Level Shift Operator
and 2nd order perturbation theory

Jan Dereziński and Rafał Früboes

Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warsaw, Poland
(e-mail: jan.derezinski@fuw.edu.pl, rafal.fruboes@fuw.edu.pl)

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Abstract

We study spectral properties of linear operators. The main tool is the so-called *Level Shift Operator*, which expresses the effects of 2nd order perturbation theory on the point spectrum. We give estimates for the location of the spectrum in a selected region and approximate formulas for the corresponding spectral projections. Presented results are quite robust—they are applicable even if the spectrum in this region has infinite multiplicity.

Keywords: spectrum, spectral projections, perturbation theory

1 Introduction

Let us first sketch the standard approach to the perturbation theory for linear operators. Suppose that L_0 is a closed operator and e is an isolated point of spectrum L_0. Let Q be a perturbation. Suppose we are interested in the spectrum of the perturbed operator $L_\lambda := L_0 + \lambda Q$ around e, for small complex λ.

Suppose for the moment that the degeneracy of e is finite (the spectral projection of L_0 onto e is finite dimensional). Then one can show that, for small λ, L_λ has a cluster of eigenvalues around e well separated from the rest of the spectrum L_λ. More precisely, one shows that, for $0 < |\lambda| < \Lambda_0$, there exists a multivalued analytic function $e(\lambda)$ with a branch point at $\lambda = 0$ that can be extended by continuity to $\lambda = 0$ by putting $e(0) = e$, and the spectrum of L_λ in a neighborhood of e is given by the values of $e(\lambda)$.

The above approach to the theory of perturbations of linear operators is well known and described e.g in [Kato, RS4]. In this paper we look at the theory of perturbations differently. We do not follow individually the eigenvalues or eigenvectors of the perturbed operator. Instead, we describe the location of its spectrum in a selected region approximately and give approximate formulas for the corresponding spectral projections. We will restrict ourselves to the 2nd order corrections to the spectrum. For simplicity, we will assume that the 1st order correction is zero.

Our method is much more robust than the standard approach to perturbation theory. Note, for instance, that if the unperturbed eigenvalue e has infinite multiplicity, then the perturbed operator may have continuous spectrum in a neighborhood of e. Thus the standard approach to perturbation theory, which consists in following the individual eigenvalues of L_λ, is unavailable. On the other hand, our approach is applicable without a change.

Let us now describe our results a little more closely. Our main object is an operator on a Banach space of the form

$$L_{\lambda,\beta} := L_0 + \lambda(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}}) + \beta Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}},$$

where the "free operator" L_0 has a cluster of isolated eigenvalues Ξ. The spectral projection of L_0 onto Ξ, denoted $\mathbf{1}_\Xi(L_0)$, gives a natural decomposition of the Banach space into a direct sum $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$. We assume that inside $\mathcal{H}^{\mathrm{v}}$ the perturbation is zero. This guarantees that there is no 1st order shift of the spectrum. $Q^{\mathrm{v}\overline{\mathrm{v}}}+Q^{\overline{\mathrm{v}}\mathrm{v}}$ is the "off-diagonal" part of the perturbation—it connects $\mathcal{H}^{\mathrm{v}}$ and $\mathcal{H}^{\overline{\mathrm{v}}}$. $Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is the "external" part of the perturbation—it acts inside $\mathcal{H}^{\overline{\mathrm{v}}}$. We are interested in what happens for small complex λ and β. We will try to estimate carefully the deviations from our predictions in terms of these two coupling constants.

It is easy to see that for small λ and β, the spectrum of $L_{\lambda,\beta}$ does not differ much from the spectrum L_0. Thus in particular, for small λ and β, for any isolated point e of Ξ, there exists a patch of spectrum of $L_{\lambda,\beta}$ located around e, which we will denote by Θ_e. (We say "a patch", not "a cluster", because we the spectrum does not have to be discrete). We will give the formula for a projection p_e that approximates the spectral projection of $L_{\lambda,\beta}$ onto Θ_e. We will show that this projection approximately diagonalizes $L_{\lambda,\beta}$. By this we mean that $L - p_e L_{\lambda,\beta} p_e - (\mathbf{1} - p_e) L_{\lambda,\beta} (\mathbf{1} - p_e)$ is small.

The above results are contained in Theorem 2.1. They are quite easy. What is more interesting is the study of the splitting of the patch Θ_e, which is the subject of Theorem 2.3—the main result of our paper.

We show that if the eigenvalue e is semisimple then the patch of the spectrum around e naturally splits into subpatches separated by a distance of order $O(|\lambda|^2)$. The subpatches will be parametrized by eigenvalues of the so-called *Level Shift Operator* (LSO). The Level Shift Operator is a certain naturally defined operator that describes the shift of spectrum under the influence of 2nd order perturbation theory. The subpatch of the spectrum of $L_{\lambda,\beta}$ around $e + \lambda^2 m$, where m is an eigenvalue of the LSO, will be denoted by $\Theta_{e,m}$. We will also give a formula for the projection $p_{e,m}$ that approximates the spectral projection of $L_{\lambda,\beta}$ onto $\Theta_{e,m}$. Finally, we will show that $p_{e,m}$ approximately diagonalizes $L_{\lambda,\beta}$.

In our paper the method of the Level Shift Operator is used to study isolated eigenvalues. A similar method can be applied to study embedded point spectrum and to describe resonances of the perturbed operator. By using the spectral deformation technique [RS4, JP1, JP2] embedded eigenvalues become isolated and thus the case of embedded eigenvalues can be reduced to that of isolated eigenvalues. In a subsequent paper we will describe in detail the method of LSO for embedded point spectrum.

The LSO appears in the mathematics and physics literature in various disguises whenever the 2nd order perturbation theory is considered. For instance, it appeared implicitly in the work [DJ1] devoted to the perturbation theory for embedded eigenvalues of the so-called Pauli-Fierz Hamiltonians. The analysis of the point spectrum given in [DJ1] is very closely related to the analysis given in our paper. Nevertheless, there are some differences. The use of the Limiting Absorption Principle and Mourre's positive commutator method in [DJ1] was a consequence of the fact that the studied eigenvalues were embedded. Another difference is the self-adjointness of the operator studied in [DJ1], whereas in our paper we do not restrict ourselves to self-adjoint operators.

Constructions similar to ours can be found in the papers of Bach-Fröhlich-Sigal [BFS1, BFS2]. The authors study the spectrum of certain operators (similar to the Pauli-Fierz operators considered in [DJ1, DJ2]) by an iterative procedure ("renormalization group"). The basic step of this procedure resembles our prescription for locating the spectrum and constructing the approximate spectral projection.

We believe that the approach of our paper has a very wide range of applications. There is, however, one class of problems that was the major motivation for us. This class of problems is the study of Pauli-Fierz systems (using the terminology of [DJ2]). These systems are supposed to model the interaction of a small quantum system interacting with radiation. The method of the Level Shift Operator was applied in particular to Pauli-Fierz Liouvilleans in [DJ1, DJ2]. It can be also applied to the generator of a Pauli-Fierz dynamics (called C-Liouvilleans in [JP3]). It is also closely related to the so-called weak coupling (van Hove) limit [Da1, Da2, DJ3].

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2 Main results

2.1 Notation

If $\Xi \subset \Theta \subset \mathbb{C}$, then we say that Ξ is an isolated subset of Θ if it is closed and open in the relative topology of Θ.

