DENSITY ESTIMATES FOR A DEGENERATE/SINGULAR PHASE-TRANSITION MODEL

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Abstract. We consider a Ginzburg-Landau type phase-transition model driven by a p-Laplacian type equation. We prove density estimates for absolute minimizers and we deduce the uniform convergence of level sets and the existence of plane-like minimizers in periodic media.

Introduction

Let $1 < p < \infty$ and $\mathcal{X} = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}) : \nabla u \in L^p_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)\}$. For a domain E in $\mathbb{R}^n$, possibly unbounded, we define the functional $\mathcal{J}_E : \mathcal{X} \longrightarrow \mathbb{R}$ as

$$\mathcal{J}_E(u) = \int_E [A(x, \nabla u) + F(x, u)]\, dx, \tag{0.1}$$

where $A(x, u) \simeq |\nabla u|^p$ and $F(x, u) \geq 0$ is a "double-well potential" vanishing at $u = \pm 1$ (see below for precise assumptions). Functionals of this type appear in the context of minimal surfaces and it has been shown by Γ-convergence methods that sequences of minimizers converge in L^1 to suitable step functions satisfying a minimal interface property (for details, see [MM77] for $p = 2$ and [Bou90] for the general case). Functionals of type (0.1) have also physical relevance, since they appear in the study of the equilibrium of elastic rods under tension (see [Ant73]), in the context of fluid jets (see [AC81] and [ACF84]) and in the van der Waals-Cahn-Hilliard and Ginzburg-Landau theories of phase transition (see, for instance, [Row79]).

In the phase-transition setting, the term $A(x, \nabla u)$ in the energy functional (0.1) can be seen as an interfacial energy contribution to the total energy, which penalizes the formation of interfaces (see [Gur85] for details).

The main purpose of this paper is to obtain Caffarelli-Cordoba [CC95] type density estimates for the absolute minimizers of $\mathcal{J}$. Roughly speaking, such density estimates state that, if u is an absolute minimizer of $\mathcal{J}$, then the set $\{|u| < 1/2\}$ behaves in measure as an $(n-1)$-dimensional set, while $\{u > 1/2\}$ and $\{u < -1/2\}$ behave in measure as n-dimensional sets (a precise statement will be given in Theorem 0.1 below). As a consequence, we obtain the uniform convergence of the level sets of minimizers to a surface of minimal "area" in the corresponding

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singular-perturbation problem, see Theorem 6.1. Another application of the density estimates is the existence of plane-like minimizers of $\mathcal{J}$ in periodic media, see Theorem 6.3.

We now state in detail the assumptions required in this paper. We assume that $F : \mathbb{R}^n \times \mathbb{R} \ni (x, u) \longrightarrow \mathbb{R}$ is a Carathéodory function, i.e., continuous in u for a.e. x and measurable in x for every u, and satisfies

$$0 \le F(x,u) \le M, \qquad F(x,\pm 1) = 0, \qquad \inf_{|u|\le\theta} F(x,u) \ge \gamma(\theta)\,, \tag{0.2}$$

where $\gamma(\theta) > 0$ for $0 \le \theta < 1$. We assume that $F \in \mathrm{Lip}_u(\mathbb{R}^n \times (-1, 1))$ and

$$F_u(x,-1+s) \ge Cs^{d-1}, \qquad F_u(x,1-s) \le -Cs^{d-1} \tag{0.3}$$

for sufficiently small $s > 0$ and certain $d \le p$ and

$$\sup_{|u|\le\theta} |F_u(x,u)| \le M(\theta) \tag{0.4}$$

for $0 \le \theta < 1$. In the case $d = p$ we also assume

(0.5) F_u is monotone increasing in u near $u = -1$ and decreasing near $u = 1$.

Next, we impose the following structural conditions on $A : \mathbb{R}^n \times \mathbb{R}^n \ni (x, \eta) \longrightarrow \mathbb{R}$. We assume that $A \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ and define

$$a(x, \eta) := D_\eta A(x, \eta).$$

We also assume $a \in C(\mathbb{R}^n \times \mathbb{R}^n) \cap C^1(\mathbb{R}^n \times \mathbb{R}^n - \{0\})$ and require that

$$A(x,0) = 0, \quad a(x,0) = 0, \tag{0.6}$$

and that for some $\Lambda > 0$

$$\begin{aligned}
\zeta \cdot D_\eta a(x,\eta)\,\zeta \quad &\ge \quad \Lambda^{-1}|\zeta|^2\,|\eta|^{p-2}\,, \text{ for any } \zeta \in \mathbb{R}^n && (0.7)\\
|D_\eta a(x,\eta)| \quad &\le \quad \Lambda\,|\eta|^{p-2} \qquad \text{and} && (0.8)\\
|D_x a(x,\eta)| \quad &\le \quad \Lambda\,|\eta|^{p-1}\,, && (0.9)\\
\eta \cdot a(x,\eta) \quad &\ge \quad \Lambda^{-1}\,|\eta|^p\,. && (0.10)
\end{aligned}$$

We will refer to the constants that appear in (0.2)–(0.10), including n and p, as the *structural constants.* A "typical" example of the functional $\mathcal{J}$, which satisfies the assumptions above, is given by

$$\int \Big(a_{i,j}(x)\,\partial_i u\,\partial_j u \Big)^{p/2} + Q(x)\,|1-u^2|^\alpha,$$

where $a_{i,j}$ is a C^1, symmetric positive definite matrix, $0 < Q_{\min} \le Q(x) \le Q_{\max}$ and $0 < \alpha \le p$. (The case $\alpha = 0$, which corresponds to $F(x, u) = Q(x)\chi_{(-1,1)}(u)$, has been treated recently in [PV03].)

Given a bounded domain Ω, we say that u is an *absolute minimizer* for $\mathcal{J}$ in Ω if $\mathcal{J}_\Omega(u) \le \mathcal{J}_\Omega(u + \phi)$, for any $\phi \in C_0^\infty(\Omega)$. We will also denote by $\mathcal{L}^n$ the standard Lebesgue measure on $\mathbb{R}^n$. In this paper, we prove the following density estimates for level sets of absolute minimizers:

Theorem 0.1. i. *For $p > 2n/(n+2)$, assume that the above mentioned hypotheses hold. Fix $\theta \in (0, 1)$. Let $|u| \le 1$ be an absolute minimizer for $\mathcal{J}$ in a domain Ω,*

$x \in \{-\theta < u < \theta\}$ and $y \in \Omega$. Then, for fixed $\delta > 0$, there exist positive r_0, c and C depending only on θ, on the structural constants and on δ such that:

$$\mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \geq c\, r^n \quad \text{and} \quad \mathcal{L}^n\big(B_r(x) \cap \{u < -\theta\}\big) \geq c\, r^n, \tag{0.11}$$

$$\begin{aligned} \mathcal{L}^n\big(B_r(x) \cap \{|u| < \theta\}\big) &\geq c\, r^{n-1} \text{ and} \\ \mathcal{L}^n\big(B_r(y) \cap \{|u| < \theta\}\big) &\leq C\, r^{n-1}, \end{aligned} \tag{0.12}$$

for any $r \geq r_0$, provided $B_{r+\delta}(x)$, $B_{r+\delta}(y) \subset\subset \Omega$.

ii. *For $1 < p \leq 2n/(n+2)$ assume additionally that $F(x, u)$ is uniformly Lipschitz continuous in u. Then the same conclusion as in part* **i** *holds.*

The density estimates of this type have been obtained originally in [CC95] for $p = 2$ and $A(x, \nabla u) = |\nabla u|^2$ and have been generalized in [Val01] to $A(x, \nabla u) = a_{i,j}\partial_i u\, \partial_j u$. The case of a general $p \in (1, \infty)$ and $A(x, \nabla u)$ satisfying the hypotheses above with $F(x, u) = Q(x)\, \chi_{(-1,1)}(u)$ has been considered in [PV03] as a model for non-Newtonian power-law fluid jets. The case treated here can be seen as a degenerate/singular phase-transition model driven by a p-Laplacian type equation.

