Notes on Rank-One Perturbed Resolvent.

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Abstract

This paper is a didactic comment (a transcription with variations) to the paper of S.R. Foguel *Finite Dimensional Perturbations in Banach Spaces*.

Addressed, mainly: postgraduates and related readers.

Subject: Suppose we have two operators, T_1, T_2, so that

$$T_2^{-1} - T_1^{-1} \text{ is rank-one.}$$

What we want to know is how we can express

$$(z - T_2)^{-1} - (z - T_1)^{-1}$$

in terms of T_1^{-1}, T_2^{-1} (and, e.g. $(z - T_1)^{-1}$).

Keywords: M.G.Krein's Formula, Finite Rank Perturbations.

Introduction

The problem I have is that my students can prove theorems, but not invent good formulae, the formulae that are worth to discuss. They, the students, can correct Leibniz, and are not capable of realizing standard and simple engineering computation of primitive engineering problem. They can *explain* and *teach* what Green's function is indeed and really, and are not able to *compute* THE Green's function.

Perhaps, I am wrong. I would be glad, if I were wrong. I hope I am wrong. However, I have expounded the situation I have observed it.

I cannot solve the problem. But as a contribution I try to tell about some old tricks which can help to *compute*, not to *substantiate the being computed and ready.*

But first of all, let us state a *primitive* problem.

Let T stands for the functions transformation defined by

$$(Tu)(x) := -\frac{\partial^2 u(x)}{\partial x^2}\,;$$

T_{DD} and T_{DN} be the restrictions of T so that T_{DD} and T_{DN} act on that functions, u, which are defined on $[0, 1]$ and, in addition:

$$\begin{aligned} u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

$$\text{in the case of } T_{DD}\,,$$

$$\begin{aligned} u(0) &= 0 \\ \left.\frac{\partial u(x)}{\partial x}\right|_{x=1} &= 0 \end{aligned}$$

$$\text{in the case of } T_{DN}\,.$$

[1] These T_{DN}^{-1} , T_{DD}^{-1} are integral operators and their integral kernels are

$$G_{DD}(x,\xi) = -\left\{ \begin{array}{lcl} x\cdot(\xi-1) & , & \text{if } x \le \xi \\ (x-1)\cdot\xi & , & \text{if } \xi \le x \end{array} \right\}$$

$$G_{DN}(x,\xi) = \left\{ \begin{array}{lcl} x & , & \text{if } x \le \xi \\ \xi & , & \text{if } \xi \le x \end{array} \right\}$$

Let us draw the attention on that fact that

$$T_{DN}^{-1} - T_{DD}^{-1}$$

is well defined and its integral kernel is

$$\begin{aligned} G_{DN}(x,\xi) - G_{DD}(x,\xi) &= \left\{ \begin{array}{lcl} x + x\cdot(\xi-1) & , & \text{if } x \le \xi \\ \xi + (x-1)\cdot\xi & , & \text{if } \xi \le x \end{array} \right\} \\ &= \left\{ \begin{array}{lcl} x\cdot\xi & , & \text{if } x \le \xi \\ x\cdot\xi & , & \text{if } \xi \le x \end{array} \right\} = x\cdot\xi \end{aligned}$$

We formulate this result a little generalizing:

$$G_{DN}(x,\xi) - G_{DD}(x,\xi) = f(x)l(\xi)$$

$$\text{where } f(x) = x\,,\ l(\xi) = \xi\,.$$

[1] take the underlying space which you like, within reason, then make corrections

So, we state:

$$T_{DN}^{-1} - T_{DD}^{-1} \quad = \quad \text{is } \textbf{rank-one .}$$

It is not very difficult to describe

$$(z - T_{DD})^{-1} \,.$$

This is an integral operator. Its integral kernel is

$$G_{DD}(x, \xi, z) \quad = \quad -\frac{1}{k \sin(k)} \left\{ \begin{array}{lcl} \sin(kx)\sin(k(1-\xi)) & , & \text{if } x \leq \xi \\ \sin(k\xi)\sin(k(1-x)) & , & \text{if } \xi \leq x \end{array} \right\}$$
$$\text{where } k \text{ is defined by } k^2 = z \,,$$

and where, of course, z is to be so, that

$$\sin(k) \neq 0 \,.$$

As for

$$(z - T_{DN})^{-1} \,,$$

it is not very difficult to describe it as well, but the primitive problem in this *toy* situation is:

to reduce (effectively) *the description of* $(z - T_{DN})^{-1}$ *to the description of* $(z - T_{DD})^{-1}$;

to construct an effective abstraction.

