Duality of compact groups and Hilbert C*-systems for C*-algebras with a nontrivial center

Hellmut Baumgärtel

Mathematical Institute, University of Potsdam,
Am Neuen Palais 10, PF 601 553,
D-14415 Potsdam, Germany.

baumg@rz.uni-potsdam.de

Fernando Lledó

Institute for Pure and Applied Mathematics,
RWTH-Aachen, Templergraben 55,
D-52062 Aachen, Germany.

lledo@iram.rwth-aachen.de

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Abstract

The new duality theory for compact groups established in [19] by Doplicher and Roberts (DR) characterizes the dual object $\widehat{\mathcal{G}}$ of a compact group $\mathcal{G}$ by means of a particular type of tensor C*-category $\mathcal{T}$ which we call a DR-category. $\mathcal{T}$ may be realized as a full subcategory of the category of endomorphisms of a suitable C*-algebra $\mathcal{A}$ with trivial center $\mathcal{Z}(\mathcal{A}) = \mathcal{Z} = \mathbb{C}\mathbb{1}$. In this context the DR-duality theory gives a bijective correspondence between the pair $\{\mathcal{A}, \mathcal{T}\}$ and a special kind of C*-dynamical system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ that, in addition, contains explicitly the representation category of the compact group $\mathcal{G}$ and which we call a Hilbert C*-system. The C*-algebra $\mathcal{A}$ is the fixed point algebra of the group action and $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$.

In the present paper we prove a duality theory for compact groups in the case when the C*-algebra $\mathcal{A}$ has a nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ and the relative commutant of the corresponding Hilbert C*-system satisfies the minimality condition

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}\,,$$

as well as a technical condition called regularity. The abstract characterization of the mentioned Hilbert C*-system is expressed by means of an inclusion of C*-categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathbb{C}}$ is a suitable DR-category and $\mathcal{T}$ a full subcategory of the category of endomorphisms of $\mathcal{A}$. Both categories have the same objects and the arrows of $\mathcal{T}$ can be generated from the arrows of $\mathcal{T}_{\mathbb{C}}$ and the center $\mathcal{Z}$.

A crucial new element that appears in the present analysis is an abelian group $\mathfrak{C}(\mathcal{G})$, which we call the chain group of $\mathcal{G}$, and that can be constructed from certain equivalence relation defined on $\widehat{\mathcal{G}}$. The chain group can be related to the character group of the center of $\mathcal{G}$ and determines the action of irreducible endomorphisms of $\mathcal{A}$ when restricted to $\mathcal{Z}$. Moreover, $\mathfrak{C}(\mathcal{G})$ encodes the possibility of defining a symmetry ϵ also for the larger category $\mathcal{T}$ of the previous inclusion.

1 Introduction

The superselection theory in algebraic quantum field theory, as stated by the Doplicher-Haag-Roberts (DHR) selection criterion [27, 13, 14], led to a profound body of work, culminating in the general Doplicher-Roberts (DR) duality theory for compact groups [19]. The DHR criterion selects a distinguished class of "admissible" representations of a quasilocal algebra $\mathcal{A}$ of observables, which has trivial center $\mathcal{Z} := \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$. This corresponds to the selection of a so-called DR-category $\mathcal{T}$, which is a full subcategory of the category of endomorphisms of the C*-algebra $\mathcal{A}$ (see Definition 3.19 below). Furthermore, from this endomorphism category $\mathcal{T}$ the DR-analysis constructs a C*-algebra $\mathcal{F} \supset \mathcal{A}$ together with a compact group action $\alpha : \mathcal{G} \ni g \to \alpha_g \in \mathrm{Aut}\,\mathcal{F}$ such that:

- $\mathcal{A}$ is the fixed point algebra of this action
- $\mathcal{T}$ coincides with the category of all "canonical endomorphisms" of $\mathcal{A}$, associated with the pair $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ (cf. Subsection 3.2).