We stress that obtaining density estimates in the case considered here is somehow harder than in [PV03] due to the *absence* of the free boundary, since, roughly speaking, a free boundary forces the function to grow at least linearly away from it. Also, the term $\chi_{(-1,1)}$ considered in [PV03] made possible to use tools from p-harmonic functions that are not available in the present case.

It is to be noted that under the assumptions that we impose on $\mathcal{J}$ additional difficulties appear in the case $1 < p < 2$. We need to require the double-well potential $F(x, u)$ to be uniformly Lipschitz continuous in u in order to obtain the desired density estimates. However, we show that *at least* for the range of the exponents $2n/(n + 2) < p < 2$, one can drop this Lipschitz continuity assumption, see §5.2.

The paper is organized as follows. In §1 we collect some short-proof-lemmata that will be of use in the sequel. A Caccioppoli-type inequality is stated and proved in §2. The proof of Theorem 0.1 is dealt with in §3 and it makes use of an auxiliary result, namely Lemma 3.1 below, which is interesting in itself and which roughly says that, as soon as the density of sublevels of minimizers is positive in some ball, it must grow as r^n in balls of bigger radius r. We devote §4 and §5 to the proof of such auxiliary result, considering the cases $p \geq 2$ and $1 < p \leq 2$ separately. In §6 we point out some consequences that can be derived from Theorem 0.1, such as uniform convergence of level sets of absolute minima to minimal interfaces and the existence of plane-like minimizers in periodic media.

1. Technical and elementary lemmata

We start this section with a recursive lemma.

Lemma 1.1. *Let $v_k \geq 0$ and $a_k \geq 0$ be two nondecreasing sequences such that $v_1 + a_1 \geq c_0$,*

$$v_k^{(n-1)/n} \leq C_0\, (v_{k+1} + a_{k+1} - v_k - a_k - c_1 a_k)^{1-\alpha}\, k^{\alpha(n-1)}$$

for any $k \in \mathbb{N}$ and certain positive constants c_0, c_1, C_0 and $0 \leq \alpha < 1/n$. Then there exists $\gamma = \gamma(c_0, c_1, C_0, \alpha) > 0$ such that

$$v_k + a_k \geq \gamma\, k^n$$

for any $k \in \mathbb{N}$.

Proof. We start with an observation that it is enough to prove the estimate for $k \geq k_0$, since

$$v_k + a_k \geq v_1 + a_1 \geq c_0 \geq (c_0/k_0^n)k^n, \quad \text{for } k \leq k_0.$$

The proof is by induction. Assume that $v_k + a_k \geq \gamma\, k^n$. Then either $v_k \geq (\gamma/2)k^n$ or $a_k \geq (\gamma/2)k^n$.

1. Assume first $v_k \geq (\gamma/2)k^n$. Then, using the recurrence relationship, we have

$$C_0\,(v_{k+1} + a_{k+1} - v_k - a_k - c_1 a_k)^{1-\alpha} \geq (\gamma/2)^{(n-1)/n}\, k^{(1-\alpha)(n-1)}$$

and consequently

$$v_{k+1} + a_{k+1} \geq \gamma\, k^n + C\gamma^{\frac{1}{1-\alpha}\cdot\frac{n-1}{n}} k^{n-1}.$$

By our assumption $\alpha < 1/n$, which implies that $\frac{1}{1-\alpha}\cdot\frac{n-1}{n} < 1$. Hence, if γ is sufficiently small,

$$v_{k+1} + a_{k+1} \geq \gamma(k^n + C_* k^{n-1})$$

for C_* as large as we wish. On the other hand, if we choose $C_* \geq 2^n$,

$$k^n + C_* k^{n-1} \geq (k+1)^n$$

and we obtain

$$v_{k+1} + a_{k+1} \geq \gamma(k+1)^n.$$

2. Assume now that $a_k \geq (\gamma/2)k^n$. Then

$$v_{k+1} + a_{k+1} \geq v_k + a_k + c_1 a_k \geq \gamma(k^n + (ck)k^{n-1}) \geq \gamma(k+1)^n$$

for sufficiently large k.

The proof is complete. □

The proof of the next lemma is similar in the spirit and can be found on page 10 in [CC95].

Lemma 1.2. *Let* $a_k \geq 0$ *be a sequence such that* $a_1 \geq c_0$,

$$\Big(\sum_{1\leq j\leq k} a_j\Big)^{(n-1)/n} \leq C_0\Big(a_{k+1} + \sum_{1\leq j\leq k} e^{-L(k+1-j)} a_j\Big),$$

for any $k \in \mathbb{N}$ *and some positive* L *and* C_0*. Then, if* L *is suitably large, there exists* $\gamma = \gamma(c_0, C_0)$ *such that*

$$a_k \geq \gamma\, k^{n-1}$$

for any $k \in \mathbb{N}$. □

Next we point out a useful inequality in the case $p \geq 2$.

Lemma 1.3. *For any* $p \geq 2$ *there is a universal constant* $c > 0$ *such that*

$$c\,|\xi' - \xi|^p \leq A(x, \xi') - A(x, \xi) - a(x, \xi)\cdot(\xi' - \xi) \tag{1.1}$$

for any ξ, $\xi' \in \mathbb{R}^n$ *and* $x \in \Omega$.

Proof. Set
$$\xi^s = s\,\xi' + (1-s)\,\xi, \quad 0 \le s \le 1.$$
Then $\xi^0 = \xi$ and $\xi^1 = \xi'$ and we have
$$\begin{aligned}
A(x,\xi') - A(x,\xi) &= \int_0^1 a(x,\xi^s)\cdot(\xi'-\xi)ds \\
&= \int_0^1 (a(x,\xi^s) - a(x,\xi))\cdot(\xi'-\xi)ds + a(x,\xi)\cdot(\xi'-\xi) \\
&= \int_0^1 (a(x,\xi^s) - a(x,\xi))\cdot(\xi^s-\xi)\frac{ds}{s} + a(x,\xi)\cdot(\xi'-\xi)
\end{aligned}$$
Now we apply the inequality (valid for $p \ge 2$)
$$(a(x,\xi^s) - a(x,\xi))\cdot(\xi^s-\xi) \ge \gamma\,|\xi^s-\xi|^p \tag{1.2}$$
with $\gamma = \gamma(n,p) > 0$, see [Tol84]. Then we obtain
$$\begin{aligned}
A(x,\xi') - A(x,\xi) &\ge \gamma \int_0^1 |\xi^s-\xi|^p\frac{ds}{s} + a(x,\xi)\cdot(\xi'-\xi) \\
&= \gamma\,|\xi'-\xi|^p \int_0^1 s^{p-1}ds + a(x,\xi)\cdot(\xi'-\xi) \\
&= c\,|\xi'-\xi|^p + a(x,\xi)\cdot(\xi'-\xi).
\end{aligned}$$
□

The analogue of the preceding Lemma 1.3 for $1 < p \le 2$ is as follows.

Lemma 1.4. *For any $1 < p \le 2$ and $M \ge 0$ there exist a universal constant $c > 0$ such that*
$$c\,M^{p-2}\,|\xi'-\xi|^2 \le A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi'-\xi) \tag{1.3}$$
for any $\xi,\ \xi' \in \mathbb{R}^n$ with $|\xi| + |\xi'| \le M$ and $x \in \Omega$.