A beautiful example of such an abstraction is that what is presented in the paper of S.R. Foguel, *Finite Dimensional Perturbations in Banach Spaces.* We will not discuss all the contents of that paper and restrict ourselves, centring on its initial part. Namely, our section 1 is a transcription of the section 1 of the Foguel's paper, and our section 2 is a variation, or modification, adapted to the described situation.

Before starting, we shall say a few words about features of our paper. The features of the notations we use are: we prefer Dirac's "bra-ket" style of expressing, in the following form:

Notation 1. If f is an element of a linear space, X, over a field, K, then $|f>$ stands for the mapping $K \to X$, defined by

$$|f> \lambda := \lambda f \quad .$$

Notation 2. If l is a functional and we wish to emphasise this factor, then we write $< l|$ instead of l. We also write $< l|f >$ instead of $< l||f > 1$, and write the terms $|f >< l|$ and $f < l|$ interchangeably:

$$< l|f > \equiv < l||f > 1 \equiv l(f) \quad , \quad f < l| \equiv |f >< l| \quad .$$

Finally, a feature of the paper is that we emphasise the algebraic aspect of the problem and try to deemphasise the topological one. We have in mind differetial and integral operators, indeed, but we avoid metioning this fact, in oder to place in the centre algebra, FORMULA.

1 Rank-One Perturbations. Abstract Formulae.

The situation we will discuss in this section is this. Suppose, we deal with two linear operators, A and B, so that $A - B$ is rank-one [2], i.e.,

$$B - A = -f_a < l_a|$$

for an element f_a and a linear functional l_a. The first question is: If A^{-1} exists and is given, what are the conditions in order that B^{-1} should exist? And if B^{-1} exists, how can we calculate

$$B^{-1} - A^{-1} \qquad ?$$

A possible answer can be found in the following way:

B is defined by solving the equation

$$Bv = w$$

with respect to v. That is, in the described situation, B is defined by solving the equation

$$Av - f_a < l_a|v >= w$$

Firstly we write this equation as

$$Av = w + f_a < l_a|v >$$

and then, –recall that A^{-1} is given,– as

$$v = A^{-1}(w + f_a < l_a|v >)$$

Thus we observe: in order to find v, it is sufficient to find

$$c_a :=< l_a|v > ;$$

we emphasise it:

$$v = A^{-1}(w + c_a f_a)\,.$$

And if we have found v, then we can found c_a:

$$c_a =< l_a|v >=< l_a|A^{-1}(w + c_a f_a) >$$

So, we have obtained the equation to c_a, and we are solving it.

It is not difficult, because A^{-1} and l_a are linear. Using this factor we deduce:

$$c_a =< l_a|v >=< l_a|A^{-1}w + c_a A^{-1} f_a >$$

$$c_a =< l_a|v >=< l_a|A^{-1}w > +c_a < l_a|A^{-1} f_a >$$

$$(1- < l_a|A^{-1} f_a >)c_a =< l_a|A^{-1}w >$$

Thus, we conclude that

$$c_a = \frac{< l_a|A^{-1}w >}{1- < l_a|A^{-1} f_a >}\ , \text{ if } 1- < l_a|A^{-1} f_a >\neq 0$$

[2] more accurately expressed, rank-one or less

and then

$$v = A^{-1}w + A^{-1}f_a \frac{< l_a|A^{-1}w >}{1- < l_a|A^{-1}f_a >} \text{ , if } 1- < l_a|A^{-1}f_a > \neq 0\,.$$

We see now that

$$\boxed{B^{-1} - A^{-1} = \frac{A^{-1}f_a < l_a|A^{-1}}{1- < l_a|A^{-1}f_a >} \text{ , if } 1- < l_a|A^{-1}f_a > \neq 0\,.}$$

So, we have partially answered the question we have formulated at the beginning of this section. Namely, we have found an answer to the question in that case where $1- < l_a|A^{-1}f_a > \neq 0$.