$\mathcal{F}$ is called a Hilbert extension of $\mathcal{A}$ in [10]. Physically, $\mathcal{F}$ is identified as a field algebra and $\mathcal{G}$ with a global gauge group of the system. The pair $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$, which we call *Hilbert C*-system* (cf. Definition 2.1; the name *crossed product* is also used), is uniquely determined by $\mathcal{T}$ up to $\mathcal{A}$-module isomorphisms. Conversely, $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ determines uniquely its category of all canonical endomorphisms. Therefore $\{\mathcal{T}, \mathcal{A}\}$ can be seen as the abstract side of the representation category of a compact group, while $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ corresponds to the concrete side of the representation category of $\mathcal{G}$, and, roughly, any irreducible representations of $\mathcal{G}$ is explicitly realized within the Hilbert C*-system. One can state the equivalence of the "selection principle", given by $\mathcal{T}$ and the "symmetry principle", given by the compact group $\mathcal{G}$. This is one of the crucial theorems of the Doplicher-Roberts theory.

In the DR-theory the center $\mathcal{Z}$ of the C*-algebra $\mathcal{A}$ plays a peculiar role: As stated above, if $\mathcal{A}$ corresponds to the inductive limit of a net of local C*-algebras indexed by open and bounded regions of Minkowski space, then the triviality of the center of $\mathcal{A}$ is a consequence of standard assumptions on the net of local C*-algebras. But, in general, the C*-algebra appearing in the DR-theorem does not need to be a quasilocal algebra and, in fact, one has to assume explicitly that $\mathcal{Z} = \mathbb{C}\mathbb{1}$ in this context (see [19, Theorem 6.1]). Finally, we quote from the introduction of the article [19]: "There is, however, no known analogue of Theorem 4.1 of [18] for a C*-algebra with a non-trivial center and hence nothing resembling a "duality" in this more general setting."

The aim of the present paper is to show that *there is* a duality theory for compact groups in the case of a nontrivial center, if the relative commutant of the corresponding Hilbert C*-system satisfies the following minimality condition:

$$\mathcal{A}' \cap \mathcal{F} = \mathcal{Z} \tag{1}$$

(cf. Theorem 4.12). The essence of the previous result is that now the abstract characterization of the mentioned Hilbert C*-system is expressed by means of an inclusion of C*-categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathbb{C}}$ is a suitable DR-category and $\mathcal{T}$ a full subcategory of the category of endomorphisms of $\mathcal{A}$. Both categories have the same objects and the arrows of $\mathcal{T}$ can be generated from the arrows of $\mathcal{T}_{\mathbb{C}}$ and the center $\mathcal{Z}$.

Several new elements appear in the generalization of the DR-theory studied here. The crucial one is an abelian group $\mathfrak{C}(\mathcal{G})$, which we call the *chain group* of $\mathcal{G}$, and that can be constructed from certain equivalence relation defined on $\widehat{\mathcal{G}}$, the dual object of the compact group $\mathcal{G}$. The chain group, which is interesting in itself and can be related to the character group of the center of $\mathcal{G}$, determines the action of irreducible endomorphisms of $\mathcal{A}$ when restricted to the center $\mathcal{Z}(\mathcal{A})$. Moreover, $\mathfrak{C}(\mathcal{G})$ appears explicitly in the construction of a family of examples realizing the inclusion of categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ mentioned above (cf. Section 7). Finally, the chain group encodes also the possibility of defining a symmetry ϵ also for the larger category $\mathcal{T}$ of the previous inclusion.

There are several reasons that motivate the generalization of the DR-theory for systems satisfying the minimality condition (1) for the relative commutant:

(i) In this context there is a nice intrinsic characterization of the Hilbert C*-systems satisfying (1) and a further technical condition called regularity (cf. Theorems 4.9 and 4.12). One can also prove several results in the spirit of the DR-theory: for example, the category

$\mathcal{T}$ is isomorphic to the category $\mathcal{M}_\mathcal{G}$ of all free right Hilbert $\mathcal{Z}$-modules generated by the algebraic Hilbert spaces in $\mathcal{T}_\mathcal{G}$ (cf. Proposition 4.4).

(ii) In the context of compact groups, the equation (1) is also convenient for technical reasons. The minimality of the relative commutant implies that irreducible endomorphisms are mutually disjoint (cf. Proposition 4.3) and this fact is crucial to have a nice decomposition of objects in terms of irreducible ones (cf. Proposition 4.5).

