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Spectral decomposition theorem for the Ishimori–II equations

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Abstract

We study the gauge transformation between spectral problems and their adjoints for the Ishimori-II and Davey–Stewartson II equations. We prove the spectral decomposition theorem for eigenfunctions of a linear operator associated with the Ishimori–II equations.

1 Introduction

Spectral decomposition theorem is a fundamental result in spectral theory of self-adjoint operators. Many nonlinear partial differential equations in the inverse scattering transform method are associated with non-self-adjoint linear operators [AC90]. Since the regular direct and inverse scattering transforms are mutually reversible, non-self-adjoint linear operators must possess complete sets of eigenfunctions which are orthogonal with respect to eigenfunctions of the adjoint operators. If initial data evolve into algebraically decaying lump solutions of nonlinear equations, the spectral decomposition involves both the continuous and discrete spectrum of linear operators. Potentials of linear operators associated with lump solutions are defined in a weaker function space, compared to that required for a regular inverse scattering transform. Nevertheless, spectral decomposition theorems for time-dependent Schrödinger and Dirac operators with lump potentials were proved in [PS00a, PS00b] by methods of complex analysis based on the Riemann–Hilbert and $\bar{\partial}$ problems. It was shown that lumps of the Kadomtsev–Petviashvili-I (KP-I) equation are structurally stable in the spectral decomposition [PS00a], while lumps of the Davey–Stewartson-II (DS-II) equations are structurally unstable and decay into continuous spectrum under generic perturbations of the initial data [PS00b]. In this paper, we study a complete set of eigenfunctions associated with the Ishimori-II (IS-II) equations,

$$\mathbf{S}_t + \frac{1}{2}\mathbf{S}\times (\mathbf{S}_{xx} - \mathbf{S}_{yy}) + \phi_x \mathbf{S}_y + \phi_y \mathbf{S}_x = 0,$$

$$\phi_{xx} + \phi_{yy} - \mathbf{S}\cdot(\mathbf{S}_x \times \mathbf{S}_y) = 0,$$

(1.1)

WHERE $\mathbf{S} = (S_1, S_2, S_3)^T = \mathbf{S}(x, y, t)$

IS A THREE-COMPONENT UNIT VECTOR ON THE PLANE (x, y)

SUCH THAT

$\mathbf{S}\cdot\mathbf{S} = 1,$

$\lim_{|x|,|y|\to\infty} \mathbf{S}(x, y, t) = (0, 0, 1)^T,$

WHILE $\phi(x, y, t)$ IS SCALAR POTENTIAL FUNCTION

GENERATED BY THE TOPOLOGICAL CHARGE DENSITY

$\mathbf{S} \cdot (\mathbf{S}_x \times \mathbf{S}_y)$.

THE IS-II EQUATIONS DESCRIBE TIME EVOLUTION OF

THE SYSTEM OF STATIC SPIN VORTICES ON THE PLANE (x, y)

[I84]. STATIC SPIN VORTICES ARE CLASSIFIED BY THE INTEGER

VALUES OF THE TOPOLOGICAL CHARGE:

$\mathrm{Q} = 1\frac{}{4\pi \iint_{R^2}}$

$\mathbf{S} \cdot(\mathbf{S}_x \times \mathbf{S}_y) dxdy.$

(1.2)

FROM MATHEMATICAL POINT OF VIEW, THE IS-II EQUATIONS

ARE REMARKABLE BECAUSE OF THEIR GAUGE EQUIVALENCE TO

THE DS-II EQUATIONS [BC88, LS89]:

$\mathrm{i}\ Q_t + \frac{1}{2}(q_{xx} - q_{yy})$
$+\ 4\ (\varphi - |q|^2)\, q = 0,$

$\varphi_{xx} + \varphi_{yy} - 2\left(|q|^2\right)_{xx}$
$=\ 0.$
(1.3)

Linear non-self-adjoint operators for the IS-II equations (1.1) have potentials multiplied by x-derivative operators, compared to linear operators for the DS-II equations (1.3). As a result, a proof of the spectral decomposition theorem is particularly difficult for the linear operators of the IS-II equations. We prove the orthogonality and completeness of eigenfunctions by using the inverse scattering transform for both IS-II and DS-II equations, with the gauge transformation between these equations. The key property in our analysis is a commutativity theorem of the gauge transformations between the linear operators and their adjoints.
The paper is organized as follows. In Section 2 we review the scattering data of the IS-II equations.
In Section 3 we study the gauge transformation between the IS-II and DS-II equations.
An adjoint problem for the IS-II equations

is introduced in Section 4. In Section 5,

we study the gauge transformation

between the adjoint problems and prove

commutativity of the gauge and adjoint transformations.

Section 6 presents applications to the spectral

decomposition and surface representation theorems.

2 Inverse scattering transform for the Ishimori-II equations

The IS–II equations are the compatibility

conditions of the over-determined linear system,

$(i\partial_y + P\partial_x)\,\mathbf{\Psi} = 0,$ (2.1)

$\left(2\partial_t - 2iP\partial_x^2 -\right.$

$\left.(iP_x - P_yP - iP\phi_x - \phi_y)\,\partial_x\right.$

$\mathbf{\Psi}=0,$

(2.2)

where $\mathbf{\Psi} =$

$(\Psi_1, \Psi_2)^T = \mathbf{\Psi}(x, y, t),$

$P = S_1\sigma_1 + S_2\sigma_2 + S_3\sigma_3 = P(x, y, t),$

and σ_1,σ_2,σ_3 are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

SUCH THAT

$$\mathrm{P} = \begin{pmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{pmatrix}.$$

WE INTRODUCE COMPLEX VARIABLES FOR ELLIPTIC PROBLEMS ON THE PLANE (x, y):

$$\mathrm{z} = 1\frac{}{2(y-ix),}$$

$$\bar{z} = \tfrac{1}{2}(y+ix),$$

(2.3)

SUCH THAT THE SPECTRAL PROBLEM (2.1)

CAN BE REWRITTEN FOR $\mathbf{\Psi}$

$= \mathbf{\Psi}(\mathrm{z})$:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \mathbf{\Psi}$$

$$= 1\frac{}{2(P(z)-\sigma_3)}$$

$(\partial_z - \partial_{\bar{z}})$

$\boldsymbol{\Psi}$,

(2.4)

ON $z \in C$. HERE AND HENCEFORTH

WE SUPPRESS THE DEPENDENCE OF $\boldsymbol{\Psi}$

FROM $\bar{z}$ AND t.

THE $\overline{\partial}$-METHOD WAS USED IN [BC88, K93]

TO SOLVE THE INVERSE SCATTERING TRANSFORM FOR THE LINEAR

SPECTRAL PROBLEM (2). WE INTRODUCE A FUNDAMENTAL

MATRIX OF SOLUTIONS OF (2),

$\boldsymbol{\Psi} =$

$\Big(\boldsymbol{\mu} e^{ikz},$

$\boldsymbol{\chi}$ E$^{-ik\bar{z}}$,

(2.5)

WHERE THE VECTORS

$\boldsymbol{\mu} = (\mu_1, \mu_2)^T =$

$\boldsymbol{\mu}$(Z,K)and

$\boldsymbol{\chi} = (\chi_1, \chi_2)^T$

$= \boldsymbol{\chi}$(Z,K)$solvetheproblems,$

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \boldsymbol{\mu} +$$

IK $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\boldsymbol{\mu} = \mathcal{P}_k \boldsymbol{\mu}$, (2.6)

$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \boldsymbol{\chi} -$

IK $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\chi} =$

$\mathrm{P}_k \boldsymbol{\chi}$,

(2.7)

WHERE

$\mathrm{P}_k = \frac{1}{2}\left(P(z) - \sigma_3\right)$

$\left(\partial_z - \partial_{\bar{z}} + ik\right)$.

(2.8)

THE EIGENVECTORS $\boldsymbol{\mu}(z,k)$

AND $\boldsymbol{\chi}(z,k)$ SATISFY

THE BOUNDARY CONDITIONS ON THE PLANE (x,y):

$\lim_{|z|\to\infty} \boldsymbol{\mu}(z,k) =$

$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$\lim_{|z|\to\infty} \boldsymbol{\chi}(z,k)$

$= \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(2.9)

IF $\mathbf{S}(z)$ IS REAL-VALUED,

THE TWO FUNDAMENTAL SOLUTIONS $\boldsymbol{\mu}(z,k)$

AND $\boldsymbol{\chi}(z,k)$ OF THE

PROBLEMS (2.6) AND (2.7) ARE RELATED

BY MEANS OF THE INVOLUTION TRANSFORMATION:

$\boldsymbol{\chi}(\mathrm{z},\bar{k}) = \sigma$

$\overline{\boldsymbol{\mu}(z,k)}$,

$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(2.10)

IN THE LIMIT $|k| \to 0$, THE EIGENVECTOR

$\boldsymbol{\mu}(z,k)$ BECOMES A

CONSTANT VECTOR IN z, WHICH IS UNIQUELY DEFINED

BY THE NORMALIZATION CONDITION (2) AS

$\lim_{|k|\to 0} \boldsymbol{\mu}(z,k) = \mathbf{e}_1$.

(2.11)

IN THE LIMIT $|k| \to \infty$, THE EIGENVECTOR

$\boldsymbol{\mu}(z,k)$ BECOMES A VECTOR

$\boldsymbol{\mu}_\infty(z)$, which solves

the algebraic problem,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boldsymbol{\mu}_\infty(z) = \frac{1}{2}$

$(P(z) - \sigma_3)$

$\boldsymbol{\mu}_\infty(z).$

(2.12)

Components of the vector $\boldsymbol{\mu}_\infty(z)$ are

related by means of the stereographical variable,

$\mu_{2\infty}\overline{\mu_{1\infty}=}$

1 - $S_3 \overline{S_1 - iS_2 = \frac{S_1+iS_2}{1+S_3}}.$

(2.13)

The stereographical variable in (2)

satisfies the same family of nonlinear problem as

the DS-II equations (1.3) [S92].