$\Theta^{\rm cl}$ denotes the closure of Θ in $\mathbb{C}$.

If L is a linear operator, ${\rm sp}L$ denotes its spectrum and ${\rm Dom}L$ its domain. If Ξ is an isolated subset of ${\rm sp}L$, then we can define the spectral projection of L onto Ξ by the formula

$$\mathbf{1}_\Xi(L) = \frac{1}{2\pi {\rm i}} \oint_\gamma (z-L)^{-1}\, dz,$$

where γ is a closed path that encircles Ξ counterclockwise.

If e is an isolated point of ${\rm sp}L$, then we write $1_e(L)$ for $1_{\{e\}}(L)$. For such e set $N_e := (L - e)\mathbf{1}_e(L)$. We say that the degree of nilpotence of e is equal to n iff $N_e^{n-1} \neq 0$ but $N_e^n = 0$. We say that e is semisimple iff $n = 1$ (i.e. $N_e = 0$).

If $\Theta \subset \mathbb{C}$ and $\epsilon > 0$, then we set

$$\mathrm{D}(\Theta, \epsilon) := \{z \in \mathbb{C} \ : \ \mathrm{dist}(z, \Theta) < \epsilon\}.$$

For $e \in \mathbb{C}$, $\mathrm{D}(e, \epsilon)$ will denote the open disc centered at e with radius ϵ. Moreover, we set $\mathrm{D}(\emptyset, \epsilon) := \emptyset$.

If $A(\lambda,\beta)$ are bounded operators, and $f(\lambda,\beta)$ a positive function, then

$$A(\lambda,\beta)=O(f(\lambda,\beta))$$

means that there exists c such that

$$\|A(\lambda,\beta)\|\leq cf(\lambda,\beta).$$

Moreover,

$$A_1(\lambda,\beta)=A_2(\lambda,\beta)+O(f(\lambda,\beta))$$

or

$$A_1(\lambda,\beta)\stackrel{O(f(\lambda,\beta))}{=}A_2(\lambda,\beta)$$

means that

$$A_1(\lambda,\beta)-A_2(\lambda,\beta)=O(f(\lambda,\beta)).$$

2.2 Assumptions

Let L_0 be a closed operator on a Banach space $\mathcal{H}$. Suppose that Ξ is an isolated subset of $\mathrm{sp}L_0$.

It will be convenient to denote $\mathbf{1}_\Xi(L_0)$ by $\mathbf{1}^{\mathrm{vv}}$ and set $\mathbf{1}^{\overline{\mathrm{vv}}}:=\mathbf{1}-\mathbf{1}^{\mathrm{vv}}$. We can also introduce the subspaces

$$\mathcal{H}^{\mathrm{v}}:=\mathbf{1}^{\mathrm{vv}}\mathcal{H},\quad \mathcal{H}^{\overline{\mathrm{v}}}:=\mathbf{1}^{\overline{\mathrm{vv}}}\mathcal{H},$$

so that $\mathcal{H}$ is decomposed into a direct sum

$$\mathcal{H}=\mathcal{H}^{\mathrm{v}}\oplus\mathcal{H}^{\overline{\mathrm{v}}}. \tag{2.1}$$

Wrt the decomposition (2.1) any operator B on $\mathcal{H}$ satisfying

$$\mathrm{Dom}(B)=\Big(\mathrm{Dom}(B)\cap\mathcal{H}^{\mathrm{v}}\Big)\oplus\Big(\mathrm{Dom}(B)\cap\mathcal{H}^{\overline{\mathrm{v}}}\Big)$$

can be written as

$$B=\left[\begin{array}{cc}B^{\mathrm{vv}} & B^{\mathrm{v}\overline{\mathrm{v}}}\\ B^{\overline{\mathrm{v}}\mathrm{v}} & B^{\overline{\mathrm{vv}}}\end{array}\right] \tag{2.2}$$

In particular, we have

$$L_0=\left[\begin{array}{cc}L_0^{\mathrm{vv}} & 0\\ 0 & L_0^{\overline{\mathrm{vv}}}\end{array}\right]. \tag{2.3}$$

It will be convenient to write E for L_0^{vv}. Note that $\mathrm{sp}(E)=\Xi$.

Assumption A *E is a bounded operator.*

Let Q be another operator, that we will treat as a perturbation of L_0. More precisely, we make the following assumptions:

Assumption B $Q^{\mathrm{vv}}=0$.

Assumption C *Off-diagonal elements of Q, i.e. $Q^{\mathrm{v}\overline{\mathrm{v}}}$ and $Q^{\overline{\mathrm{v}}\mathrm{v}}$, are bounded.*

We will also use either one of the following two assumption:

Assumption D *$Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is an operator bounded perturbation of $L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ [Kato]*

Assumption E *$\mathcal{H}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is a Hilbert space, $L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is selfadjoint, bounded from below and $Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is a form bounded perturbation of $L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ [Kato].*

Let $\lambda, \beta \in \mathbb{C}$. Note that under Assumption D or E the operator $L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}} + \beta Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ is well defined for small enough β [Kato]. Likewise,

$$L_{\lambda,\beta} := L_0 + \lambda Q^{\mathrm{v}\overline{\mathrm{v}}} + \lambda Q^{\overline{\mathrm{v}}\mathrm{v}} + \beta Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}} = \begin{bmatrix} E & \lambda Q^{\mathrm{v}\overline{\mathrm{v}}} \\ \lambda Q^{\overline{\mathrm{v}}\mathrm{v}} & L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}} + \beta Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}.$$

is well defined for small enough β. For simplicity we will write L instead of $L_{\lambda,\beta}$.

Fix an open subset $\Omega \subset \mathbb{C}$ such that $\Omega^{\mathrm{cl}} \cap \mathrm{sp} L_0 = \Xi$ and $\Xi \subset \Omega$. Note that there exists β_0 such that, for $|\beta| \leq \beta_0$, $\mathrm{sp}(L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}} + \beta Q^{\overline{\mathrm{v}}\overline{\mathrm{v}}}) \cap \Omega^{\mathrm{cl}} = \emptyset$. We fix β_0 satisfying these conditions.

2.3 Results

The main results of our paper are stated in the following two theorems. Note that Theorem 2.1 is quite easy and basically describes the well known stability of spectrum under a perturbation. Theorem 2.3 is more difficult—it describes the splitting of the spectrum according to 2nd order perturbation theory. In that theorem, an important role is played by the Level Shift Operator. Note that we tried to make the two theorems as parallel as possible.

Theorem 2.1 *Suppose that Assumptions A, B and C hold. We also suppose that either Assumption D or E is satisfied. Then the following is true:*

1. *There exists a continuous and increasing function*

$$[0, \infty[\ni x \mapsto \delta(x) \in [0, \infty],$$

such that $\lim_{x \to 0} \delta(x) = 0$, and for $|\beta| < \beta_0$ and $|\lambda| < \lambda_0$, for some $\lambda_0 > 0$, we have:

$$\mathrm{sp}(L) \cap \Omega \subset \mathrm{D}(\mathrm{sp} E, \delta(|\lambda|)). \tag{2.4}$$

2. *In what follows we assume that $\mathcal{E}$ is an isolated subset of $\mathrm{sp} E$. Clearly, 1. implies that there exists $0 < \lambda_{\mathcal{E}}$ such that, for $|\lambda| < \lambda_{\mathcal{E}}$,*

$$\Theta_{\mathcal{E}} := \mathrm{D}(\mathcal{E}, \delta(|\lambda|)) \cap \mathrm{sp} L \tag{2.5}$$

is an isolated subset of $\mathrm{sp} L$ and $\Theta_{\mathcal{E}} \subset \Omega$.

