \leq 1, \quad x \in [0,\Omega].
\end{aligned} \tag{4.59}$$

Moreover, we introduce the sequence $\{h_n\}_{n\in\mathbb{N}} \in L^2(\mathbb{R}; dx)$ by

$$h_n(x) = \begin{cases} 1, & |x| \leq (n-1)\Omega, \\ g(n\Omega - |x|), & (n-1)\Omega \leq |x| \leq n\Omega, \\ 0, & |x| \geq n\Omega \end{cases} \tag{4.60}$$

and the sequence $\{f_n(\lambda)\}_{n\in\mathbb{N}} \in L^2(\mathbb{R}; dx)$ by

$$f_n(\lambda, x) = d_n(\lambda)\psi(\lambda, x)h_n(x), \quad x \in \mathbb{R},\ d_n(\lambda) > 0,\ n \in \mathbb{N}. \tag{4.61}$$

Here $d_n(\lambda)$ is determined by the requirement

$$\|f_n(\lambda)\|_2 = 1, \quad n \in \mathbb{N}. \tag{4.62}$$

One readily verifies that

$$f_n(\lambda,\cdot) \in \mathrm{dom}(H) = H^{2,2}(\mathbb{R}), \quad n \in \mathbb{N}. \tag{4.63}$$

Next, we note that as a consequence of Theorem 3.3 (ix),

$$\int_{-T}^{T} dx\, |\psi(\lambda, x)|^2 \underset{T\to\infty}{=} 2\big\langle |\psi(\lambda,\cdot)|^2 \big\rangle T + o(T) \tag{4.64}$$

with

$$\big\langle |\psi(\lambda,\cdot)|^2 \big\rangle > 0. \tag{4.65}$$

Thus, one computes

$$\begin{aligned}
1 = \|f_n(\lambda)\|_2^2 &= d_n(\lambda)^2 \int_{\mathbb{R}} dx\, |\psi(\lambda,x)|^2 h_n(x)^2 \\
&= d_n(\lambda)^2 \int_{|x|\leq n\Omega} dx\, |\psi(\lambda,x)|^2 h_n(x)^2 \geq d_n(\lambda)^2 \int_{|x|\leq (n-1)\Omega} dx\, |\psi(\lambda,x)|^2 \\
&\geq d_n(\lambda)^2 \big[\big\langle |\psi(\lambda,\cdot)|^2 \big\rangle (n-1)\Omega + o(n)\big].
\end{aligned} \tag{4.66}$$

Consequently,

$$d_n(\lambda) \underset{n\to\infty}{=} O\big(n^{-1/2}\big). \tag{4.67}$$

Next, one computes

$$(H-\lambda I)f_n(\lambda,x) = -d_n(\lambda)[2\psi'(\lambda,x)h_n'(x) + \psi(\lambda,x)h_n''(x)] \tag{4.68}$$

and hence

$$\|(H-\lambda I)f_n\|_2 \leq d_n(\lambda)[2\|\psi'(\lambda)h_n'\|_2 + \|\psi(\lambda)h_n''\|_2], \quad n\in\mathbb{N}. \tag{4.69}$$

Using (4.58) and (4.60) one estimates

$$\begin{aligned}\|\psi'(\lambda)h_n'\|_2^2 &= \int_{(n-1)\Omega\leq|x|\leq n\Omega} dx\,|\psi'(\lambda,x)|^2|h_n'(x)|^2 \leq 2\|\psi'(\lambda)\|_\infty^2 \int_0^\Omega dx\,|g'(x)|^2\\ &\leq 2\Omega\|\psi'(\lambda)\|_\infty^2\|g'\|_{L^\infty([0,\Omega];dx)}^2,\end{aligned} \tag{4.70}$$

and similarly,

$$\begin{aligned}\|\psi(\lambda)h_n''\|_2^2 &= \int_{(n-1)\Omega\leq|x|\leq n\Omega} dx\,|\psi(\lambda,x)|^2|h_n''(x)|^2 \leq 2\|\psi(\lambda)\|_\infty^2 \int_0^\Omega dx\,|g''(x)|^2\\ &\leq 2\Omega\|\psi(\lambda)\|_\infty^2\|g''\|_{L^\infty([0,\Omega];dx)}^2.\end{aligned} \tag{4.71}$$

Thus, combining (4.67) and (4.69)–(4.71) one infers

$$\lim_{n\to\infty}\|(H-\lambda I)f_n\|_2 = 0, \tag{4.72}$$

and hence $\lambda\in\sigma_{\rm ap}(H)=\sigma(H)$ by (4.23) and (4.35).

Relation (4.46) is clear from (4.4) and the fact that by (2.45) there exists a distributional solution $\psi((E_m,0),\cdot,x_0)\in L^\infty(\mathbb{R};dx)$ of $H\psi=E_m\psi$ for all $m=0,\dots,2n$.

Finally, $\sigma(H)$ contains no isolated points since those would necessarily be essential singularities of the resolvent of H, as H has no eigenvalues by (4.34) (cf. [35, Sect. III.6.5]). An explicit investigation of the Green's function of H reveals at most a square root singularity at the points $\{E_m\}_{m=0}^{2n}$ and hence excludes the possibility of an essential singularity of $(H-zI)^{-1}$. □

In the special self-adjoint case where V is real-valued, the result (4.44) is equivalent to the vanishing of the Lyapunov exponent of H which characterizes the (purely absolutely continous) spectrum of H as discussed by Kotani [36], [37], [38], [39] (see also [12, p. 372]). In the case where V is periodic and complex-valued, this has also been studied by Kotani [39].

The explicit formula for Σ in (4.3) permits a qualitative description of the spectrum of H as follows. We recall (3.22) and write

$$\frac{d}{dz}\langle g(z,\cdot)^{-1}\rangle = -2\langle g(z,\cdot)\rangle = -i\frac{\prod_{j=1}^n\big(z-\widetilde\lambda_j\big)}{\big(\prod_{m=0}^{2n}(z-E_m)\big)^{1/2}}, \quad z\in\Pi, \tag{4.73}$$

for some constants

$$\{\widetilde\lambda_j\}_{j=1}^n\subset\mathbb{C}. \tag{4.74}$$

As in similar situations before, (4.73) extends to either side of the cuts in $\mathcal{C}$ by continuity with respect to z.

Theorem 4.9. *Assume Hypothesis* 3.4. *Then the spectrum $\sigma(H)$ of H has the following properties:*
(i) $\sigma(H)$ is contained in the semi-strip

$$\sigma(H)\subset\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)\in[M_1,M_2],\ \mathrm{Re}(z)\geq M_3\}, \tag{4.75}$$

where

$$M_1 = \inf_{x\in\mathbb{R}}[\mathrm{Im}(V(x))], \quad M_2 = \sup_{x\in\mathbb{R}}[\mathrm{Im}(V(x))], \quad M_3 = \inf_{x\in\mathbb{R}}[\mathrm{Re}(V(x))]. \tag{4.76}$$

(ii) *$\sigma(H)$ consists of finitely many simple analytic arcs and one simple semi-infinite arc. These analytic arcs may only end at the points $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_n$, $E_0,\dots,E_{2n}$, and at infinity. The semi-infinite arc, σ_∞, asymptotically approaches the half-line $L_{\langle V\rangle} = \{z\in\mathbb{C}\,|\,z=\langle V\rangle + x,\, x\geq 0\}$ in the following sense: asymptotically, σ_∞ can be parameterized by*

$$\sigma_\infty = \big\{z\in\mathbb{C}\,\big|\, z = R + i\,\mathrm{Im}(\langle V\rangle) + O\big(R^{-1/2}\big) \text{ as } R\uparrow\infty\big\}. \tag{4.77}$$

(iii) *Each E_m, $m=0,\dots,2n$, is met by at least one of these arcs. More precisely, a particular E_{m_0} is hit by precisely $2N_0+1$ analytic arcs, where $N_0\in\{0,\dots,n\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with E_{m_0}. Adjacent arcs meet at an angle $2\pi/(2N_0+1)$ at E_{m_0}. (Thus, generically, $N_0=0$ and precisely one arc hits E_{m_0}.)*
(iv) *Crossings of spectral arcs are permitted. This phenomenon and takes place precisely when for a particular $j_0\in\{1,\dots,n\}$, $\widetilde{\lambda}_{j_0}\in\sigma(H)$ such that*

$$\mathrm{Re}\big(\big\langle g(\widetilde{\lambda}_{j_0},\cdot)^{-1}\big\rangle\big) = 0 \text{ for some } j_0\in\{1,\dots,n\} \text{ with } \widetilde{\lambda}_{j_0}\notin\{E_m\}_{m=0}^{2n}. \tag{4.78}$$