We define the matrix $m_\infty = m_\infty(z)$ as

$\mathrm{M}_\infty(z) = (\boldsymbol{\mu}_\infty(z),$

$\sigma\overline{\boldsymbol{\mu}_\infty(z)} =$

$$\begin{pmatrix} \mu_{1\infty} & -\overline{\mu_{2\infty}} \\ \mu_{2\infty} & \overline{\mu_{1\infty}} \end{pmatrix},$$
(2.14)

such that $\det(m_\infty) = |\mu_{1\infty}|^2 +$

$—\mu_{2\infty}|^2$. As a result,

$$\mathrm{P(z)} = \mathrm{M}_\infty(z)\sigma_3 m_\infty^{-1}(z),$$
(2.15)

or explicitly,

$\mathrm{S}_1 - iS_2 =$

$\overline{\mu_{2\infty}}|\mu_{1\infty}|^2 + |\mu_{2\infty}|^2,$

$\mathrm{S}_3 =$

$—\mu_{1\infty}|^2 + |\mu_{2\infty}|^2.$

(2.16)

The differential problem (2.6) can be transformed to

the linear integral equation by means of the Green function,

$\mu_1(z,k) = 1 + \frac{1}{2\pi i}$

$\iint \frac{dz'\wedge d\bar{z}'}{z'-z}(\mathcal{P}_k$

$\boldsymbol{\mu})_1(z'),$

$\mu_2(z,k) = \frac{1}{2\pi i}$

$\iint \frac{dz'\wedge d\bar{z}'}{\bar{z}'-\bar{z}}$

$(\mathrm{P}_k\boldsymbol{\mu})_2(z')$

$\mathrm{E}^{ik(z'-z)+i\bar{k}(\bar{z}'-\bar{z})}.$

(2.17)

The same Fredholm integral equations (2.17) occur in the spectral problem for the DS-II equations [PS00b]. The integral equations (2.17) have a unique solution if the potential function $P(z)$ in $L^2(R^2)\cap$ $\mathrm{L}^1(R^2)$ [BC84]. If the potential function $P(z)$ in $L^2(R^2)$ but not in $L^1(R^2)$, the integral equations (2.17) may have homogeneous solutions, corresponding to bound states [FA84, APP89]. We denote bound states as $\boldsymbol{\Phi}_j(z)$, as decaying solutions of the equation,

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}$$

$$\Phi_j + ik_j$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Phi_j = \mathcal{P}_{k_j}\boldsymbol{\Phi}_j, \tag{2.18}$$

SUBJECT TO THE BOUNDARY CONDITIONS,

$\lim_{|z|\to\infty} z\mathbf{\Phi}_j(z) = \mathbf{e}_1.$
(2.19)

EACH EIGENVALUE k_j IS DOUBLE, SINCE

THERE EXISTS A LINEARLY INDEPENDENT

EIGENFUNCTION $\mathbf{\Phi}'_j(z)$ SOLVING

THE SAME EQUATION (2)

AND SATISFYING THE BOUNDARY CONDITIONS

$\lim_{|z|\to\infty} z\mathbf{\Phi}'_j(z)$
$\mathrm{E}^{ik_jz+i\overline{k_jz}} = \mathbf{e}_2.$
(2.20)

THE TWO LINEARLY INDEPENDENT EIGENFUNCTIONS

ARE RELATED AS

$\Phi'_j(z) = \sigma\overline{\mathbf{\Phi}_j(z)}e^{-i(k_jz+\overline{k_jz})}.$
(2.21)

We assume that there is a finite number (say n)

of eigenvalues k_j on the plane $k \in C$,

such that $k_j \neq 0,\ j = 1, ..., n$.

It follows from (2.17) that the

eigenfunction $\boldsymbol{\mu}(z, k)$

satisfies the $\bar{\partial}$ problem in the domain

$\mathcal{D} = \{k \in C : k \neq k_j,$

j = 1,...,n , such that

$\bar{k} = b(k)\mathbf{N}_{\mu}(z, k),$
(2.22)

where the spectral data $b(k)$ are defined as

b(k)=1$_{\overline{2\pi \iint dz \wedge d\bar{z}}}$
$(\mathcal{P}_k \boldsymbol{\mu})_2\,(z)$
e$^{i(kz+\overline{kz})}$
(2.23)

and the linear independent eigenfunction

$\mathbf{N}_{\mu}(z, k)$ solves the same integral equation

(2.17) but with a different (oscillatory)

behavior at infinity:

$\mathrm{N}_{\mu 1}(z,k) = \frac{1}{2\pi i}$

$\iint \frac{dz' \wedge d\bar{z}'}{z'-z}$

$(\mathrm{P}_k \mathbf{N}_\mu)_1(z')$

$\mathrm{N}_{\mu 2}(z,k) = e^{-i(kz+\overline{kz})} +$

$1 \overline{2\pi i}$

$\iint$

K(Z'-Z)$+\overline{k(z'-z)})\bar{z}' - \bar{z}$

$(\mathrm{P}_k \mathbf{N}_\mu)_2(z')$.

(2.24)

For real-valued $\mathbf{S}(z)$, there exists a local relation between two linearly independent eigenfunctions $\boldsymbol{\mu}(z,k)$ and $\mathbf{N}_\mu(z,k)$:

$\mathbf{N}_\mu(z,k) = \sigma \overline{\boldsymbol{\mu}(z,k)}$

E$^{-i(kz+\overline{kz})}$.

(2.25)

When the eigenfunction $\boldsymbol{\mu}(z,k)$ approaches to the points $k = k_j,\ j = 1, ..., n,$ where the bound state exist, it has a singular (pole) behavior in the limit $k \to k_j$. We assume the following limiting behavior of $\boldsymbol{\mu}(z,k)$:

$\lim_{k\to k_j} (\boldsymbol{\mu}(z,k)$

- I$\overline{k-k_j \boldsymbol{\Phi}_j(z)}$

$= (z+z_j)\boldsymbol{\Phi}_j(z) + c_j \boldsymbol{\Phi}'_j(z),$
(2.26)

WHERE z_j, c_j ARE ARBITRARY CONSTANTS.

WE NOTICE THAT THIS LIMITING BEHAVIOR INTRODUCES

A CONSTRAINT ON THE CLASS OF POTENTIALS

AS FOR THE DS-II EQUATIONS IN [PS00B].

THE CONSTRAINT IS ALWAYS SATISFIED FOR

LUMP POTENTIALS.

NO SUCH CONSTRAINTS WERE INTRODUCED IN THE

ORIGINAL PAPER [APP89], WHERE

A DIFFERENT LIMITING BEHAVIOR OF $\boldsymbol{\mu}(z,k)$

WAS SUGGESTED AS $k \to k_j$:

$\lim_{k\to k_j} \frac{\partial}{\partial k}$

$((k-k_j)\boldsymbol{\mu}(z,k))$

$= (\mathrm{z} + \mathrm{z}_j)\boldsymbol{\Phi}_j(z) + c_j \boldsymbol{\Phi}'_j(z),$
(2.27)

WE NEGLECT THIS DISCREPANCY BETWEEN

(2) AND (2), SINCE

THE SPECTRAL DECOMPOSITION AND STRUCTURAL

INSTABILITY OF POINTS $k = k_j$ DOES NOT DEPEND

ON THE TYPES OF SINGULAR BEHAVIOR OF

$\boldsymbol{\mu}(z,k)$ AS $k \to k_j$.

SOLVING THE $\bar{\partial}$ PROBLEM (2)

ON PLANE $k \in C$ SUBJECT TO THE BOUNDARY

CONDITIONS $\lim_{|k|\to\infty} \boldsymbol{\mu}(z,k)$

$= \boldsymbol{\mu}_\infty(z)$ AND

USING THE LIMITING CONDITIONS (2)

AT THE POLES $k = k_j,\ j = 1, ..., n,$

WE DERIVE THE EXPANSION FORMULA,

$\boldsymbol{\mu}$(Z,K)=

$\boldsymbol{\mu}_\infty(z)$+

$\sum_{j=1}^{n} \frac{i}{k-k_j}$

$\Phi_j(z) + \frac{1}{2\pi i} \iint$

dk'$\wedge d\bar{k}' \frac{}{k'-kb(k')}$

$\mathbf{N}_\mu(z,k')$.

(2.28)

USING THE LIMITING CONDITION (2)

AS $k \to 0$, WE EXPRESS THE LIMITING

FUNCTION $\mu_\infty(z)$ EXPLICITLY AS

$\boldsymbol{\mu}_\infty(z) =$

$\mathbf{e}_1 + \sum_{j=1}^{n} \frac{i}{k_j}$

$\Phi_j(z) - \frac{1}{2\pi i} \iint$

dk'$\wedge d\bar{k}' \frac{}{k'b(k')\mathbf{N}_\mu(z,k')}$

(2.29)

As a result, we find an explicit expansion formula for $\boldsymbol{\mu}(z,k)$ as

$\boldsymbol{\mu}$(z,k) $= \mathbf{e}_1 +$

$\sum_{j=1}^{n} \frac{ik}{k_j(k-k_j)} \mathbf{\Phi}_j(z) +$

k$\overline{2\pi i}$

$\iint \frac{dk' \wedge d\bar{k}'}{k'(k'-k)}$

b(k') $\mathbf{N}_\mu(z,k')$.

(2.30)

Using the limiting relations as $k \to k_j$, we close the system (2) with a set of algebraic equations for $j = 1, ..., n$:

$\left(z + z_j - \frac{i}{k_j}\right) \mathbf{\Phi}_j(z)$

$+$ c$_j \mathbf{\Phi}'_j(z) = \mathbf{e}_1 + \sum_{l \neq j}^{n}$

ik$_j \overline{k_l(k_j-k_l)\mathbf{\Phi}_l(z)}$

$+$ k$_j \overline{2\pi i \iint}$

d$\bar{k}' k'(k'-k_j) b(k') \mathbf{N}_\mu(z,k')$.