3. *For $|\lambda| < \lambda_{\mathcal{E}}$ we have*

$$\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0) = O(|\lambda|). \tag{2.6}$$

4. *For $|\lambda| < \lambda_{\mathcal{E}}$ we have*

$$\dim \mathbf{1}_{\Theta_{\mathcal{E}}}(L) = \dim \mathbf{1}_{\mathcal{E}}(L_0). \tag{2.7}$$

5. *In what follows we assume that* e *is an isolated point of* $\mathrm{sp}E$. *We will write* Θ_e *for* $\Theta_{\{e\}}$. *If the degree of nilpotence of* e *as an eigenvalue of* E *is equal to* n, *then there exists* C_e *such that*
$$\Theta_e \subset \mathrm{D}(e, C_e|\lambda|^{\frac{2}{n}}).$$

6. *For*
$$|\lambda| < \|\mathbf{1}_e(L_0)Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(L_0)\|^{-\frac{1}{2}} =: \widehat{\lambda}_e, \tag{2.8}$$
we set
$$\begin{aligned} p_e := \quad & \Big(\mathbf{1}_e(L_0) + \lambda(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(L_0)\Big) \\ & \times\Big(\mathbf{1}_e(L_0) + \lambda^2\mathbf{1}_e(L_0)Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(L_0)\Big)^{-1} \\ & \times\Big(\mathbf{1}_e(L_0) + \lambda\mathbf{1}_e(L_0)Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\Big). \end{aligned} \tag{2.9}$$
Then p_e *is a projection. Moreover:*

   (a)
   $$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|); \tag{2.10}$$

   (b) *if* e *is a semisimple eigenvalue of* E *then*
   $$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|^2 + |\lambda\beta|); \tag{2.11}$$

   (c) *if, in addition,* $\mathrm{sp}(E) = \{e\}$, *then*
   $$\mathbf{1}_{\Theta_e}(L) - p_e = O(|\lambda|^3 + |\lambda\beta|). \tag{2.12}$$

7. (a) *For* $|\lambda| < \widehat{\lambda}_e$ *we have:*
   $$L - p_eLp_e - (\mathbf{1} - p_e)L(\mathbf{1} - p_e) = O(|\lambda|); \tag{2.13}$$

   (b) *if* e *is a semisimple eigenvalue of* E *then*
   $$L - p_eLp_e - (\mathbf{1} - p_e)L(\mathbf{1} - p_e) = O(|\lambda|^2 + |\lambda\beta|); \tag{2.14}$$

   (c) *if, moreover,* $\mathrm{sp}(E) = \{e\}$ *then*
   $$L - p_eLp_e - (\mathbf{1} - p_e)L(\mathbf{1} - p_e) = O(|\lambda|^3 + |\lambda\beta|). \tag{2.15}$$

Note that in Equation 2.9 we use the notation $(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}$ for the inverse of the operator $e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$ *restricted to* $\mathcal{H}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}$. In what follows we will often use similar notation without a comment.

Let us now assume that $\mathrm{sp}E$ is a finite set.

Definition 2.2 *We define the Level Shift Operator (LSO) as*
$$\Gamma := \sum_{e\in\mathrm{sp}(E)} \mathbf{1}_e(E)Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_e(E).$$

>From now on we will write for shortness $\Gamma^{ee} := \mathbf{1}_e(E)\Gamma\mathbf{1}_e(E)$.

Now we are ready to state our main theorem.

Theorem 2.3 *Suppose that Assumptions A, B and C hold. We also assume either Assumption D or E. Assume also that* $\mathrm{sp}E$ *is a finite set consisting of semisimple eigenvalues. Then the following is true:*

1. *There exists a continuous and increasing function*
$$[0,\infty[\ni x \mapsto \delta(x) \in [0,\infty],$$
such that $\lim_{x\to 0}\delta(x) = 0$*, and, for* $|\beta| < \beta_0$*,* $|\lambda| < \lambda_0$*, for some* $\lambda_0 > 0$*, we have*
$$\begin{aligned} \mathrm{sp}(L)\cap\Omega \ &\subset \mathrm{D}\big(\mathrm{sp}(E+\lambda^2\Gamma), \lambda^2\delta(|\lambda|+|\beta|)\big) \\ &= \bigcup_{e\in\mathrm{sp}(E)} \mathrm{D}\big(e+\lambda^2\mathrm{sp}(\Gamma^{ee}), \lambda^2\delta(|\lambda|+|\beta|)\big). \end{aligned} \tag{2.16}$$

2. *In what follows we fix* $e \in \mathrm{sp}E$*, and* $\mathcal{M}$ *is an isolated subset of* $\mathrm{sp}\Gamma^{ee}$*. Clearly, 1. implies that there exists* $0 < \lambda_{e,\mathcal{M}}$ *and* $0 < \beta_{e,\mathcal{M}}$ *such that for* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$
$$\Theta_{e,\mathcal{M}} := \mathrm{D}(e+\lambda^2\mathcal{M}, \lambda^2\delta(|\lambda|+|\beta|)) \cap \mathrm{sp}L$$
is an isolated subset of $\mathrm{sp}L$ *and* $\Theta_{e,\mathcal{M}} \subset \Omega$*.*

3. *For* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$ *we have*
$$\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) - \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}) = O(|\lambda|+|\beta|). \tag{2.17}$$

4. *For* $|\lambda| < \lambda_{e,\mathcal{M}}$ *and* $|\beta| < \beta_{e,\mathcal{M}}$ *we have*
$$\dim \mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) = \dim \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}). \tag{2.18}$$

5. *Assume now that* m *is an isolated point of* $\mathrm{sp}\Gamma^{ee}$*. We will write* $\Theta_{e,m}$ *for* $\Theta_{e,\{m\}}$*. Suppose that the degree of nilpotence of* m *as an eigenvalue of* Γ^{ee} *is equal to* n*. Then*
$$\Theta_{e,m} \subset \mathrm{D}\big(e+\lambda^2 m, C_{e,m}\lambda^2(|\lambda|+|\beta|)^{\frac{1}{n}}\big),$$
for some $C_{e,m} > 0$*.*

6. *For*
$$|\lambda| < \|\mathbf{1}_m(\Gamma^{ee})Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_m(\Gamma^{ee})\|^{-\frac{1}{2}} =: \widehat{\lambda_{e,m}}, \tag{2.19}$$
we set
$$\begin{aligned} p_{e,m} := \ &\Big(\mathbf{1}_m(\Gamma^{ee}) + \lambda(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_m(\Gamma^{ee})\Big) \\ &\times\Big(\mathbf{1}_m(\Gamma^{ee}) + \lambda^2\mathbf{1}_m(\Gamma^{ee})Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}_m(\Gamma^{ee})\Big)^{-1} \\ &\times\Big(\mathbf{1}_m(\Gamma^{ee}) + \lambda\mathbf{1}_m(\Gamma^{ee})Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\Big). \end{aligned} \tag{2.20}$$
Then $p_{e,m}$ *is a projection. Moreover:*

