The Ground, a String, Two Elastic Springs.

Simple Exactly Solvable Models of One-Dimensional Scalar Fields with Concentrated Factors.

S.A. Chorošavin

Abstract

This paper is an electronic application to my set of lectures, subject:‘Formal methods in solving differential equations and constructing models of physical phenomena’. Addressed, mainly: postgraduates and related readers. Content: a discussion of the simple models (I would rather say, toy models) of the interaction based on equation arrays of the kind:

$$\begin{aligned}
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma_a c\delta(x-x_a)Q_a(t) - 4\gamma_b c\delta(x-x_b)Q_b(t),\\
Q_a(t) &= u(t,x_a)\\
Q_b(t) &= u(t,x_b)
\end{aligned}$$

Central mathematical points: d’Alembert-Kirchhoff-like Formulae, Finite Rank Perturbations.

Introduction

As long as the researcher takes an interest in resolvent formulae of finite rank perturbed operators, in other words, as long as he or she, the researcher, prefers to remain in the space-FREQUENCE framework, so long he has an infinite series of good papers, articles, books, manuals...

But as soon as the researcher is turning his attention to the problems of the "space-TIME", in other words, as soon as he has need to solve the associated wave equation, as soon as he has need of a suitable d'Alembert-like formula, or so, - in that moment the situation is changing dramatically.

I cannot say "there is no paper on the subject at all", I cannot say "I have not seen it", but if anyone asks me "where have you seen it?, where is this 'there'?", I will become very "pensativo", and I am in doubt that I will be "solitario" in this state.

So, I tried, try and (I hope) will try to collect the suitable examples of simple exactly solvable models of wave'n'particle.

Whether my collection is worth to discuss it, you solve.

1 Models of Two-Point Interaction with an only one-dimensional Scalar Field

1.1 Preliminaries

In this paper we fix measure units and let x be dimensionless position parameter, i.e.,

$$\text{physical position coordinate } = [\text{ length unit }] \times x + const\,.$$

Otherwise a confusion can ocurr, in relating to the definition

$$\int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)\,.$$

We assume the standard foramalism, where

$$\delta(x-x_0) = \frac{\partial 1_+(x-x_0)}{\partial x}$$

and where 1_+ stands for a unit step function (Heaviside function):

$$1_+(\xi) := 1(\xi \geq 0) := \begin{cases} 1 \quad , & \text{if } \xi \geq 0\,, \\ 0 \quad , & \text{if } \xi < 0\,. \end{cases}$$

Another feature of the notations is this. We will handle the functions which have a special variable, t, that means no doubt "time", and we will be interested in the case where $t \geq 0$. So, we could consider the restrictions of these functions, onto positive t-half-line. But it will be **technically** more convenient to **redefine** the functions, putting them zero on negative t-half-line. For more details of this feature of the notations see **The** $\cdot 1_+()$ **Convention** , in the next subsection.

A few words about the models: Recently I have already presented some models of ONE particle of FINITE mass, interacting with scalar field, and the interaction has been concentrated at one, only ONE, point.

Now I discuss models of TWO-point concentrated interaction, but the mass of the particle (or particles) I assume to be INFINITE, or, more precisely, I assume the particle(s) to be of infinite mass and immobile: motionless, fixed.

A naive formulation of the situation is: let us look at the Ground (the infinite immobile mass), the Earth or the Moon say, connected with a String (very-very long thin tensed cord) by means of two Ideal (we never approve Imperfection, don't we?) Springs. What will we tell then, ideally?

1.2 D'Alembert-Kirchhoff-like formulae

Recall that a standard D'Alembert-Kirchhoff-like formula reads: if

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f\,, \quad u = u(t,x)\,, \quad f = f(t,x)\,, \quad (t \geq 0) \qquad (*)$$

$$f = -4\gamma_a c\delta(x - x_a)\Big(F_{src\ a}(t)\Big) - 4\gamma_b c\delta(x - x_b)\Big(F_{src\ b}(t)\Big) \qquad x_a \leq x_b$$

and given initial data, $u(0,\cdot)$ and $\frac{\partial u(t,\xi)}{\partial t})\Big|_{t=0}$, then, for $t \geq 0$,

$$\begin{aligned} u(t,x) \quad &= \quad -2\gamma_a \int_0^{t-|x-x_a|/c} \Big(F_{src\ a}(\tau)\Big)d\tau \cdot 1_+(t - |x - x_a|/c) \\ &\quad - 2\gamma_b \int_0^{t-|x-x_b|/c} \Big(F_{src\ b}(\tau)\Big)d\tau \cdot 1_+(t - |x - x_b|/c) \\ &\quad + u_0(t,x) \end{aligned}$$

where

$$\begin{aligned} u_0(t,x) \quad &:= \quad c_+(x + ct) + c_-(x - ct) \\ &= \quad \frac{1}{2}\Big(u(0, x + ct) + u(0, x - ct)\Big) \\ &\quad + \frac{1}{2c}\Big(\widetilde{u}(0, x + ct) - \widetilde{u}(0, x - ct)\Big) \end{aligned}$$

and where $\widetilde{u}$ stands for any function defined by

$$\frac{\partial \widetilde{u}(0,\xi)}{\partial \xi} = \Big(\frac{\partial u(t,\xi)}{\partial t}\Big)\Big|_{t=0}.$$

[1]

If we put

$$\begin{aligned} F_a(t) \quad &:= \quad -2\gamma_a \int_0^t \Big(F_{src\ a}(\tau)\Big)d\tau \cdot 1_+(t)\,, \\ F_b(t) \quad &:= \quad -2\gamma_b \int_0^t \Big(F_{src\ b}(\tau)\Big)d\tau \cdot 1_+(t)\,, \end{aligned}$$

for short, then

[1] Note that

$$\widetilde{u}(0, x + ct) - \widetilde{u}(0, x - ct)$$

does not depend on what the primitive is which one has chosen!!! Moreover, we need only $\widetilde{u}|_{t=0}$ and not $\dot{u}|_{t=0}$ itself!

$$\boxed{u(t,x) \quad = \quad F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t,x) \qquad (t \geq 0)}$$

Another and a little more correct form of this expression is:

$$\boxed{u(t,x) \cdot 1_+(t) \quad = \quad F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t,x) \cdot 1_+(t)}$$