Proof. The proof repeats the one for Lemma 1.3, only instead of (1.2) we use
$$(a(x,\xi^s) - a(x,\xi))\cdot(\xi^s-\xi) \ge \gamma\,(|\xi^s| + |\xi|)^{p-2}|\xi'-\xi|^2\,,$$
with $\gamma = \gamma(n,p) > 0$, valid for $1 < p \le 2$; see [Tol84]. Then, we will obtain
$$c\,(|\xi| + |\xi'|)^{p-2}|\xi'-\xi|^2 \le A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi'-\xi),$$
which implies (1.3) if $|\xi| + |\xi'| \le M$. □

The following result is elementary and we omit the proof:

Lemma 1.5. *Let $d \ge 1$. There exists $c_d > 0$ such that*
$$(u+1)^d - (u'+1)^d \ge c_d\,(u-u')^d\,,$$
for any $u \ge u' \ge -1$.

Next, we deduce an estimate on the double-well potential.

Lemma 1.6. *There exists $c > 0$ such that, for any $-1 \le u' \le u \le \theta$,*
$$F(x,u) - F(x,u') \ge c\,(u-u')^d\,,$$
provided $1 + \theta > 0$ is small enough.

We omit the proof of Lemma 1.6, which easily follows from (0.3) and Lemma 1.5. The proof of the next two lemmata is also elementary:

Lemma 1.7. *Let us assume (0.5). Then, there exists $c > 0$ so that, for any $-1 \le u' \le u \le \theta$, $F(x, u) - F(x, u') \ge c\,(u' + 1)^{d-1}\,(u - u')$, provided $1 + \theta > 0$ is small enough.*

Lemma 1.8. *Let us assume that F is uniformly Lipschitz continuous in u. Then, there exists $c > 0$ so that, for any $u \in [-1, 1]$, $F(x, u) \le c(1 + u)$.*

2. A Caccioppoli-type inequality

We now state and prove a weaker version of the Caccioppoli inequality:

Lemma 2.1. *Fix $\delta > 0$. Let $|u| \le 1$ be an absolute minimizer for $\mathcal{J}$ in a domain Ω. Then, there exists $C > 0$, depending only on δ and on the structural constants, such that*

$$\int_{B_r(x_0)} |\nabla u|^p \,\le\, C\,(r + \delta)^n,$$

for any $r > 0$ and any $x_0 \in \Omega$, provided $B_{r+\delta}(x_0) \subset \Omega$.

Proof. We start with the claim that

$$\int_\Omega a(x, \nabla u) \cdot \nabla\phi \,+\, \int_{\Omega\cap\{|u|\neq 1\}} F_u(x, u)\,\phi\,dx \ge\, 0\,, \tag{2.1}$$

for any nonnegative $\phi \in C_0^\infty(\Omega \cap \{u > -1\})$ and

$$\int_\Omega a(x, \nabla u) \cdot \nabla\psi \,+\, \int_{\Omega\cap\{|u|\neq 1\}} F_u(x, u)\,\psi\,dx \le\, 0\,, \tag{2.2}$$

for any nonnegative $\psi \in C_0^\infty(\Omega \cap \{u < 1\})$. Let us show (2.2), the proof of (2.1) being analogous. For ψ as above and a small $\varepsilon > 0$, let

$$\psi_\varepsilon(x) = \psi(x)\chi_\varepsilon(u(x)),$$

where

$$\chi_\varepsilon(u) = \begin{cases} 0, & \text{if } u \le -1 + \varepsilon \\ (u + 1)/\varepsilon - 1 & \text{if } -1 + \varepsilon < u < -1 + 2\varepsilon \\ 1 & \text{if } 1 + u \ge 2\varepsilon. \end{cases}$$

Then $\psi_\varepsilon \in \mathcal{X}$, $\operatorname{supp}\psi_\varepsilon \subset \Omega \cap \{|u| < 1\}$ and therefore

$$\int_\Omega a(x, \nabla u) \cdot \nabla\psi_\varepsilon + F_u(x, u)\,\psi_\varepsilon = 0.$$

On the other hand,

$$\begin{aligned} &\int_\Omega a(x, \nabla u) \cdot \nabla\psi_\varepsilon \\ =\ &\int_\Omega [a(x, \nabla u) \cdot \nabla\psi]\chi_\varepsilon(u) + \frac{1}{\varepsilon}\int_{\Omega\cap\{\varepsilon<u+1<-2\varepsilon\}} [a(x, \nabla u) \cdot \nabla u]\psi \\ \ge\ &\int_\Omega [a(x, \nabla u) \cdot \nabla\psi]\chi_\varepsilon(u) \to \int_\Omega a(x, \nabla u) \cdot \nabla\psi \end{aligned}$$

as $\varepsilon \to 0+$ and

$$\int_\Omega F_u(x, u)\psi_\varepsilon \to \int_{\Omega\cap\{u>-1\}} F_u(x, u)\psi.$$

The passage to the limit is legitimate, since $0 \leq \psi_\varepsilon \leq \psi$,

$$\int_\Omega |a(x, \nabla u) \cdot \nabla \psi| < \infty,$$

and $F_u(x, u) \geq 0$ by (0.3) for u close to -1. Collecting the estimates above, we obtain (2.2).

Fix now $0 < \theta < 1$. If θ is sufficiently close to 1, the assumptions (0.3)–(0.4) and (2.1)–(2.2) above will imply that

$$\int_\Omega a(x, \nabla u) \cdot \nabla \phi + K\, \phi \geq 0 \quad \text{and} \quad \int_\Omega a(x, \nabla u) \cdot \nabla \psi - K\, \psi \leq 0, \tag{2.3}$$

for any nonnegative $\phi \in C_0^\infty(\Omega \cap \{u > -\theta\})$ and $\psi \in C_0^\infty(\Omega \cap \{u < \theta\})$ with $K = M(\theta)$ as in (0.4).

Next, we observe that in light of [GG82], the distance of the level sets $\{u = -\theta\}$ and $\{u = \theta\}$ in $B_{r+\delta/2}(x_0)$ is bounded from below by some universal constant (depending only on θ, δ and the structural constants). Therefore, by partition of unity, there exist two smooth functions η_- and η_+, supported in $B_{r+\delta/2}(x_0)$, so that $0 \leq \eta_-(x), \eta_+(x) \leq 1$ for any $x \in \Omega$, whose gradients are uniformly bounded by a universal constant, and which satisfy:

$$\begin{gathered}
\eta_-(x) + \eta_+(x) \;=\; 1\,, \;\; \forall x \in B_r(x_0)\,, \\
\operatorname{supp} \eta_- \subseteq \{-1 \leq u < \theta\}\,, \\
\operatorname{supp} \eta_+ \subseteq \{-\theta < u \leq 1\}\,, \\
\eta_-(x) + \eta_+(x) \leq 1\,, \;\forall x \in \Omega\,.
\end{gathered}$$

We set $\phi := (1-u)\, \eta_+^p$ and $\psi := (1+u)\, \eta_-^p$. By (2.3), our assumptions on $a(x, \nabla u)$ and (a suitably scaled) Young inequality, repeating the standard arguments in the proof of the Caccioppoli inequality (e.g., see Lemma 3.27 in [HKM93]), one infers that

$$\int_\Omega |\nabla u|^p \eta_-^p \;\leq\; C\,(r+\delta)^n \qquad \text{and} \qquad \int_\Omega |\nabla u|^p \eta_+^p \;\leq\; C\,(r+\delta)^n.$$

Therefore,

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u|^p \;&=\; \int_{B_r(x_0)} |\nabla u|^p (\eta_- + \eta_+)^p \\
\leq\;\; C \int_{B_r(x_0)} |\nabla u|^p \eta_-^p &+ C \int_{B_r(x_0)} |\nabla u|^p \eta_+^p \;\leq\; C\,(r+\delta)^n\,.
\end{aligned}$$