(a)

$$\mathbf{1}_{\Theta_{e,m}}(L) - p_{e,m} = O(|\lambda| + |\beta|). \tag{2.21}$$

(b) if m is a semisimple eigenvalue of Γ^{ee} then

$$\mathbf{1}_{\Theta_{e,m}}(L) - p_{e,m} = O(|\lambda|^2 + |\beta|). \tag{2.22}$$

7. *For $|\lambda| < \widehat{\lambda_{e,m}}$ we have*

   (a)

$$L - p_{e,m} L p_{e,m} - (\mathbf{1} - p_{e,m}) L (\mathbf{1} - p_{e,m}) = O(|\lambda|^2 + |\lambda\beta|).$$

   (b) if $\mathbf{1}^{\mathrm{vv}} = \mathbf{1}_m(\Gamma^{ee})$ then

$$L - p_{e,m} L p_{e,m} - (\mathbf{1} - p_{e,m}) L (\mathbf{1} - p_{e,m}) = O(|\lambda|^3 + |\lambda\beta|).$$

Remark 2.4 *Note that in both theorems 6) describes how close the projections p_e and $p_{e,m}$ are to the correspomding spectral projections of L and 7) describes how well they diagonalize L.*

In Theorem 2.1, we have the same order of smallness in 6) and 7). We will see from the proof, that 7) essentially follows from 6).

On the other hand, in Theorem 2.3, the order of smallness of 7) is much better than that of 6). Thus 7) requires a separate proof.

3 Proofs

Let us begin with a general fact about the stability of the spectrum of bounded operators [Kato].

Theorem 3.1 *Let $A \in B(\mathcal{H})$. Then there exists an increasing and continuous function*

$$[0, \infty[\ni x \mapsto \mu_A(x) \in [0, \infty],$$

such that $\lim_{x \to 0} \mu_A(x) = 0$ and for any $B \in B(\mathcal{H})$ we have $\mathrm{sp}(A + B) \subset \mathrm{D}(\mathrm{sp}(A), \mu_A(\|B\|))$.

If $a \in \mathrm{sp}(A)$ is an isolated eigenvalue with the degree of nilpotence equal to n. Then there exists $\epsilon > 0$ such that for $z \in \mathrm{D}(a, \epsilon) \backslash \{a\}$ we have

$$\|(z - A)^{-1}\| \leq C |z - a|^{-n}, \tag{3.23}$$

for some $C > 0$. Moreover, for $|\lambda| < \Lambda_a$ for some $\Lambda_a > 0$ we have

$$\mathrm{sp}(A + \lambda B) \cap \mathrm{D}(a, \epsilon) \subset \mathrm{D}(a, c|\lambda|^{\frac{1}{n}}), \tag{3.24}$$

where $c = (C\|B\|)^{\frac{1}{n}}$.

Proof. We prove only the last statement. Let $|\lambda| < \epsilon^n (C\|B\|)^{-1}$. If $z \in \mathrm{D}(a, \epsilon) \backslash \mathrm{D}(a, (C\|\lambda B\|)^{\frac{1}{n}})$ then

$$|z - a| > (C\|\lambda B\|)^{\frac{1}{n}},$$

so by the inequality (3.23)

$$1 > C\|\lambda B\||z-a|^{-n} \geq \|\lambda B\|\|(z-A)^{-1}\| \geq \|\lambda B(z-A)^{-1}\|.$$

This shows that $z - A - \lambda B$ is invertible and hence $z \notin \mathrm{sp}(A + \lambda B)$, so we get (3.24). □

Let us comment on some additional notation that we will use. For $\mathcal{E}$ an isolated subset of $\mathrm{sp}(E)$ we will write

$$\mathbf{1}^{\mathcal{E}\mathcal{E}} := \mathbf{1}_{\mathcal{E}}(E) = \mathbf{1}_{\mathcal{E}}(L_0), \qquad \mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} := \mathbf{1} - \mathbf{1}^{\mathcal{E}\mathcal{E}}, \qquad \mathbf{1}^{\underline{\mathcal{E}\mathcal{E}}} := \mathbf{1}^{\mathrm{vv}} - \mathbf{1}^{\mathcal{E}\mathcal{E}}$$

$$\mathcal{H}^{\mathcal{E}} := \mathrm{Ran}\mathbf{1}_{\mathcal{E}}(E), \qquad \mathcal{H}^{\overline{\mathcal{E}}} := \mathrm{Ran}\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}}, \qquad \mathcal{H}^{\underline{\mathcal{E}}} := \mathrm{Ran}\mathbf{1}^{\underline{\mathcal{E}\mathcal{E}}}$$

Now the Banach space $\mathcal{H}$ can be decomposed in the following way

$$\mathcal{H} = \mathcal{H}^{\mathcal{E}} \oplus \mathcal{H}^{\overline{\mathcal{E}}} = \mathcal{H}^{\mathcal{E}} \oplus \mathcal{H}^{\underline{\mathcal{E}}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}.$$

and operator L can be written as

$$L = \begin{bmatrix} E^{\mathcal{E}\mathcal{E}} & \lambda Q^{\mathcal{E}\overline{\mathcal{E}}} \\ \lambda Q^{\overline{\mathcal{E}}\mathcal{E}} & L^{\overline{\mathcal{E}\mathcal{E}}} \end{bmatrix} = \begin{bmatrix} E^{\mathcal{E}\mathcal{E}} & 0 & \lambda Q^{\mathcal{E}\overline{\mathrm{v}}} \\ 0 & E^{\underline{\mathcal{E}\mathcal{E}}} & \lambda Q^{\underline{\mathcal{E}}\overline{\mathrm{v}}} \\ \lambda Q^{\overline{\mathrm{v}}\mathcal{E}} & \lambda Q^{\overline{\mathrm{v}}\underline{\mathcal{E}}} & L_0^{\overline{\mathrm{vv}}} + \beta Q^{\overline{\mathrm{vv}}} \end{bmatrix}. \tag{3.25}$$

If e is an isolated point of $\mathrm{sp}(E)$ then we write $\mathbf{1}^{ee}$ for $\mathbf{1}^{\{e\}\{e\}}$, $\mathcal{H}^{e}$ for $\mathcal{H}^{\{e\}}$, etc. Note that $E^{\overline{\mathcal{E}\mathcal{E}}} = E^{\underline{\mathcal{E}\mathcal{E}}}$.

We will use the following theorem for several operators and for various decompositions of the space $\mathcal{H}$.