Now we set

$$T := |x_a - x_b|/c$$

and concentrate on

$$x = x_a, x_b$$

by writing

$$u(t,x_a) \quad = \quad F_a(t) + F_b(t-T) + u_0(t,x_a) \qquad (t \geq 0)$$

$$u(t,x_b) \quad = \quad F_a(t-T) + F_b(t) + u_0(t,x_b) \qquad (t \geq 0)$$

We transform these expressions into

$$\boxed{\begin{aligned} F_a(t) \quad &= \quad \Big(u(t,x_a) - u_0(t,x_a) - F_b(t-T)\Big) \cdot 1_+(t) \\[2em] F_b(t) \quad &= \quad \Big(u(t,x_b) - u_0(t,x_b) - F_a(t-T)\Big) \cdot 1_+(t) \end{aligned}}$$

and then

$$\begin{aligned} F_a(t) \quad = \quad & \Big(u(t,x_a) - u_0(t,x_a)\Big) \cdot 1_+(t) \\ & - \Big(u(t-T,x_b) - u_0(t-T,x_b)\Big) \cdot 1_+(t-T) \\ & + F_a(t-2T) \cdot 1_+(t-T) \end{aligned}$$

$$\begin{aligned} F_b(t) \quad = \quad & \Big(u(t,x_b) - u_0(t,x_b)\Big) \cdot 1_+(t) \\ & - \Big(u(t-T,x_a) - u_0(t-T,x_a)\Big) \cdot 1_+(t-T) \\ & + F_b(t-2T) \cdot 1_+(t-T) \end{aligned}$$

$$
\begin{aligned}
F_a(t) \quad = \quad & \Big(u(t,x_a) - u_0(t,x_a)\Big) \cdot 1_+(t) \\
& - \Big(u(t-T,x_b) - u_0(t-T,x_b)\Big) \cdot 1_+(t-T) \\
& + \Big(u(t-2T,x_a) - u_0(t-2T,x_a)\Big) \cdot 1_+(t-2T) \\
& - \Big(u(t-3T,x_b) - u_0(t-3T,x_b)\Big) \cdot 1_+(t-3T) \\
& + F_a(t-4T) \cdot 1_+(t-3T)
\end{aligned}
$$

$$
\begin{aligned}
F_b(t) \quad = \quad & \Big(u(t,x_b) - u_0(t,x_b)\Big) \cdot 1_+(t) \\
& - \Big(u(t-T,x_a) - u_0(t-T,x_a)\Big) \cdot 1_+(t-T) \\
& + \Big(u(t-2T,x_b) - u_0(t-2T,x_b)\Big) \cdot 1_+(t-2T) \\
& - \Big(u(t-3T,x_a) - u_0(t-3T,x_a)\Big) \cdot 1_+(t-3T) \\
& + F_b(t-4T) \cdot 1_+(t-3T)
\end{aligned}
$$

and so on.

Take here in account that

$$F_a(t) = 0\,,\ F_b(t) = 0\,,\ 1_+(t) = 0 \text{ falls } t < 0\,;$$

in any case we see that

$$
\begin{aligned}
1_+(t-NT) = 0\,, \\
\text{if } t < NT\,.
\end{aligned}
$$

It follows that

$$\text{if } T \neq 0\,, \text{ then}$$

$$
\begin{aligned}
F_a(t) \quad = \quad & \Big(u(t,x_a) - u_0(t,x_a)\Big) \cdot 1_+(t) \\
& - \Big(u(t-T,x_b) - u_0(t-T,x_b)\Big) \cdot 1_+(t-T) \\
& + \Big(u(t-2T,x_a) - u_0(t-2T,x_a)\Big) \cdot 1_+(t-2T) \\
& - \Big(u(t-3T,x_b) - u_0(t-3T,x_b)\Big) \cdot 1_+(t-3T) \\
& + \cdots
\end{aligned}
$$

$$
\begin{aligned}
F_b(t) \quad = \quad & \Big(u(t,x_b) - u_0(t,x_b)\Big) \cdot 1_+(t) \\
& - \Big(u(t-T,x_a) - u_0(t-T,x_a)\Big) \cdot 1_+(t-T) \\
& + \Big(u(t-2T,x_b) - u_0(t-2T,x_b)\Big) \cdot 1_+(t-2T) \\
& - \Big(u(t-3T,x_a) - u_0(t-3T,x_a)\Big) \cdot 1_+(t-3T) \\
& + \cdots
\end{aligned}
$$

The $\cdot 1_+()$ Convention.

In the following text, we will often redefine functions of t by multiplying them by $1_+(t)$, i.e., by resetting, e.g.,

$$\begin{aligned} u_0(t,\cdots) &:= u_0(t,\cdots)\cdot 1_+(t) \\ u(t,\cdots) &:= u(t,\cdots)\cdot 1_+(t) \\ u_0(t-T,\cdots) &:= u_0(t-T,\cdots)\cdot 1_+(t-T) \\ &\text{and so on.} \end{aligned}$$

Nevertheless we will sometimes write the multiplier $1_+()$, although we will mostly do it for emphasis, or there, where the formula does not become too long.

To take an example of this convention implementation, we say that the recent relations concerning F_a, F_b, we may write them as

$$\begin{aligned} F_a(t) &= u(t,x_a) - u_0(t,x_a) - F_b(t-T) \\ F_b(t) &= u(t,x_b) - u_0(t,x_b) - F_a(t-T) \end{aligned}$$

In addition to The $\cdot 1_+()$ Convention, we will

assume that $T \neq 0$.

2 Particular Cases. Recurrence Relations

2.1 The Case of $\gamma_b = 0$

We begin the analysis of this case turning to the following relations, which we have seen in the previous section:

$$\begin{aligned} u(t,x) &= F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t,x) \qquad (t \geq 0) \\ F_a(t) &= \Big(u(t,x_a) - u_0(t,x_a) - F_b(t-T)\Big) \cdot 1_+(t) \\ F_b(t) &= \Big(u(t,x_b) - u_0(t,x_b) - F_a(t-T)\Big) \cdot 1_+(t) \end{aligned}$$

Thus, we see, that in the case of $\gamma_b = 0$, these relations become

$$u(t,x) = F_a(t - |x - x_a|/c) + u_0(t,x) \qquad (t \geq 0)$$

$$\begin{aligned} u(t,x) = &\Big(u(t - |x - x_a|/c, x_a) - u_0(t - |x - x_a|/c, x_a)\Big) \cdot 1_+(t - |x - x_a|/c) \\ &+ u_0(t,x) \qquad (t \geq 0) \end{aligned}$$

Take here into account that

$$u(t,x_a) = F_a(t) + u_0(t,x_a) \qquad (t \geq 0)$$

In particular, if

$$u(t,x_a) \equiv 0 \qquad (t \geq 0)\,,$$

then

$$F_a(t) = -u_0(t,x_a)$$

and, hence,

if

$$u(t,x_a) \equiv 0 \qquad (t \geq 0)\,,$$

then

$$u(t,x) = -u_0(t - |x - x_a|/c, x_a) \cdot 1_+(t - |x - x_a|/c) + u_0(t,x) \qquad (t \geq 0)$$