(iii) The nontriviality of the center gives also the possibility to a more geometrical interpretation of the DR-theory. Indeed, from Gelfand's theorem we have $\mathcal{Z} \cong C(\Gamma)$, Γ a compact Hausdorff space, and in certain situations the Hilbert C*-system $\{\mathcal{F}, \alpha_\mathcal{G}\}$ is a bundle over Γ, where the Hilbert C*-system corresponding to the base point $\lambda \in \Gamma$ is of a DR-type with the same group $\mathcal{G}$. Here the chain group plays again an important role. This more geometrical line of research has lead to recent developments in the context of vector bundles (cf. [36, 37, 35]).

(iv) There are physically relevant examples that satisfy the condition (1). For example, this equation is presented in [31] as a "new principle". Moreover, the elements of the center $\mathcal{Z}$ of $\mathcal{A}$ may be interpreted as classical observables contained in the quasilocal algebra.

(v) The present generalization of the DR-theory in the context minimal and regular Hilbert C*-systems has also found application in the context of superselection theory for systems carrying quantum constraints (see [5] as well as [24, 26] for a C*-algebraic formulation of the theory of quantum constraints).

The paper is structured in 9 sections: in Section 2 we introduce the notion of a Hilbert C*-system (cf. Definition 2.1) and give a detailed account of its properties. Hilbert C*-systems are special types of C*-dynamical systems $\{\mathcal{F}, \alpha_\mathcal{G}\}$ that, in addition, contain the information of the representation category of $\mathcal{G}$. They also satisfy important properties, which are interesting in themselves, as for example: $\mathcal{F}$ is simple iff the fixed point algebra $\mathcal{A}$ is simple (cf. Proposition 2.7); one can naturally introduce spectral subspaces of $\mathcal{F}$ and prove Parseval-type equations for a suitable $\mathcal{A}$-valued scalar product on $\mathcal{F}$ (cf. Proposition 2.5). Finally, Hilbert C*-systems provide a natural and concrete frame to describe the DR-theory as well as the generalization to the nontrivial center situation that we study here. In Section 3 we study the important relation between two C*-categories $\mathcal{T}_\mathcal{G}$ and $\mathcal{T}$ that are naturally associated with a Hilbert C*-system. In general, $\mathcal{T}_\mathcal{G}$ is a subcategory of $\mathcal{T}$ and this inclusion turns out to be characteristic for the inverse result stated in Theorem 4.12. In Section 4 the main duality theorems are stated in the context of minimal and regular Hilbert C*-systems. The next section defines the notion of an irreducible object and introduces the chain group of $\mathcal{G}$, denoted by $\mathfrak{C}(\mathcal{G})$. We give examples of chain groups for several finite and compact Lie groups and state the conjecture that the chain group is isomorphic to the character group of the center of $\mathcal{G}$. There is a close relation between the chain group and the set of irreducible canonical endomorphisms: an irreducible canonical endomorphism of $\mathcal{A}$ restricted to the center $\mathcal{Z}$ turns out to be an automorphism of $\mathcal{Z}$. We show that there is a group homomorphism between the chain group and the subgroup of $\operatorname{aut}\mathcal{Z}$ generated by irreducible objects (cf. Theorem 5.6). One of the typical difficulties in the context of a nontrivial center is that $\mathcal{Z}$ is not stable under the action of a general canonical endomorphism σ, i.e.