Theorem 3.2 *Let H be a closed operator on a Banach space $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$. Assume that off-diagonal elements of H i.e. $H^{\mathrm{v}\overline{\mathrm{v}}}$ and $H^{\overline{\mathrm{v}}\mathrm{v}}$ are bounded. For $z \in \mathbb{C}\backslash\mathrm{sp}(H^{\overline{\mathrm{vv}}})$ define*

$$G_{\mathrm{v}}(z) := z\mathbf{1}^{\mathrm{vv}} - H^{\mathrm{vv}} - H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Then for $z \notin \mathrm{sp}(H^{\overline{\mathrm{vv}}})$ we have

1. *$z \in \mathrm{sp}(H)$ iff $0 \in \mathrm{sp}(G_{\mathrm{v}}(z))$,*
2. *if $0 \notin \mathrm{sp}(G_{\mathrm{v}}(z))$ then*

$$(z-H)^{-1} = (z\mathbf{1}^{\overline{\mathrm{vv}}}-H^{\overline{\mathrm{vv}}})^{-1}+(\mathbf{1}^{\mathrm{vv}}+(z\mathbf{1}^{\overline{\mathrm{vv}}}-H^{\overline{\mathrm{vv}}})^{-1}H^{\overline{\mathrm{v}}\mathrm{v}})G_{\mathrm{v}}^{-1}(z)(\mathbf{1}^{\mathrm{vv}}+H^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}}-H^{\overline{\mathrm{vv}}})^{-1})$$

The last equation is often called the Feshbach formula. We will keep this name. For more information about the above theorem the reader is referred to [DJ1, GGK].

Lemma 3.3 *Let $\mathcal{E}$ be an isolated subset of $\mathrm{sp}(E)$ and fix $r > 0$. Then there exists $0 < \Lambda_{\mathcal{E}}$ such that for $|\lambda| < \Lambda_{\mathcal{E}}$, $|\beta| < \beta_0$ we have $\Big(\Omega\backslash\mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}\mathcal{E}}}), r)\Big) \cap \mathrm{sp}(L^{\overline{\mathcal{E}\mathcal{E}}}) = \emptyset$ and*

$$\sup_{\substack{z\in\Omega\backslash\mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}\mathcal{E}}}),r) \\ |\lambda|<\Lambda_{\mathcal{E}},\ |\beta|<\beta_0}} \|(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1}\| < \infty. \tag{3.26}$$

Proof. If $z \in \Omega$ and $|\beta| < \beta_0$ then $z \notin \mathrm{sp}(L^{\overline{\mathrm{vv}}})$ and hence we can use Theorem 3.2 for the operator $L^{\overline{\mathcal{E}\mathcal{E}}}$ and for decomposition $\mathcal{H}^{\overline{\mathcal{E}}} := \mathcal{H}^{\underline{\mathcal{E}}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$. We obtain that, for some $\Lambda_{\mathcal{E}} > 0$, and $|\lambda| < \Lambda_{\mathcal{E}}$ and for $z \in \Omega \backslash \mathrm{D}(\mathrm{sp}(E^{\underline{\mathcal{E}\mathcal{E}}}), r)$,

$$G_{\underline{\mathcal{E}}}(z) = z\mathbf{1}^{\underline{\mathcal{E}\mathcal{E}}} - E^{\underline{\mathcal{E}\mathcal{E}}} - \lambda^2 Q^{\underline{\mathcal{E}}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\underline{\mathcal{E}}},$$

is also invertible and hence $z \notin \mathrm{sp}(L^{\overline{\mathcal{E}\mathcal{E}}})$. Moreover, for such z, $G_{\underline{\mathcal{E}}}(z)$ has a uniformly bounded inverse. Therefore, the Feshbach formula implies (3.26).□

Proof of the Theorem 2.1

1) By Theorem 3.2, $z \in \mathrm{sp}(L) \cap \Omega$ iff $z \in \mathrm{sp}(E + \lambda^2 Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}) \cap \Omega$. By Theorem 3.1

$$\mathrm{sp}(E + \lambda^2 Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}) \subset \mathrm{D}(\mathrm{sp}(E), \mu_E(\lambda^2 c)),$$

where

$$c = \sup_{z\in\Omega, |\beta|<\beta_0} \|Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}\|,$$

(which as we know is finite), and $\mu_E : [0, \infty[\to [0, \infty[$ is a continuous increasing function with $\lim_{x\to 0} \mu_E(x) = 0$. Thus $\mathrm{sp}(L) \cap \Omega \subset \mathrm{D}(\mathrm{sp}(E), \delta(|\lambda|))$, where $\delta(x) = \mu_E(x^2 c)$.

2) A simple consequence of 1).

3) For some $0 < \lambda_{\mathcal{E}}$, $|\lambda| < \lambda_{\mathcal{E}}$ and $|\beta| < \beta_0$, there exists a closed path $\gamma \subset \Omega$ that encircles $\Theta_{\mathcal{E}}$ counterclockwise, but no other parts of $\mathrm{sp}(L)$. We have

$$\sup_{\substack{z\in\gamma,\ |\beta|<\beta_0 \\ |\lambda|<\lambda_{\mathcal{E}}}} \|(z - L)^{-1}\| < \infty.$$

Besides,

$$(2\pi\mathrm{i})^{-1} \oint_\gamma (z\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1} dz = (2\pi\mathrm{i})^{-1} \oint_\gamma (z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} dz = 0.$$

Therefore,

$$\begin{aligned}
\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0) &= (2\pi\mathrm{i})^{-1} \oint_\gamma \big((z\mathbf{1} - L)^{-1} - (z\mathbf{1}^{\mathrm{vv}} - E)^{-1} - (z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1}\big) dz \\
&= \lambda(2\pi\mathrm{i})^{-1} \oint_\gamma (z\mathbf{1} - L)^{-1}(Q^{\mathrm{v}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\mathrm{v}})\Big((z\mathbf{1}^{\mathrm{vv}} - E)^{-1} + (z\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1}\Big) dz \\
&= O(|\lambda|)
\end{aligned}$$

4) Equation (2.6) implies that for $|\lambda|$ sufficiently small we have $\|\mathbf{1}_{\Theta_{\mathcal{E}}}(L) - \mathbf{1}_{\mathcal{E}}(L_0)\| < 1$ so by a well known theorem [Kato] equation (2.7) holds for $|\lambda|$ small. But $\lambda \mapsto \dim \mathbf{1}_{\Theta_{\mathcal{E}}}(L) \in \mathbb{N}$ is a continuous function so (2.7) holds for all $|\lambda| < \lambda_{\mathcal{E}}$.

5) Let $\mathcal{E}$, r and $\Lambda_{\mathcal{E}}$ be the same as in the Lemma 3.3. For $|\lambda| < \Lambda_{\mathcal{E}}$, $|\beta| < \beta_0$, we can use Theorem 3.2, which implies that for $z \in \Omega \backslash \mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}\mathcal{E}}}), r)$ we have $z \in \mathrm{sp}(L)$ iff

$$z \in \mathrm{sp}(E^{\mathcal{E}\mathcal{E}} + \lambda^2 Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1} Q^{\overline{\mathcal{E}}\mathcal{E}}).$$

By Theorem 3.1 we get

$$\mathrm{sp}(E^{\mathcal{E}\mathcal{E}} + \lambda^2 Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}\mathcal{E}}} - L^{\overline{\mathcal{E}\mathcal{E}}})^{-1} Q^{\overline{\mathcal{E}}\mathcal{E}}) \subset \mathrm{D}(\mathcal{E}, \mu_{E^{\mathcal{E}\mathcal{E}}}(\lambda^2 c)), \tag{3.27}$$

where

$$c := \sup_{\substack{z\in\Omega\setminus \mathrm{D}(\mathrm{sp}(E^{\overline{\mathcal{E}}\overline{\mathcal{E}}}),r)\\ |\beta|<\beta_0,\ |\lambda|<\Lambda_{\mathcal{E}}}} \|Q^{\mathcal{E}\overline{\mathcal{E}}}(z\mathbf{1}^{\overline{\mathcal{E}}\overline{\mathcal{E}}}-L^{\overline{\mathcal{E}}\overline{\mathcal{E}}})^{-1}Q^{\overline{\mathcal{E}}\mathcal{E}}\|$$

is finite by Lemma 3.3.