The restriction $u(t, x_a) \equiv 0 \quad (t \geq 0)$ states that the string is absolutely motionless, fixed at the point $x = x_a$. If we replace this restriction by

$$\begin{aligned} F_a(t) &:= -2\gamma_a \int_0^t \Big(F_{src\ a}(\tau)\Big) d\tau \cdot 1_+(t)\,, \\ &:= -2\gamma_a \int_0^t u(\tau, x_a) d\tau \cdot 1_+(t)\,, \end{aligned}$$

then we obtain:

$$u(t, x_a) = -2\gamma_a \int_0^t u(\tau, x_a) d\tau \cdot 1_+(t) + u_0(t, x_a) \qquad (t \geq 0)$$

and, finally,

$$\boxed{\begin{aligned} u(t, x_a) &= e^{-2\gamma_a t} u_0(0, x_a) + \int_0^t e^{-2\gamma_a (t-\tau)} d_\tau u_0(\tau, x_a) \cdot 1_+(t)\,, \qquad (t \geq 0) \\ u(t, x) &= \Big(u(t - |x - x_a|/c, x_a) - u_0(t - |x - x_a|/c, x_a)\Big) \cdot 1_+(t - |x - x_a|/c) \\ &\quad + u_0(t, x) \qquad (t \geq 0) \end{aligned}}$$

2.2 The Case of $u(t,x_a)=0\,,\quad u(t,x_b)=0$

We have yet observed that in any case

$$\begin{aligned}
F_a(t) \;=\; & \Big(u(t,x_a)-u_0(t,x_a)\Big)\cdot 1_+(t)\\
& -\Big(u(t-T,x_b)-u_0(t-T,x_b)\Big)\cdot 1_+(t-T)\\
& +\Big(u(t-2T,x_a)-u_0(t-2T,x_a)\Big)\cdot 1_+(t-2T)\\
& -\Big(u(t-3T,x_b)-u_0(t-3T,x_b)\Big)\cdot 1_+(t-3T)\\
& +\cdots
\end{aligned}$$

$$\begin{aligned}
F_b(t) \;=\; & \Big(u(t,x_b)-u_0(t,x_b)\Big)\cdot 1_+(t)\\
& -\Big(u(t-T,x_a)-u_0(t-T,x_a)\Big)\cdot 1_+(t-T)\\
& +\Big(u(t-2T,x_b)-u_0(t-2T,x_b)\Big)\cdot 1_+(t-2T)\\
& -\Big(u(t-3T,x_a)-u_0(t-3T,x_a)\Big)\cdot 1_+(t-3T)\\
& +\cdots
\end{aligned}$$

Thus we infer that, if

$$u(t,x_a)=0\,,\quad u(t,x_b)=0\,,$$

then exactly

$$\begin{aligned}
& u(t,x)\\
& \;=\; u_0(t,x)\\
& \qquad -u_0(t-|x-x_a|/c,x_a)-u_0(t-|x-x_b|/c,x_b)\\
& \qquad +u_0(t-|x-x_a|/c-T,x_b)+u_0(t-|x-x_b|/c-T,x_a)\\
& \qquad -u_0(t-|x-x_a|/c-2T,x_a)-u_0(t-|x-x_b|/c-2T,x_b)\\
& \qquad +u_0(t-|x-x_a|/c-3T,x_b)+u_0(t-|x-x_b|/c-3T,x_a)\\
& \qquad \cdots \qquad (t\geq 0)
\end{aligned}$$

Needs there a comment?

2.3 The Case of $\quad u(t,x_a)=0\,, \quad \gamma_b \neq \infty$

It this case, we observe [2] that

$$\begin{aligned} F_a(t) &= 0 - u_0(t,x_a) - F_b(t-T) \\ F_b(t) &= u(t,x_b) - u_0(t,x_b) - F_a(t-T) \end{aligned}$$

$$\begin{aligned} F_a(t) &= 0 - u_0(t,x_a) \\ &\quad - u(t-T,x_b) + u_0(t-T,x_b) \\ &\quad + F_a(t-2T) \end{aligned}$$

$$\begin{aligned} F_b(t) &= u(t,x_b) - u_0(t,x_b) \\ &\quad - 0 + u_0(t-T,x_a) \\ &\quad + F_b(t-2T) \end{aligned}$$

Thus we infer that, if

$$\begin{aligned} F_b(t) &:= -2\gamma_b \int_0^t \Big(F_{src\ b}(\tau)\Big) d\tau \cdot 1_+(t)\,, \\ &:= -2\gamma_b \int_0^t u(\tau,x_b) d\tau \cdot 1_+(t)\,, \end{aligned}$$

then

$$\begin{aligned} F_a(t) &= -u_0(t,x_a) \\ &\quad - u(t-T,x_b) + u_0(t-T,x_b) \\ &\quad + F_a(t-2T) \end{aligned}$$

$$\begin{aligned} -2\gamma_b \int_0^t u(\tau,x_b) d\tau \cdot 1_+(t) &= u(t,x_b) - u_0(t,x_b) \\ &\quad + u_0(t-T,x_a) \\ &\quad - 2\gamma_b \int_0^{t-2T} u(\tau,x_b) d\tau \cdot 1_+(t-2T) \end{aligned}$$

We now focus on the latter relation, which we write in the following terms:

$$\begin{aligned} u(t,x_b) &= -2\gamma_b \int_0^t u(\tau,x_b) d\tau + 2\gamma_b \int_0^{t-2T} u(\tau,x_b) d\tau \\ &\quad + u_0(t,x_b) - u_0(t-T,x_a) \end{aligned}$$

[2] exploiting the **The** $\cdot 1_+()$ **Convention** ,

Next, let us define

$$Q_b(t) := u(t,x_b)\,;\; Q_{0b}(t) := u_0(t,x_b)\,;\; Q_{0a}(t) := u_0(t,x_a)\,;$$

so that the former relation becomes

$$\begin{aligned} Q_b(t) \quad &= \quad -2\gamma_b \int_0^t Q_b(\tau)d\tau + 2\gamma_b \int_0^{t-2T} Q_b(\tau)d\tau \\ &\qquad + Q_{0b}(t) - Q_{0a}(t-T) \end{aligned}$$

$$\begin{aligned} Q_b(t) \quad &= \quad -2\gamma_b \int_{t-2T}^t Q_b(\tau)d\tau \\ &\qquad + Q_{0b}(t) - Q_{0a}(t-T) \end{aligned}$$

$$\begin{aligned} \frac{\partial Q_b(t)}{\partial t} \quad &= \quad -2\gamma_b Q_b(t) + 2\gamma_b Q_b(t-2T) \\ &\qquad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t-T))}{\partial t} \end{aligned}$$

Let [3]

$$Q_b(t) = e^{-2\gamma_b t} C(t)\,.$$

Then we obtain:

$$\begin{aligned} Q_b(t-2T) \quad &= \quad e^{-2\gamma_b t} e^{2\gamma_b T} C(t-2T)\,, \\ \frac{\partial Q_b(t)}{\partial t} \quad &= \quad -2\gamma_b e^{-2\gamma_b t} C(t) + e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} \\ &= \quad -2\gamma_b Q_b(t) + e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} \end{aligned}$$

$$\begin{aligned} e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} \quad &= \quad 2\gamma_b Q_b(t-2T) \\ &\qquad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t-T))}{\partial t} \end{aligned}$$