(2.31)

We conclude this section with an explicit

EXAMPLE OF A SINGLE LUMP SOLUTION

OF THE IS–II EQUATIONS. LET $b(k) = 0$

AND $n = 1$ IN THE EQUATIONS ABOVE. THEN THE ALGEBRAIC SYSTEM

(2.31) IS DECOUPLED FROM (2)

AND IT HAS A UNIQUE SOLUTION:

$\Phi_{1j}(z) =$

—z+z$_j - \frac{i}{k_j}|^2 + |c_j|^2,$

$\Phi_{2j}(z) =$

—z+z$_j - \frac{i}{k_j}|^2 + |c_j|^2,$

(2.32)

THE EIGENFUNCTION $\boldsymbol{\mu}(z, k)$

HAS A POLE DECOMPOSITION REPRESENTATION,

$\boldsymbol{\mu}$(Z,K)

$= \mathbf{e}_1 + \frac{ik}{k_j(k-k_j)}\mathbf{\Phi}_j(z),$

$\boldsymbol{\mu}_\infty(z) =$

$\mathbf{e}_1 + \frac{i}{k_j}\mathbf{\Phi}_j(z).$

(2.33)

THE SINGLE LUMP SOLUTION FOR THE VECTOR

$\mathbf{S}(z)$ TAKES THE EXPLICIT FORM:

$S_1 - iS_2 =$

$(\ \bar{z} + \bar{z}_j + \frac{i}{k_j}) + |c_j|^2$

$E^{i(k_j z + \overline{k_j z})} \bar{k}_j$

$(|z + z_j|^2 + |c_j|^2)(|z + z_j - \frac{i}{k_j}|^2 + |c_j|^2)$ (2.34)

$S_3 = 1 -$

$(|z + z_j|^2 + |c_j|^2)(|z + z_j - \frac{i}{k_j}|^2 + |c_j|^2).$

(2.35)

We notice that the single lump solution of the IS-II equations is different from the single vortex solution found in [I84] with the Hirota's method. However, the single vortex solution in [I84] can be recovered from the single lump solution (2)–(2.35) in the singular limit $k_j \to 0$, with the appropriate shifts of z_j and $\bar{z}_j$. In our paper, we will be working only with the lump solutions which are derived in the inverse scattering transform for a general non-singular case $k_j \neq 0, \ j = 1, ..., n$.

3 Gauge transform between the DS-II and IS-II equations

The DS-II equations (1.3) in complex coordinates (2) are related with the following linear operator,

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}$$

$\mathbf{\Psi}_{DS} =$

Q(z) $\boldsymbol{\Psi}_{DS}$,
(3.1)

WHERE

$$\mathrm{Q(z)} = \begin{pmatrix} 0 & -q(z) \\ \bar{q}(z) & 0 \end{pmatrix}.$$
(3.2)

THE FUNDAMENTAL MATRIX OF THE LINEAR OPERATOR

FOR THE DS-II EQUATIONS IS $\boldsymbol{\Psi}_{DS}$

$= \Big(\boldsymbol{\mu}_{DS} e^{ikz},$

$\boldsymbol{\chi}_{DS} e^{-ik\bar{z}},$

WHERE $\boldsymbol{\chi}_{DS}(z, \bar{k}) =$

$\sigma \overline{\boldsymbol{\mu}_{DS}(z, k)}$

AND $\boldsymbol{\mu}_{DS}(z, k)$ SATISFIES THE PROBLEM:

$$\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \boldsymbol{\mu}_{DS}$$

$$+ \text{ IK} \Bigg(\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$$

$\boldsymbol{\mu}_{DS} = Q(z)$

$\boldsymbol{\mu}_{DS}$.
(3.3)

Linear spectral problems (2) and (3)
for the IS-II and DS-II equations are related
with the gauge transformation reported in [BC88, LS89].

Lemma 3.1 *Let $m_\infty(z)$ be defined by (2)*
such that $P(z) = m_\infty \sigma_3 m_\infty^{-1}$. Solutions
of the linear system (2)
are equivalent to solutions of the
linear system (3) with the gauge transformation
$\mathbf{\Psi}_{DS}(z,k) = m_\infty^{-1}(z)$
$\mathbf{\Psi}$ *(z,k), suchthat*

$$Q(z) = (\mathcal{D}_z m_\infty^{-1})\, m_\infty,$$
$$D_z = \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \tag{3.4}$$

Proof.
We first show that if $\mathbf{m} = (\boldsymbol{\mu}(z,k), \boldsymbol{\chi}(z,k))$ with $\boldsymbol{\mu}$, $\boldsymbol{\chi}$ satisfying (2.6),
(2.7) and $\mathbf{m}_{DS} = m_\infty^{-1}\mathbf{m} =$
$(\boldsymbol{\mu}_{DS}, \boldsymbol{\chi}_{DS})$,
then $\boldsymbol{\mu}_{DS}$ solves the problem
(3), where the matrix $Q(z)$ has zero diagonal
entries, while $\boldsymbol{\chi}_{DS}(z,\bar k) =$
$\sigma\overline{\boldsymbol{\mu}_{DS}(z,k)}$.
For simplicity, we consider only the case $n = 0$. We
rewrite (2) in the operator form,

$\mathbf{m} = \textsc{m}_\infty + \mathcal{C}\mathcal{T}(\mathbf{m}),$
(3.5)

WHERE $\mathcal{C}$ IS THE CAUCHY INTEGRAL OPERATOR,

$\mathrm{C}(\mathbf{f}) =$

$1\overline{2\pi i \iint \frac{dk' \wedge d\bar{k}'}{k'-k}}$

$\mathbf{f}$(K'),
(3.6)

AND $\mathcal{T}$ IS THE RIGHT MULTIPLICATIVE OPERATOR,

SUCH THAT

$\mathrm{T}(\boldsymbol{f}) =$

$\mathrm{T}\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}(z,k) =$

$\boldsymbol{f}(\textsc{z},\bar{k})\mathbf{S}_z(k).$

(3.7)

HERE THE SCATTERING MATRIX $\mathbf{S}_z(k)$ IS

$$\mathbf{S}_z(k) = \begin{pmatrix} 0 & e^{i(\bar{k}z+k\bar{z})}\overline{b(\bar{k})} \\ e^{-i(kz+\overline{kz})}b(k) & 0 \end{pmatrix},$$
(3.8)

AND WE HAVE USED (2)

AND (2). MULTIPLYING

(3) WITH m_∞^{-1}, WE FIND THAT

$\text{M}_\infty^{-1}\mathbf{m} = I + \mathcal{C}$

$\text{M}_\infty^{-1}\mathcal{T}(\mathbf{m})$

$= \text{I} + \text{C T } (\text{M}_\infty^{-1}\mathbf{m}), (3.9)$

WHERE I IS THE 2-BY-2 IDENTITY MATRIX. WE APPLY THE OPERATOR,

$\text{D}_k\boldsymbol{f} =$

$\text{D}_z\boldsymbol{f} + k(\text{ad}\mathcal{J}$

$\boldsymbol{f}) = \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}$

$\boldsymbol{f} - \frac{i}{2}k$

$[\sigma_3, \boldsymbol{f}],$

(3.10)

TO BOTH SIDES OF (3) AND USE THE

COMMUTATIVITY PROPERTY,

$[\mathcal{D}_k, \mathcal{T}]$

$\boldsymbol{f}=0.$

(3.11)

As a result, we obtain

$\mathrm{D}_k(m_\infty^{-1}\mathbf{m}) =$

$\mathrm{D}_k\mathcal{C}\mathcal{T}\,(m_\infty^{-1}$

$\mathbf{m}= \mathcal{C}\mathcal{T}\mathcal{D}_k\,($

$\mathrm{M}_\infty^{-1}\mathbf{m} + Q(z),$

where

$\mathrm{Q(z)} \ = \ [\mathcal{D}_k, \mathcal{C}]$

$\mathrm{T}\ (\mathrm{M}_\infty^{-1}\mathbf{m}) =$

$\text{-}1\overline{{}_{2\pi i \mathrm{ad}\mathcal{J}\iint}}$

$\mathrm{DK}\ \wedge d\bar{k}\ \mathcal{T}(m_\infty^{-1}\mathbf{m})(z,k)$

$= \ \ 1\overline{{}_{2\pi\iint dk\wedge d\bar{k}}}$

$$\begin{pmatrix} 0 & (\mathcal{T}(m_\infty^{-1}\mathbf{m}))_{12}(z,k) \\ (\mathcal{T}(m_\infty^{-1}\mathbf{m}))_{21}(z,\bar{k}) & 0 \end{matrix}$$

.

(3.12)

WE USE THE RELATION (3) AND

THE COMMUTATIVITY RELATION,

$$[Q(z), \mathcal{T}]\,\mathbf{f} = 0, \tag{3.13}$$

AND FINISH THE FIRST PART OF THE PROOF,

$$\begin{aligned}
&\mathrm{D}_k(m_\infty^{-1}\mathbf{m}) = \\
&(I - \mathcal{C}\mathcal{D}_k)^{-1}\,Q(z) = \\
&\mathrm{Q(z)}\ (I - \mathcal{C}\mathcal{D}_k)^{-1}\,I = \\
&\mathrm{Q(z)}\ \mathrm{M}_\infty^{-1}\mathbf{m}.
\end{aligned} \tag{3.14}$$

IN THE SECOND PART OF THE PROOF, WE SHOW THAT

IF $\mathbf{\Psi}$ SOLVES (2)

WITH $P(z) = m_\infty \sigma_3 m_\infty^{-1}$

AND $\mathbf{\Psi} = m_\infty \mathbf{\Psi}_{DS}$,

THEN $\mathbf{\Psi}_{DS}$ SOLVES (3),

WHERE $Q(z)$ IS RELATED TO m_∞ BY MEANS OF (3.1).

WE REWRITE (2) IN THE FORM:

$$\mathrm{D}_z\,(m_\infty \mathbf{\Psi}_{DS})$$

$= 1\frac{}{2\big(m_\infty \sigma_3 m_\infty^{-1} -}$

$\sigma_3\,(\partial_z - \partial_{\bar z}$

$m_\infty \mathbf{\Psi}_{DS}.$

(3.15)

Straightforward differentiations transform (3)

to the form:

$$\begin{pmatrix} \partial_{\bar z}\mu_{1\infty} & -\partial_{\bar z}\overline{\mu_{2\infty}} \\ \partial_z \mu_{2\infty} & \partial_z \overline{\mu_{1\infty}} \end{pmatrix} \mathbf{\Psi}_{DS} +$$

$\mathrm{M}_\infty \mathcal{D}_z \mathbf{\Psi}_{DS}$

$= 1\frac{}{\det(m_\infty)}$

$$\begin{pmatrix} -|\mu_{2\infty}|^2 & \mu_{1\infty}\overline{\mu_{2\infty}} \\ \overline{\mu_{1\infty}}\mu_{2\infty} & |\mu_{2\infty}|^2 \end{pmatrix} ($$

$(\partial_z - \partial_{\bar z}$

$m_\infty \mathbf{\Psi}_{DS},$

(3.16)

which further transforms to the form:

$\mathrm{M}_\infty \mathcal{D}_z \mathbf{\Psi}_{DS}$

$= \mathrm{M}_\infty Q(z) \mathbf{\Psi}_{DS},$

(3.17)

where

Q(z) $= 1\frac{}{\det(m_\infty)}$

$$\begin{pmatrix} -\overline{\mu_{1\infty}}\partial_{\bar z}\mu_{1\infty} - \overline{\mu_{2\infty}}\partial_{\bar z}\mu_{2\infty} & \overline{\mu_{1\infty}}\partial_{\bar z}\overline{\mu_{2\infty}} - \overline{\mu_{2\infty}}\partial_{\bar z}\overline{\mu_{1\infty}} \\ \mu_{2\infty}\partial_z\mu_{1\infty} - \mu_{1\infty}\partial_z\mu_{2\infty} & -\mu_{2\infty}\partial_z\overline{\mu_{2\infty}} - \mu_{1\infty}\partial_z\overline{\mu_{1\infty}} \end{pmatrix}.$$

(3.18)

Explicit differentiations in (3.1)

shows that representations (3.1) and

(3) are the same.