In this case $2M_0+2$ analytic arcs are converging toward $\widetilde{\lambda}_{j_0}$, where $M_0\in\{1,\dots,n\}$ denotes the number of $\widetilde{\lambda}_j$ that coincide with $\widetilde{\lambda}_{j_0}$. Adjacent arcs meet at an angle $\pi/(M_0+1)$ at $\widetilde{\lambda}_{j_0}$. (Thus, generically, $M_0=1$ and two arcs cross at a right angle.)
(v) *The resolvent set $\mathbb{C}\backslash\sigma(H)$ of H is path-connected.*

Proof. Item (i) follows from (4.32) and (4.35) by noting that

$$(f,Hf) = \|f'\|^2 + (f,\mathrm{Re}(V)f) + i(f,\mathrm{Im}(V)f), \quad f\in H^{2,2}(\mathbb{R}). \tag{4.79}$$

To prove (ii) we first introduce the meromorphic differential of the second kind

$$\Omega^{(2)} = \langle g(P,\cdot)\rangle dz = \frac{i\langle F_n(z,\cdot)\rangle dz}{2y} = \frac{i}{2}\frac{\prod_{j=1}^n\big(z-\widetilde{\lambda}_j\big)dz}{R_{2n+1}(z)^{1/2}}, \quad P=(z,y)\in\mathcal{K}_n\backslash\{P_\infty\} \tag{4.80}$$

(cf. (4.74)). Then, by Lemma 3.6,

$$\big\langle g(P,\cdot)^{-1}\big\rangle = -2\int_{Q_0}^P \Omega^{(2)} + \big\langle g(Q_0,\cdot)^{-1}\big\rangle, \quad P\in\mathcal{K}_n\backslash\{P_\infty\} \tag{4.81}$$

for some fixed $Q_0\in\mathcal{K}_n\backslash\{P_\infty\}$, is holomorphic on $\mathcal{K}_n\backslash\{P_\infty\}$. By (4.73), (4.74), the characterization (4.44) of the spectrum,

$$\sigma(H) = \big\{\lambda\in\mathbb{C}\,\big|\,\mathrm{Re}\big(\big\langle g(\lambda,\cdot)^{-1}\big\rangle\big) = 0\big\}, \tag{4.82}$$

and the fact that $\mathrm{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big)$ is a harmonic function on the cut plane Π, the spectrum $\sigma(H)$ of H consists of analytic arcs which may only end at the points $\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_n$, $E_0,\dots,E_{2n}$, and possibly tend to infinity. (Since $\sigma(H)$ is independent of the chosen set of cuts, if a spectral arc crosses or runs along a part of one of the cuts in $\mathcal{C}$, one can slightly deform the original set of cuts to extend an analytic arc along or across such an original cut.) To study the behavior of spectral arcs near infinity we first note that

$$g(z,x) \underset{|z|\to\infty}{=} \frac{i}{2z^{1/2}} + \frac{i}{4z^{3/2}}V(x) + O\big(|z|^{-3/2}\big), \tag{4.83}$$

combining (2.4), (2.12), (2.16), and (3.14). Thus, one computes

$$g(z,x)^{-1} \underset{|z|\to\infty}{=} -2iz^{1/2} + \frac{i}{z^{1/2}} V(x) + O\big(|z|^{-3/2}\big) \tag{4.84}$$

and hence

$$\big\langle g(z,\cdot)^{-1}\big\rangle \underset{|z|\to\infty}{=} -2iz^{1/2} + \frac{i}{z^{1/2}} \langle V\rangle + O\big(|z|^{-3/2}\big). \tag{4.85}$$

Writing $z = Re^{i\varphi}$ this yields

$$0 = \mathrm{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big) \underset{R\to\infty}{=} 2\mathrm{Im}\big\{R^{1/2}e^{i\varphi/2} - 2^{-1}R^{-1/2}e^{-i\varphi/2}\langle V\rangle + O\big(R^{-3/2}\big)\big\} \tag{4.86}$$

implying

$$\varphi \underset{R\to\infty}{=} \mathrm{Im}(\langle V\rangle)R^{-1} + O\big(R^{-3/2}\big) \tag{4.87}$$

and hence (4.77). In particular, there is precisely one analytic semi-infinite arc σ_∞ that tends to infinity and asymptotically approaches the half-line $L_{\langle V\rangle}$. This proves item *(ii)*.

To prove *(iii)* one first recalls that by Theorem 4.8 the spectrum of H contains no isolated points. On the other hand, since $\{E_m\}_{m=0}^{2n} \subset \sigma(H)$ by (4.46), one concludes that at least one spectral arc meets each E_m, $m = 0, \dots, 2n$. Choosing $Q_0 = (E_{m_0}, 0)$ in (4.81) one obtains

$$\begin{aligned}
&\big\langle g(z,\cdot)^{-1}\big\rangle = -2\int_{E_{m_0}}^{z} dz'\, \langle g(z',\cdot)\rangle + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle \\
&= -i\int_{E_{m_0}}^{z} dz' \frac{\prod_{j=1}^{n} \big(z'-\widetilde\lambda_j\big)}{\big(\prod_{m=0}^{2n}(z'-E_m)\big)^{1/2}} + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle \\
&\underset{z\to E_{m_0}}{=} -i\int_{E_{m_0}}^{z} dz'\, (z'-E_{m_0})^{N_0-(1/2)}[C + O(z'-E_{m_0})] + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle \qquad (4.88)\\
&\underset{z\to E_{m_0}}{=} -i[N_0+(1/2)]^{-1}(z-E_{m_0})^{N_0+(1/2)}[C + O(z-E_{m_0})] + \big\langle g(E_{m_0},\cdot)^{-1}\big\rangle, \\
&\hspace{30em} z\in\Pi
\end{aligned}$$

for some $C = |C|e^{i\varphi_0} \in \mathbb{C}\backslash\{0\}$. Using

$$\mathrm{Re}\big(\big\langle g(E_m,\cdot)^{-1}\big\rangle\big) = 0, \quad m = 0, \dots, 2n, \tag{4.89}$$

as a consequence of (4.46), $\mathrm{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big) = 0$ and $z = E_{m_0} + \rho e^{i\varphi}$ imply

$$0 \underset{\rho\downarrow 0}{=} \sin[(N_0+(1/2))\varphi + \varphi_0]\rho^{N_0+(1/2)}[|C| + O(\rho)]. \tag{4.90}$$

This proves the assertions made in item *(iii)*.

To prove *(iv)* it suffices to refer to (4.73) and to note that locally, $d\big\langle g(z,\cdot)^{-1}\big\rangle/dz$ behaves like $C_0(z-\widetilde\lambda_{j_0})^{M_0}$ for some $C_0 \in \mathbb{C}\backslash\{0\}$ in a sufficiently small neighborhood of $\widetilde\lambda_{j_0}$.

Finally we will show that all arcs are simple (i.e., do not self-intersect each other). Assume that the spectrum of H contains a simple closed loop γ, $\gamma \subset \sigma(H)$. Then

$$\mathrm{Re}\big(\big\langle g(P,\cdot)^{-1}\big\rangle\big) = 0, \quad P \in \Gamma, \tag{4.91}$$

where the closed simple curve $\Gamma \subset \mathcal{K}_n$ denotes the lift of γ to $\mathcal{K}_n$, yields the contradiction

$$\mathrm{Re}\big(\langle g(P,\cdot)^{-1}\rangle\big) = 0 \text{ for all } P \text{ in the interior of } \Gamma \tag{4.92}$$

by Corollary 8.2.5 in [5]. Therefore, since there are no closed loops in $\sigma(H)$ and precisely one semi-infinite arc tends to infinity, the resolvent set of H is connected and hence path-connected, proving (v). □

Remark 4.10. For simplicity we focused on $L^2(\mathbb{R};dx)$-spectra thus far. However, since $V \in L^\infty(\mathbb{R};dx)$, H in $L^2(\mathbb{R};dx)$ is the generator of a C_0-semigroup $T(t)$ in $L^2(\mathbb{R};dx)$, $t>0$, whose integral kernel $T(t,x,x')$ satisfies the Gaussian upper bound (cf., e.g., [4])

$$\big|T(t,x,x')\big| \leq C_1 t^{-1/2} e^{C_2 t} e^{-C_3|x-x'|^2/t}, \quad t>0,\ x,x' \in \mathbb{R} \tag{4.93}$$

for some $C_1>0$, $C_2 \geq 0$, $C_3>0$. Thus, $T(t)$ in $L^2(\mathbb{R};dx)$ defines, for $p \in [1,\infty)$, consistent C_0-semigroups $T_p(t)$ in $L^p(\mathbb{R};dx)$ with generators denoted by H_p (i.e., $H=H_2$, $T(t)=T_2(t)$, etc.). Applying Theorem 1.1 of Kunstman [40] one then infers the p-independence of the spectrum,

$$\sigma(H_p) = \sigma(H), \quad p \in [1,\infty). \tag{4.94}$$

Actually, since $\mathbb{C}\backslash\sigma(H)$ is connected by Theorem 4.9 (v), (4.94) also follows from Theorem 4.2 of Arendt [3].