□

3. Proof of Theorem 0.1

First of all, we point out that, since u is an absolute minimizer,

$$\mathcal{J}_{B_r(y)}(u) \leq C\, r^{n-1}\,, \tag{3.1}$$

for any $r \geq r_0$, for a suitable universal r_0, provided $B_{r+\delta}(y) \subset \Omega$. To prove (3.1), with no loss of generality assume $y = 0$ and proceed as follows. Let h be a smooth function so that $h\big|_{B_{r-1}} = -1$ and $h\big|_{\partial B_r} = 1$. Let $u^* = \min\{u, h\}$. Then,

$$\begin{aligned}
\mathcal{J}_{B_r}(u) &\leq \mathcal{J}_{B_r}(u^*) \leq \\
&\leq C \int_{B_r - B_{r-1}} (|\nabla u|^p + |\nabla h|^p) + r^{n-1} \leq \\
&\leq C \int_{B_r - B_{r-1}} |\nabla u|^p + r^{n-1}.
\end{aligned}$$

Covering $B_r - B_{r-1}$ with balls of radius $\delta/2$, $\mathcal{B}_1, \dots, \mathcal{B}_N$, with $N \leq C\, r^{n-1}$ and applying Lemma 2.1, we obtain

$$\int_{B_r - B_{r-1}} |\nabla u|^p \leq \sum_{i=1}^{N} \int_{\mathcal{B}_i} |\nabla u|^p \leq C\, r^{n-1}\,.$$

This completes the proof of (3.1).

We now focus our attention on the <ins>proof of (0.11)</ins>. We will deal only with the first inequality in (0.11), the proof of the second one being analogous. To this end, we state the following result, the proof of which is deferred to §4:

Lemma 3.1. *Let us assume the same hypotheses on A and F as in Theorem 0.1. Fix $\theta \in (-1, 1)$ and let u be an absolute minimizer for $\mathcal{J}$ in a domain Ω. Assume that there exist $\mu_1, \mu_2 > 0$ so that $B_{\mu_1}(x) \subset \Omega$ and $\mathcal{L}^n\big(B_{\mu_1}(x) \cap \{u > \theta\}\big) \geq \mu_2$. Then, for fixed $\delta > 0$, there exist positive r_0 and c depending only on θ, μ_1, μ_2, δ and on the structural constants, such that $\mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \geq c\, r^n$, for any $r \geq r_0$ provided $B_{r+\delta}(x) \subset\subset \Omega$.*

Analogously, if $\mathcal{L}^n\big(B_{\mu_1}(x) \cap \{u < \theta\}\big) \geq \mu_2$, then $\mathcal{L}^n\big(B_r(x) \cap \{u < \theta\}\big) \geq c\, r^n$, for any $r \geq r_0$ provided $B_{r+\delta}(x) \subset\subset \Omega$.

We now use the above result to prove (0.11). Let $\theta^\star = (1 + \theta)/2 \in (\theta, 1)$. Since u is uniformly continuous (see [GG82]) and $|u(x)| < \theta$, we have that $|u(x')| < \theta^\star$ for any $x' \in B_{\mu^\star}(x)$, for a suitable universal $\mu^\star > 0$. Hence, in view of Lemma 3.1, $\mathcal{L}^n\big(B_r(x) \cap \{u > -\theta^\star\}\big) \geq c\, r^n$ and $\mathcal{L}^n\big(B_r(x) \cap \{u < \theta^\star\}\big) \geq c\, r^n$. Therefore, by (0.2) and (3.1),

$$\begin{aligned}
& c\, r^n - \mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \\
\leq\;& \mathcal{L}^n\big(B_r(x) \cap \{u > -\theta^\star\}\big) - \mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \\
\leq\;& \mathcal{L}^n\big(B_r(x) \cap \{-\theta^\star < u \leq \theta\}\big) \\
\leq\;& C \int_{\{-\theta^\star < u \leq \theta\} \cap B_r(x)} F \\
\leq\;& C\, \mathcal{J}_{B_r(x)}(u) \;\leq\; C\, r^{n-1}\,.
\end{aligned}$$

Hence, if r is suitably large, $\mathcal{L}^n\big(B_r(x) \cap \{u > \theta\}\big) \geq c\, r^n$, thus proving (0.11).

We now deal with the <ins>proof of (0.12)</ins>. The second inequality follows from (3.1), hence we focus on the proof of the first one. Let

$$u^\star(x) = \begin{cases} u(x) & \text{if} \quad |u(x)| < \theta\,, \\ \theta & \text{if} \quad u(x) \geq \theta\,, \\ -\theta & \text{if} \quad u(x) \leq -\theta\,, \end{cases} \;.$$

Using a standard notation in geometric measure theory, we denote by $\mathcal{P}(E,U)$ the perimeter of the Borel set E in an open set U. Then, using the coarea formula, the isoperimetric inequality and (0.11), we have:

$$\begin{aligned}
& \int_{B_r(x)} |\nabla u^\star| \\
\geq\ & \int_{-\theta}^{\theta} \mathcal{P}\Big(\{u^\star < s\},\, B_r(x) \Big)\, ds \\
\geq\ & c \int_{-\theta}^{\theta} \min \Big\{ \mathcal{L}^n \left(B_r(x) \cap \{u^\star < s\}\right) ,\, \mathcal{L}^n \left(B_r(x) \cap \{u^\star \geq s\}\right) \Big\}^{\frac{n-1}{n}} ds \\
=\ & c \int_{-\theta}^{\theta} \min \Big\{ \mathcal{L}^n \left(B_r(x) \cap \{u < s\}\right) ,\, \mathcal{L}^n \left(B_r(x) \cap \{u \geq s\}\right) \Big\}^{\frac{n-1}{n}} ds \\
\geq\ & c \int_{-\theta}^{\theta} \min \Big\{ \mathcal{L}^n \left(B_r(x) \cap \{u < -\theta\}\right) ,\, \mathcal{L}^n \left(B_r(x) \cap \{u \geq \theta\}\right) \Big\}^{\frac{n-1}{n}} ds \\
\geq\ & c\, r^{n-1} .
\end{aligned}$$

Let us now fix a suitably large parameter $K > 0$. In view of the above estimate, denoting by p' the conjugated exponent of p, using Young Inequality and (3.1), we deduce that

$$\begin{aligned}
c\, r^{n-1} &\leq \frac{1}{K^p} \int_{B_r(x)} |\nabla u|^p + K^{p'} \mathcal{L}^n \left(B_r(x) \cap \{|u| < \theta\}\right) \\
&\leq \frac{C}{K^p} \mathcal{J}_{B_r(x)}(u) \,+\, K^{p'} \mathcal{L}^n \left(B_r(x) \cap \{|u| < \theta\}\right) \\
&\leq \frac{C}{K^p} r^{n-1} \,+\, K^{p'} \mathcal{L}^n \left(B_r(x) \cap \{|u| < \theta\}\right) .
\end{aligned}$$

Then, (0.12) follows by choosing K large enough here above.

4. Proof of Lemma 3.1. The case $p \geq 2$

We will prove the first claim in Lemma 3.1, the second claim being analogous. We point out that it is enough to show the validity of the first claim of Lemma 3.1 for θ as close to -1 as we wish. Indeed, let us assume that the claim holds true for $\theta_\star$ and $-1 < \theta_\star < \theta < 1$. Then, if $\mathcal{L}^n \left(B_{\mu_1}(x) \cap \{u > \theta\}\right) \geq \mu_2$, then of course $\mathcal{L}^n \left(B_{\mu_1}(x) \cap \{u > \theta_\star\}\right) \geq \mu_2$, and so, since the claim holds for $\theta_\star$, $\mathcal{L}^n \left(B_r(x) \cap \{u > \theta_\star\}\right) \geq c\, r^n$. Thus, using (0.12),

$$\begin{aligned}
& \mathcal{L}^n \left(B_r(x) \cap \{u > \theta\}\right) \\
\geq\ & \mathcal{L}^n \left(B_r(x) \cap \{u > \theta_\star\}\right) \,-\, \mathcal{L}^n \left(B_r(x) \cap \{\theta_\star \leq u < \theta\}\right) \\
\geq\ & c\, r^n \,-\, C\, r^{n-1} \,\geq\, \tilde{c}\, r^n ,
\end{aligned}$$

if r is sufficiently big. This shows that we only need to prove the first claim of Lemma 3.1 for θ close to -1. Thus, we may assume that (0.3) is satisfied for $0 < s \leq \theta + 1$.