Now set $\mathcal{E}=\{e\}$ and assume that e has a degree of nilpotence equal to n. Then by Theorem 3.1 we can take $\mu_{E^{ee}}(x) := c_1 x^{1/n}$.

6) For $|\lambda| < \widehat{\lambda_e}$

$$\mathbf{1}^{ee}+\lambda^2 Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}e},$$

is an invertible operator so the expression for p_e makes sense. Direct computations show that $p_e^2=p_e$. Note that

$$(\mathbf{1}^{ee}+\lambda^2 Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}e})^{-1} = \mathbf{1}^{ee}-\lambda^2 Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}e}+O(\lambda^4)$$

so

$$\begin{aligned} p_e = \mathbf{1}^{ee} + \lambda\Big(Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}+(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}e}\Big) \\ +\lambda^2\Big((e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}e}Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}-Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-2}Q^{\overline{\mathrm{v}}e}\Big)+O(|\lambda|^3). \end{aligned} \tag{3.28}$$

We have

$$\begin{aligned} \mathbf{1}_{\Theta_e}(L) = \;& \mathbf{1}_e(L_0) \\ & +\lambda(2\pi\mathrm{i})^{-1}\oint_\gamma\Big((z\mathbf{1}^{\mathrm{vv}}-E)^{-1}Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} \\ & \qquad +(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}\Big)dz \\ & +\lambda^2(2\pi\mathrm{i})^{-1}\oint_\gamma\Big((z\mathbf{1}^{\mathrm{vv}}-E)^{-1}Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}(z\mathbf{1}^{\mathrm{vv}}-E)^{-1} \\ & \qquad +(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\Big)dz \\ & +O(|\lambda|^3+|\lambda\beta|). \end{aligned}$$

e is the only one eigenvalue of E inside γ so $(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}$ has only one pole inside γ. All points on and inside γ are not in $\mathrm{sp}(L_0^{\overline{e}\overline{e}})$ so $(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}$ is analytic inside and continuous on γ. If e is semisimple then $(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}=(z-e)^{-1}\mathbf{1}^{ee}+\mathit{analytic\ part}$ and hence

$$\begin{aligned} \mathbf{1}_{\Theta_e}(L) = \;& \mathbf{1}_e(L_0) \\ & +\lambda\Big(Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}+(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}e}\Big) \\ & +\lambda^2\Big((2\pi\mathrm{i})^{-1}\oint_\gamma(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}}(z\mathbf{1}^{\mathrm{vv}}-E)^{-1}dz \\ & \qquad +(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}e}Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}-L_0^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}\Big) \\ & +O(|\lambda|^3+|\lambda\beta|). \end{aligned} \tag{3.29}$$

Now part *(b)* (e.q (2.11)) is a simple consequence of (3.29) and (3.28). In general, when e is not semisimple, terms of order $O(|\lambda|)$ will not cancel so part *(a)* (e.q (2.10)) cannot be improved.

If $\mathrm{sp}(E) = \{e\}$ and e is semisimple then $(z\mathbf{1}^{\mathrm{vv}} - E)^{-1} = (z-e)^{-1}\mathbf{1}^{ee} = (z-e)^{-1}\mathbf{1}^{\mathrm{vv}}$. Now

$$(2\pi\mathrm{i})^{-1}\oint_\gamma (z\mathbf{1}^{\mathrm{vv}} - E)^{-1} Q^{\mathrm{v}\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}(z\mathbf{1}^{\mathrm{vv}} - E)^{-1} dz = -Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}e}$$

Now part *(c)* (e.q. (2.12)) is a simple consequence of (3.29) and (3.28).

7) The proof of 6) in the cases (a), (b) and (c) shows actually slightly improved results:

$$(\mathbf{1}_{\Theta_e}(L) - p_e)L = O(|\lambda|),\ \ O(|\lambda|^2 + |\lambda\beta|) \text{ and } \ O(|\lambda|^3 + |\lambda\beta|);$$

$$L(\mathbf{1}_{\Theta_e}(L) - p_e) = O(|\lambda|),\ \ O(|\lambda|^2 + |\lambda\beta|) \text{ and } \ O(|\lambda|^3 + |\lambda\beta|).$$

To obtain 7) we use

$$\begin{aligned} L - p_e L p_e - (\mathbf{1} - p_e)L(\mathbf{1} - p_e) &= -[p_e, [p_e, L]] \\ &= -[p_e, [p_e - \mathbf{1}_e(L), L]]. \end{aligned}$$

□

Proof of the Theorem 2.3

1) Let $e \in \mathrm{sp}(E)$. Let $\mathcal{E} = \{e\}$, $\Lambda_e = \Lambda_{\mathcal{E}}$ and r be the same as in the Lemma 3.3 and in the proof of 5) of the previous theorem. For $|\lambda| < \Lambda_e$, $|\beta| < \beta_0$ and for $z \in \Omega\backslash \mathrm{D}(\mathrm{sp}(E^{\overline{ee}}), r)$ we can use Theorem 3.2 for the operator L and for decomposition $\mathcal{H} := \mathcal{H}^e \oplus \mathcal{H}^{\overline{e}}$. If $z \in \mathrm{sp}(L) \cap \Omega\backslash \mathrm{D}(\mathrm{sp}(E^{\overline{ee}}), r)$ then

$$0 \in \mathrm{sp}(z\mathbf{1}^{ee} - E^{ee} - \lambda^2 Q^{e\overline{e}}(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{e}e}). \tag{3.30}$$

Note that $e \in \mathrm{sp}(E)$ is semisimple so $E^{ee} = e\mathbf{1}^{ee}$ and moreover, we have $Q^{\overline{e}e} = Q^{\overline{\mathrm{v}}e}$ and $Q^{e\overline{e}} = Q^{e\overline{\mathrm{v}}}$. Now (3.30) can be written as

$$\frac{z-e}{\lambda^2} \in \mathrm{sp}(Q^{e\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{\mathrm{v}}e}). \tag{3.31}$$

Note that

$$\begin{aligned} (z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} = \ & (e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} + (e-z)(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}(e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} \\ = \ & (e\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} \\ & + (e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}\Big(\lambda(Q^{\underline{e}\overline{\mathrm{v}}} + Q^{\overline{\mathrm{v}}\underline{e}}) + \beta Q^{\overline{\mathrm{vv}}}\Big)(e\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} \\ & + (e-z)(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}(e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}, \end{aligned} \tag{3.32}$$

and $(e\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} = (e\mathbf{1}^{\underline{ee}} - E^{\underline{ee}})^{-1} + (e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1}$. Now we can write

$$Q^{e\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{\mathrm{v}}e} = \Gamma^{ee} + \mathbf{I} + \mathbf{II} + \mathbf{III}, \tag{3.33}$$

where

$$\begin{aligned} \mathbf{I} = \ & \beta Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{\mathrm{vv}}}(e\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} Q^{\overline{\mathrm{v}}e}, \\ \mathbf{II} = \ & \lambda Q^{e\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}(Q^{\overline{\mathrm{v}}\underline{e}} + Q^{\underline{e}\overline{\mathrm{v}}})(e\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} Q^{\overline{\mathrm{v}}e}, \\ \mathbf{III} = \ & (e-z) Q^{e\overline{\mathrm{v}}}(z\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}(e\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{\mathrm{v}}e}. \end{aligned} \tag{3.34}$$