□

Corollary 3.2 *Components of the transformation matrix* $m_\infty(z)$

in (2) satisfy the constraints:

q(z) $=$ *1*$\frac{}{\det(m_\infty)}$

$(\overline{\mu_{2\infty}}$

$\partial_{\bar z}\overline{\mu_{1\infty}}$

- $\overline{\mu_{1\infty}}\partial_{\bar z}$

$\overline{\mu_{2\infty}}$, (3.19)

0 $=$ $\mu_{1\infty}\partial_z$

$\overline{\mu_{1\infty}} + \mu_{2\infty}$

$\partial_z\overline{\mu_{2\infty}}$.

(3.20)

WE CONTINUE THE EXPLICIT EXAMPLE (2)–(2.35)

AND COMPUTE THE GAUGE TRANSFORMATION BETWEEN THE SINGLE LUMP

SOLUTIONS OF THE IS-II AND DS-II EQUATIONS.

IT FOLLOWS FROM (2) THAT

$\mu_{1\infty} = 1+$

$\mathrm{I}\overline{k_j)}$

$\mathrm{K}_j(|z + z_j - \frac{i}{k_j}|^2 + |c_j|^2),$

$\mu_{2\infty} =$

$\overline{k_j z})$

$\mathrm{K}_j(|z + z_j - \frac{i}{k_j}|^2 + |c_j|^2),$
(3.21)

SUCH THAT

$\det(\mathrm{M}_\infty) =$

$—\mathrm{Z} + \mathrm{Z}_j - \frac{i}{k_j}|^2 + |c_j|^2.$
(3.22)

CONSTRAINTS (3.19) AND

(3.20) ARE SATISFIED WITH

$\mathrm{Q} = \bar{c}_j e^{i(k_j z + \overline{k_j z})} \overline{|z+z_j|^2+|c_j|^2,}$
(3.23)

Therefore, the gauge transformation (3.1)

generates a single lump solution of the DS-II equations (1.3)

from a single lump solution of the IS-II equations (1.1).

The eigenfunctions of the Lax operator for the DS-II equations

are generated by the gauge transformation

$\boldsymbol{\mu}_{DS} = m_{\infty}^{-1}\boldsymbol{\mu}$ to the form:

$\boldsymbol{\mu}_{DS}(z,k) = \mathbf{e}_1 + \frac{i}{k-k_j}$

$\boldsymbol{\Phi}_{DS}(z)$

(3.24)

and

$\Phi_{DS1}(z) = \frac{\bar{z}+\bar{z}_j}{|z+z_j|^2+|c_j|^2}$,

$\Phi_{DS2}(z) =$

—z+z$_j|^2 + |c_j|^2$.

(3.25)

4 The adjoint linear problem

Let $\mathcal{L}$ be a linear operator that maps $L^2(R^2)$

to $L^2(R^2)$. We define the adjoint operator $\mathcal{L}^a$

BY THE INNER PRODUCT FOR TWO VECTOR FUNCTIONS

$f, g \in L^2(R^2)$:

$$\langle \mathbf{f} | \mathcal{L} \mathbf{g} \rangle = \langle \mathcal{L}^{+} \mathbf{f} | \mathbf{g} \rangle, \qquad \mathrm{L}^{+} = \overline{\mathcal{L}^{a}}, \tag{4.1}$$

WHERE THE STANDARD INNER PRODUCT IS

$$\langle \mathbf{f} | \mathbf{g} \rangle = \iint (\bar{f}_1 g_1 + \bar{f}_2 g_2 dz \wedge d\bar{z}. \tag{4.2}$$

IF $\mathcal{L}\boldsymbol{\Psi} = \mathbf{0}$ IS A LINEAR PROBLEM WITH AN EIGENFUNCTION $\boldsymbol{\Psi} \in \mathrm{L}^2(R^2)$, THEN $\mathcal{L}^a \boldsymbol{\Psi}^a = \mathbf{0}$ IS THE ADJOINT PROBLEM WITH AN ADJOINT EIGENFUNCTION $\boldsymbol{\Psi}^a \in L^2(R^2)$, WHICH TAKES THE FORM:

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \mathbf{\Psi}^a = -\frac{1}{2} \left(\partial_z - \partial_{\bar{z}} \right.$$

$$\left(P^+(z) - \sigma_3\right) \mathbf{\Psi}^a,$$

(4.3)

WHERE $P^+(z) = P(z)$.

THE FUNDAMENTAL MATRIX OF

ADJOINT EIGENFUNCTIONS FOR THE IS-II EQUATIONS IS

$$\mathbf{\Psi}^a(z) =$$

$$\Big(\boldsymbol{\chi}^a(z,k) e^{ik\overline{z}},$$

$$\boldsymbol{\mu}^a(z,k) e^{-ikz},$$

(4.4)

WHERE $\boldsymbol{\chi}^a(z,k)$ AND

$\boldsymbol{\mu}^a(z,k)$ SOLVES THE

ADJOINT PROBLEMS TO (2.6) AND (2.7), RESPECTIVELY:

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \boldsymbol{\chi}^a$$

$$+ \text{ IK} \left(\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right.$$

$$\boldsymbol{\chi}^a = -\mathcal{P}_k^a$$

$$\boldsymbol{\chi}^a, (4.5)$$

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix} \boldsymbol{\mu}^a$$

- IK $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}^a =$

- $\mathrm{P}_k^a \boldsymbol{\mu}^a$,

(4.6)

WHERE

$\mathrm{P}_k^a = \frac{1}{2} (\partial_z$

- $\partial_{\bar{z}} - ik$

$(P^+(z) - \sigma_3)\,.$

(4.7)

THE EIGENVECTORS

$\boldsymbol{\chi}^a(z,k)$ AND

$\boldsymbol{\mu}^a(z,k)$ SATISFY

THE BOUNDARY CONDITIONS ON THE PLANE (x,y):

$\lim_{|z|\to\infty} \boldsymbol{\chi}^a(z,k) =$

$\mathbf{e}_1$,

$\lim_{|z|\to\infty} \boldsymbol{\mu}^a(z,k)$

$= \mathbf{e}_2$.

(4.8)

IF $\mathbf{S}(z)$ IS REAL-VALUED,

THE TWO FUNDAMENTAL SOLUTIONS $\boldsymbol{\chi}^a(z,k)$

AND $\boldsymbol{\mu}^a(z,k)$ OF THE

PROBLEMS (4.5) AND (4.6) ARE RELATED

BY MEANS OF THE INVOLUTION TRANSFORMATION:

$$\boldsymbol{\chi}^a(z,\bar{k}) = -\sigma \overline{\boldsymbol{\mu}^a(z,k)}. \tag{4.9}$$

IN THE LIMIT $|k| \to \infty$, THE EIGENVECTOR

$\boldsymbol{\mu}^a(z,k)$ BECOMES A VECTOR

$\boldsymbol{\mu}^a_\infty(z)$, WHICH SOLVES

THE ALGEBRAIC PROBLEM,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\mu}^a_\infty(z) = -\frac{1}{2} (P(z) - \sigma_3) \boldsymbol{\mu}^a_\infty(z). \tag{4.10}$$

COMPONENTS OF THE VECTOR $\boldsymbol{\mu}^a_\infty(z)$ ARE

RELATED BY MEANS OF THE ADJOINT STEREOGRAPHICAL VARIABLE,

$\mu^a_{2\infty} \overline{\mu^a_{1\infty}=-}$

$1 + S_3 \overline{{}_{S_1 - iS_2 = -\frac{S_1+iS_2}{1-S_3}}}.$

(4.11)

WE DEFINE THE ADJOINT MATRIX $m^a_\infty = m^a_\infty(z)$ AS

$\mathrm{M}^a_\infty(z) = (-\sigma$

$\overline{\boldsymbol{\mu}^a_\infty(z)},$

$\boldsymbol{\mu}^a_\infty(z) =$

$\begin{pmatrix} \overline{\mu^a_{2\infty}} & \mu^a_{1\infty} \\ -\overline{\mu^a_{1\infty}} & \mu^a_{2\infty} \end{pmatrix},$

(4.12)

SUCH THAT $\det(m^a_\infty) = |\mu^a_{1\infty}|^2 +$

$—\mu^a_{2\infty}|^2$. AS A RESULT,

$\mathrm{P}^+(z) = m^a_\infty(z)\sigma_3(m^a_\infty)^{-1}(z),$

(4.13)

OR EXPLICITLY,

$S_1 - iS_2 =$

$\overline{\mu^a_{2\infty}}|\mu^a_{1\infty}|^2 + |\mu^a_{2\infty}|^2,$

$S_3 =$

$—\mu^a_{1\infty}|^2 + |\mu^a_{2\infty}|^2.$

(4.14)

THE DIFFERENTIAL PROBLEM (4.6) CAN BE TRANSFORMED TO

THE LINEAR INTEGRAL EQUATION BY MEANS OF THE GREEN FUNCTION,

$\mu^a_1(z,k) = -\frac{1}{2\pi i}$

$\iint \frac{dz'\wedge d\bar{z}'}{\bar{z}'-\bar{z}} (\mathcal{P}^a_k$

$\boldsymbol{\mu}^a)_1(z')$

$\mathrm{E}^{-ik(z'-z)-i\bar{k}(\bar{z}'-\bar{z})},$

$\mu^a_2(z,k) = 1 - \frac{1}{2\pi i}$

$\iint \frac{dz'\wedge d\bar{z}'}{z'-z}$

$(\mathrm{P}^a_k\boldsymbol{\mu}^a)_2(z').$

(4.15)