In this section we will assume $p \geq 2$. We will distinguish the case $d < p$ and $d = p$, where d is the exponent that appears in (0.3).

4.1. **The case $d<p$.** For θ close to -1 let

$$\mathcal{V}_r=\mathcal{L}^n\Big(\{u\geq\theta\}\cap B_r\Big),\qquad \mathcal{A}_r=\int_{B_r}F(x,u)dx\,. \tag{4.1}$$

Then, we claim that

$$\mathcal{V}_r^{(n-1)/n}+\mathcal{A}_r\leq C_0\left(\mathcal{V}_{r+1}+\mathcal{A}_{r+1}-\mathcal{V}_r-\mathcal{A}_r\right). \tag{4.2}$$

With no loss of generality, we can take $r_0\geq\mu_1$ and $\delta\geq 2$. Thus, by assumption, $\mathcal{V}_{r_0}\geq\mu_2>0$. Therefore, by means of Lemma 1.1, the above inequality implies that

$$\mathcal{V}_r\geq c\,r^n,$$

for $r\geq 1$.

We now prove (4.2). We use a barrier function $h\in C^2(B_{r+1})$ such that

$$h\big|_{\partial B_{r+1}}=1,\qquad h\big|_{B_r}=-1.$$

Let $\varepsilon=1+\theta$ and define $u^*=\min(u,h)$ and $\beta=\min(u-u^*,\varepsilon)$. Using the Sobolev inequality applied to β^p and then the Young inequality, we have:

$$\begin{aligned}
&\Big(\int_{B_{r+1}}|\beta|^{pn/(n-1)}\Big)^{(n-1)/n}\\
\leq\;& C\int_{B_{r+1}\cap\{u-u^*<\varepsilon\}}|\beta|^{p-1}|\nabla\beta|\\
\leq\;& C\,K^p\int_{B_{r+1}\cap\{u-u^*<\varepsilon\}}|\nabla(u-u^*)|^p\\
&+\frac{C}{K^{p'}}\int_{B_{r+1}\cap\{u-u^*<\varepsilon\}}(u-u^*)^p\,.
\end{aligned} \tag{4.3}$$

Here above, $K>0$ is a free parameter that will be conveniently chosen in what follows. As customary, we also denoted the conjugated exponent of p by p'. Since $u^*=-1$ in B_r, $u-u^*\geq\varepsilon$ in $B_r\cap\{u\geq\theta\}$, the left hand side of the inequality above is estimated from below by

$$c\,\mathcal{L}^n(\{u\geq\theta\}\cap B_r)^{(n-1)/n}=c\,\mathcal{V}_r^{(n-1)/n}.$$

Next, we apply (1.1) with $\xi=\nabla u^*$ and $\xi'=\nabla u$ to estimate $|\nabla(u-u^*)|^p$ in the right hand side of (4.3). Thus, we obtain

$$\begin{aligned}
\mathcal{V}_r^{(n-1)/n}\;\leq\;& C\,K^p\int_{B_{r+1}}A(x,\nabla u)-A(x,\nabla u^*)\\
+\;& C\,K^p\int_{B_{r+1}}a(x,\nabla u^*)\cdot\nabla(u-u^*)\\
+\;& \frac{C}{K^{p'}}\int_{B_{r+1}\cap\{u-u^*<\varepsilon\}}(u-u^*)^p.
\end{aligned} \tag{4.4}$$

Since $\operatorname{supp}(u-u^*)\subset B_{r+1}\subset\subset\Omega$, the minimality of u implies that $\mathcal{J}(u)\leq\mathcal{J}(u^*)$ or, equivalently,

$$\int_{B_{r+1}}A(x,\nabla u)-A(x,\nabla u^*)\leq\int_{B_{r+1}}F(x,u^*)-F(x,u).$$

Using this, and integrating by parts the term $a(x,\nabla u^*)\cdot\nabla(u-u^*)$ in the right hand side of (4.4), we obtain

$$
\begin{aligned}
\mathcal{V}_r^{(n-1)/n} &\leq C\,K^p \int_{B_{r+1}} F(x,u^*)-F(x,u)\\
&+ C\,K^p \int_{B_{r+1}} \operatorname{div} a(x,\nabla u^*)(u-u^*)\\
&+ \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p.
\end{aligned} \tag{4.5}
$$

To proceed, we recall that by (0.8) and (0.10) we have

$$\operatorname{div} a(x,\nabla h) \leq C|\nabla h|^{p-2}(|\nabla h|+|D^2 h|). \tag{4.6}$$

Let now h be a C^2 radial function, defined by

$$h(x) = -1 + 2(|x|-r)_+^\alpha,$$

with some fixed

$$\alpha > \max\left\{\frac{p}{p-d},2\right\}.$$

Then

$$|\nabla h| \leq C(h+1)^{(\alpha-1)/\alpha}, \qquad |D^2 h| \leq C(h+1)^{(\alpha-2)/\alpha}.$$

Hence,

$$\operatorname{div} a(x,\nabla h) \leq C(h+1)^{(p-2)(\alpha-1)/\alpha+(\alpha-2)/\alpha} \leq C(h+1)^{d-1} \tag{4.7}$$

and consequently

$$
\begin{aligned}
\mathcal{V}_r^{(n-1)/n} &\leq C\,K^p \int_{B_{r+1}} [F(x,u^*)-F(x,u)]\\
&+ C\,K^p \int_{B_{r+1}} (u^*+1)^{d-1}\,(u-u^*)\\
&+ \frac{C}{K^{p'}} \int_{B_{r+1}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p.
\end{aligned} \tag{4.8}
$$

We now split the right hand side of the above inequality into three parts, namely: the contribution in B_r, the one in $\{u<\theta\}\cap(B_{r+1}-B_r)$ and the one in $\{u\geq\theta\}\cap(B_{r+1}-B_r)$.

1. The contribution in B_r. Here the second integrand in the right hand side of (4.8) vanishes, as well as the term $F(x,u^*)$ of the first integrand. Besides, the third integral is taken over the region, where $u-u^*<\varepsilon$. In B_r, the latter condition is equivalent to $u<\theta$, since $u^*=-1$. Furthermore, if K is sufficiently large, for $u<\theta$ we have

$$-C\,K^p F(x,u) + C\,K^{-p'}(u+1)^p \leq -c\,F(x,u),$$

since by our assumption $F(x,u)\geq c(u+1)^d \geq c(u+1)^p$ for $-1\leq u<\theta$. Hence, the contribution of the right hand side of (4.8) in B_r is bounded from above by $-c\mathcal{A}_r$.

2. The contribution in $\{u<\theta\}\cap(B_{r+1}-B_r)$. Since $-1\leq u^*\leq u<\theta$, from Lemma 1.6 we have that

$$F(x,u^*)-F(x,u) \leq -c\,(u-u^*)^d.$$

Since both $u^*+1 \le u+1$ and $u-u^* \le u+1$, we also have

$$(u^*+1)^{d-1}(u-u^*) \le (u+1)^d \le C\,F(x,u).$$

Thus,

$$K^p[F(x,u^*)-F(x,u)+(u^*+1)^{d-1}(u-u^*)]+K^{-p'}(u-u^*)^p \le C\,F(x,u)$$

and the total contribution of the right hand side of (4.8) in $\{u<\theta\}\cap(B_{r+1}-B_r)$ is bounded from above by $C(\mathcal{A}_{r+1}-\mathcal{A}_r)$.