Clearly, $\|\mathbf{I}\| \le C_{\mathrm{I}}|\beta|$ and $\|\mathbf{II}\| \le C_{\mathrm{II}}|\lambda|$. Moreover, Theorem 2.1 implies that $|z-e| < C\lambda^2$ and hence by the Lemma 3.3 (eq. (3.26)) we get $\|\mathbf{III}\| \le C_{\mathrm{III}}\lambda^2$. So for $|\lambda| < \Lambda_e$ and $|\beta| < \beta_0$ we have

$$\|\mathbf{I}+\mathbf{II}+\mathbf{III}\| < C_e(|\lambda|+|\beta|).$$

for some $C_e > 0$. Now we can apply the Theorem 3.1 to the expression (3.31) and get for $|\lambda| < \Lambda_e$ and $|\beta| < \beta_0$

$$\frac{z-e}{\lambda^2} \in \mathrm{D}(\mathrm{sp}(\Gamma^{ee}), \mu_{\Gamma^{ee}}(C_e(|\lambda|+|\beta|))). \tag{3.35}$$

where functions $\mu_{\Gamma^{ee}} : [0,\infty[\to [0,\infty]$ are continuous, increasing and $\lim_{x\to 0}\mu_{\Gamma^{ee}}(x) = 0$. This implies

$$\mathrm{sp}(L)\cap\Omega\backslash\mathrm{D}(\mathrm{sp}(E^{\overline{ee}}), r) \subset \mathrm{D}(e+\lambda^2\mathrm{sp}(\Gamma^{ee}), \lambda^2\mu_{\Gamma^{ee}}(C_e(|\lambda|+|\beta|))),$$

and hence for $|\lambda| < \lambda_0 := \min_{e\in\mathrm{sp}(E)} \Lambda_e$ and $|\beta| < \beta_0$ we have

$$\mathrm{sp}(L)\cap\Omega \subset \bigcup_{e\in\mathrm{sp}(E)} \mathrm{D}(e+\lambda^2\mathrm{sp}(\Gamma^{ee}), \lambda^2\delta(|\lambda|+|\beta|)),$$

where we denoted $\delta(x) := \max_{e\in\mathrm{sp}(E)}\Big(\mu_{\Gamma^{ee}}(C_e x)\Big)$.

2) a simple consequence of 1)

3) Let γ be a closed path such that γ encircles $\mathcal{M}$ but no other parts of $\mathrm{sp}(\Gamma^{ee})$. By 1) and 2), for small enough λ and β, the translated and rescaled path $e+\lambda^2\gamma$ encircles only $\Theta_{e,\mathcal{M}}$ but no other parts of $\mathrm{sp}(L)$. Now

$$\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) = (2\pi\mathrm{i})^{-1}\oint_{e+\lambda^2\gamma}\frac{1}{\eta\mathbf{1}-L}\,d\eta,$$

For all $\eta\in e+\lambda^2\gamma$ we can use Feshbach formula for the operator L and for the decomposition $\mathcal{H} = \mathcal{H}^e\oplus\mathcal{H}^{\overline{e}}$. We get

$$\begin{aligned}\oint_{e+\lambda^2\gamma}(\eta\mathbf{1}-L)^{-1}\,d\eta = \;& \oint_{e+\lambda^2\gamma}\big(\mathbf{1}^{ee}+(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\lambda Q^{\overline{e}e}\big)G_e^{-1}(\eta)\\ &\times\big(\mathbf{1}^{ee}+\lambda Q^{e\overline{e}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\big)\,d\eta.\end{aligned}$$

where

$$G_e(\eta) = \eta\mathbf{1}^{ee} - e\mathbf{1}^{ee} - \lambda^2 Q^{e\overline{e}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}Q^{\overline{e}e}.$$

Note that

$$\begin{aligned}&((\eta-e)\mathbf{1}^{ee}-\lambda^2\Gamma^{ee})^{-1} - G_e^{-1}(\eta)\\ &= \lambda^2((\eta-e)\mathbf{1}^{ee}-\lambda^2\Gamma^{ee})^{-1}Q^{e\overline{v}}\Big((e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}-\mathbf{1}^{\overline{vv}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\mathbf{1}^{\overline{vv}}\Big)Q^{\overline{v}e}G_e^{-1}(\eta)\end{aligned} \tag{3.36}$$

where we used $Q^{\overline{v}e} = Q^{\overline{e}e}$ and $Q^{e\overline{v}} = Q^{e\overline{e}}$. Moreover,

$$\begin{aligned}(e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1} - \mathbf{1}^{\overline{vv}}(\eta\mathbf{1}^{\overline{ee}}-L^{\overline{ee}})^{-1}\mathbf{1}^{\overline{vv}} &= (e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1} - (\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}\\ &\quad -\lambda^2(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}Q^{\overline{v}\underline{e}}G_{\underline{e}}^{-1}(\eta)Q^{\underline{e}\overline{v}}(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}\\ &= (e\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}})^{-1}\Big((\eta-e)\mathbf{1}^{\overline{vv}}+\beta Q^{\overline{vv}}\Big)(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}\\ &\quad -\lambda^2(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}Q^{\overline{v}\underline{e}}G_{\underline{e}}^{-1}(\eta)Q^{\underline{e}\overline{v}}(\eta\mathbf{1}^{\overline{vv}}-L_0^{\overline{vv}}-\beta Q^{\overline{vv}})^{-1}\end{aligned} \tag{3.37}$$

where

$$G_{\underline{e}}(\eta) = \eta \mathbf{1}^{\underline{ee}} - E^{\underline{ee}} - \lambda^2 Q^{\underline{e}\overline{\mathrm{v}}}(\eta \mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\underline{e}}.$$

If we change the variable $\eta = e + \lambda^2 z$ and use the equations (3.36) and (3.37) we get

$$\begin{aligned} \oint_{e+\lambda^2\gamma} (\eta\mathbf{1} - L)^{-1}\, d\eta = \ & \oint_\gamma \Big(\mathbf{1}^{ee} + ((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}\lambda Q^{\overline{e}e}\Big)\big(z\mathbf{1}^{ee} - \Gamma^{ee}\big)^{-1} \\ & \times \Big(\mathbf{1}^{ee} + \lambda Q^{e\overline{e}}((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1}\Big)\, dz \\ & + O(|\lambda|^2 + |\beta|). \end{aligned} \tag{3.38}$$

Since

$$(2\pi \mathrm{i})^{-1} \oint_\gamma (z\mathbf{1}^{ee} - \Gamma^{ee})^{-1}\, dz = \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}).$$

we get $\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) - \mathbf{1}_{\mathcal{M}}(\Gamma^{ee}) = O(|\lambda| + |\beta|)$

4) Equation (2.17) implies that for $|\lambda|$ and $|\beta|$ sufficiently small we have $\|\mathbf{1}_{\Theta_{e,\mathcal{M}}}(L) - \mathbf{1}_{\mathcal{M}}(\Gamma^{ee})\| < 1$ so by a well known theorem [Kato] equality (2.18) holds. But $\dim \mathbf{1}_{\Theta_{\mathcal{E}}}(L) \in \mathbb{N}$ is a continuous function of λ and β so (2.18) holds for all $|\lambda| < \lambda_{e,\mathcal{M}}$ and $|\beta| < \beta_{e,\mathcal{M}}$