Of course, these results apply to the special case of algebro-geometric complex-valued periodic potentials (see [9], [10], [64], [65]) and we briefly point out the corresponding connections between the algebro-geometric approach and standard Floquet theory in Appendix C. But even in this special case, items (iii) and (iv) of Theorem 4.9 provide additional new details on the nature of the spectrum of H. We briefly illustrate the results of this section in Example C.1 of Appendix C.

The methods of this paper extend to the case of algebro-geometric non-self-adjoint second order finite difference (Jacobi) operators associated with the Toda lattice hierarchy and to the case of Dirac-type operators related to the focusing nonlinear Schrödinger hierarchy. Moreover, they extend to the infinite genus limit $n \to \infty$ using the approach in [25]. This will be studied elsewhere.

Appendix A. Hyperelliptic curves and their theta functions

We provide a brief summary of some of the fundamental notations needed from the theory of hyperelliptic Riemann surfaces. More details can be found in some of the standard textbooks [21] and [50], as well as in monographs dedicated to integrable systems such as [7, Ch. 2], [26, App. A, B]. In particular, the following material is taken from [26, App. A, B].

Fix $n \in \mathbb{N}$. We intend to describe the hyperelliptic Riemann surface $\mathcal{K}_n$ of genus n of the KdV-type curve (2.24), associated with the polynomial

$$\begin{aligned} &\mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n} (z-E_m), \quad \{E_m\}_{m=0}^{2n} \subset \mathbb{C}. \end{aligned} \tag{A.1}$$

To simplify the discussion we will assume that the affine part of $\mathcal{K}_n$ is nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 0, \dots, 2n \tag{A.2}$$

throughout this appendix. Introducing an appropriate set of (nonintersecting) cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$, $j = 1, \dots, n$, and $\mathcal{C}_{n+1}$, joining E_{2n} and ∞, we denote

$$\mathcal{C} = \bigcup_{j=1}^{n+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.3}$$

Define the cut plane Π by

$$\Pi = \mathbb{C} \backslash \mathcal{C}, \tag{A.4}$$

and introduce the holomorphic function

$$R_{2n+1}(\cdot)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \bigg(\prod_{m=0}^{2n} (z - E_m) \bigg)^{1/2} \tag{A.5}$$

on Π with an appropriate choice of the square root branch in (A.5). Define

$$\mathcal{M}_n = \{ (z, \sigma R_{2n+1}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{1, -1\} \} \cup \{P_\infty\} \tag{A.6}$$

by extending $R_{2n+1}(\cdot)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_n$ is then the set $\mathcal{M}_n$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_n$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_n)$ of $\mathcal{K}_n$ is given by

$$\mathcal{B}(\mathcal{K}_n) = \{ (E_m, 0) \}_{m=0}^{2n}. \tag{A.7}$$

Points $P \in \mathcal{K}_n \backslash \{P_\infty\}$ are denoted by

$$P = (z, \sigma R_{2n+1}(z)^{1/2}) = (z, y), \tag{A.8}$$

where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0$ and

$$y(P) \underset{\zeta \to 0}{=} \bigg(1 - \frac{1}{2} \bigg(\sum_{m=0}^{2n} E_m \bigg) \zeta^2 + O(\zeta^4) \bigg) \zeta^{-2n-1} \text{ as } P \to P_\infty, \qquad \zeta = \sigma'/z^{1/2},\ \sigma' \in \{1, -1\} \tag{A.9}$$

(i.e., we abbreviate $y(P) = \sigma R_{2n+1}(z)^{1/2}$). Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_n \backslash \{ \mathcal{B}(\mathcal{K}_n) \cup \{P_\infty\} \}$ are given by $\zeta_{P_0} = z - z_0$, near P_∞ by $\zeta_{P_{\infty_\pm}} = 1/z^{1/2}$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_n)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The compact hyperelliptic Riemann surface $\mathcal{K}_n$ resulting in this manner has topological genus n.

Moreover, we introduce the holomorphic sheet exchange map (involution)

$$* \colon \mathcal{K}_n \to \mathcal{K}_n, \quad P = (z, y) \mapsto P^* = (z, -y),\ P_\infty \mapsto P_\infty^* = P_\infty \tag{A.10}$$

and the two meromorphic projection maps

$$\tilde{\pi} \colon \mathcal{K}_n \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto z,\ P_\infty \mapsto \infty \tag{A.11}$$

and

$$y \colon \mathcal{K}_n \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto y,\ P_\infty \mapsto \infty. \tag{A.12}$$

The map $\tilde{\pi}$ has a pole of order 2 at P_∞, and y has a pole of order $2n + 1$ at P_∞. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P), \quad y(P^*) = -y(P), \quad P \in \mathcal{K}_n. \tag{A.13}$$

Thus $\mathcal{K}_n$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$ ($\cong \mathbb{C}\cup\{\infty\}$) branched at the $2n+2$ points $\{(E_m,0)\}_{m=0}^{2n}, P_\infty$.

We introduce the upper and lower sheets $\Pi_\pm$ by

$$\Pi_\pm = \{(z, \pm R_{2n+1}(z)^{1/2}) \in \mathcal{M}_n \mid z \in \Pi\} \tag{A.14}$$

and the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P \mapsto z. \tag{A.15}$$

Next, let $\{a_j, b_j\}_{j=1}^n$ be a homology basis for $\mathcal{K}_n$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j,k = 1,\dots,n. \tag{A.16}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1}^n$ we also recall the canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_n$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_n$ given by

$$\partial\widehat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}. \tag{A.17}$$

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$, respectively. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_n)$ at a point $Q \in \mathcal{K}_n$ is defined by

$$\operatorname{res}_Q(\nu) = \frac{1}{2\pi i} \int_{\gamma_Q} \nu, \tag{A.18}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1,\dots,n. \tag{A.19}$$

If $\omega^{(2)}_{P_1,m}$ is a differential of the second kind on $\mathcal{K}_n$ whose only pole is $P_1 \in \widehat{\mathcal{K}}_n$ with principal part $\zeta^{-m-2}\, d\zeta$, $m \in \mathbb{N}_0$, near P_1 and $\omega_j = \big(\sum_{q=0}^\infty d_{j,q}(P_1)\zeta^q\big) d\zeta$ near P_1, then

$$\frac{1}{2\pi i} \int_{b_j} \omega^{(2)}_{P_1,m} = \frac{d_{j,m}(P_1)}{m+1}, \quad m \in \mathbb{N}_0,\ j = 1,\dots,n. \tag{A.20}$$

Using the local chart near P_∞, one verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $2(n-1)$ at P_∞ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1,\dots,n, \tag{A.21}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Upon introduction of the invertible matrix C in $\mathbb{C}^n$,

$$C = \big(C_{j,k}\big)_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{A.22}$$

$$\underline{c}(k) = (c_1(k),\dots,c_n(k)), \quad c_j(k) = \big(C^{-1}\big)_{j,k}, \quad j,k = 1,\dots,n, \tag{A.23}$$

the normalized differentials ω_j for $j = 1, \dots, n$,

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \dots, n, \tag{A.24}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_n$.

In the chart $(U_{P_\infty}, \zeta_{P_\infty})$ induced by $1/\tilde{\pi}^{1/2}$ near P_∞ one infers,

$$\begin{aligned} \underline{\omega} = (\omega_1, \dots, \omega_n) &= -2\bigg(\sum_{j=1}^{n} \frac{\underline{c}(j)\zeta^{2(n-j)}}{\big(\prod_{m=0}^{2n}(1-\zeta^2 E_m)\big)^{1/2}} \bigg) d\zeta \\ &= -2\bigg(\underline{c}(n) + \bigg(\frac{1}{2}\underline{c}(n) \sum_{m=0}^{2n} E_m + \underline{c}(n-1) \bigg)\zeta^2 + O(\zeta^4) \bigg) d\zeta \text{ as } P \to P_\infty, \\ &\qquad \zeta = \sigma/z^{1/2},\ \sigma \in \{1,-1\}, \end{aligned} \tag{A.25}$$

where $\underline{E} = (E_0, \dots, E_{2n})$ and we used (A.9). Given (A.25), one computes for the vector $\underline{U}_0^{(2)}$ of b-periods of $\omega_{P_\infty,0}^{(2)}/(2\pi i)$, the normalized differential of the second kind, holomorphic on $\mathcal{K}_n \backslash \{P_\infty\}$, with principal part $\zeta^{-2} d\zeta/(2\pi i)$,

$$\underline{U}_0^{(2)} = \big(U_{0,1}^{(2)}, \dots, U_{0,n}^{(2)}\big), \quad U_{0,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_\infty,0}^{(2)} = -2c_j(n),\ j = 1, \dots, n. \tag{A.26}$$