[3] Method of Variation of Constants

$$\begin{aligned} e^{-2\gamma_b t}\frac{\partial C(t)}{\partial t} &= 2\gamma_b e^{-2\gamma_b t} e^{2\gamma_b T} C(t-2T) \\ &\quad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t-T))}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial C(t)}{\partial t} &= 2\gamma_b e^{2\gamma_b T} C(t-2T) \\ &\quad + e^{2\gamma_b t}\frac{\partial (Q_{0b}(t) - Q_{0a}(t-T))}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial C(t)}{\partial t} &= 2\gamma_b e^{2\gamma_b t} Q_b(t-2T) \\ &\quad + e^{2\gamma_b t}\frac{\partial (Q_{0b}(t) - Q_{0a}(t-T))}{\partial t} \end{aligned}$$

$$\begin{aligned} C(t) &= 2\gamma_b \int_0^t e^{2\gamma_b \tau} Q_b(\tau - 2T) d\tau \\ &\quad + \int_0^t e^{2\gamma_b \tau} \frac{\partial (Q_{0b}(\tau) - Q_{0a}(\tau - T))}{\partial \tau} d\tau + C(0) \end{aligned}$$

$$\begin{aligned} Q_b(t) &= e^{-2\gamma_b t} 2\gamma_b \int_0^t e^{2\gamma_b \tau} Q_b(\tau - 2T) d\tau \\ &\quad + e^{-2\gamma_b t} \int_0^t e^{2\gamma_b \tau} \frac{\partial (Q_{0b}(\tau) - Q_{0a}(\tau - T))}{\partial \tau} d\tau + e^{-2\gamma_b t} Q_b(0) \end{aligned}$$

$$\begin{aligned} Q_b(t) &= 2\gamma_b \int_0^t e^{-2\gamma_b (t-\tau)} Q_b(\tau - 2T) d\tau \\ &\quad + \int_0^t e^{-2\gamma_b (t-\tau)} d_\tau (Q_{0b}(\tau) - Q_{0a}(\tau - T)) + e^{-2\gamma_b t} Q_b(0) \end{aligned}$$

$$\begin{aligned} Q_b(t) &= 2\gamma_b \int_0^t e^{-2\gamma_b (t-\tau)} Q_b(\tau - 2T) d\tau \\ &\quad + I_{0b}(t)\,, \end{aligned}$$

where

$$I_{0b}(t) := \int_0^t e^{-2\gamma_b (t-\tau)} d_\tau (Q_{0b}(\tau) - Q_{0a}(\tau - T)) + e^{-2\gamma_b t} Q_b(0)$$

and then, after returning to the terms of u , u_0, we infer that

$$\boxed{\begin{aligned} u(t,x_b) \quad &= \quad 2\gamma_b \int_{2T}^{t} e^{-2\gamma_b(t-\tau)} u(\tau-2T,x_b)d\tau \\ &\quad + I_{0b}(t)\,, \\ &\text{where} \\ I_{0b}(t) \quad &:= \quad \int_{0+0}^{t} e^{-2\gamma_b(t-\tau)} d_\tau(u_0(\tau,x_b) - u_0(\tau-T,x_a)1_+(\tau-T)) + e^{-2\gamma_b t}u(0,x_b) \end{aligned}}$$

Finally, recall that

$$\boxed{u(t,x) \quad = \quad F_a(t-|x-x_a|/c) - F_b(t-|x-x_b|/c) + u_0(t,x) \qquad (t \geq 0)}$$

where

$$\begin{aligned} F_a(t) \quad &= \quad \Big(0-u_0(t,x_a)\Big)\cdot 1_+(t) \\ &\quad - \Big(u(t-T,x_b)-u_0(t-T,x_b)\Big)\cdot 1_+(t-T) \\ &\quad + \Big(0-u_0(t-2T,x_a)\Big)\cdot 1_+(t-2T) \\ &\quad - \Big(u(t-3T,x_b)-u_0(t-3T,x_b)\Big)\cdot 1_+(t-3T) \\ &\quad + \cdots \end{aligned}$$

$$\begin{aligned} F_b(t) \quad &= \quad \Big(u(t,x_b)-u_0(t,x_b)\Big)\cdot 1_+(t) \\ &\quad - \Big(0-u_0(t-T,x_a)\Big)\cdot 1_+(t-T) \\ &\quad + \Big(u(t-2T,x_b)-u_0(t-2T,x_b)\Big)\cdot 1_+(t-2T) \\ &\quad - \Big(0-u_0(t-3T,x_a)\Big)\cdot 1_+(t-3T) \\ &\quad + \cdots \end{aligned}$$

2.4 The Case of $\gamma_a \neq \infty\,, \quad \gamma_b \neq \infty$

In this case [4]

$$\begin{aligned} u(t,x_a) &= -2\gamma_a \int_0^t \Big(F_{src\ a}(\tau)\Big)d\tau - 2\gamma_b \int_0^{t-T} \Big(F_{src\ b}(\tau)\Big)d\tau \cdot 1_+(t-T) \\ &\quad + u_0(t,x_a) \\ u(t,x_b) &= -2\gamma_a \int_0^{t-T} \Big(F_{src\ a}(\tau)\Big)d\tau \cdot 1_+(t-T) - 2\gamma_b \int_0^{t} \Big(F_{src\ b}(\tau)\Big)d\tau \\ &\quad + u_0(t,x_b) \end{aligned}$$

We put, for short,

$$Q_a(t) := u(t,x_a)\,;\; Q_{0a}(t) := u_0(t,x_a)\,;\; Q_b(t) := u(t,x_b)\,;\; Q_{0b}(t) := u_0(t,x_b)\,;$$

so that the relationships become

$$\begin{aligned} Q_a(t) &= -2\gamma_a \int_0^t \Big(F_{src\ a}(\tau)\Big)d\tau - 2\gamma_b \int_0^{t-T} \Big(F_{src\ b}(\tau)\Big)d\tau \cdot 1_+(t-T) \\ &\quad + Q_{0a}(t) \\ Q_b(t) &= -2\gamma_a \int_0^{t-T} \Big(F_{src\ a}(\tau)\Big)d\tau \cdot 1_+(t-T) - 2\gamma_b \int_0^{t} \Big(F_{src\ b}(\tau)\Big)d\tau \\ &\quad + Q_{0b}(t) \end{aligned}$$

and, hence,

$$\begin{aligned} \frac{\partial Q_a(t)}{\partial t} &= -2\gamma_a F_{src\ a}(t) - 2\gamma_b F_{src\ b}(t-T)\cdot 1_+(t-T) \\ &\quad + Q_{0a}(t) \\ \frac{\partial Q_b(t)}{\partial t} &= -2\gamma_a F_{src\ a}(t-T)\cdot 1_+(t-T) - 2\gamma_b F_{src\ b}(t) \\ &\quad + Q_{0b}(t)\,. \end{aligned}$$