And what can we state, if

$$1- < l_a|A^{-1}f_a > = 0 \qquad ?$$

In this case,

$$(A - f_a < l_a|)A^{-1}f_a = AA^{-1}f_a - f_a < l_a|A^{-1}f_a > = f_a - f_a = 0$$

i.e.,

$$\boxed{\text{if } 1- < l_a|A^{-1}f_a > = 0 \text{ then } BA^{-1}f_a = 0}$$

An interesting detail is: Let

$$Bv_0 = 0$$

It means that

$$Av_0 - f_a < l_a|v_0 > = 0$$

and it implies that

$$v_0 = A^{-1}f_a < l_a|v_0 >$$

Hence

$$< l_a|v_0 > = < l_a|A^{-1}f_a > < l_a|v_0 >$$

and

$$0 = Bv_0 \equiv BA^{-1}f_a < l_a|v_0 > = 0\,.$$

Thus we infer:

$$\boxed{\text{if } Bv_0 = 0 \text{ and } v_0 \neq 0 \text{ then } < l_a|v_0 > = 0\,,\ 1- < l_a|A^{-1}f_a > = 0\,,\ BA^{-1}f_a = 0\,.}$$

So, we have completely answered the question we have formulated at the beginning of this section.

2 Rank-One Perturbed Resolvent.

Suppose we have two operators, T_1, T_2, so that

$$T_2^{-1} - T_1^{-1} = |f >< l|\,.$$

What we want to know is how we can express

$$(z - T_2)^{-1} - (z - T_1)^{-1}$$

in terms of T_1^{-1}, T_2^{-1} and $(z - T_1)^{-1}$.

We begin the analysis with a general (and quite standard) argumentation:

$$\begin{aligned}
&(z - T_2)^{-1} - (z - T_1)^{-1} \\
&\quad= T_2^{-1}(zT_2^{-1} - I)^{-1} - T_1^{-1}(zT_1^{-1} - I)^{-1} \\
&\quad= \frac{1}{z}\Big(zT_2^{-1}(zT_2^{-1} - I)^{-1} - zT_1^{-1}(zT_1^{-1} - I)^{-1}\Big) \\
&\quad= \frac{1}{z}\Big((zT_2^{-1} - I + I)(zT_2^{-1} - I)^{-1} - (zT_1^{-1} - I + I)(zT_1^{-1} - I)^{-1}\Big) \\
&\quad= \frac{1}{z}\Big((zT_2^{-1} - I)^{-1} - (zT_1^{-1} - I)^{-1}\Big)\,.
\end{aligned}$$

Notice now, that

$$\Big(zT_2^{-1} - I\Big) - \Big(zT_1^{-1} - I\Big) \quad = \quad -(-z)|f >< l|$$

$$\Big(zT_2^{-1} - I\Big) \quad = \quad \Big(zT_1^{-1} - I\Big) - (-z)|f >< l| \quad .$$

Now recall the relation, which we have seen in the previous section:

$$\boxed{(A - f_a < l_a|)^{-1} - A^{-1} = \frac{A^{-1}f_a < l_a|A^{-1}}{1- < l_a|A^{-1}f_a >}\,, \text{ if } 1- < l_a|A^{-1}f_a > \neq 0}$$