Next, define the matrix $\tau = \big(\tau_{j,\ell}\big)_{j,\ell=1}^{n}$ by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j,\ell = 1, \dots, n. \tag{A.27}$$

Then

$$\operatorname{Im}(\tau) > 0, \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j,\ell = 1, \dots, n. \tag{A.28}$$

Associated with τ one introduces the period lattice

$$L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \underline{n}\tau,\ \underline{m}, \underline{n} \in \mathbb{Z}^n\} \tag{A.29}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^n} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^n, \tag{A.30}$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\,\underline{v}^\top = \sum_{j=1}^{n} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \tag{A.31}$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \underline{n}\tau)\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n. \tag{A.32}$$

Next we briefly study some consequences of a change of homology basis. Let

$$\{a_1, \dots, a_n, b_1, \dots, b_n\} \tag{A.33}$$

be a canonical homology basis on $\mathcal{K}_n$ with intersection matrix satisfying (A.16) and let

$$\{a_1', \dots, a_n', b_1', \dots, b_n'\} \tag{A.34}$$

be a homology basis on $\mathcal{K}_n$ related to each other by

$$\begin{pmatrix} \underline{a'}^\top \\ \underline{b'}^\top \end{pmatrix} = X \begin{pmatrix} \underline{a}^\top \\ \underline{b}^\top \end{pmatrix}, \tag{A.35}$$

where

$$\underline{a}^\top = (a_1, \dots, a_n)^\top, \quad \underline{b}^\top = (b_1, \dots, b_n)^\top,$$
$$\underline{a}'^\top = (a_1', \dots, a_n')^\top, \quad \underline{b}'^\top = (b_1', \dots, b_n')^\top, \tag{A.36}$$
$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A.37}$$

with A, B, C, and D being $n \times n$ matrices with integer entries. Then (A.34) is also a canonical homology basis on $\mathcal{K}_n$ with intersection matrix satisfying (A.16) if and only if

$$X \in \mathrm{Sp}(n, \mathbb{Z}), \tag{A.38}$$

where

$$\mathrm{Sp}(n, \mathbb{Z}) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, X \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} X^\top = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \det(X) = 1 \right\} \tag{A.39}$$

denotes the symplectic modular group (here A, B, C, D in X are again $n \times n$ matrices with integer entries). If $\{\omega_j\}_{j=1}^n$ and $\{\omega_j'\}_{j=1}^n$ are the normalized bases of holomorphic differentials corresponding to the canonical homology bases (A.33) and (A.34), with τ and τ' the associated b and b'-periods of $\omega_1, \dots, \omega_n$ and $\omega_1', \dots, \omega_n'$, respectively, one computes

$$\underline{\omega}' = \underline{\omega}(A + B\tau)^{-1}, \quad \tau' = (C + D\tau)(A + B\tau)^{-1}, \tag{A.40}$$

where $\underline{\omega} = (\omega_1, \dots, \omega_n)$ and $\underline{\omega}' = (\omega_1', \dots, \omega_n')$.

Fixing a base point $Q_0 \in \mathcal{K}_n \backslash \{P_\infty\}$, one denotes by $J(\mathcal{K}_n) = \mathbb{C}^n / L_n$ the Jacobi variety of $\mathcal{K}_n$, and defines the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_n \to J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n \right) \pmod{L_n}, \quad P \in \mathcal{K}_n. \tag{A.41}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \mathrm{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{A.42}$$

where $\mathrm{Div}(\mathcal{K}_n)$ denotes the set of divisors on $\mathcal{K}_n$. Here $\mathcal{D} \colon \mathcal{K}_n \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_n$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_n$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_n)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.) For subsequent use in Remark A.4 we also introduce

$$\widehat{\underline{A}}_{Q_0} \colon \widehat{\mathcal{K}}_n \to \mathbb{C}^n, \tag{A.43}$$
$$P \mapsto \widehat{\underline{A}}_{Q_0}(P) = \big(\widehat{A}_{Q_0,1}(P), \dots, \widehat{A}_{Q_0,n}(P)\big) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n \right)$$

and

$$\widehat{\underline{\alpha}}_{Q_0} \colon \mathrm{Div}(\widehat{\mathcal{K}}_n) \to \mathbb{C}^n, \quad \mathcal{D} \mapsto \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}) = \sum_{P \in \widehat{\mathcal{K}}_n} \mathcal{D}(P) \widehat{\underline{A}}_{Q_0}(P). \tag{A.44}$$

In connection with divisors on $\mathcal{K}_n$ we shall employ the following (additive) notation,

$$\mathcal{D}_{Q_0 \underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \tag{A.45}$$

$$\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m \mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n, \ m \in \mathbb{N},$$

where for any $Q \in \mathcal{K}_n$,

$$\mathcal{D}_Q \colon \mathcal{K}_n \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \backslash \{Q\}, \end{cases} \tag{A.46}$$

and $\operatorname{Sym}^m \mathcal{K}_n$ denotes the mth symmetric product of $\mathcal{K}_n$. In particular, $\operatorname{Sym}^m \mathcal{K}_n$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$ of degree $m \in \mathbb{N}$.

For $f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\}$ and $\omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0, \deg((\omega)) = 2(n-1), \ f \in \mathcal{M}(\mathcal{K}_n)\backslash\{0\}, \omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}, \tag{A.47}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\}, \ r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.48}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_n) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \ i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}) \tag{A.49}$$

(with $i(\mathcal{D})$ the index of specialty of $\mathcal{D}$), one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $\omega \in \mathcal{M}^1(\mathcal{K}_n)\backslash\{0\}$. Then*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad n \in \mathbb{N}_0. \tag{A.50}$$

The Riemann-Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - n + 1, \quad n \in \mathbb{N}_0. \tag{A.51}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $n \in \mathbb{N}$, is principal if and only if

$$\deg(\mathcal{D}) = 0 \ \textit{and} \ \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.52}$$

Finally, assume $n \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n)$ is surjective (Jacobi's inversion theorem).

Theorem A.2. *Let $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^n \mathcal{K}_n$, $\underline{Q} = \{Q_1, \dots, Q_n\}$. Then*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \tag{A.53}$$

if and only if there are s pairs of the type $\{P, P^*\} \subseteq \{Q_1, \dots, Q_n\}$ (this includes, of course, branch points for which $P = P^*$). Obviously, one has $s \leq n/2$.

Next, denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,n}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, n. \tag{A.54}$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n \mathcal{K}_n$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{A.55}$$

Remark A.4. In Section 2 we dealt with theta function expressions of the type

$$\psi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\bigg(- c \int_{Q_0}^{P} \Omega^{(2)} \bigg), \quad P \in \mathcal{K}_n, \tag{A.56}$$

where $\mathcal{D}_j \in \operatorname{Sym}^n \mathcal{K}_n$, $j = 1, 2$, are nonspecial positive divisors of degree n, $c \in \mathbb{C}$ is a constant, and $\Omega^{(2)}$ is a normalized differential of the second kind with a prescribed singularity at P_∞. Even though we agree to always choose identical paths of integration from P_0 to P in all Abelian integrals (A.56), this is not sufficient to render ψ single-valued on $\mathcal{K}_n$. To achieve single-valuedness one needs to replace $\mathcal{K}_n$ by its simply connected canonical dissection $\widehat{\mathcal{K}}_n$ and then replace $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ in (A.56) with $\underline{\widehat{A}}_{Q_0}$ and $\underline{\widehat{\alpha}}_{Q_0}$ as introduced in (A.43) and (A.44). In particular, one regards a_j, b_j, $j = 1, \dots, n$, as curves (being a part of $\partial\widehat{\mathcal{K}}_n$, cf. (A.17)) and not as homology classes. Similarly, one then replaces $\underline{\Xi}_{Q_0}$ by $\underline{\widehat{\Xi}}_{Q_0}$ (replacing $\underline{A}_{Q_0}$ by $\underline{\widehat{A}}_{Q_0}$ in (A.54), etc.). Moreover, in connection with ψ, one introduces the vector of b-periods $\underline{U}^{(2)}$ of $\Omega^{(2)}$ by

$$\underline{U}^{(2)} = (U_1^{(2)}, \dots, U_n^{(2)}), \quad U_j^{(2)} = \frac{1}{2\pi i} \int_{b_j} \Omega^{(2)}, \quad j = 1, \dots, n, \tag{A.57}$$

and then renders ψ single-valued on $\widehat{\mathcal{K}}_n$ by requiring

$$\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_2) = c\, \underline{U}^{(2)} \tag{A.58}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = c\, \underline{U}^{(2)} \pmod{L_n}$). Actually, by (A.32),

$$\underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_2) - c\, \underline{U}^{(2)} \in \mathbb{Z}^n, \tag{A.59}$$

suffices to guarantee single-valuedness of ψ on $\widehat{\mathcal{K}}_n$. Without the replacement of $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ by $\underline{\widehat{A}}_{Q_0}$ and $\underline{\widehat{\alpha}}_{Q_0}$ in (A.56) and without the assumption (A.58) (or (A.59)), ψ is a multiplicative (multi-valued) function on $\mathcal{K}_n$, and then most effectively discussed by introducing the notion of characters on $\mathcal{K}_n$ (cf. [21, Sect. III.9]). For simplicity, we decided to avoid the latter possibility and throughout this paper will always tacitly assume (A.58) or (A.59).