3. Finally, the contribution in $\{u\ge\theta\}\cap(B_{r+1}-B_r)$ is easily estimated by

$$C\,\mathcal{L}^n(\{u\ge\theta\}\cap(B_{r+1}-B_r)) = C(\mathcal{V}_{r+1}-\mathcal{V}_r),$$

since the terms inside the integrals are bounded.

Collecting the estimates from **1**–**3**, we obtain (4.2), which completes the proof of Lemma 3.1 in the case $p\ge 2$, $d<p$.

4.2. **The case $d=p$.** The proof is a refinement of the one in the case $d<p$. We use here suitable positive parameters Θ and T: we will fix Θ small enough and then choose T suitably large (and in fact ΘT suitably large).

We define $\mathcal{V}_r$ as in (4.1) and set $a_k=\mathcal{V}_{kT}-\mathcal{V}_{(k-1)T}$. Then, we claim that

$$\Big(\sum_{1\le j\le k} a_j\Big)^{(n-1)/n} \le C\Big(a_{k+1}+\sum_{1\le j\le k} e^{-L(k+1-j)}a_j\Big), \tag{4.9}$$

with L as large as we wish. With no loss of generality, we can take $r_0\ge\mu_1$. Thus, by assumption, $\mathcal{V}_{r_0}\ge\mu_2>0$. Therefore, by means of Lemma 1.2, the above inequality implies that

$$\mathcal{V}_r \ge c\,r^n,$$

for $r\ge 1$.

We now prove (4.9). We use a barrier function $h=h_k\in C^2(B_{(k+1)T})$ so that $-1\le h\le 1$, $h=1$ on $\partial B_{(k+1)T}$,

$$(h+1)+|\nabla h|+|D^2 h| \le C(h+1) \le Ce^{-\Theta T(k+1-j)} \tag{4.10}$$

in $B_{jT}-B_{(j-1)T}$, and

$$|\nabla h|+|D^2h| \le C\Theta(h+1) \tag{4.11}$$

in $B_{(k+1)T}$. For the explicit construction of h, see [Val01].

Fix $\varepsilon\in(0,1+\theta)$. Define $u^*=\min(u,h)$ and $\beta=\min(u-u^*,\varepsilon)$. Using the Sobolev inequality applied to β^p and then the Young inequality, we have:

$$\begin{aligned}(4.12)\qquad &\Big(\int_{B_{(k+1)T}} |\beta|^{pn/(n-1)}\Big)^{(n-1)/n}\\ \le\;& C\,K^p\int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} |\nabla(u-u^*)|^p\\ &+\frac{C}{K^{p'}}\int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^p\end{aligned}$$

with the parameter $K>0$ to be chosen later. From (4.10),

$$u-u^* \ge \theta+1-Ce^{-\Theta T} > \varepsilon\,, \tag{4.13}$$

in $B_{kT} \cap \{u \geq \theta\}$, provided ΘT is conveniently large, hence the left hand side of the inequality above is estimated from below by

$$c\,\mathcal{L}^n(\{u \geq \theta\} \cap B_{kT})^{(n-1)/n} = c\,\mathcal{V}_{kT}^{(n-1)/n}.$$

Now, combining (4.12) and (1.1), using the minimality of u, and integrating by parts the term $a(x, \nabla u^*) \cdot \nabla(u - u^*)$ (for more details, see the respective part in §4.1), we obtain

$$\begin{aligned}\mathcal{V}_{kT}^{(n-1)/n} &\leq C\,K^p \int_{B_{(k+1)T}} F(x, u^*) - F(x, u) \\ &+ C\,K^p \int_{B_{(k+1)T}} \operatorname{div} a(x, \nabla u^*)(u - u^*) \\ &+ \frac{C}{K^{p'}} \int_{B_{(k+1)T} \cap \{u - u^* < \varepsilon\}} (u - u^*)^p.\end{aligned}$$

Recalling (0.8) and (0.10), we have that

$$\operatorname{div} a(x, \nabla h) \leq C|\nabla h|^{p-2}(|\nabla h| + |D^2 h|). \tag{4.14}$$

Hence,

$$\begin{aligned}\mathcal{V}_{kT}^{(n-1)/n} &\leq C\,K^p \int_{B_{(k+1)T}} F(x, u^*) - F(x, u) \\ &+ C\,K^p \int_{B_{(k+1)T}} |\nabla u^*|^{p-2} \left(|\nabla u^*| + |D^2 u^*|\right)(u - u^*) \\ &+ \frac{C}{K^{p'}} \int_{B_{(k+1)T} \cap \{u - u^* < \varepsilon\}} (u - u^*)^p.\end{aligned} \tag{4.15}$$

We now split the right hand side of the above inequality into three parts, namely: the contribution in $\{u < \theta\}$, the one in $\{u \geq \theta\} \cap (B_{(k+1)T} - B_{kT})$ and the one in $\{u \geq \theta\} \cap B_{kT}$.

1. The contribution in $\{u < \theta\}$ is estimated using (4.11) and Lemmata 1.6 and 1.7. We actually show that such contribution is negative. Indeed, using the above mentioned results and taking K suitably big (so to kill the last term with the first one) and Θ suitably small (so to kill the constant c in Lemma 1.7), the contribution in $\{u < \theta\}$ is bounded by

$$C \int_{B_{(k+1)T} \cap \{u < \theta\}} F(x, u^*) - F(x, u) + C\Theta(u^* + 1)^{d-1}(u - u^*) \leq 0\,.$$

2. The contribution in $\{u \geq \theta\} \cap (B_{(k+1)T} - B_{kT})$ of the right hand side of (4.15) can be easily bounded by $C\,a_k$, since the terms inside the integrals are bounded.

3. We now estimate the contribution of the right hand side of (4.15) in $\{u \geq \theta\} \cap B_{kT}$. First, notice that, by (4.13),

$$\int_{B_{kT} \cap \{u - u^* < \varepsilon\} \cap \{u \geq \theta\}} (u - u^*)^p = \int_{\emptyset} (u - u^*)^p = 0\,.$$

Also, from (0.4) and (0.5), it follows that F is uniformly Lipschitz continuous in u, thus, from Lemma 1.8, $F(x, h) \leq c\,(1 + h)$. Therefore, by (4.10), we bound the contribution in $\{u \geq \theta\} \cap B_{kT}$ by

$$
\begin{aligned}
& C\Big(\sum_{j=1}^{k}\int_{(B_{jT}-B_{(j-1)T})\cap\{u\geq\theta\}} F(x,h)+|\nabla h|+|D^2 h|\Big)\\
\leq\ & C\Big(\sum_{j=1}^{k} e^{-\Theta T(k+1-j)}\,\mathcal{L}^n\big((B_{jT}-B_{(j-1)T})\cap\{u\geq\theta\}\big)\Big).
\end{aligned}
$$

In light of **1**–**3**, we bound the right hand side of (4.15) by

$$
C\Big(a_{k+1}+\sum_{1\leq j\leq k} e^{-L(k+1-j)}a_j\Big)\,.
$$

This proves (4.9) and completes the proof of Theorem 0.1 in the case $p \geq 2$.

5. Proof of Lemma 3.1. The case $1 < p < 2$

5.1. **The case of a Lipschitz** F**.** If we assume that F_u is uniformly bounded, the absolute minimizers of $\mathcal{J}$ will be locally uniformly Lipschitz continuous, even $C^{1,\alpha}$ regular, see [Tol84]. Then the proof of Lemma 3.1 in the case $1 < p \leq 2$ under this hypothesis is a slight variation of the one for $p \geq 2$. The main difference is that we use Lemma 1.4 instead of Lemma 1.3. Technically, we should separately consider the cases $d < p$ and $d = p$. However, since the changes from the case $p \geq 2$ are similar in both cases, we sketch only the proof for the more subtle case $d = p$.