IF THE POTENTIAL FUNCTION $P(z)$ IN $L^2(R^2)$

BUT NOT IN $L^1(R^2)$,

THE INTEGRAL EQUATIONS (4.15) MAY

HAVE HOMOGENEOUS SOLUTIONS, CORRESPONDING TO

BOUND STATES. THE BOUND

STATES $\boldsymbol{\Phi}^a_j(z)$ ARE

DECAYING SOLUTIONS OF THE EQUATION,

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar z} \end{pmatrix}$$

$$\Phi_j^a - ik_j^a$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Phi_j^a = -\mathcal{P}^a_{k_j^a}\boldsymbol{\Phi}_j^a,$$

(4.16)

SUBJECT TO THE BOUNDARY CONDITIONS,

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}_j^a(z) = \mathbf{e}_2.$$

(4.17)

EACH EIGENVALUE k_j^a OF THE ADJOINT PROBLEM

IS DOUBLE, SINCE THERE EXISTS A LINEARLY INDEPENDENT

EIGENFUNCTION $\boldsymbol{\Phi}_j^{a\prime}(z)$ SOLVING

THE SAME EQUATION (4)

AND RELATED TO $\boldsymbol{\Phi}_j^a(z)$ BY MEANS OF THE INVOLUTION:

$$\Phi_j^{a\prime}(z) = -\sigma\overline{\boldsymbol{\Phi}_j^a(z)}e^{i(k_j^a z + \overline{k_j^a z})}.$$

(4.18)

IT FOLLOWS FROM (4.15) THAT THE

EIGENFUNCTION $\boldsymbol{\mu}^a(z,k)$

SATISFIES THE $\bar\partial$ PROBLEM IN THE DOMAIN

$\mathcal{D}^a = \Big\{ k \in C : k \neq k_j^a,$

J $= 1,\ldots,$N , SUCH THAT

$\bar{k} = -b^a(k)\mathbf{N}_\mu^a(z,k),$

(4.19)

WHERE THE SPECTRAL DATA $b^a(k)$ ARE DEFINED AS

B$^a(k) = \frac{1}{2\pi} \iint dz \wedge d\bar{z}$

$\left(\mathcal{P}_k^a \boldsymbol{\mu}^a\right)_1 (z)$

E$^{-i(kz+\overline{kz})}$

(4.20)

AND THE LINEAR INDEPENDENT EIGENFUNCTION

$\mathbf{N}_\mu^a(z,k)$ SOLVES THE SAME INTEGRAL EQUATION

(4.15) BUT WITH A DIFFERENT (OSCILLATORY)

BEHAVIOR AT INFINITY,

N$_{\mu 1}^a(z,k) = e^{i(kz+\overline{kz})}$

- 1$\overline{_{2\pi i}}$

$\iint \frac{dz' \wedge d\bar{z}'}{\bar{z}'-\bar{z}}$

(P$_k^a \mathbf{N}_\mu^a)_1(z')$

E$^{-i(k(z'-z)+\overline{k(z'-z)})}$,

$\mathrm{N}^a_{\mu 2}(z,k) = -\frac{1}{2\pi i}$

$\iint \frac{dz' \wedge d\bar{z}'}{z'-z}$

$(\mathrm{P}^a_k \mathbf{N}^a_\mu)_2(z')$.

(4.21)

For real-valued $\mathbf{S}(z)$, there exists a local relation between two linearly independent eigenfunctions $\boldsymbol{\mu}^a(z,k)$ and $\mathbf{N}^a_\mu(z,k)$:

$\mathbf{N}^a_\mu(z,k) = -\sigma$

$\overline{\boldsymbol{\mu}^a(z,k)}$

$\mathrm{E}^{i(kz+\overline{kz})}$.

(4.22)

When the eigenfunction $\boldsymbol{\mu}(z,k)$ approaches to the points $k = k^a_j,\ j = 1, ..., n,$ where the bound states exist, it has a singular (pole) behavior in the limit $k \to k^a_j$. We assume the following limiting behavior of $\boldsymbol{\mu}^a(z,k)$:

$\lim_{k \to k^a_j} (\boldsymbol{\mu}^a(z,k)$

$+ \mathrm{I}\overline{{}_{k-k^a_j \boldsymbol{\Phi}^a_j(z)}}$

$= (z + z^a_j)\boldsymbol{\Phi}^a_j(z) + c^a_j$

$\Phi^{a\prime}_j(z),$

(4.23)

WHERE z_j^a, c_j^a ARE ARBITRARY CONSTANTS.

SOLVING THE $\bar{\partial}$ PROBLEM (4)

ON PLANE $k \in C$ SUBJECT TO THE BOUNDARY

CONDITIONS $\lim_{|k|\to\infty} \boldsymbol{\mu}^a(z,k)$

$= \boldsymbol{\mu}^a_\infty(z)$ AND

USING THE LIMITING CONDITIONS (4)

AT THE POLES $k = k_j^a,\ j = 1, ..., n,$

WE DERIVE THE EXPANSION FORMULA,

$\boldsymbol{\mu}^a(z,k) =$

$\boldsymbol{\mu}^a_\infty(z) -$

$\sum_{j=1}^n \frac{i}{k-k_j^a}$

$\Phi_j^a(z) - \frac{1}{2\pi i} \iint$

$\mathbf{dk'} \wedge d\bar{k}' \frac{}{k'-kb^a(k')}$

$\mathbf{N}^a_\mu(z,k')$.

(4.24)

WITH THE USE OF THE LIMITING RELATIONS AS $k \to k_j^a$,

WE CLOSE THE SYSTEM (4)

WITH A SET OF ALGEBRAIC EQUATIONS FOR $j = 1, ..., n$:

$\left(z + z_j^a\right) \boldsymbol{\Phi}_j^a(z)$

$+\ \mathrm{c}_j^a \boldsymbol{\Phi}_j^{a\prime}(z) =$

$\boldsymbol{\mu}^a_\infty(z) - \sum_{l\neq j}^n$

I$\overline{k_j^a - k_l^a \boldsymbol{\Phi}_l^a(z)}$

- $1\overline{2\pi i}$

$\iint$

D$\bar{k}'k' - k_j^a b^a(k')\mathbf{N}^a_\mu(z,k')$.

(4.25)

The system of equations (4)

and (4.25) is incomplete, since

$\boldsymbol{\mu}^a_\infty(z)$ is unknown.

The adjoint problem (4.6) does not exhibit

any simple solution at $k=0$, on contrary to the

direct problem (2.6).

We complete the system

(4)–(4.25)

by deriving the relation between m^a_∞

and m_∞. The relation

appears as the commutativity condition

between the gauge and adjoint transformations.

transformations]Commutativity between the gauge and adjoint

transformations

The adjoint problem for the DS-II equations is

$$\begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}$$

$\boldsymbol{\Psi}^a_{DS} =$

- Q$^+(z)\boldsymbol{\Psi}^a_{DS}$,

(4.26)

WHERE $Q^{+}(z) = -Q(z)$. THE FUNDAMENTAL MATRIX OF THE ADJOINT

OPERATOR FOR THE DS-II EQUATIONS IS $\boldsymbol{\Psi}^{a}_{DS} = (\boldsymbol{\chi}^{a}_{DS}$

E$^{ik\bar{z}}, \boldsymbol{\mu}^{a}_{DS} e^{-ikz},$

WHERE $\boldsymbol{\chi}^{a}_{DS}(z, \bar{k}) = -\sigma$

$\overline{\boldsymbol{\mu}^{a}_{DS}(z,k)}$. ADJOINT SPECTRAL

PROBLEMS (4) AND (4) FOR

THE IS-II AND DS-II EQUATIONS ARE RELATED WITH THE ADJOINT GAUGE

TRANSFORMATION.

Lemma 4.1 *Let $m^{a}_{\infty}(z)$ be defined by*

(4) such that $P^{+}(z) = m^{a}_{\infty}\sigma_3$

$(m^{a}_{\infty})^{-1}$. Solutions of the linear system

(4) are equivalent to solutions of the linear

system (4) with the gauge transformation

$\boldsymbol{\Psi}^{a}_{DS}(z,k) = (m^{a}_{\infty})^{-1}(z)$

$\boldsymbol{\Psi}^{a}(z,k)$, such that

$Q^{+}(z) = (m^{a}_{\infty})^{-1} (D_{\bar{z}}$

$(m^{a}_{\infty})^{TT},$

$$D_{\bar{z}} = \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar{z}} \end{pmatrix}.$$

(4.27)

Proof.

We show by direct differentiation that if

$\mathbf{\Psi}^a$ satisfies (4)

and $\mathbf{\Psi}^a = m^a_\infty$

$\mathbf{\Psi}^a_{DS}$, then

$\mathbf{\Psi}^a_{DS}$ satisfies

(4) with $Q^+(z)$ given in (4.1).

Using (4), we notice that

- $1 \overline{\scriptstyle 2(P^+(z)-\sigma_3)m^a_\infty}$

$$= \begin{pmatrix} 0 & \mu^a_{1\infty} \\ \overline{\mu^a_{1\infty}} & 0 \end{pmatrix}. \tag{4.28}$$

Substitution $\mathbf{\Psi}^a = m^a_\infty$

$\mathbf{\Psi}^a_{DS}$ reduces (4)

to the form:

$\mathrm{D}_{\bar z} m^a_\infty \mathbf{\Psi}^a_{DS} = (\partial_z - \partial_{\bar z}$

$$\begin{pmatrix} 0 & \mu^a_{1\infty} \\ \overline{\mu^a_{1\infty}} & 0 \end{pmatrix}$$

$$\mathbf{\Psi}^a_{DS}. \tag{4.29}$$

Differentiation expands (4)

to the form:

$\mathrm{M}^a_\infty \mathcal{D}_{\bar z}$

$\mathbf{\Psi}^a_{DS} =$

- $\begin{pmatrix} \partial_z \overline{\mu^a_{2\infty}} & \partial_{\bar z} \mu^a_{1\infty} \\ -\partial_z \overline{\mu^a_{1\infty}} & \partial_{\bar z} \mu^a_{2\infty} \end{pmatrix}$

$\mathbf{\Psi}^a_{DS}.$

(4.30)

INVERTING m^a_∞ AND MATCHING WITH (4),

WE EXPRESS $Q^+(z)$ IN TERMS OF m^a_∞ AS

$$\mathrm{Q}^+(z) = \frac{1}{\det(m^a_\infty)}$$

$$\begin{pmatrix} \mu^a_{2\infty}\partial_z \overline{\mu^a_{2\infty}} + \mu^a_{1\infty}\partial_z \overline{\mu^a_{1\infty}} & \mu^a_{2\infty}\partial_{\bar z}\mu^a_{1\infty} - \mu^a_{1\infty}\partial_{\bar z}\mu^a_{2\infty} \\ \overline{\mu^a_{1\infty}}\partial_z \overline{\mu^a_{2\infty}} - \overline{\mu^a_{2\infty}}\partial_z \overline{\mu^a_{1\infty}} & \overline{\mu^a_{1\infty}}\partial_{\bar z}\mu^a_{1\infty} + \overline{\mu^a_{2\infty}}\partial_{\bar z}\mu^a_{2\infty} \end{pmatrix}. \quad (4.31)$$

DIRECT DIFFERENTIATION SHOWS EQUIVALENCE OF REPRESENTATIONS

(4.1) AND (4).