Appendix B. Restrictions on $\underline{B} = i\underline{U}_0^{(2)}$

The purpose of this appendix is to prove the result (2.70), $\underline{B} = i\underline{U}_0^{(2)} \in \mathbb{R}^n$, for some choice of homology basis $\{a_j, b_j\}_{j=1}^n$ on $\mathcal{K}_n$ as recorded in Remark 2.8.

To this end we first recall a few notions in connection with periodic meromorphic functions of p complex variables.

Definition B.1. Let $p \in \mathbb{N}$ and $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ be meromorphic (i.e., a ratio of two entire functions of p complex variables). Then,
(i) $\underline{\omega} = (\omega_1, \dots, \omega_p) \in \mathbb{C}^p \backslash \{0\}$ is called a *period* of F if

$$F(\underline{z} + \underline{\omega}) = F(\underline{z}) \tag{B.1}$$

for all $\underline{z} \in \mathbb{C}^p$ for which F is analytic. The set of all periods of F is denoted by $\mathcal{P}_F$.
(ii) F is called *degenerate* if it depends on less than p complex variables; otherwise, F is called *nondegenerate*.

Theorem B.2. *Let $p \in \mathbb{N}$, $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ be meromorphic, and $\mathcal{P}_F$ be the set of all periods of F. Then either*
(i) $\mathcal{P}_F$ has a finite limit point,
or
(ii) $\mathcal{P}_F$ has no finite limit point.
In case (i), $\mathcal{P}_F$ contains infinitesimal periods $($i.e., sequences of nonzero periods converging to zero$)$. In addition, in case (i) each period is a limit point of periods and hence $\mathcal{P}_F$ is a perfect set.
Moreover, F is degenerate if and only if F admits infinitesimal periods. In particular, for nondegenerate functions F only alternative (ii) applies.

Next, let $\underline{\omega}_q \in \mathbb{C}^p\backslash\{0\}$, $q = 1, \dots, r$ for some $r \in \mathbb{N}$. Then $\underline{\omega}_1, \dots, \underline{\omega}_r$ are called *linearly independent over* $\mathbb{Z}$ (*resp.* $\mathbb{R}$) if

$$\begin{aligned} &\nu_1\underline{\omega}_1 + \cdots + \nu_r\underline{\omega}_r = 0, \quad \nu_q \in \mathbb{Z} \text{ (resp., } \nu_q \in \mathbb{R}),\ q = 1, \dots, r, \\ &\text{implies } \nu_1 = \cdots = \nu_r = 0. \end{aligned} \tag{B.2}$$

Clearly, the maximal number of vectors in $\mathbb{C}^p$ linearly independent over $\mathbb{R}$ equals $2p$.

Theorem B.3. *Let $p \in \mathbb{N}$.*
(i) If $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$ is a nondegenerate meromorphic function with periods $\underline{\omega}_q \in \mathbb{C}^p\backslash\{0\}$, $q = 1, \dots, r$, $r \in \mathbb{N}$, linearly independent over $\mathbb{Z}$, then $\underline{\omega}_1, \dots, \underline{\omega}_r$ are also linearly independent over $\mathbb{R}$. In particular, $r \leq 2p$.
(ii) A nondegenerate entire function $F\colon \mathbb{C}^p \to \mathbb{C}$ cannot have more than p periods linearly independent over $\mathbb{Z}$ $($or $\mathbb{R})$.

For $p = 1$, $\exp(z)$, $\sin(z)$ are examples of entire functions with precisely one period. Any non-constant doubly periodic meromorphic function of one complex variable is elliptic (and hence has indeed poles).

Definition B.4. Let $p, r \in \mathbb{N}$. A system of periods $\underline{\omega}_q \in \mathbb{C}^p\backslash\{0\}$, $q = 1, \dots, r$ of a nondegenerate meromorphic function $F\colon \mathbb{C}^p \to \mathbb{C} \cup \{\infty\}$, linearly independent over $\mathbb{Z}$, is called *fundamental* or a *basis* of periods for F if every period $\underline{\omega}$ of F is of the form

$$\underline{\omega} = m_1\underline{\omega}_1 + \cdots + m_r\underline{\omega}_r \text{ for some } m_q \in \mathbb{Z},\ q = 1, \dots, r. \tag{B.3}$$

The representation of $\underline{\omega}$ in (B.3) is unique since by hypothesis $\underline{\omega}_1, \dots, \underline{\omega}_r$ are linearly independent over $\mathbb{Z}$. In addition, $\mathcal{P}_F$ is countable in this case. (This rules out case (i) in Theorem B.2 since a perfect set is uncountable. Hence, one does not have to assume that F is nondegenerate in Definition B.4.)

This material is standard and can be found, for instance, in [47, Ch. 2].

Next, returning to the Riemann theta function $\theta(\underline{\cdot})$ in (A.30), we introduce the vectors $\{\underline{e}_j\}_{j=1}^n, \{\underline{\tau}_j\}_{j=1}^n \subset \mathbb{C}^n\backslash\{0\}$ by

$$\underline{e}_j = (0, \dots, 0, \underbrace{1}_{j}, 0, \dots, 0), \quad \underline{\tau}_j = \underline{e}_j\tau, \quad j = 1, \dots, n. \tag{B.4}$$

Then

$$\{\underline{e}_j\}_{j=1}^n \tag{B.5}$$

is a basis of periods for the entire (nondegenerate) function $\theta(\underline{\cdot})\colon \mathbb{C}^n \to \mathbb{C}$. Moreover, fixing $k, k' \in \{1, \dots, n\}$, then

$$\{\underline{e}_j, \underline{\tau}_j\}_{j=1}^n \tag{B.6}$$

is a basis of periods for the meromorphic function $\partial^2_{z_k z_{k'}} \ln\big(\theta(\underline{\cdot})\big)\colon \mathbb{C}^n \to \mathbb{C}\cup\{\infty\}$ (cf. (A.32) and [21, p. 91]).

Next, let $\underline{A} \in \mathbb{C}^n$, $\underline{D} = (D_1, \dots, D_n) \in \mathbb{R}^n$, $D_j \in \mathbb{R}\backslash\{0\}$, $j = 1, \dots, n$ and consider

$$\begin{aligned} f_{k,k'}\colon \mathbb{R} \to \mathbb{C}, \quad f_{k,k'}(x) &= \partial^2_{z_k z_{k'}} \ln\big(\theta(\underline{A} + \underline{z})\big)\big|_{\underline{z}=\underline{D}x} \\ &= \partial^2_{z_k z_{k'}} \ln\big(\theta(\underline{A} + \underline{z}\operatorname{diag}(\underline{D}))\big)\big|_{\underline{z}=(x,\dots,x)}. \end{aligned} \tag{B.7}$$

Here $\operatorname{diag}(\underline{D})$ denotes the diagonal matrix

$$\operatorname{diag}(\underline{D}) = \big(D_j \delta_{j,j'}\big)_{j,j'=1}^n. \tag{B.8}$$

Then the quasi-periods D_j^{-1}, $j = 1, \dots, n$, of $f_{k,k'}$ are in a one-to-one correspondence with the periods of

$$F_{k,k'}\colon \mathbb{C}^n \to \mathbb{C}\cup\{\infty\}, \quad F_{k,k'}(\underline{z}) = \partial^2_{z_k z_{k'}} \ln\big(\theta(\underline{A} + \underline{z}\operatorname{diag}(\underline{D})\big) \tag{B.9}$$

of the special type

$$\underline{e}_j\big(\operatorname{diag}(\underline{D})\big)^{-1} = \big(0, \dots, 0, \underbrace{D_j^{-1}}_{j}, 0, \dots, 0\big). \tag{B.10}$$

Moreover,

$$f_{k,k'}(x) = F_{k,k'}(\underline{z})|_{\underline{z}=(x,\dots,x)}, \quad x \in \mathbb{R}. \tag{B.11}$$