We consider suitable positive parameters Θ, T and K (playing the same rôle as in §4.2) and we define h, u^* and β as we did in §4.2 above. In analogy with (4.12), using the Sobolev inequality applied to β^2 and then the Young inequality, we have:

$$
\begin{aligned}
&\Big(\int_{B_{(k+1)T}} |\beta|^{2n/(n-1)}\Big)^{(n-1)/n}\\
\leq\ & C\,K^2\int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} |\nabla(u-u^*)|^2\\
&+\frac{C}{K^2}\int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}}(u-u^*)^2\,.
\end{aligned}
\tag{5.1}
$$

Arguing as in (4.13), we get that the left hand side of the inequality above is estimated from below by

$$
c\,\mathcal{L}^n(\{u\geq\theta\}\cap B_{kT})^{(n-1)/n} = c\,\mathcal{V}_{kT}^{(n-1)/n}.
$$

Notice that $|\nabla u|$ is uniformly bounded by means of Theorem 1 in [Tol84] (and, indeed, u is $C^{1,\alpha}$ with uniform estimates in the interior of Ω). Thus, using (1.3), the minimality of u and an integration by parts, we infer from (5.1) the following inequality:

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} \ \leq\ & C\,K^2\int_{B_{(k+1)T}} F(x,u^*)-F(x,u)\\
+\ & C\,K^2\int_{B_{(k+1)T}} \operatorname{div} a(x,\nabla u^*)(u-u^*)\\
+\ & \frac{C}{K^2}\int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}}(u-u^*)^2.
\end{aligned}
$$

In light of (4.14), we deduce that

$$
\begin{aligned}
\mathcal{V}_{kT}^{(n-1)/n} &\leq C\,K^2 \int_{B_{(k+1)T}} F(x,u^*) - F(x,u) \\
&+ C\,K^2 \int_{B_{(k+1)T}} |\nabla u^*|^{p-2}\,(|\nabla u^*| + |D^2 u^*|)\,(u-u^*) \qquad (5.2)\\
&+ \frac{C}{K^2} \int_{B_{(k+1)T}\cap\{u-u^*<\varepsilon\}} (u-u^*)^2 .
\end{aligned}
$$

As done in §4.2, one splits the right hand side of the above inequality into three parts, namely: the contribution in $\{u<\theta\}$, the one in $\{u \geq \theta\} \cap (B_{(k+1)T} - B_{kT})$ and the one in $\{u \geq \theta\} \cap B_{kT}$. Such estimates follow the lines of §4.2. Namely, the contribution in $\{u \leq \theta\}$ is estimated by using (4.11) and Lemmata 1.6 and 1.7, obtaining, for big K and small $\Theta > 0$ the bound

$$
C \int_{B_{(k+1)T}\cap\{u<\theta\}} F(x,u^*) - F(x,u) + C\Theta^{p-1}(u^*+1)^{d-1}(u-u^*) \leq 0\,,
$$

which is negative. The contribution in $\{u \geq \theta\} \cap (B_{(k+1)T} - B_{kT})$ of the right hand side of (4.15) can be easily bounded by $C\,a_k$. As above, the contribution in $\{u \geq \theta\} \cap B_{kT}$ is bounded by using Lemma 1.8 and (4.10), obtaining

$$
C\Big(\sum_{j=1}^{k} e^{-(p-1)\Theta T(k+1-j)}\,\mathcal{L}^n\big((B_{jT} - B_{(j-1)T}) \cap \{u \geq \theta\}\big)\Big).
$$

This proves (4.9) and hence completes the proof of Theorem 0.1 in the case $1 < p \leq 2$ for uniformly Lipschitz continuous in u potentials F.

5.2. **The case of a non-Lipschitz F.** We now show that the density estimate in Lemma 3.1 can be obtained at least for $2n/(n+2) < p < 2$ without the technical assumption of Lipschitz continuity in u of the double-well potential F. This will be achieved with a more effective use of the inequality

$$
c(|\xi'| + |\xi|)^{p-2}|\xi' - \xi|^2 \leq A(x,\xi') - A(x,\xi) - a(x,\xi)\cdot(\xi'-\xi) \qquad (5.3)
$$

for $1 < p < 2$, see the proof of Lemma 1.4.

Without loss of generality we may assume that $d < p$. Indeed, the additional hypothesis (0.5) in the case $d = p$ makes F uniformly Lipschitz continuous in u for $u \in [-1,1]$, contrary to our assumption.

We revisit the proof of Lemma 3.1 in §4.1, now with $1 < p < 2$, and let h, u^* and β be the same as there. We also introduce the weight

$$
\omega = (|\nabla u| + |\nabla u^*|)^{1-p/2}.
$$

Integrating (5.3) with $\xi = \nabla u$ and $\xi' = \nabla u^*$, we obtain that

$$
c\int_{B_{r+1}} \frac{|\nabla\beta|^2}{\omega^2} \leq \int_{B_{r+1}} A(x,\nabla u) - A(x,\nabla u^*) - a(x,\nabla u^*)\cdot\nabla(u-u^*). \qquad (5.4)
$$

To avoid complications, related to the vanishing of ω, we also introduce its "regularization"

$$
\omega_\varepsilon = (|\nabla u| + |\nabla u^*| + \varepsilon)^{1-p/2}, \quad \varepsilon > 0.
$$

Observe that we always have $\omega_\varepsilon > \omega$ and $\omega_\varepsilon \searrow \omega$ as $\varepsilon \searrow 0$.

Analyzing the proof in §4.1, we realize that one can improve the step when the Sobolev and Young inequalities are applied to the function β^p, see (4.3). Indeed, let κ and $\lambda(x)$ be a certain positive number and a function, to be chosen later. Then

$$c\left(\int_{B_{r+1}}\beta^{\kappa\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\leq\int_{B_{r+1}}\beta^{\kappa-1}|\nabla\beta|=\int_{B_{r+1}}\left(\frac{|\nabla\beta|}{\omega_\varepsilon}\lambda\right)\left(\beta^{\kappa-1}\frac{\omega_\varepsilon}{\lambda}\right),$$

where in the first step we have applied the Sobolev inequality to the function β^κ. Now we use the Young inequality, with a certain parameter q, $1<q<2$, and its conjugate $q'=q/(q-1)$. We obtain

$$c\int_{B_{r+1}}\left(\frac{|\nabla\beta|}{\omega_\varepsilon}\lambda\right)\left(\beta^{\kappa-1}\frac{\omega_\varepsilon}{\lambda}\right)\leq K^q\int_{B_{r+1}}\frac{|\nabla\beta|^q}{\omega_\varepsilon^q}\lambda^q+K^{-q'}\int_{B_{r+1}}\beta^{(\kappa-1)q'}\frac{\omega_\varepsilon^{q'}}{\lambda^{q'}}.$$

Applying the Hölder inequality with exponents $2/q$ and $2/(2-q)$ in both integrals on the right hand side of the inequality above, we estimate it by

$$\begin{aligned}&K^q\left(\int_{B_{r+1}}\frac{|\nabla\beta|^2}{\omega_\varepsilon^2}\right)^{q/2}\left(\int_{B_{r+1}}\lambda^{2q/(2-q)}\right)^{1-q/2}\\+\quad&K^{-q'}\left(\int_{B_{r+1}}\beta^{(\kappa-1)q'(2/q)}\right)^{q/2}\left(\int_{B_{r+1}}\left(\frac{\omega_\varepsilon}{\lambda}\right)^{2q'/(2-q)}\right)^{1-q/2}.\end{aligned}$$

Let us now choose λ so that

$$\lambda^{2q/(2-q)}=\left(\frac{\omega_\varepsilon}{\lambda}\right)^{2q'/(2-q)}.$$