Thus, letting us put

$$A := zT_1^{-1} - I\,,$$

we infer:

$$\begin{aligned}
&(z - T_2)^{-1} - (z - T_1)^{-1} \\
&\quad= \frac{1}{z}\Big((zT_2^{-1} - I)^{-1} - (zT_1^{-1} - I)^{-1}\Big) \\
&\quad= \frac{1}{z}\,\frac{(zT_1^{-1} - I)^{-1}| - zf >< l|(zT_1^{-1} - I)^{-1}}{1- < l|(zT_1^{-1} - I)^{-1}(-zf) >} \\
&\quad= -\frac{(zT_1^{-1} - I)^{-1}|f >< l|(zT_1^{-1} - I)^{-1}}{1 + z < l|(zT_1^{-1} - I)^{-1}f >} \\
&\quad= -\frac{T_1(z - T_1)^{-1}|f >< l|T_1(z - T_1)^{-1}}{1 + z < l|T_1(z - T_1)^{-1}f >} \\
&\quad= -\frac{\big(-I + z(z - T_1)^{-1}\big)|f >< l|\big(-I + z(z - T_1)^{-1}\big)}{1 + z < l|\big(-I + z(z - T_1)^{-1}\big)f >} \\
&\quad\text{if } 1 + z < l|\big(-I + z(z - T_1)^{-1}\big)f > \neq 0\,.
\end{aligned}$$

A result is:

$$\boxed{\begin{aligned}(z-T_2)^{-1}-(z-T_1)^{-1} &= -\frac{\big(-I+z(z-T_1)^{-1}\big)|f><l|\big(-I+z(z-T_1)^{-1}\big)}{1+z<l|\big(-I+z(z-T_1)^{-1}\big)f>} \\ &\quad \text{if } 1+z<l|\big(-I+z(z-T_1)^{-1}\big)f>\neq 0\,.\end{aligned}}$$

Now suppose, that although we know that

$$T_2^{-1}-T_1^{-1}=|f><l|\,,$$

however we do not separately know f or l. Let us try to find a method to calculate the value of the expression

$$<l|\big(-I+z(z-T_1)^{-1}\big)f>$$

without specifying f and l separately. A way is this. Let f_0 be an element, *arbitrary* taken from the domain of $T_2^{-1}-T_1^{-1}\equiv|f><l|$, and l_0 be an (*arbitrary*, but linear!) functional, which is defined on the range of $T_2^{-1}-T_1^{-1}\equiv|f><l|$. In other words, f_0 is an element of the domain of l, and l_0 is defined at f.

Then we infer:

$$T_2^{-1}-T_1^{-1}=|f><l|\,,$$

$$\Big(T_2^{-1}-T_1^{-1}\Big)f_0=|f><l|f_0>\,,$$

$$<l_0|\Big(T_2^{-1}-T_1^{-1}\Big)=<l_0|f><l|\,,$$

$$<l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0>=<l_0|f><l|f_0>\ .$$

Hence,

$$\begin{aligned}\Big(T_2^{-1}-T_1^{-1}\Big)f_0<l_0|\Big(T_2^{-1}-T_1^{-1}\Big) &= |f><l|f_0><l_0|f><l| \\ &= |f><l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0><l| \\ &= <l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0>|f><l| \\ &= <l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0>\Big(T_2^{-1}-T_1^{-1}\Big)\,.\end{aligned}$$

We have now seen: if we are able to find f_0 and l_0 so that

$$<l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0>\neq 0\,,$$

then

$$T_2^{-1}-T_1^{-1}=|f_1><l_1|\,,$$

where f_1 and g_1 are defined, e.g., by:

$$\begin{aligned}<l_1| &:= <l_0|\Big(T_2^{-1}-T_1^{-1}\Big)\,, \\ f_1 &:= \frac{\Big(T_2^{-1}-T_1^{-1}\Big)f_0}{<l_0|\Big(T_2^{-1}-T_1^{-1}\Big)f_0>}\quad .\end{aligned}$$

Thus we have actually no need to know *separetely* f or l.