□

Corollary 4.2 *Components of the transformation matrix $m^a_\infty(z)$ in (4) satisfy the constraints:*

$q(z) \ = \ 1\overline{\overline{\mu^a_{2\infty}}}$

$\partial_{\bar z}\mu^a_{1\infty}$, (4.32)

$0 \ = \ \mu^a_{1\infty}\partial_z\overline{\mu^a_{1\infty}}$

$+ \ \mu^a_{2\infty}\partial_z\overline{\mu^a_{2\infty}}.$

(4.33)

if $Q^+ = -Q$.

WE PLOT THE DIAGRAM OF THE GAUGE AND ADJOINT TRANSFORMS FOR THE IS-II AND DS-II EQUATIONS:

$$\begin{array}{ccc} \text{IS-II (2) for } P(z) & \rightarrow & \text{DS-II (3) for } Q(z) \\ \downarrow & & \vdots \\ \text{adjoint IS-II (4) for } P^+(z) & \rightarrow & \text{adjoint DS-II (4) for } Q^+(z) \end{array}$$

(4.34)

THE GAUGE TRANSFORM FROM IS-II TO DS-II IS GIVEN BY LEMMA 3.1. THE ADJOINT TRANSFORM FROM IS-II TO ADJOINT IS-II IS DESCRIBED IN SECTION 4 FOR $P^+ = P$.

THE GAUGE TRANSFORM FROM ADJOINT IS-II TO ADJOINT DS-II IS GIVEN BY LEMMA 5.1.

IT REMAINS TO SHOW THAT THE SYSTEM (4) IS INDEED THE ADJOINT SYSTEM OF (3) WITH $Q^+ = -Q$. THIS WOULD SERVE AS A COMMUTATIVITY CONDITION BETWEEN THE GAUGE AND ADJOINT TRANSFORMS.

WE PROVE THE COMMUTATIVITY CONDITION AND CLOSE THE DIAGRAM (4)

WITH AN IMPORTANT RELATION

BETWEEN $m^a_\infty(z)$ AND $m_\infty(z)$.

Theorem 4.3 *The gauge transform and the adjoint transform*

are commutative if and only if

$m^a_\infty = (m^{-1}_\infty)^+ =$

$m_\infty \overline{\det(m_\infty),}$

(4.35)

such that

$\mu^a_{1\infty} = -$

$+ -\mu_{2\infty}|^2,$

$\mu^a_{2\infty} =$

$-\mu_{1\infty}|^2 + |\mu_{2\infty}|^2,$

(4.36)

and $\det(m^a_\infty)\det(m_\infty) = 1.$

Proof.

A RELATION BETWEEN m_∞ AND m^a_∞ FOLLOWS

FROM COMPARISON OF REPRESENTATIONS (2)

AND (4):

$$\mathrm{P}^{+}(z) = (m_{\infty}^{-1})^{+}\sigma_3 m_{\infty}^{+} =$$
$$\mathrm{M}_{\infty}^{a}\sigma_3(m_{\infty}^{a})^{-1}.$$
(4.37)

ANOTHER RELATION FOLLOWS FROM COMPARISON OF REPRESENTATIONS (3.1) AND (4.1):

$$\mathrm{Q}^{+} = m_{\infty}^{+}\Big(D_{\bar{z}}(m_{\infty}^{-1})^{+T}$$
$${}^{T} = (m_{\infty}^{a})^{-1}\Big(D_{\bar{z}}(m_{\infty}^{a})^{T}$$
$${}^{T}.$$
(4.38)

THE CONSTRAINT (4.3) IS A SUFFICIENT CONDITION FOR THE RELATIONS (4) AND (4). WE PROVE THAT THIS IS ALSO A NECESSARY CONDITION FOR COMMUTABILITY OF THE GAUGE AND ADJOINT TRANSFORMATIONS. BY COMPARING (2) AND (4), WE DERIVE A RELATION BETWEEN $\boldsymbol{\mu}_{\infty}$ AND $\boldsymbol{\mu}_{\infty}^{a}$:

$$\mu_{1\infty}\overline{\mu_{1\infty}^{a}} +$$
$$\mu_{2\infty}\overline{\mu_{2\infty}^{a}} = 0.$$
(4.39)

Using combinations of constraints

(3.19), (3.20),

(4.32), and

(4.33), combined

with the relation (4),

we also derive that

$\mu^a_{1\infty}\mu_{2\infty} = \overline{\mu^a_{1\infty}}$

$\overline{\mu_{2\infty}}, \qquad \mu^a_{2\infty}\mu_{1\infty} = \overline{\mu^a_{2\infty}}$

$\overline{\mu_{1\infty}}.$

(4.40)

The relations (4) and (4.40) are satisfied if and only if

$\mu^a_{1\infty} = -c(z)\overline{\mu_{2\infty}},$

$\mu^a_{2\infty} = c(z)\overline{\mu_{1\infty}},$

(4.41)

where $c \in R$. Using constraints

(3.19), (4.32),

combined with the relation (4),

we find that

$\mathrm{c(z)} = \mathrm{c}_0 \overline{\det(m_\infty),}$

(4.42)

WHERE c_0 IS CONSTANT. IT FOLLOWS FROM THE ASYMPTOTIC LIMITS,

$$\lim_{|z|\to\infty} m_\infty(z) = I, \qquad \lim_{|z|\to\infty} m_\infty^a(z) = I, \tag{4.43}$$

THAT $c_0 = 1$ AND THE FORMULA (4.3) IS THUS PROVED.

□

WE CAN DEVELOP AN ALTERNATIVE PROOF OF OF THEOREM 4.3, WITHOUT USING LEMMA 4.1 AND COROLLARY 4.2. THE DIAGRAM OF THE GAUGE AND ADJOINT TRANSFORMS FOR THE IS-II AND DS-II EQUATIONS CAN BE CLOSED AS:

$$\begin{array}{ccc} \text{IS-II (2) for } P(z) & \rightarrow & \text{DS-II (3) for } Q(z) \\ \downarrow & & \downarrow \\ \text{adjoint IS-II (4) for } P^+(z) & \cdots & \text{adjoint DS-II (4) for } Q^+(z) \end{array} \tag{4.44}$$

HERE WE WILL USE THE ADJOINT SYSTEM FOR DS-II EQUATIONS (4) WITH $Q^+ = -Q$ AND SHOW THAT THE SYSTEM IS RELATED TO THE ADJOINT SYSTEM FOR IS-II EQUATIONS (4) WITH $P^+ = P$ BY MEANS OF THE GAUGE TRANSFORM:

$$\mathbf{\Psi}^a(z,k) = \frac{m_\infty(z)}{\text{DET}(m_\infty(z))}\mathbf{\Psi}^a_{DS}(z,k). \tag{4.45}$$

This relation is equivalent to the gauge transform between the adjoint IS-II and DS-II equations in Lemma 4.1, where $m^a_\infty(z)$ is given by (4.3).

The left-hand-side of (4) is written explicitly in the form,

$$\begin{aligned}
&D_{\bar z}\left(\frac{m_\infty}{\text{DET}m_\infty}\begin{pmatrix}\Psi^a_{1,DS}\\ \Psi^a_{2,DS}\end{pmatrix}\right)\\
&= \frac{1}{\det m_\infty}\begin{pmatrix}(\partial_z\mu_{1\infty})\Psi^a_{1,DS} + \mu_{1\infty}\partial_z\Psi^a_{1,DS} - (\partial_z\bar\mu_{2\infty})\Psi^a_{2,DS} - \bar\mu_{2\infty}\partial_z\Psi^a_{2,DS}\\ (\partial_{\bar z}\mu_{2\infty})\Psi^a_{1,DS} + \mu_{2\infty}\partial_{\bar z}\Psi^a_{1,DS} + (\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS} + \bar\mu_{1\infty}\partial_{\bar z}\Psi^a_{2,DS}\end{pmatrix}\\
&-\frac{1}{(\det m_\infty)^2}\begin{pmatrix}(\mu_{1\infty}\Psi^a_{1,DS} - \bar\mu_{2\infty}\Psi^a_{2,DS})[(\partial_z\mu_{1\infty})\bar\mu_{1\infty} + (\partial_z\mu_{2\infty})\bar\mu_{2\infty}]\\ (\mu_{2\infty}\Psi^a_{1,DS} + \bar\mu_{1\infty}\Psi^a_{2,DS})[\mu_{1\infty}\partial_{\bar z}\bar\mu_{1\infty} + \mu_{2\infty}\partial_{\bar z}\bar\mu_{2\infty}]\end{pmatrix}.
\end{aligned} \tag{4.46}$$