We now assume that

$$F_{src\ a}(t) := u(t,x_a)\,,\; F_{src\ b}(t) := u(t,x_b)\,.$$

Then

$$\begin{aligned} \frac{\partial Q_a(t)}{\partial t} &= -2\gamma_a Q_a(t) - 2\gamma_b Q_b(t-T)\cdot 1_+(t-T) \\ &\quad + \frac{\partial Q_{0a}(t)}{\partial t} \\ \frac{\partial Q_b(t)}{\partial t} &= -2\gamma_a Q_a(t-T)\cdot 1_+(t-T) - 2\gamma_b Q_b(t) \\ &\quad + \frac{\partial Q_{0b}(t)}{\partial t} \end{aligned}$$

Let [5]

$$Q_a(t) = e^{-2\gamma_a t} C_a(t)\,,\, Q_b(t) = e^{-2\gamma_b t} C_b(t)\,.$$

[4] Recall and take into account "The $\cdot 1_+()$ Convention"
[5] Method of Variation of Constants

Then we infer that:

$$\begin{aligned}\frac{\partial Q_a(t)}{\partial t} &= -2\gamma_a e^{-2\gamma_a t} C_a(t) + e^{-2\gamma_a t}\frac{\partial C_a(t)}{\partial t}\\ &= -2\gamma_a Q_a(t) + e^{-2\gamma_a t}\frac{\partial C_a(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial Q_b(t)}{\partial t} &= -2\gamma_b e^{-2\gamma_b t} C_b(t) + e^{-2\gamma_b t}\frac{\partial C_b(t)}{\partial t}\\ &= -2\gamma_b Q_b(t) + e^{-2\gamma_b t}\frac{\partial C_b(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}e^{-2\gamma_a t}\frac{\partial C_a(t)}{\partial t} &= -2\gamma_b Q_b(t-T)\cdot 1_+(t-T)\\ &\quad + \frac{\partial Q_{0a}(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}e^{-2\gamma_b t}\frac{\partial C_b(t)}{\partial t} &= -2\gamma_a Q_a(t-T)\cdot 1_+(t-T)\\ &\quad + \frac{\partial Q_{0b}(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial C_a(t)}{\partial t} &= -2\gamma_b e^{2\gamma_a t} Q_b(t-T)\cdot 1_+(t-T)\\ &\quad + e^{2\gamma_a t}\frac{\partial Q_{0a}(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}\frac{\partial C_b(t)}{\partial t} &= -2\gamma_a e^{2\gamma_b t} Q_a(t-T)\cdot 1_+(t-T)\\ &\quad + e^{2\gamma_b t}\frac{\partial Q_{0b}(t)}{\partial t}\end{aligned}$$

$$\begin{aligned}C_a(t) &= -2\gamma_b \int_0^t e^{2\gamma_a \tau} Q_b(\tau-T)d\tau\cdot 1_+(t-T)\\ &\quad + \int_0^t e^{2\gamma_a \tau}\frac{\partial Q_{0a}(\tau)}{\partial \tau}d\tau + C_a(0)\end{aligned}$$

$$\begin{aligned}C_b(t) &= -2\gamma_a \int_0^t e^{2\gamma_b \tau} Q_a(\tau-T)d\tau\cdot 1_+(t-T)\\ &\quad + \int_0^t e^{2\gamma_b \tau}\frac{\partial Q_{0b}(\tau)}{\partial \tau}d\tau + C_b(0)\end{aligned}$$

$$\begin{aligned} Q_a(t) &= -2\gamma_b \int_0^t e^{-2\gamma_a(t-\tau)} Q_b(\tau - T) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_a(t-\tau)} d_\tau Q_{0a}(\tau) + e^{-2\gamma_a t} Q_a(0) \end{aligned}$$

$$\begin{aligned} Q_b(t) &= -2\gamma_a \int_0^t e^{-2\gamma_b(t-\tau)} Q_a(\tau - T) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_b(t-\tau)} d_\tau Q_{0b}(\tau) + e^{-2\gamma_b t} Q_b(0) \end{aligned}$$

Thus, after returning to the terms of u , u_0, we conclude that

$$\boxed{\begin{aligned} u(t,x_a) &= -2\gamma_b \int_T^t e^{-2\gamma_a(t-\tau)} u(\tau - T, x_b) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_a(t-\tau)} d_\tau u_0(\tau, x_a) + e^{-2\gamma_a t} u_0(0, x_a) \\ u(t,x_b) &= -2\gamma_a \int_T^t e^{-2\gamma_b(t-\tau)} u(\tau - T, x_a) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_b(t-\tau)} d_\tau u_0(\tau, x_b) + e^{-2\gamma_b t} u_0(0, x_b) \end{aligned}}$$

Finally, recall that

$$\boxed{u(t,x) = F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t,x) \qquad (t \geq 0)}$$

where

$$\begin{aligned} F_a(t) &= \Big(u(t,x_a) - u_0(t,x_a)\Big) \cdot 1_+(t) \\ &\quad - \Big(u(t-T,x_b) - u_0(t-T,x_b)\Big) \cdot 1_+(t-T) \\ &\quad + \Big(u(t-2T,x_a) - u_0(t-2T,x_a)\Big) \cdot 1_+(t-2T) \\ &\quad - \Big(u(t-3T,x_b) - u_0(t-3T,x_b)\Big) \cdot 1_+(t-3T) \\ &\quad + \cdots \end{aligned}$$

$$\begin{aligned} F_b(t) &= \Big(u(t,x_b) - u_0(t,x_b)\Big) \cdot 1_+(t) \\ &\quad - \Big(u(t-T,x_a) - u_0(t-T,x_a)\Big) \cdot 1_+(t-T) \\ &\quad + \Big(u(t-2T,x_b) - u_0(t-2T,x_b)\Big) \cdot 1_+(t-2T) \\ &\quad - \Big(u(t-3T,x_a) - u_0(t-3T,x_a)\Big) \cdot 1_+(t-3T) \\ &\quad + \cdots \end{aligned}$$

3 Explicit Relations

3.1 The Algebra of $\mathbf{I}_{\gamma,T_0}$

In the previous section we have seen the recurrence relations, which normally contain "retarded integral operations". When solving such relations, a usual machinery involves a treatment of the various compositions of the operations that build the relations. So, we try to find an effective representation of the referred compositions.