Another way to calculate the value of the expression of the form $< l|Sf >$ is:

$$< l_0|\Big(T_2^{-1} - T_1^{-1}\Big)S\Big(T_2^{-1} - T_1^{-1}\Big)f_0 > \quad = \quad < l_0|f >< l|Sf >< l|f_0 >,$$

$$< l_0|\Big(T_2^{-1} - T_1^{-1}\Big)f_0 > \quad = \quad < l_0|f >< l|f_0 >,$$

$$< l|Sf > \quad = \quad \frac{< l_0|\Big(T_2^{-1} - T_1^{-1}\Big)S\Big(T_2^{-1} - T_1^{-1}\Big)f_0 >}{< l_0|\Big(T_2^{-1} - T_1^{-1}\Big)f_0 >} \quad .$$

Of course, we have here assumed that

$$< l_0|\Big(T_2^{-1} - T_1^{-1}\Big)f_0 > \neq 0 .$$

Finally, notice that the formula of

$$(z - T_2)^{-1} - (z - T_1)^{-1}$$

can be written as:

$$(z - T_2)^{-1} - (z - T_1)^{-1} \quad = \quad -\frac{\big(-I + z(z - T_1)^{-1}\big)\big(T_2^{-1} - T_1^{-1}\big)\big(-I + z(z - T_1)^{-1}\big)}{1 + z < l|\big(-I + z(z - T_1)^{-1}\big)f >}$$
$$\text{if } 1 + z < l|\big(-I + z(z - T_1)^{-1}\big)f > \neq 0 .$$

3 $(z - T_{DN})^{-1} - (z - T_{DD})^{-1}$. (Only Computations)

Let

$$f_z := z(z - T_{DD})^{-1} f\,,$$

i.e.,

$$\begin{aligned} z f_z(x) + \frac{\partial^2 f_z(x)}{\partial x^2} &= zx\,, \\ f_z(0) &= 0\,, \\ f_z(1) &= 0\,. \end{aligned}$$

Hence

$$f_z(x) = x - \frac{\sin(kx)}{\sin(k)}$$

$$\text{where } k \text{ is defined by } k^2 = z\,,$$

and where, recall, z is such that

$$\sin(k) \neq 0\,.$$

Thus we deduce:

$$\begin{aligned} \Big((-I + z(z - T_{DD})^{-1})f\Big)(x) &= -f(x) + f_z(x) \\ &= -x + \Big(x - \frac{\sin(kx)}{\sin(k)}\Big) \\ &= -\frac{\sin(kx)}{\sin(k)} \\ &\quad \text{where } k \text{ is defined by } k^2 = z\,, \end{aligned}$$

$$\begin{aligned} < l|(-I + z(z - T_{DD})^{-1})f > &= -\int_0^1 \xi \frac{\sin(k\xi)}{\sin(k)} d\xi \\ &= \int_{\xi=0}^1 \xi \frac{d\cos(k\xi)}{k\sin(k)} \\ &= \frac{\cos(k)}{k\sin(k)} - \int_{\xi=0}^1 \frac{\cos(k\xi)}{k\sin(k)} d\xi \\ &= \frac{\cos(k)}{k\sin(k)} - \frac{1}{k^2} \end{aligned}$$

$$\begin{aligned} 1 + z < l|(-I + z(z - T_{DD})^{-1})f > &= 1 + k^2\Big(\frac{\cos(k)}{k\sin(k)} - \frac{1}{k^2}\Big) \\ &= k\frac{\cos(k)}{\sin(k)} \end{aligned}$$

We conclude :

The new eigenvalues, z_n, are defined by

$$1 + z_n < l|(-I + z_n(z_n - T_{DD})^{-1})f >= 0\,,$$

i.e., by

$$\cos(k_n) = 0\,,$$

and the associated eigenfunctions are

$$\Big((-I + z_n(z_n - T_{DD})^{-1})f\Big)(x) \quad = \quad -\frac{\sin(k_n x)}{\sin(k_n)}\,.$$

Finally, the integral kernel of

$$(z - T_{DN})^{-1} - (z - T_{DD})^{-1}$$

is

$$G_{DN}(x,\xi,z) - G_{DD}(x,\xi,z) \quad = \quad -\frac{\sin(kx)\sin(k\xi)}{k\sin(k)\cos(k)}$$

References

[Fog] S.R. Foguel, *Finite Dimensional Perturbations in Banach Spaces*, American Journal of Mathematics, Volume 82, Issue 2 (Apr., 1960), 260-270