Theorem B.5. *Suppose V in (2.65) (or (2.66)) to be quasi-periodic. Then there exists a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that the vector $\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)}$ with $\underline{\widetilde U}_0^{(2)}$ the vector of $\tilde b$-periods of the corresponding normalized differential of the second kind, $\widetilde\omega^{(2)}_{P_\infty,0}$, satisfies the constraint*

$$\underline{\widetilde B} = i\underline{\widetilde U}_0^{(2)} \in \mathbb{R}^n. \tag{B.12}$$

Proof. By (A.26), the vector of b-periods $\underline{U}_0^{(2)}$ associated with a given homology basis $\{a_j, b_j\}_{j=1}^n$ on $\mathcal{K}_n$ and the normalized differential of the 2nd kind, $\omega^{(2)}_{P_\infty,0}$, is continuous with respect to $E_0, \dots, E_{2n}$. Hence, we may assume in the following that

$$B_j \neq 0,\ j = 1, \dots, n, \quad \underline{B} = (B_1, \dots, B_n) \tag{B.13}$$

by slightly altering $E_0, \dots, E_{2n}$, if necessary. By comparison with the Its–Matveev formula (2.66), we may write

$$\begin{aligned} V(x) &= \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)) \\ &= \Lambda_0 + 2\sum_{j,k=1}^n U^{(2)}_{0,j} U^{(2)}_{0,k} \partial^2_{z_k z_j} \ln\big(\theta(\underline{A} + \underline{z})\big)\big|_{\underline{z}=\underline{B}x}. \end{aligned} \tag{B.14}$$

Introducing the meromorphic (nondegenerate) function $\mathcal{V}\colon \mathbb{C}^n \to \mathbb{C}\cup\{\infty\}$ by

$$\mathcal{V}(\underline{z}) = \Lambda_0 + 2\sum_{j,k=1}^n U^{(2)}_{0,j} U^{(2)}_{0,k} \partial^2_{z_k z_j} \ln\big(\theta(\underline{A} + \underline{z}\operatorname{diag}(\underline{B}))\big), \tag{B.15}$$

one observes that

$$V(x) = \mathcal{V}(\underline{z})|_{\underline{z}=(x,\dots,x)}. \tag{B.16}$$

In addition, $\mathcal{V}$ has a basis of periods

$$\Big\{\underline{e}_j\big(\operatorname{diag}(\underline{B})\big)^{-1}, \underline{\tau}_j\big(\operatorname{diag}(\underline{B})\big)^{-1}\Big\}_{j=1}^{n} \tag{B.17}$$

by (B.6), where

$$\underline{e}_j\big(\operatorname{diag}(\underline{B})\big)^{-1} = \big(0,\dots,0,\underbrace{B_j^{-1}}_{j},0,\dots,0\big), \quad j=1,\dots,n, \tag{B.18}$$

$$\underline{\tau}_j\big(\operatorname{diag}(\underline{B})\big)^{-1} = \big(\tau_{j,1}B_1^{-1},\dots,\tau_{j,n}B_n^{-1}\big), \quad j=1,\dots,n. \tag{B.19}$$

By hypothesis, V in (B.14) is quasi-periodic and hence has n real (scalar) quasi-periods. The latter are not necessarily linearly independent over $\mathbb{Q}$ from the outset, but by slightly changing the locations of branchpoints $\{E_m\}_{m=0}^{2n}$ into, say, $\{\widetilde{E}_m\}_{m=0}^{2n}$, one can assume they are. In particular, since the period vectors in (B.17) are linearly independent and the (scalar) quasi-periods of V are in a one-one correspondence with vector periods of $\mathcal{V}$ of the special form (B.18) (cf. (B.9), (B.10)), there exists a homology basis $\{\tilde a_j, \tilde b_j\}_{j=1}^n$ on $\mathcal{K}_n$ such that the vector $\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)}$ corresponding to the normalized differential of the second kind, $\widetilde{\omega}_{P_\infty,0}^{(2)}$ and this particular homology basis, is real-valued. By continuity of $\underline{\widetilde{U}}_0^{2}$ with respect to $\widetilde{E}_0,\dots,\widetilde{E}_{2m}$, this proves (B.12). $\square$

Remark B.6. Given the existence of a homology basis with associated real vector $\underline{\widetilde{B}} = i\underline{\widetilde{U}}_0^{(2)}$, one can follow the proof of Theorem 10.3.1 in [42] and show that each μ_j, $j=1,\dots,n$, is quasi-periodic with the same quasi-periods as V.

Appendix C. Floquet theory and an explicit example

In this appendix we discuss the special case of algebro-geometric complex-valued periodic potentials and we briefly point out the connections between the algebro-geometric approach and standard Floquet theory. We then conclude with the explicit genus $n=1$ example which illustrates both, the algebro-geometric as well as the periodic case.

We start with the periodic case. Suppose V satisfies

$$V \in CP(\mathbb{R}) \text{ and for all } x\in\mathbb{R},\ V(x+\Omega) = V(x) \tag{C.1}$$

for some period $\Omega > 0$. In addition, we suppose that V satisfies Hypothesis 3.4.

Under these assumptions the Riemann surface associated with V, which by Floquet theoretic arguments, in general, would be a two-sheeted Riemann surface of infinite genus, can be reduced to the compact hyperelliptic Riemann surface corresponding to $\mathcal{K}_n$ induced by $y^2 = R_{2n+1}(z)$. Moreover, the corresponding Schrödinger operator H is then defined as in (4.1) and one introduces the fundamental system of distributional solutions $c(z,\cdot,x_0)$ and $s(z,\cdot,x_0)$ of $H\psi = z\psi$ satisfying

$$c(z,x_0,x_0) = s_x(z,x_0,x_0) = 1, \tag{C.2}$$

$$c_x(z,x_0,x_0) = s(z,x_0,x_0) = 0, \quad z\in\mathbb{C} \tag{C.3}$$

with $x_0 \in \mathbb{R}$ a fixed reference point. For each $x, x_0 \in \mathbb{R}$, $c(z, x, x_0)$ and $s(z, x, x_0)$ are entire with respect to z. The monodromy matrix $\mathcal{M}(z, x_0)$ is then given by

$$\mathcal{M}(z, x_0) = \begin{pmatrix} c(z, x_0 + \Omega, x_0) & s(z, x_0 + \Omega, x_0) \\ c_x(z, x_0 + \Omega, x_0) & s_x(z, x_0 + \Omega, x_0) \end{pmatrix}, \quad z \in \mathbb{C} \tag{C.4}$$

and its eigenvalues $\rho_\pm(z)$, the Floquet multipliers (which are x_0-independent), satisfy

$$\rho_+(z)\rho_-(z) = 1 \tag{C.5}$$

since $\det(\mathcal{M}(z, x_0)) = 1$. The Floquet discriminant $\Delta(\cdot)$ is then defined by

$$\Delta(z) = \operatorname{tr}(\mathcal{M}(z, x_0))/2 = [c(z, x_0 + \Omega, x_0) + s_x(z, x_0 + \Omega, x_0)]/2 \tag{C.6}$$

and one obtains

$$\rho_\pm(z) = \Delta(z) \pm [\Delta(z)^2 - 1]^{1/2}. \tag{C.7}$$

We also note that

$$|\rho_\pm(z)| = 1 \text{ if and only if } \Delta(z) \in [-1, 1]. \tag{C.8}$$

The Floquet solutions $\psi_\pm(z, x, x_0)$, the analog of the functions in (4.48), are then given by

$$\psi_\pm(z, x, x_0) = c(z, x, x_0) + s(z, x, x_0)[\rho_\pm(z) - c(z, x_0 + \Omega, x_0)]s(z, x_0 + \Omega, x_0)^{-1}, \\ z \in \Pi \backslash \{\mu_j(x_0)\}_{j=1,\dots,n} \tag{C.9}$$

and one verifies (for $x, x_0 \in \mathbb{R}$),

$$\psi_\pm(z, x + \Omega, x_0) = \rho_\pm(z)\psi_\pm(z, x, x_0), \quad z \in \Pi \backslash \{\mu_j(x_0)\}_{j=1,\dots,n}, \tag{C.10}$$

$$\psi_+(z, x, x_0)\psi_-(z, x, x_0) = \frac{s(z, x + \Omega, x)}{s(z, x_0 + \Omega, x_0)}, \quad z \in \mathbb{C} \backslash \{\mu_j(x_0)\}_{j=1,\dots,n}, \tag{C.11}$$

$$W(\psi_+(z, \cdot, x_0), \psi_-(z, \cdot, x_0)) = -\frac{2[\Delta(z)^2 - 1]^{1/2}}{s(z, x_0 + \Omega, x_0)}, \quad z \in \Pi \backslash \{\mu_j(x_0)\}_{j=1,\dots,n}, \tag{C.12}$$

$$g(z, x) = -\frac{s(z, x + \Omega, x)}{2[\Delta(z)^2 - 1]^{1/2}} = \frac{iF_n(z, x)}{2R_{2n+1}(z)^{1/2}}, \quad z \in \Pi. \tag{C.13}$$