Now, let $\mathbf{I}_{\gamma,T_0}$ denote the operation defined by [6]

$$\begin{aligned}\Big(\mathbf{I}_{\gamma,T_0}f\Big)(t) \quad &:= \quad \int_{T_0}^{t} e^{-2\gamma(t-\tau)} f(\tau - T_0)d\tau \\ &\equiv \int_{T_0}^{t} e^{-2\gamma(t-\tau)} f(\tau - T_0)d\tau \cdot 1_{+}(t-T_0) \\ &= \int_{0}^{t-T_0} e^{-2\gamma(t-T_0-\tau)} f(\tau)d\tau \cdot 1_{+}(t-T_0) \\ &\equiv \int_{0}^{t-T_0} e^{-2\gamma(t-T_0-\tau)} f(\tau)d\tau \cdot\end{aligned}$$

Firstly, notice that

$$\Big(\mathbf{I}_{\gamma,T_0}f\Big)(t) \quad = \quad \int_{0}^{t} e^{-2\gamma(t-T_0-\tau)} f(\tau)\cdot 1_{+}(t-T_0-\tau)d\tau \cdot 1_{+}(t)$$

In particular, the operations under consideration are usual convolution operations, and as such they are commutative:

$$\mathbf{I}_{\gamma_1,T_1}\mathbf{I}_{\gamma_2,T_2} \quad = \quad \mathbf{I}_{\gamma_2,T_2}\mathbf{I}_{\gamma_1,T_1}\,.$$

In addition notice that

$$\begin{aligned}\Big(\mathbf{I}_{\gamma,T_0}f\Big)(t+T_0) \quad &= \quad \int_{0}^{t} e^{-2\gamma(t-\tau)} f(\tau)d\tau \cdot 1_{+}(t) \\ &= \quad \Big(\mathbf{I}_{\gamma,0}f\Big)(t)\end{aligned}$$

Next, let us calculate

$$\mathbf{I}_{\gamma,T_1}\mathbf{I}_{\gamma,T_2}$$

The usual way is:

$$\begin{aligned}&\Big(\mathbf{I}_{\gamma,T_1}\mathbf{I}_{\gamma,T_2}f\Big)(t) \\ &\quad = \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \Big(\mathbf{I}_{\gamma,T_2}f\Big)(\tau)d\tau \cdot 1_{+}(t-T_1) \\ &\quad = \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \Big(\mathbf{I}_{\gamma,T_2}f\Big)(\tau)\cdot 1_{+}(\tau-T_2)d\tau \cdot 1_{+}(t-T_1) \\ &\quad = \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \Big(\mathbf{I}_{\gamma,T_2}f\Big)(\tau)\cdot 1_{+}(\tau-T_2)d\tau \cdot 1_{+}(t-T_1-T_2) \\ &\quad = \int_{T_2}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \Big(\mathbf{I}_{\gamma,T_2}f\Big)(\tau)d\tau \cdot 1_{+}(t-T_1-T_2) \\ &\quad = \int_{0}^{t-T_1-T_2} e^{-2\gamma(t-T_1-T_2-\tau)} \Big(\mathbf{I}_{\gamma,T_2}f\Big)(\tau+T_2)d\tau \cdot 1_{+}(t-T_1-T_2)\end{aligned}$$

[6] recall that, in the context, $\int_a^b$ means $\int_a^b \cdot 1_{+}(b-a)$

$$= \int_0^{t-T_1-T_2} e^{-2\gamma(t-T_1-T_2-\tau)} \Big(\mathbf{I}_{\gamma,0} f\Big)(\tau) d\tau \cdot 1_+(t-T_1-T_2)$$
$$= \Big(\mathbf{I}_{\gamma,T_1+T_2}\mathbf{I}_{\gamma,0} f\Big)(t)$$

Thus, we have seen that

$$\mathbf{I}_{\gamma,T_1}\mathbf{I}_{\gamma,T_2} \quad = \quad \mathbf{I}_{\gamma,T_1+T_2}\mathbf{I}_{\gamma,0} \,.$$

Consequences which we need:

$$\mathbf{I}^N_{\gamma_b,2T} \quad = \quad \mathbf{I}_{\gamma_b,2NT}\mathbf{I}^{N-1}_{\gamma_b,0} \,,$$

$$(\mathbf{I}_{\gamma_b,T})^{2N} \quad = \quad \mathbf{I}_{\gamma_b,2NT}\mathbf{I}^{2N-1}_{\gamma_b,0} \,,$$

$$(\mathbf{I}_{\gamma_b,T})^{2N+1} \quad = \quad \mathbf{I}_{\gamma_b,2NT}\mathbf{I}^{2N}_{\gamma_b,0} \,,$$

$$\begin{aligned} (\mathbf{I}_{\gamma_a,T}\mathbf{I}_{\gamma_b,T})^N \quad &= \quad (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^N \\ &= \quad \mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T}\mathbf{I}^{N-1}_{\gamma_b,0}\mathbf{I}^{N-1}_{\gamma_a,0} \,. \end{aligned}$$

In addition, let E_γ denote the operation defined by

$$\Big(E_\gamma f\Big)(t) \quad := \quad e^{\gamma t} f(t) \,.$$

Then we can write

$$\mathbf{I}_{\gamma,0} \quad = \quad E_\gamma^{-1}\mathbf{I}_{0,0}E_\gamma \,,$$

and then

$$\begin{aligned} \mathbf{I}^2_{\gamma,0} \quad &= \quad E_\gamma^{-1}\mathbf{I}_{0,0}E_\gamma E_\gamma^{-1}\mathbf{I}_{0,0}E_\gamma \\ &= E_\gamma^{-1}\mathbf{I}^2_{0,0}E_\gamma \,. \end{aligned}$$

Iterating this kind of arguments we infer that

$$\mathbf{I}^N_{\gamma,T_0} \quad = \quad \mathbf{I}_{\gamma,T_0}E_\gamma^{-1}\mathbf{I}^{N-1}_{0,0}E_\gamma$$

and then

$$\begin{aligned} (\mathbf{I}^N_{\gamma,T_0} f)(t) \quad &= \quad \int_0^{t-NT_0}\int_0^{\tau}\int_0^{\tau_1}\cdots\int_0^{\tau_{N-2}} e^{-2\gamma(t-NT_0-\tau_{N-2})} f(\tau_{N-2}) d\tau_{N-2}\cdots d\tau_1 d\tau \\ &= \quad \int_0^{t-NT_0} \frac{(t-NT_0-\tau)^{N-1}}{(N-1)!} e^{-2\gamma(t-NT_0-\tau)} f(\tau) d\tau \\ &\equiv \quad \int_0^{t} \frac{(t-NT_0-\tau)^{N-1}}{(N-1)!} e^{-2\gamma(t-NT_0-\tau)} f(\tau)\cdot 1_+(t-NT_0-\tau) d\tau \,. \end{aligned}$$

Now, we go on to see consequences of these consequences.