We apply the gauge transformation $Q = \mathcal{D}_z(m_\infty^{-1})m_\infty$ to the adjoint DS-II system (4) and find that

$$\begin{aligned}
&D_{\bar z}\left(\frac{m_\infty}{\text{DET}(m_\infty)}\begin{pmatrix}\Psi^a_{1,DS}\\ \Psi^a_{2,DS}\end{pmatrix}\right)\\
&= \frac{1}{\text{DET}m_\infty}\begin{pmatrix}(\partial_z\mu_{1\infty})\Psi^a_{1,DS} - (\partial_z\bar\mu_{2\infty})\Psi^a_{2,DS} - \bar\mu_{2\infty}\partial_z\Psi^a_{2,DS}\\ (\partial_{\bar z}\mu_{2\infty})\Psi^a_{1,DS} + \mu_{2\infty}(\partial_{\bar z}\Psi^a_{1,DS}) + (\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}\end{pmatrix}\\
&+\frac{1}{(\text{DET}m_\infty)^2}\begin{pmatrix}\mu_{1\infty}(\bar\mu_{1\infty}\partial_{\bar z}\bar\mu_{2\infty} - \bar\mu_{2\infty}\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}\\ \bar\mu_{1\infty}(\mu_{2\infty}\partial_z\mu_{1\infty} - \mu_{1\infty}\partial_z\mu_{2\infty})\Psi^a_{1,DS}\end{pmatrix}\\
&+\frac{1}{(\text{DET}m_\infty)^2}\begin{pmatrix}-|\mu_{1\infty}|^2(\partial_z\mu_{1\infty})\Psi^a_{1,DS} + \bar\mu_{1\infty}\bar\mu_{2\infty}(\partial_z\mu_{1\infty})\Psi^a_{2,DS}\\ -|\mu_{1\infty}|^2(\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS} - \mu_{2\infty}\mu_{1\infty}(\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{1,DS}\end{pmatrix}\\
&+\frac{1}{(\text{DET}m_\infty)^2}\begin{pmatrix}-\mu_{1\infty}\bar\mu_{2\infty}(\partial_z\mu_{2\infty})\Psi^a_{1,DS} + \bar\mu^2_{2\infty}(\partial_z\mu_{2\infty})\Phi^a_{21}\\ -\mu_{2\infty}\bar\mu_{1\infty}(\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{2,DS} - \mu^2_{2\infty}(\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{1,DS}\end{pmatrix}.
\end{aligned} \tag{4.47}$$

On the other hand, the right-hand-side of (4) can be rewritten as

$$\begin{aligned}
&-\frac{1}{2}\left(\partial_z - \partial_{\bar z}\right)\left(P(z) - \sigma_3\right)\left(\frac{m_\infty}{\text{DET}(m_\infty)}\begin{pmatrix}\Psi^a_{1,DS}\\ \Psi^a_{2,DS}\end{pmatrix}\right)\\
&= \frac{1}{\det m_\infty}\begin{pmatrix}(\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{2,DS} + \bar\mu_{2\infty}\partial_{\bar z}\Psi^a_{2,DS} - (\partial_z\bar\mu_{2\infty})\Psi^a_{2,DS} - \bar\mu_{2\infty}\partial_z\Psi^a_{2,DS}\\ (\partial_{\bar z}\mu_{2\infty})\Psi^a_{1,DS} + \mu_{2\infty}\partial_{\bar z}\Psi^a_{1,DS} - (\partial_z\mu_{2\infty})\Psi^a_{1,DS} - \mu_{2\infty}\partial_z\Psi^a_{1,DS}\end{pmatrix}\\
&-\frac{1}{(\det m_\infty)^2}\begin{pmatrix}\bar\mu_{2\infty}\Psi^a_{2,DS}\left(\mu_{1\infty}\partial_{\bar z}\bar\mu_{1\infty} + \mu_{2\infty}\partial_{\bar z}\bar\mu_{2\infty} - (\partial_z\mu_{1\infty})\bar\mu_{1\infty} - (\partial_z\mu_{2\infty})\bar\mu_{2\infty}\right)\\ \mu_{2\infty}\Psi^a_{1,DS}\left(\mu_{1\infty}\partial_{\bar z}\bar\mu_{1\infty} + \mu_{2\infty}\partial_{\bar z}\bar\mu_{2\infty} - (\partial_z\mu_{1\infty})\bar\mu_{1\infty} - (\partial_z\mu_{2\infty})\bar\mu_{2\infty}\right)\end{pmatrix}.
\end{aligned} \tag{4.48}$$

We prove that the right-hand-sides of (4.47) and (4.48) are equal by showing that the

FOLLOWING TWO EXPRESSIONS ARE EQUAL:

$$\frac{1}{\det m_\infty}\begin{pmatrix} (\partial_z\mu_{1\infty})\Psi^a_{1,DS}+\mu_{1\infty}\frac{(\bar\mu_{1\infty}\partial_{\bar z}\bar\mu_{2\infty}-\bar\mu_{2\infty}\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}}{\det m_\infty} \\ (\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}+\bar\mu_{1\infty}\frac{(\mu_{2\infty}\partial_z\mu_{1\infty}-\mu_{1\infty}\partial_z\mu_{2\infty})\Psi^a_{1,DS}}{\det m_\infty}\end{pmatrix}$$
$$+\frac{1}{(\det m_\infty)^2}\begin{pmatrix} -|\mu_{1\infty}|^2(\partial_z\mu_{1\infty})\Psi^a_{1,DS}-\mu_{1\infty}\bar\mu_{2\infty}(\partial_z\mu_{2\infty})\Psi^a_{1,DS} \\ -|\mu_{1\infty}|^2(\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}-\mu_{2\infty}\bar\mu_{1\infty}(\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{2,DS}\end{pmatrix}, \tag{4.49}$$

AND

$$\frac{1}{\det m_\infty}\begin{pmatrix} (\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{2,DS}+\bar\mu_{2\infty}\frac{(\mu_{2\infty}\partial_z\mu_{1\infty}-\mu_{1\infty}\partial_z\mu_{2\infty})\Psi^a_{1,DS}}{\det m_\infty} \\ -(\partial_z\mu_{2\infty})\Psi^a_{1,DS}-\mu_{2\infty}\frac{(\bar\mu_{1\infty}\partial_{\bar z}\bar\mu_{2\infty}-\bar\mu_{2\infty}\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS}}{\det m_\infty}\end{pmatrix}$$
$$-\frac{1}{(\det m_\infty)^2}\begin{pmatrix} |\mu_{2\infty}|^2(\partial_{\bar z}\bar\mu_{2\infty})\Psi^a_{2,DS}+\mu_{1\infty}\bar\mu_{2\infty}(\partial_{\bar z}\bar\mu_{1\infty})\Psi^a_{2,DS} \\ -|\mu_{2\infty}|^2(\partial_z\mu_{2\infty})\Psi^a_{1,DS}-\bar\mu_{1\infty}\mu_{2\infty}(\partial_z\mu_{1\infty})\Psi^a_{1,DS}\end{pmatrix} \tag{4.50}$$

THEREFORE, THE GAUGE AND ADJOINT TRANSFORMS ARE COMMUTATIVE UNDER THE CONDITION (4.3).

Lemma 4.4 *Spectral data of the adjoint problem (4)*

are related to the spectral data of the direct problem

(2) as

$b^a(k) = \overline{b(k)},$

$k^a_j = k_j,$

$z^a_j = z_j,$

$c^a_j = -\bar c_j.$

(4.51)

Proof.

APPLYING THE GAUGE TRANSFORMS OF LEMMA 3.1 AND 5.1

TO SPECTRAL REPRESENTATIONS (2)

AND (4) FOR $\boldsymbol{\mu}(z,k)$

AND $\boldsymbol{\mu}^a(z,k)$, WE OBTAIN

THE SPECTRAL REPRESENTATIONS FOR $\boldsymbol{\mu}_{DS}(z,k)$ AND $\boldsymbol{\mu}^a_{DS}(z,k)$:

$\boldsymbol{\mu}_{DS}(z,k) =$

$\mathbf{e}_1 + \sum_{j=1}^{n}$

I$\overline{k-k_j(m_\infty)^{-1}\boldsymbol{\Phi}_j(z)}$

$+\ 1\overline{2\pi i \iint \frac{dk'\wedge d\bar{k}'}{k'-k}}$

B(K')(M$_\infty)^{-1}\mathbf{N}_\mu(z,k'), (4.52)$

$\boldsymbol{\mu}^a_{DS}(z,k) =$

$\mathbf{e}_2 - \sum_{j=1}^{n} \frac{i}{k-k^a_j}$

(M$^a_\infty)^{-1}\boldsymbol{\Phi}^a_j(z)-$

$1\overline{2\pi i \iint}$

D$\bar{k}'k' - kb^a(k')(m^a_\infty)^{-1}\mathbf{N}^a_\mu(z,k').$

(4.53)

SIMILARLY, APPLYING GAUGE TRANSFORMS TO THE LIMITING FORMULA (2) AND (4), WE OBTAIN

$\lim_{k\to k_j} (\boldsymbol{\mu}_{DS}(z,k)$

- I$\overline{k-k_j(m_\infty)^{-1}\boldsymbol{\Phi}_j(z)}$

$= (z+z_j)(m_\infty)^{-1}\boldsymbol{\Phi}_j(z) + c_j$

(M$_\infty)^{-1}\boldsymbol{\Phi}'_j(z), (4.54)$

$\lim_{k\to k^a_j} \Big(\boldsymbol{\mu}^a_{DS}(z,k) + \frac{i}{k-k^a_j}(m^a_\infty)^{-1}$

$\Phi^a_j(z) = (z+z^a_j)(m^a_\infty)^{-1}$

$\Phi^a_j(z) + c^a_j(m^a_\infty)^{-1}\boldsymbol{\Phi}^{a\prime}_j(z).$

(4.55)

Since eigenfunctions of the DS-II and IS-II equations are related by the gauge transform as

$$
\begin{array}{ll}
(m_\infty)^{-1}\mathbf{N}_\mu = \mathbf{N}_{\mu,DS}, & (m^a_\infty)^{-1}\mathbf{N}^a_\mu = \mathbf{N}^a_{\mu,DS}, \\
(m_\infty)^{-1}\mathbf{\Phi}_j = \mathbf{\Phi}_{j,DS}, & (m^a_\infty)^{-1}\mathbf{\Phi}^a_j = \mathbf{\Phi}^a_{j,DS}, \\
(m_\infty)^{-1}\mathbf{\Phi}'_j = \mathbf{\Phi}'_{j,DS}, & (m^a_\infty)^{-1}\mathbf{\Phi}'^a_j = \mathbf{\Phi}'^a_{j,DS}.
\end{array}
\quad (4.56)
$$

we find the representations (4) and (4.52) and limiting formulas (4.53) and (4.55) for the DS-II equations with the same spectral data $\{b(k), k_j, c_j, z_j\}$ and $\{b^a(k), k^a_j, c^a_j, z^a_j\}$. The symmetry relations (4.4) were proved for the DS-II equations in [PS00b].