Moreover, one computes

$$\begin{aligned} \frac{d\Delta(z)}{dz} &= -s(z, x_0 + \Omega, x_0)\frac{1}{2}\int_{x_0}^{x_0+\Omega} dx\, \psi_+(z, x, x_0)\psi_-(z, x, x_0) \\ &= \Omega[\Delta(z)^2 - 1]^{1/2}\langle g(z, \cdot)\rangle, \quad z \in \mathbb{C} \end{aligned} \tag{C.14}$$

and hence

$$\frac{d\Delta(z)/dz}{[\Delta(z)^2 - 1]^{1/2}} = \frac{d}{dz}\big\{ \ln\big[\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\big]\big\} = \Omega\langle g(z, \cdot)\rangle, \quad z \in \Pi. \tag{C.15}$$

Here the mean value $\langle f\rangle$ of a periodic function $f \in CP(\mathbb{R})$ of period $\Omega > 0$ is simply given by

$$\langle f\rangle = \frac{1}{\Omega}\int_{x_0}^{x_0+\Omega} dx\, f(x), \tag{C.16}$$

independent of the choice of $x_0 \in \mathbb{R}$. Thus, applying (3.22) one obtains

$$\int_{z_0}^{z} \frac{dz'\,[d\Delta(z')/dz']}{[\Delta(z')^2 - 1]^{1/2}} = \ln\left(\frac{\Delta(z) + [\Delta(z)^2 - 1]^{1/2}}{\Delta(z_0) + [\Delta(z_0)^2 - 1]^{1/2}}\right)$$

$$= \Omega \int_{z_0}^{z} dz' \, \langle g(z',\cdot)\rangle = -(\Omega/2)\big[\big\langle g(z,\cdot)^{-1}\big\rangle - \big\langle g(z_0,\cdot)^{-1}\big\rangle\big], \quad z, z_0 \in \Pi \tag{C.17}$$

and hence

$$\ln\big[\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\big] = -(\Omega/2)\big\langle g(z,\cdot)^{-1}\big\rangle + C. \tag{C.18}$$

Letting $|z| \to \infty$ one verifies that $C = 0$ and thus

$$\ln\big[\Delta(z) + [\Delta(z)^2 - 1]^{1/2}\big] = -(\Omega/2)\big\langle g(z,\cdot)^{-1}\big\rangle, \quad z \in \Pi. \tag{C.19}$$

We note that by continuity with respect to z, equations (C.12), (C.13), (C.15), (C.17), and (C.19) all extend to either side of the set of cuts in $\mathcal{C}$. Consequently,

$$\Delta(z) \in [-1,1] \text{ if and only if } \operatorname{Re}\big(\big\langle g(z,\cdot)^{-1}\big\rangle\big) = 0. \tag{C.20}$$

In particular, our characterization of the spectrum of H in (4.44) is thus equivalent to the standard Floquet theoretic characterization of H in terms of the Floquet discriminant,

$$\sigma(H) = \{\lambda \in \mathbb{C} \,|\, \Delta(\lambda) \in [-1,1]\}. \tag{C.21}$$

The result (C.21) was originally proven in [55] and [57] for complex-valued periodic (not necessarily algebro-geometric) potentials (cf. also [60], and more recently, [61], [62]).

We will end this appendix by providing an explicit example of the simple yet nontrivial genus $n = 1$ case which illustrates the periodic case as well as some of the general results of Sections 2–4 and Appendix B. For more general elliptic examples we refer to [29], [30] and the references therein.

By $\wp(\cdot) = \wp(\cdot \,|\, \Omega_1, \Omega_3)$ we denote the Weierstrass $\wp$-function with fundamental half-periods Ω_j, $j = 1,3$, $\Omega_1 > 0$, $\Omega_3 \in \mathbb{C}\backslash\{0\}$, $\operatorname{Im}(\Omega_3) > 0$, $\Omega_2 = \Omega_1 + \Omega_3$, and invariants g_2 and g_3 (cf. [1, Ch. 18]). By $\zeta(\cdot) = \zeta(\cdot\,|\Omega_1,\Omega_3)$ and $\sigma(\cdot) = \sigma(\cdot\,|\Omega_1,\Omega_3)$ we denote the Weierstrass zeta and sigma functions, respectively. We also denote $\tau = \Omega_3/\Omega_1$ and hence stress that $\operatorname{Im}(\tau) > 0$.

Example C.1. Consider the genus one ($n = 1$) Lamé potential

$$V(x) = 2\wp(x + \Omega_3) \tag{C.22}$$

$$= -2\bigg\{\ln\bigg[\theta\bigg(\frac{1}{2} + \frac{x}{2\Omega_1}\bigg)\bigg]\bigg\}'' - 2\frac{\zeta(\Omega_1)}{\Omega_1}, \quad x \in \mathbb{R}, \tag{C.23}$$

where

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp\big(2\pi i n z + \pi i n^2 \tau\big), \quad z \in \mathbb{C}, \ \tau = \Omega_3/\Omega_1, \tag{C.24}$$

and introduce

$$L = -\frac{d^2}{dx^2} + 2\wp(x+\Omega_3), \quad P_3 = -\frac{d^3}{dx^3} + 3\wp(x+\Omega_3)\frac{d}{dx} + \frac{3}{2}\wp'(x+\Omega_3). \tag{C.25}$$

Then one obtains

$$[L, P_3] = 0 \tag{C.26}$$

which yields the elliptic curve

$$\begin{aligned} &\mathcal{K}_1 \colon \mathcal{F}_1(z,y) = y^2 - R_3(z) = y^2 - \big(z^3 - (g_2/4)z + (g_3/4)\big) = 0, \\ &R_3(z) = \prod_{m=0}^{2}(z - E_m) = z^3 - (g_2/4)z + (g_3/4), \\ &E_0 = -\wp(\Omega_1), \ E_1 = -\wp(\Omega_2), \ E_2 = -\wp(\Omega_3). \end{aligned} \tag{C.27}$$

Moreover, one has

$$F_1(z,x) = z + \wp(x+\Omega_3), \quad \mu_1(x) = -\wp(x+\Omega_3), \tag{C.28}$$

$$H_2(z,x) = z^2 - \wp(x+\Omega_3)z + \wp(x+\Omega_3)^2 - (g_2/4), \tag{C.29}$$

$$\nu_\ell(x) = \big[\wp(x+\Omega_3) - (-1)^\ell [g_2 - 3\wp(x+\Omega_3)^2]^{1/2}\big]/2, \quad \ell = 0,1$$

and

$$\text{s-}\widehat{\text{KdV}}_1(V) = 0, \tag{C.30}$$

$$\text{s-}\widehat{\text{KdV}}_2(V) - (g_2/8)\,\text{s-}\widehat{\text{KdV}}_0(V) = 0, \text{ etc.} \tag{C.31}$$

In addition, we record

$$\psi_\pm(z,x,x_0) = \frac{\sigma(x+\Omega_3 \pm b)\sigma(x_0+\Omega_3)}{\sigma(x+\Omega_3)\sigma(x_0+\Omega_3 \pm b)} e^{\mp\zeta(b)(x-x_0)}, \tag{C.32}$$

$$\psi_\pm(z,x+2\Omega_1,x_0) = \rho_\pm(z)\psi_\pm(z,x,x_0), \quad \rho_\pm(z) = e^{\pm[(b/\Omega_1)\zeta(\Omega_1)-\zeta(b)]2\Omega_1} \tag{C.33}$$

with Floquet parameter corresponding to Ω_1-direction given by

$$k_1(b) = i[\zeta(b)\Omega_1 - \zeta(\Omega_1)b]/\Omega_1. \tag{C.34}$$

Here

$$\begin{aligned} P &= (z,y) = (-\wp(b), -(i/2)\wp'(b)) \in \Pi_+, \\ P^* &= (z,-y) = (-\wp(b), (i/2)\wp'(b)) \in \Pi_-, \end{aligned} \tag{C.35}$$

where b varies in the fundamental period parallelogram spanned by the vertices 0, $2\Omega_1$, $2\Omega_2$, and $2\Omega_3$. One then computes

$$\Delta(z) = \cosh[2(\Omega_1\zeta(b) - b\zeta(\Omega_1))], \tag{C.36}$$

$$\langle\mu_1\rangle = \zeta(\Omega_1)/\Omega_1, \quad \langle V\rangle = -2\zeta(\Omega_1)/\Omega_1, \tag{C.37}$$

$$g(z,x) = -\frac{z+\wp(x+\Omega_3)}{\wp'(b)}, \tag{C.38}$$

$$\frac{d}{dz}\big\langle g(z,\cdot)^{-1}\big\rangle = 2\frac{z - [\zeta(\Omega_1)/\Omega_1]}{\wp'(b)} = -2\langle g(z,\cdot)\rangle, \tag{C.39}$$

$$\big\langle g(z,\cdot)^{-1}\big\rangle = -2[\zeta(b) - (b/\Omega_1)\zeta(\Omega_1)], \tag{C.40}$$

where $(z,y) = (-\wp(b), -(i/2)\wp'(b)) \in \Pi_+$. The spectrum of the operator H with potential $V(x) = 2\wp(x+\Omega_3)$ is then determined as follows

$$\sigma(H) = \{\lambda \in \mathbb{C} \,|\, \Delta(\lambda) \in [-1,1]\} \tag{C.41}$$

$$= \big\{\lambda \in \mathbb{C} \,\big|\, \mathrm{Re}\big(\big\langle g(\lambda,\cdot)^{-1}\big\rangle\big) = 0\big\} \tag{C.42}$$

$$= \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}[\Omega_1\zeta(b) - b\zeta(\Omega_1)] = 0,\, \lambda = -\wp(b)\}. \tag{C.43}$$