3.2 Explicit Relations: The Case of $u(t, x_a) = 0\,, \quad \gamma_b \neq \infty$

We have seen that

$$
\begin{aligned}
u(t, x_b) \quad &= \quad 2\gamma_b \int_{2T}^{t} e^{-2\gamma_b(t-\tau)} u(\tau - 2T, x_b) d\tau \\
&\quad + I_{0b}(t)\,, \\
&\text{where} \\
I_{0b}(t) \quad &:= \quad \int_{0+0}^{t} e^{-2\gamma_b(t-\tau)} d_\tau (u_0(\tau, x_b) - u_0(\tau - T, x_a) 1_+(\tau - T)) + e^{-2\gamma_b t} u(0, x_b)\,.
\end{aligned}
$$

Recall, $\mathbf{I}_{\gamma, T_0}$ denote the operation defined by

$$
\begin{aligned}
\Big(\mathbf{I}_{\gamma, T_0} f\Big)(t) \quad &:= \quad \int_{T_0}^{t} e^{-2\gamma(t-\tau)} f(\tau - T_0) d\tau \\
&= e^{2\gamma T_0} \int_{0}^{t-T_0} e^{-2\gamma(t-\tau)} f(\tau) d\tau \\
&\equiv e^{2\gamma T_0} \int_{-\infty}^{\infty} e^{-2\gamma_b(t-\tau)} f(\tau) \cdot 1_+(t - T_0 - \tau) \cdot 1_+(\tau) d\tau \cdot 1_+(t - T_0)\,.
\end{aligned}
$$

Thus, we can write

$$
u(t, x_b) \quad = \quad I_{0b}(t)\,, \quad (\text{ if } 0 \le t < 2T)
$$

$$
u(t, x_b) \quad = \quad \Big(\Big(1 + (2\gamma_b)\mathbf{I}_{\gamma_b, 2T}\Big) I_{0b}\Big)(t)\,, \quad (\text{ if } 0 \le t < 4T)
$$

$$
u(t, x_b) \quad = \quad \Big(\Big(1 + (2\gamma_b)\mathbf{I}_{\gamma_b, 2T} + (2\gamma_b)^2 \mathbf{I}^2_{\gamma, 2T} + \cdots + (2\gamma_b)^N \mathbf{I}^N_{\gamma_b, 2T}\Big) I_{0b}\Big)(t)\,, \quad (\text{ if } 0 \le t < (N+1)2T)
$$

So, iterating,

$$
u(t, x_b) \quad = \quad \Big(\Big(1 + (2\gamma_b)\mathbf{I}_{\gamma_b, 2T} + (2\gamma_b)^2 \mathbf{I}^2_{\gamma, 2T} + \cdots + (2\gamma_b)^N \mathbf{I}^N_{\gamma_b, 2T} \cdots\Big) I_{0b}\Big)(t)
$$

Finally, by recalling that

$$
(\mathbf{I}^N_{\gamma, T_0} f)(t) \quad = \quad \int_0^t \frac{(t - NT_0 - \tau)^{N-1}}{(N-1)!} e^{-2\gamma(t - NT_0 - \tau)} f(\tau) \cdot 1_+(t - NT_0 - \tau) d\tau
$$

and after introducing

$$
\begin{aligned}
&Exp(\lambda, T_0, t) \\
&\quad := \quad 1 \cdot 1_+(t) + \lambda(t - T_0) \cdot 1_+(t - T_0) + \lambda^2 \frac{(t - 2T_0)^2}{2!} \cdot 1_+(t - 2T_0) \\
&\qquad + \cdots + \lambda^N \frac{(t - NT_0)^N}{N!} \cdot 1_+(t - NT_0) \cdots
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
u(t, x_b) \quad &= \quad I_{0b}(t) \\
&\quad + (2\gamma_b) \int_{2T}^{t} e^{-2\gamma_b(t-\tau)} I_{0b}(\tau - 2T) d\tau
\end{aligned}
$$

$$
\begin{aligned}
& + (2\gamma_b)^2 \int_{4T}^{t} \int_{4T}^{\tau} e^{-2\gamma_b(t-\tau_1)} I_{0b}(\tau_1 - 4T) d\tau_1 d\tau \\
& + \cdots \\
= \; & I_{0b}(t) \\
& + (2\gamma_b) \int_0^{t-2T} e^{-2\gamma_b(t-2T-\tau)} I_{0b}(\tau) d\tau \\
& + (2\gamma_b)^2 \int_0^{t-4T} \int_0^{\tau} e^{-2\gamma_b(t-4T-\tau_1)} I_{0b}(\tau_1) d\tau_1 d\tau \\
& + \cdots \\
= \; & I_{0b}(t) + 2\gamma_b e^{4\gamma_b T} \int_0^t Exp(2\gamma_b e^{4\gamma_b T}, 2T, t-\tau-2T) e^{-2\gamma_b(t-\tau)} I_{0b}(\tau) d\tau
\end{aligned}
$$

where

$$
I_{0b}(t) \;:=\; \int_{0+0}^{t} e^{-2\gamma_b(t-\tau)} d_\tau (u_0(\tau, x_b) - u_0(\tau - T, x_a) 1_+(\tau - T)) + e^{-2\gamma_b t} u(0, x_b)\,.
$$

3.3 Explicit Relations: The Case of $\gamma_a \neq \infty\,, \quad \gamma_b \neq \infty$

Imitating the arguments of the previous subsections, we observe that

$$\begin{aligned} u(t,x_a) &= -2\gamma_b \int_T^t e^{-2\gamma_a(t-\tau)} u(\tau - T, x_b) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_a(t-\tau)} d_\tau u_0(\tau, x_a) + e^{-2\gamma_a t} u_0(0,x_a) \end{aligned}$$

$$\begin{aligned} u(t,x_b) &= -2\gamma_a \int_T^t e^{-2\gamma_b(t-\tau)} u(\tau - T, x_a) d\tau \cdot 1_+(t-T) \\ &\quad + \int_{0+0}^t e^{-2\gamma_b(t-\tau)} d_\tau u_0(\tau, x_b) + e^{-2\gamma_b t} u_0(0,x_b) \end{aligned}$$

and then after defining the operation $\mathbf{I}_{\gamma,T}$ by

$$\Big(\mathbf{I}_{\gamma,T} f\Big)(t) := \int_T^t e^{-2\gamma_b(t-\tau)} f(\tau - T) d\tau$$

we infer that

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \begin{pmatrix} 0 & -2\gamma_b \mathbf{I}_{\gamma,T} \\ -2\gamma_a \mathbf{I}_{\gamma,T} & 0 \end{pmatrix} \begin{pmatrix} Q_a \\ Q_b \end{pmatrix} + \begin{pmatrix} u_{0,a,\gamma_a} \\ u_{0,b,\gamma_b} \end{pmatrix}$$

where

$$\begin{pmatrix} Q_a(t) \\ Q_b(t) \end{pmatrix} := \begin{pmatrix} u(t,x_a) \\ u(t,x_b) \end{pmatrix},$$

$$\begin{pmatrix} u_{0,a,\gamma_a}(t) \\ u_{0,b,\gamma_b}(t) \end{pmatrix} := \begin{pmatrix} \int_{0+0}^t e^{-2\gamma_a(t-\tau)} d_\tau u_0(\tau, x_a) + e^{-2\gamma_a t} u_0(0,x_a) \\ \int_{0+0}^t e^{-2\gamma_b(t-\tau)} d_\tau u_0(\tau, x_b) + e^{-2\gamma_b t} u_0(0,x_b) \end{pmatrix}.$$