□

The relation (4.3) between $m^a_\infty(z)$ and $m_\infty(z)$ closes the system of equations (4) and (4.25) for eigenfunctions of the adjoint problem. For the explicit example of the single lump solution of the IS-II equations, we find from (3), (3), and (4.3) that

$\mu^a_{1\infty} =$

$\mathrm{E}^{i(k_j z + \bar{k_j} z)}$

$\bar{k}_j(|z + z_j|^2 + |c_j|^2),$

$\mu^a_{2\infty} = 1-$

$\mathrm{K}_j(|z + z_j|^2 + |c_j|^2).$

(4.57)

The algebraic system (4.25)

for $b(k) = 0$ and $n = 1$ becomes closed

with (4.3) and (4.4)

and it has an exact solution:

$\Phi^a_{1j}(z) =$

$\mathrm{E}^{i(k_j z + \overline{k_j z})}$

$\Big[|z + z_j|^2 + |c_j|^2 + \frac{i}{k_j}$

$(z + z_j - \bar{z} - \bar{z}_j)$

$(\text{—Z+Z}_j|^2 + |c_j|^2)^2,$ (4.58)

$\Phi^a_{2j}(z) =$

$\text{—C}_j|^2) - \frac{i}{k_j}((\bar{z} + \bar{z}_j)^2 + |c_j|^2)$

$(\text{—Z+Z}_j|^2 + |c_j|^2)^2.$

(4.59)

Using the gauge transformation

$\boldsymbol{\mu}^a_{DS} = (m^a_\infty)^{-1}$

$\boldsymbol{\mu}^a$, WE FIND THE EIGENFUNCTION OF

THE ADJOINT PROBLEM FOR DS-II EQUATIONS FROM (4.4)

AND (4.53):

$\boldsymbol{\mu}^a_{DS}(z,k) = \mathbf{e}_2 - \frac{i}{k-k_j}$

$\boldsymbol{\Phi}^a_{j,DS}(z),$

(4.60)

WHERE THE BOUND STATE OF THE ADJOINT PROBLEM

$\boldsymbol{\Phi}^a_{j,DS}(z)$ IS

$\Phi^a_{1j,DS}(z) =$

—z+z$_j|^2 + |c_j|^2$,

$\Phi^a_{2j,DS}(z) = \frac{\bar{z}+\bar{z}_j}{|z+z_j|^2+|c_j|^2}.$

(4.61)

5 Applications

WE USE THE MAIN RESULTS OF OUR ANALYSIS TO STUDY

THE SPECTRAL DECOMPOSITION AND SURFACE REPRESENTATION THEOREMS

FOR THE ISHIMORI–II EQUATIONS.

THE SPECTRAL DECOMPOSITION THEOREM FOR THE DS-II EQUATIONS

WAS PROVED IN [PS00B]. STRUCTURAL INSTABILITY OF LUMP

SOLUTIONS OF THE DS-II EQUATIONS WAS STUDIED BY MEANS OF perturbation series expansions over a complete set of eigenfunctions of the linear problem (3).

Using orthogonality and completeness of eigenfunctions of the linear problem for the DS-II equations and the gauge transformation between DS-II and IS-II equations, we construct a basis of the spectral decomposition for the IS-II equations. Similar to [PS00b], the perturbation series expansions can be used to prove structural instability of lump solutions of the IS-II equations.

Theorem 5.1 *An arbitrary scalar function* $f(z) \in L^2(R^2)$ *can be decomposed through the set of eigenfunctions* $S = \{[m_\infty^{-1}\mathbf{N}_\mu]_2(z,k),$ $[m_\infty^{-1}\mathbf{\Phi}_j]_2, j = 1, ..., n\}$ *as follows*

$$f(z) = \iint dk \wedge d\bar{k}\, \alpha(k)[m_\infty^{-1}\mathbf{N}_\mu]_2(z,k) + \sum_{j=1}^{n} \alpha_j [m_\infty^{-1}\Phi_j]_2(z), \tag{5.1}$$

where

$\alpha(k) = -\frac{1}{4\pi^2}\,([(m^a_\infty)^{-1}$

$\mathbf{N}^a_\mu]_1(k), f,$

$\alpha_j = \frac{1}{2\pi i}\,($

$[(m^a_\infty)^{-1}\mathbf{\Phi}^a_j]_1, f,$

(5.2)

and the inner product for scalar functions is defined by

$(g(k'), f(k)) =$

$\iint g(z,k')f(z,k)dz \wedge d\bar{z}.$

(5.3)

Proof.

The proof follows from Propositions 3.3–3.5 of

[PS00b].

□

Orthogonality and completeness relations

for eigenfunctions of the IS-II equations

can be formulated explicitly as follows.

Lemma 5.2 *Eigenfunctions*

$[m_\infty^{-1}\mathbf{N}_\mu]_2(z,k)$ *and*

$[m_\infty^{-1}\mathbf{\Phi}_j]_2$

are orthogonal with respect to the eigenfunctions

$[(m^a_\infty)^{-1}\mathbf{N}^a_\mu]_1(z,k)$ *and*

$[(m^a_\infty)^{-1}\mathbf{\Phi}^a_j]_1$

according to the orthogonality relations:

$\Big([(m^a_\infty)^{-1}\mathbf{N}^a_\mu]_1(k'),$

$[m^{-1}_\infty\mathbf{N}_\mu]_2(k)$

$=$ - $2\ \pi^2 i\delta(k'-k), (5.4)$

$\Big([(m^a_\infty)^{-1}\mathbf{N}^a_\mu]_1(k'),$

$[m^{-1}_\infty\mathbf{\Phi}_j]_2 = 0, (5.5)$

$\Big([(m^a_\infty)^{-1}\mathbf{\Phi}^a_j]_1,$

$[m^{-1}_\infty\mathbf{N}_\mu]_2(k) = 0, (5.6)$

$([(m^a_\infty)^{-1}\mathbf{\Phi}^a_l]_1,$

$[m^{-1}_\infty\mathbf{\Phi}_j]_2 =$

$2\pi i\delta_{j,l}.$

(5.7)

Corollary 5.3 *The orthogonality relations for eigenfunctions of the IS-II equations take the form of weighted inner vector products defined by (4):*

$([(m^a_\infty)^{-1}\mathbf{g}^a]_1,$

$[m^{-1}_\infty\mathbf{f}]_2(k) =$

$\langle\sigma\overline{g^a_1}\mathbf{e}_1|f_2\mathbf{e}_2\rangle$

- $1_{\overline{2\langle\sigma\mathbf{g}^a|}}$

$(\ P$ - $\sigma_3)\mathbf{f}\rangle.$

(5.8)

Remark 5.4 *The orthogonality relations*

(5.4)–(5.7) for

the IS-II equations can be proved directly

from the linear systems (2)

and (4), with the use

of weighted inner products in the

right-hand-side of (5.3).

Lemma 5.5 *Eigenfunctions*

$[m_\infty^{-1}\mathbf{N}_\mu]_2(z,k)$ *and*

$[m_\infty^{-1}\mathbf{\Phi}_j]_2$

are complete with respect to the adjoint eigenfunctions

$[(m_\infty^a)^{-1}\mathbf{N}_\mu^a]_1(z,k)$ *and*

$[(m_\infty^a)^{-1}\mathbf{\Phi}_j^a]_1$

according to the completeness relation:

$\delta(z'-z) = -\frac{1}{2\pi^2 i}$

$\iint dk \wedge d\bar{k}$

$[(m_\infty^a)^{-1}\mathbf{N}_\mu^a]_1(z',k)$

$[m_\infty^{-1}\mathbf{N}_\mu]_2(z,k)$

- $1 \frac{}{\pi \sum_{j=1}^{n} [(m_\infty^a)^{-1}}$

$\Phi_j^a]_1(z')[m_\infty^{-1}\mathbf{\Phi}_j]_2(z).$

(5.9)

The linear problem for the DS-II equations and its adjoint

was used in [K96, K00] to parametrize surfaces and

integrable evolutions of surfaces in R^3 and

R^4. Their results

generalize the classical Weierstrass-Enneper representation for

conformal parametrizations of minimal surfaces in R^3, or

stationary Lagragian surfaces in R^4 [H02, HR03].

Similarly, we formulate the surface representation theorem

by using the gauge transformation between the DS-II and IS-II

equations.

Theorem 5.6 *Define the generalized Weierstrass formulae*

$X^1 + iX^2 = \int_\Gamma$

$(-\phi_1\phi_2 dz' + \psi_1\psi_2 d\bar{z}')$,

$X^1 - iX^2 = \int_\Gamma$

$(\bar{\psi}_1\bar{\psi}_2 dz' - \bar{\phi}_1\bar{\phi}_2 d\bar{z}')$,

$X^3 + iX^4 = \int_\Gamma (\bar{\psi}_2\phi_1 dz' + \psi_1\bar{\phi}_2 d\bar{}$

z'),

$X^3 - iX^4 = \int_\Gamma (\bar{\psi}_1\phi_2 dz' + \psi_2\bar{}$

$\phi_1 d\bar{z}')$,

where

$(\phi_1, \psi_1)^T = m_\infty^{-1}$

$\boldsymbol{\mu}$*(z,k)* e^{ikz},

$(\psi_2, -\phi_2)^T = (m_\infty^a)^{-1}$

$\boldsymbol{\mu}^a(z,k)e^{-ikz}$

(5.10)

and Γ *is a contour in* C.

Then, the Weierstrass formulas define a

conformal immersion of a surface into

$R^4 : X^i(z, \bar{z}) : C \mapsto$

R^4. *The induced metric of the surface is*

$$ds^2 = u_1 u_2 dz d\bar{z},$$
(5.11)

where $u_\alpha = |\phi_\alpha|^2 + |\psi_\alpha|^2$,

$\alpha = 1, 2$, *and the Gaussian and mean curvatures are*

respectively:

$K =$ - $2\frac{}{u_1 u_2}$

$\bar{z} \log(u_1 u_2)$,

$H^2 = \frac{4}{u_1 u_2}$

-$\bar{\mu}_{2\infty} \partial_{\bar{z}} \bar{\mu}_{1\infty}|^2$

$(-\mu_{1\infty}|^2 + |\mu_{2\infty}|^2)^2$.
(5.12)

Proof.

Proof follows from Theorem 3 and Proposition 1 in

[K00].

□

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