A simple computation shows that

$$\lambda=\omega_\varepsilon^{q'/(q+q')}=\omega_\varepsilon^{1/q}.$$

Then the expression above transforms to

$$\left[K^q\left(\int_{B_{r+1}}\frac{|\nabla\beta|^2}{\omega_\varepsilon^2}\right)^{q/2}+K^{-q'}\left(\int_{B_{r+1}}\beta^{(\kappa-1)q'(2/q)}\right)^{q/2}\right]\left(\int_{B_{r+1}}\omega_\varepsilon^{1/(1-q/2)}\right)^{1-q/2}$$

which is bounded from above by

$$C\left[K^2\int_{B_{r+1}}\frac{|\nabla\beta|^2}{\omega_\varepsilon^2}+K^{-2q'/q}\int_{B_{r+1}}\beta^{(\kappa-1)q'(2/q)}\right]^{q/2}\left(\int_{B_{r+1}}\omega_\varepsilon^{1/(1-q/2)}\right)^{1-q/2}.$$

Collecting the estimates above and then letting $\varepsilon\to0$, we will arrive at the inequality

$$\begin{aligned}&\left(\int_{B_{r+1}}\beta^{\kappa\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\leq\\&C\left[K^2\int_{B_{r+1}}\frac{|\nabla\beta|^2}{\omega^2}+K^{-2q'/q}\int_{B_{r+1}}\beta^{(\kappa-1)q'(2/q)}\right]^{q/2}\left(\int_{B_{r+1}}\omega^{1/(1-q/2)}\right)^{1-q/2}.\end{aligned}$$

Now observe that the term inside square brackets can be estimated similarly as in §4.1, recalling also (5.4). We now have arrived at a point, when we have to choose q. For that purpose we turn our attention to the term $\omega^{1/(1-q/2)}$. If we knew that ω is bounded, we could let $q\nearrow 2$. This is so, for instance, when F is uniformly

Lipschitz in u, and we recover the proof in §5.1 above. However, for non-Lipschitz F, we *a priori* know only the L^p integrability of $|\nabla u|$ and $|\nabla u^*|$. Moreover, we have

$$\int_{B_{r+1}} |\nabla u|^p + |\nabla u^*|^p \leq C\, r^{n-1},$$

by (3.1), for sufficiently large r. Thus, in order to obtain the desired density estimate, we choose q so that $\omega^{1/(1-q/2)} \simeq (|\nabla u|^p + |\nabla u^*|^p)$. Since $\omega = (|\nabla u| + |\nabla u^*|)^{1-p/2}$, we require

$$(1-p/2)/(1-q/2) = p \quad \Longleftrightarrow \quad q = 3 - \frac{2}{p}.$$

Observe that the condition $1 < q < 2$ is satisfied for $1 < p < 2$. As for the value of κ, we choose it to have

$$(\kappa - 1)q'(2/q) = p \quad \Longleftrightarrow \quad \kappa = p.$$

Thus, with this choice of constants we obtain

$$\left(\int_{B_{r+1}} \beta^{p\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C\left[K^2 \int_{B_{r+1}} \frac{|\nabla\beta|^2}{\omega^2} + K^{-p'} \int_{B_{r+1}} \beta^p\right]^{q/2} \left(r^{n-1}\right)^{1-q/2}.$$

Using (5.4) and repeating the arguments as in §4.1, we can deduce the following recursive inequality

$$c\,\mathcal{V}_r^{\frac{n-1}{n}} \leq [\mathcal{V}_{r+1} - \mathcal{V}_r + \mathcal{A}_{r+1} - \mathcal{A}_r - c\mathcal{A}_r]^{q/2}\,(r^{n-1})^{1-q/2}. \tag{5.5}$$

The question is now, if we can infer from (5.5) that

$$\mathcal{V}_r \geq c\, r^n$$

for $r \geq 1$, if $\mathcal{V}_1 \geq \mu > 0$. The answer is affirmative when

$$\frac{n-1}{n} \cdot \frac{2}{q} < 1 \quad \Longleftrightarrow \quad p > \frac{2n}{n+2}.$$

This follows from Lemma 1.1 with $\alpha = 1 - q/2$. (Unfortunately, (5.5) alone does not imply the density estimate for $p \leq 2n/(n+2)$, since we need $\alpha < 1/n$ in Lemma 1.1.)

Summarizing, we obtain that for the range of the exponents $2n/(n+2) < p < 2$, one can drop the assumption of the uniform Lipschitz continuity of the double-well potential F to prove Lemma 3.1. This completes the proof of Theorem 0.1.

6. Consequences of the density estimates

We briefly show in this section two consequences that can be easily derived from Theorem 0.1, thanks to the techniques developed in the last years.

The first consequence is that level sets of absolute minimizers converges, up to subsequence, to minimal interfaces in L^∞_{loc}. More precisely, it has been proved in [Bou90] that minimizers u_ε of

$$\mathcal{J}^\varepsilon(u) = \int A(x, \varepsilon\,\nabla u) + F(x,u) \tag{6.1}$$

converge, up to subsequence, in L^1_{loc} to a step function u_0 which has a minimal interface with respect to a suitably weighted area. Indeed, from the above density estimates we have that level sets converge in L^∞_{loc}:

Theorem 6.1. *Fix $\theta \in (0,1)$. Let $|u_\varepsilon| \le 1$ be an absolute minimizer of (6.1) in a bounded domain D. Assume that, as ε tends to zero, u_ε converges in $L^1_{\rm loc}$ to*

$$u_0 := \chi_E - \chi_{D-E}$$

for a suitable $E \subset D$. Then, $\{|u_\varepsilon| \le \theta\}$ converges locally uniformly to ∂E.

The latter convergence is understood in the sense that dist $(x, \partial E) \to 0$ uniformly for $x \in \{|u_\varepsilon| \le \theta\} \cap K$ for any $K \subset\subset D$.

The proof of the above result is analogous to the one given in [PV03] at the end of §6 and it is omitted here.

Remark 6.2. We point out the following particular case of the theorem above. Let $F(x,u) = |1-u^2|^\alpha$ with $\alpha > 2$. Then it is unknown whether there is a uniform convergence of the level sets of minimizers in the singular perturbation problem

$$\int \varepsilon^2 |\nabla u|^2 + |1-u^2|^\alpha, \qquad \varepsilon \to 0+ .$$

However, if one perturbs with $\varepsilon^p |\nabla u|^p$ with $p \ge \alpha$, the uniform convergence follows from Theorem 6.1.

The second consequence of the density estimates is the existence of plane-like minimizers in the periodic setting. We say that u is a class A minimizer for $\mathcal{J}$ if it is an absolute minimizer for $\mathcal{J}_B$ for any ball B. With this setting, we can prove:

Theorem 6.3. *Assume that $A(x+e,\eta) = A(x,\eta)$ and $F(x+e,\eta) = F(x,\eta)$ for any $e \in \mathbb{Z}^n$. Fix $\theta \in (0,1)$. Then, there exists a positive constant M_0, depending only on θ and on the structural constants, such that, given any $\omega \in \mathbb{R}^n - \{0\}$, there exists a class A minimizer $u = u_\omega$ for the functional $\mathcal{J}$ for which the set $\{|u| \le \theta\}$ is constrained in the strip $\{x \cdot \omega \in [0, M_0|\omega|]\}$.*

Furthermore, such u enjoys the following property of "quasi-periodicity": if $\omega \in \mathbb{Q}^n - \{0\}$, then u is periodic (with respect to the identification induced by ω, i.e. $u(x+k) = u(x)$ for any $k \in \mathbb{Z}^n \cap \omega^\perp$); if $\omega \in \mathbb{R}^n - \mathbb{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.

Notice that M_0 above is *independent* on the frequency ω. These kind of plane-like structures have been considered in [CdlL01] in the minimal surfaces case, and generalized to fluid jets and Ginzburg-Landau models in [Val01] and [PV03]. See also [Tor02] for a case with a degenerate metric. The proof of Theorem 6.3 is analogous to the one presented in §8 of [PV03], with minor obvious changes, and we therefore omit the details.

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