So,

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \sum_{N=0}^{\infty} \begin{pmatrix} 0 & -2\gamma_b \mathbf{I}_{\gamma_b,T} \\ -2\gamma_a \mathbf{I}_{\gamma_a,T} & 0 \end{pmatrix}^N \begin{pmatrix} u_{0,a,\gamma_a} \\ u_{0,b,\gamma_b} \end{pmatrix}$$

Since

$$\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T} = \mathbf{I}_{\gamma_a,T}\mathbf{I}_{\gamma_b,T}\,,$$

$$\begin{pmatrix} 0 & -2\gamma_b \mathbf{I}_{\gamma_b,T} \\ -2\gamma_a \mathbf{I}_{\gamma_a,T} & 0 \end{pmatrix}^{2n} = (4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -2\gamma_b \mathbf{I}_{\gamma_b,T} \\ -2\gamma_a \mathbf{I}_{\gamma_a,T} & 0 \end{pmatrix}^{2n+1} = -(4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} 0 & 2\gamma_b \mathbf{I}_{\gamma_b,T} \\ 2\gamma_a \mathbf{I}_{\gamma_a,T} & 0 \end{pmatrix},$$

we infer that

$$\begin{aligned} \begin{pmatrix} Q_a \\ Q_b \end{pmatrix} &= \sum_{n=0}^{\infty} (4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{0,a,\gamma_a} \\ u_{0,b,\gamma_b} \end{pmatrix} \\ &\quad - \sum_{n=0}^{\infty} (4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} 0 & 2\gamma_b \mathbf{I}_{\gamma_b,T} \\ 2\gamma_a \mathbf{I}_{\gamma_a,T} & 0 \end{pmatrix} \begin{pmatrix} u_{0,a,\gamma_a} \\ u_{0,b,\gamma_b} \end{pmatrix} \\ &= \sum_{n=0}^{\infty} (4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} u_{0,a,\gamma_a} \\ u_{0,b,\gamma_b} \end{pmatrix} \\ &\quad - \sum_{n=0}^{\infty} (4\gamma_a\gamma_b)^n (\mathbf{I}_{\gamma_b,T}\mathbf{I}_{\gamma_a,T})^n \begin{pmatrix} 2\gamma_b \mathbf{I}_{\gamma_b,T} u_{0,b,\gamma_b} \\ 2\gamma_a \mathbf{I}_{\gamma_a,T} u_{0,a,\gamma_a} \end{pmatrix} \end{aligned}$$

Unfortunately, I do not know a good representation of this formala, in the general case. Nevertheless, if

$$\gamma_a = \gamma_b =: \gamma_0 \,,$$

then the situation becomes very similar to that of the previous subsection: we infer that

$$\begin{aligned}
\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} &= \sum_{n=0}^{\infty} (2\gamma_0)^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{I}^{2n}_{\gamma_0,T} \begin{pmatrix} u_{0,a,\gamma_0} \\ u_{0,b,\gamma_0} \end{pmatrix} \\
&\quad - \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{I}^{2n+1}_{\gamma_0,T} \begin{pmatrix} u_{0,a,\gamma_0} \\ u_{0,b,\gamma_0} \end{pmatrix} \\
&= \sum_{n=0}^{\infty} (2\gamma_0)^{2n} \mathbf{I}^{2n}_{\gamma_0,T} \begin{pmatrix} u_{0,a,\gamma_0} \\ u_{0,b,\gamma_0} \end{pmatrix} \\
&\quad - \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \mathbf{I}^{2n+1}_{\gamma_0,T} \begin{pmatrix} u_{0,b,\gamma_0} \\ u_{0,a,\gamma_0} \end{pmatrix}
\end{aligned}$$

Finally, as in the previous subsection, by recalling that

$$(\mathbf{I}^{2n}_{\gamma,T_0} f)(t) = \int_0^t \frac{(t-2nT_0-\tau)^{2n-1}}{(2n-1)!} e^{-2\gamma(t-2nT_0-\tau)} f(\tau)\cdot 1_+(t-2nT_0-\tau)d\tau$$

$$(\mathbf{I}^{2n+1}_{\gamma,T_0} f)(t) = \int_0^t \frac{(t-(2n+1)T_0-\tau)^{2n}}{(2n)!} e^{-2\gamma(t-(2n+1)T_0-\tau)} f(\tau)\cdot 1_+(t-(2n+1)T_0-\tau)d\tau$$

and after introducing

$$\begin{aligned}
&Sinh(\lambda, T_0, t) \\
&\quad := \lambda(t-T_0)\cdot 1_+(t-T_0) + \lambda^3 \frac{(t-3T_0)^3}{3!} \cdot 1_+(t-3T_0) \\
&\qquad + \cdots + \lambda^{2n+1} \frac{(t-(2n+1)T_0)^{2n+1}}{(2n+1)!} \cdot 1_+(t-(2n+1)T_0) \cdots \\
&Cosh(\lambda, T_0, t) \\
&\quad := 1\cdot 1_+(t) + \lambda^2 \frac{(t-2T_0)^2}{2!} \cdot 1_+(t-2T_0) \\
&\qquad + \cdots + \lambda^{2n} \frac{(t-2nT_0)^{2n}}{(2n)!} \cdot 1_+(t-2nT_0) \cdots
\end{aligned}$$

we observe that

$$\Big(\sum_{n=0}^{\infty} (2\gamma_0)^{2n} \mathbf{I}^{2n}_{\gamma_0,T} f\Big)(t) = f(t) + 2\gamma_0 e^{2\gamma_0 T} \int_0^t Sinh(2\gamma_0 e^{2\gamma_0 T}, T, t-T-\tau) e^{-2\gamma(t-\tau)} f(\tau)d\tau$$

$$\Big(\sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \mathbf{I}^{2n+1}_{\gamma_0,T} f\Big)(t) = 2\gamma_0 e^{2\gamma_0 T} \int_0^t Cosh(2\gamma_0 e^{2\gamma_0 T}, T, t-T-\tau) e^{-2\gamma(t-\tau)} f(\tau)d\tau$$

The conclusion is evident.

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