Generically (cf. [61]), $\sigma(H)$ consists of one simple analytic arc (connecting two of the three branch points E_m, $m = 0,1,2$) and one simple semi-infinite analytic arc (connecting the remaining of the branch points and infinity). The semi-infinite arc σ_∞ asymptotically approaches the half-line $L_{\langle V\rangle} = \{z \in \mathbb{C} \,|\, z = -2\zeta(\Omega_1)/\Omega_1 + x,\, x \geq 0\}$ in the following sense: asymptotically, σ_∞ can be parameterized by

$$\sigma_\infty = \big\{z \in \mathbb{C} \,\big|\, z = R - 2i\,[\mathrm{Im}(\zeta(\Omega_1))/\Omega_1] + O\big(R^{-1/2}\big) \text{ as } R \uparrow \infty\big\}. \tag{C.44}$$

We note that a slight change in the setup of Example C.1 permits one to construct crossing spectral arcs as shown in [28]. One only needs to choose complex conjugate fundamental half-periods $\widehat{\Omega}_1 \notin \mathbb{R}$, $\widehat{\Omega}_3 = \overline{\widehat{\Omega}_1}$ with real period $\Omega = 2(\widehat{\Omega}_1 + \widehat{\Omega}_3) > 0$ and consider the potential $V(x) = 2\wp(x + a\,|\widehat{\Omega}_1, \widehat{\Omega}_3)$, $0 < \mathrm{Im}(a) < 2|\mathrm{Im}(\widehat{\Omega}_1)|$.

Finally, we briefly consider a change of homology basis and illustrate Theorem B.5. Let $\Omega_1 > 0$ and $\Omega_3 \in \mathbb{C}$, $\mathrm{Im}(\Omega_3) > 0$. We choose the homology basis $\{\tilde{a}_1, \tilde{b}_1\}$ such that $\tilde{b}_1$ encircles E_0 and E_1 counterclockwise on Π_+ and $\tilde{a}_1$ starts near E_1, intersects $\tilde{b}_1$ on Π_+, surrounds E_2 clockwise and then continues on Π_- back to its initial point surrounding E_1 such that (A.16) holds. Then,

$$\omega_1 = c_1(1)\,dz/y, \quad c_1(1) = (4i\Omega_1)^{-1}, \tag{C.45}$$

$$\int_{\tilde{a}_1} \omega_1 = 1, \quad \int_{\tilde{b}_1} \omega_1 = \tau, \quad \tau = \Omega_3/\Omega_1, \tag{C.46}$$

$$\widetilde{\omega}^{(2)}_{P_\infty,0} = -\frac{(z-\lambda_1)dz}{2y}, \quad \lambda_1 = \zeta(\Omega_1)/\Omega_1, \tag{C.47}$$

$$\int_{\tilde{a}_1} \widetilde{\omega}^{(2)}_{P_\infty,0} = 0, \quad \frac{1}{2\pi i}\int_{\tilde{b}_1} \widetilde{\omega}^{(2)}_{P_\infty,0} = -2c_1(1) = \widetilde{U}_{0,1}, \tag{C.48}$$

$$\widetilde{U}_{0,1} = \frac{i}{2\Omega_1} \in i\mathbb{R}, \tag{C.49}$$

$$\begin{aligned}\int_{Q_0}^{P} \widetilde{\omega}^{(2)}_{P_\infty,0} - \tilde{e}^{(2)}_0(Q_0) &\underset{b\to 0}{=} \frac{i}{b} + O(b) \\ &\underset{\zeta\to 0}{=} -\zeta^{-1} + O(\zeta), \quad \zeta = \sigma/z^{1/2},\ \sigma \in \{1,-1\},\end{aligned} \tag{C.50}$$

$$\tilde{e}^{(2)}_0(Q_0) = -i[\zeta(b_0)\Omega_1 - \zeta(\Omega_1)b_0]/\Omega_1, \tag{C.51}$$

$$\begin{aligned} &i\bigg[\int_{Q_0}^{P} \widetilde{\omega}^{(2)}_{P_\infty,0} - \tilde{e}^{(2)}_0(Q_0)\bigg] = [\zeta(\Omega_1)b - \zeta(b)\Omega_1]/\Omega_1, \\ &P = (-\wp(b), -(i/2)\wp'(b)),\ Q_0 = (-\wp(b_0), -(i/2)\wp'(b_0)).\end{aligned} \tag{C.52}$$

The change of homology basis (cf. (A.33)–(A.39))

$$\begin{pmatrix}\tilde{a}_1\\ \tilde{b}_1\end{pmatrix} \mapsto \begin{pmatrix}a'_1\\ b'_1\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\tilde{a}_1\\ \tilde{b}_1\end{pmatrix} = \begin{pmatrix}A\tilde{a}_1 + B\tilde{b}_1\\ Ca_1 + Db_1\end{pmatrix}, \tag{C.53}$$

$$A, B, C, D \in \mathbb{Z}, \quad AD - BC = 1, \tag{C.54}$$

then implies

$$\omega'_1 = \frac{\omega_1}{A + B\tau}, \tag{C.55}$$

$$\tau' = \frac{\Omega'_3}{\Omega'_1} = \frac{C + D\tau}{A + B\tau}, \tag{C.56}$$

$$\Omega'_1 = A\Omega_1 + B\Omega_3, \quad \Omega'_3 = C\Omega_1 + D\Omega_3, \tag{C.57}$$

$$\omega^{(2)\,\prime}_{P_\infty,0} = -\frac{(z-\lambda'_1)dz}{2y}, \quad \lambda'_1 = \lambda_1 - \frac{\pi i B}{2\Omega_1\Omega'_1}, \tag{C.58}$$

$$\int_{a'_1} \omega^{(2)\,\prime}_{P_\infty,0} = 0, \quad \frac{1}{2\pi i}\int_{b'_1} \omega^{(2)\,\prime}_{P_\infty,0} = -\frac{2c_1(1)}{A + B\tau} = U'_{0,1}, \tag{C.59}$$

$$U'_{0,1} = \frac{\widetilde{U}_{0,1}}{A + B\tau} = \frac{i}{2\Omega'_1}. \tag{C.60}$$

Moreover, one infers

$$\begin{aligned}&\psi_\pm(z,x+2\Omega_1',x_0)=\rho_\pm(z)'\psi_\pm(z,x,x_0),\\&\rho_\pm(z)'=e^{\pm[(b/\Omega_1')(A\zeta(\Omega_1)+B\zeta(\Omega_3))-\zeta(b)]2\Omega_1'}\end{aligned}\tag{C.61}$$

with Floquet parameter $k_1(b)'$ corresponding to Ω_1'-direction given by

$$k_1(b)'=i\bigg[\zeta(b)\Omega_1-\zeta(\Omega_1)b+\frac{\pi iB}{2\Omega_1'}b\bigg]\bigg/\Omega_1.\tag{C.62}$$

Acknowledgments. F. G. is particularly indebted to Vladimir A. Marchenko for renewing his interest in the spectral theoretic questions addressed in this paper and for the discussions we shared on this topic in June of 2000 at the Department of Mathematical Sciences of the Norwegian University of Science and Technology in Trondheim, Norway.

We thank Helge Holden and Kwang Shin for many discussions on topics related to this paper and Kwang Shin for a critical reading of our manuscript. Moreover, we are indebted to Norrie Everitt and Igor Verbitsky for pointing out the origin of Lemma 4.5 and to Jürgen Voigt for pointing out references [3] and [40] to us.

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