5) If the degree of nilpotence of m as an eigenvalue of Γ^{ee} is n then due to the Theorem 3.1 equation (3.35) can be written as

$$\frac{z-e}{\lambda^2} \in \mathrm{D}(m, C'_e(|\lambda| + |\beta|)^{1/n}) \cup \mathrm{D}(\mathrm{sp}(\Gamma^{ee}) \backslash \{m\}, \mu_{\Gamma^{ee}}(C_e(|\lambda| + |\beta|))).$$

6) For $|\lambda| < \widehat{\lambda_{e,m}}$

$$\mathbf{1}_m(\Gamma^{ee}) + \lambda^2 \mathbf{1}_m(\Gamma^{ee}) Q^{\mathrm{v}\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}\mathrm{v}} \mathbf{1}_m(\Gamma^{ee})$$

is an invertible operator so the expression for $p_{e,m}$ makes sense. Direct computations show that $p_{e,m}^2 = p_{e,m}$.

In the proof of the part 3) we showed that $\mathbf{1}_{\Theta_{e,m}}(L) = \mathbf{1}_m(\Gamma^{ee}) + O(|\lambda| + |\beta|)$ so (a) is already done. To show (b), we use the approximation for $\mathbf{1}_{\Theta_{e,m}}(L)$ given by (3.38):

$$\begin{aligned} \mathbf{1}_{\Theta_{e,m}}(L) \ & \overset{O(|\lambda^2| + |\beta|)}{=} \ \tfrac{1}{2\pi \mathrm{i}} \oint_\gamma \Big(\lambda((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} Q^{\overline{e}e} + \mathbf{1}^{ee}\Big)(z\mathbf{1}^{ee} - \Gamma^{ee})^{-1} \\ & \qquad \times \Big(\lambda Q^{\overline{e}e}((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} + \mathbf{1}^{ee}\Big)\, \mathrm{d}z \\ & \overset{O(|\lambda^2| + |\lambda\beta|)}{=} \ \tfrac{1}{2\pi \mathrm{i}} \oint_\gamma \Big(\lambda(e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} Q^{\overline{e}e} + \mathbf{1}^{ee}\Big)(z\mathbf{1}^{ee} - \Gamma^{ee})^{-1} \\ & \qquad \times \Big(\lambda Q^{e\overline{e}}((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} + \mathbf{1}^{ee}\Big)\, \mathrm{d}z \\ & = \ \Big(\lambda((e+\lambda^2 m)\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} Q^{\overline{e}e} + 1^{ee}\Big)\mathbf{1}_m(\Gamma^{ee}) \\ & \qquad \times \Big(\lambda Q^{e\overline{e}}((e+\lambda^2 m)\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} + \mathbf{1}^{ee}\Big) \\ & \overset{O(|\lambda|^3)}{=} \ p_{e,m}. \end{aligned}$$

where we used

$$((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L_0^{\overline{ee}})^{-1} - ((e+\lambda^2 z)\mathbf{1}^{\overline{ee}} - L^{\overline{ee}})^{-1} = O(|\lambda| + |\beta|).$$

7) Let us denote $\mathbf{1}^{mm} := \mathbf{1}_m(\Gamma^{ee})$, $Q^{m\overline{\mathrm{v}}} := \mathbf{1}^{mm} Q^{\mathrm{v}\overline{\mathrm{v}}}$ and $Q^{\overline{\mathrm{v}}m} := Q^{\overline{\mathrm{v}}\mathrm{v}}\mathbf{1}^{mm}$, so that

$$\begin{aligned} p_{e,m} = \quad & \Big(\mathbf{1}^{mm} + \lambda(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m}\Big)\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}m}\Big)^{-1} \\ & \times\Big(\mathbf{1}^{mm} + \lambda Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1}\Big). \end{aligned}$$

We compute:

$$\begin{aligned} p_{e,m}(L-e) = \quad & \Big(\mathbf{1}^{mm} + \lambda(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m}\Big)\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}m}\Big)^{-1} \\ & \times\Big(\lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}} + \lambda\beta Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{vv}}}\Big) \\ & = O(|\lambda|^2 + |\lambda\beta|), \end{aligned}$$

$$\begin{aligned} p_{e,m} L p_{e,m} - e p_{e,m} = \quad & \Big(\mathbf{1}^{mm} + \lambda(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m}\Big)\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}m}\Big)^{-1} \\ & \times\Big(\lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m} \\ & + \lambda^2\beta Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{vv}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m}\Big) \\ & \times\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-2} Q^{\overline{\mathrm{v}}m}\Big)^{-1}\Big(\mathbf{1}^{mm} + \lambda Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L_0^{\overline{\mathrm{vv}}})^{-1}\Big) \\ & = O(|\lambda|^2 + |\lambda\beta|). \end{aligned}$$

Thus

$$p_{e,m}(L-e)(1-p_{e,m}) = O(|\lambda|^2 + |\lambda\beta|).$$

Similarly,

$$(1-p_{e,m})(L-e)p_{e,m} = O(|\lambda|^2 + |\lambda\beta|).$$

Finally, we use

$$\begin{aligned} & L - p_{e,m} L p_{e,m} - (\mathbf{1} - p_{e,m}) L (\mathbf{1} - p_{e,m}) \\ & = p_{e,m}(L-e)(1-p_{e,m}) + (1-p_{e,m})(L-e)p_{e,m}. \end{aligned}$$

This proves (a).

Assume now that $\mathrm{sp}(E) = \{e\}$. Then

$$\begin{aligned} Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}} \quad & = Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}e} \\ & = Q^{m\overline{\mathrm{v}}}(e\mathbf{1}^{\overline{\mathrm{vv}}} - L^{\overline{\mathrm{vv}}})^{-1} Q^{\overline{\mathrm{v}}m}. \end{aligned}$$

(The first identity follows from $\mathrm{sp}(E) = \{e\}$, the second is a consequence of the definition of $\mathbf{1}^{mm}$). Using this we get

$$\begin{aligned} p_{e,m}(L-e) = \; & \Big(\mathbf{1}^{mm} + \lambda(e\mathbf{1}^{\overline{v}\overline{v}} - L_0^{\overline{v}\overline{v}})^{-1}Q^{\overline{v}m}\Big)\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{v}\overline{v}} - L_0^{\overline{v}\overline{v}})^{-2}Q^{\overline{v}m}\Big)^{-1} \\ & \times\lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{v}\overline{v}} - L^{\overline{v}\overline{v}})^{-1}Q^{\overline{v}m} \\ & \times\Big(\mathbf{1}^{mm} + \lambda^2 Q^{m\overline{v}}(e\mathbf{1}^{\overline{v}\overline{v}} - L_0^{\overline{v}\overline{v}})^{-2}Q^{\overline{v}m}\Big)^{-1}\Big(\mathbf{1}^{mm} + \lambda Q^{m\overline{v}}(e\mathbf{1}^{\overline{v}\overline{v}} - L_0^{\overline{v}\overline{v}})^{-1}\Big) \\ & +O(|\lambda|^3). \end{aligned}$$

This proves (b). $\Box$

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