$$\sigma(\mathcal{Z}) \not\subset \mathcal{Z}\,.$$

In this section we also give an explicit formula in terms of isotypical projections that describes the action of reducible endomorphisms restricted to the center (cf. Theorem 5.8). In Section 6 we construct a family of examples that satisfy the requirements of the pair of categories $\mathcal{T}_\mathbb{C} < \mathcal{T}$ considered in Theorem 4.12. In Section 7 we analyze the situation where the homomorphism between the chain group and the subgroup of aut $\mathcal{Z}$ generated by irreducible objects is trivial. In this case $\mathcal{Z}$ becomes the common center of $\mathcal{A}$ *and* $\mathcal{F}$. We can therefore decompose these algebras, which in this section are assumed to be separable, w.r.t. $\mathcal{Z}$. Then the Hilbert C*-system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ becomes a bundle over $\Gamma := \operatorname{spec}\mathcal{Z}$ and the fibre Hilbert C*-system corresponding to the base point $\lambda \in \Gamma$ is of a DR-type with the same group $\mathcal{G}$. That means, in particular, that the fixed point algebra associated with λ has a trivial center. Another simplifying condition of the present situation is the fact that any canonical endomorphism acts trivially on the center, i.e. $\rho\restriction\mathcal{Z} = \mathrm{id}\restriction\mathcal{Z}$. Moreover, we show that in this case the minimality condition already implies the regularity of the corresponding Hilbert C*-system (cf. Corollary 7.6). The special situation studied in this section is also related to the notion of extention of C*-categories by abelian C*-algebras (cf. [35]).

Some conclusions connecting the present analysis to related lines of research are stated in Section 8. Finally, the paper contains an appendix recalling the decomposition of a C*-algebra w.r.t. its center.

2 Basic properties of Hilbert C*-systems

In this section we summarize the structures from superselection theory which we need. For proofs, we refer to the literature if possible, otherwise proofs are included in this paper.

Below $\mathcal{F}$ will always denote a unital C*-algebra. A Hilbert space $\mathcal{H} \subset \mathcal{F}$ is called *algebraic* if the scalar product $\langle\cdot,\cdot\rangle$ of $\mathcal{H}$ is given by $\langle A, B\rangle\mathbb{1} := A^*B$ for $A,\ B \in \mathcal{H}$. Henceforth, we consider only finite-dimensional algebraic Hilbert spaces. The support $\operatorname{supp}\mathcal{H}$ of $\mathcal{H}$ is defined by $\operatorname{supp}\mathcal{H} := \sum_{j=1}^{d} \Phi_j\Phi_j^*$, where $\{\Phi_j \mid j = 1, \ldots, d\}$ is any orthonormal basis of $\mathcal{H}$. Unless otherwise specified, we assume below that each considered algebraic Hilbert space $\mathcal{H}$ satisfies $\operatorname{supp}\mathcal{H} = \mathbb{1}$.

We also fix a compact C*-dynamical system $\{\mathcal{F}, \mathcal{G}, \alpha\}$, i.e. $\mathcal{G}$ is a compact group and $\alpha : \mathcal{G} \ni g \to \alpha_g \in \operatorname{Aut}\mathcal{F}$ is a pointwise norm-continuous morphism. For $D \in \widehat{\mathcal{G}}$ (the dual of $\mathcal{G}$) its *spectral projection* $\Pi_D \in \mathcal{L}(\mathcal{F})$ is defined by

$$\begin{aligned} \Pi_D(F) \quad &:= \quad \int_{\mathcal{G}} \overline{\chi_D(g)}\,\alpha_g(F)\,dg \quad \text{for all} \quad F \in \mathcal{F}, \\ \text{where:} \qquad \chi_D(g) \quad &:= \quad \dim D \cdot \operatorname{Tr}\pi(g), \quad \pi \in D\,. \end{aligned}$$

The spectrum of $\alpha_{\mathcal{G}}$ can then be defined by

$$\operatorname{spec}\alpha_{\mathcal{G}} := \left\{ D \in \widehat{\mathcal{G}} \;\mid\; \Pi_D \neq 0 \right\}.$$

Note that $\operatorname{spec}\alpha_{\mathcal{G}}$ coincides with the so-called Arveson spectrum of $\alpha_{\mathcal{G}}$ (see e.g. [1]).

Our central object of study is:

2.1 Definition *The compact C*-dynamical system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ is called a* **Hilbert C*-system** *if for each $D \in \widehat{\mathcal{G}}$ there is an algebraic Hilbert space $\mathcal{H}_D \subset \Pi_D\mathcal{F}$, such that $\alpha_{\mathcal{G}}$ acts invariantly on $\mathcal{H}_D$, and the unitary representation $\alpha_{\mathcal{G}}\restriction\mathcal{H}_D$ is in the equivalence class $D \in \widehat{\mathcal{G}}$.*

We are mainly interested in Hilbert C*-systems whose fixed point algebras coincide such that they appear as extensions of it.

2.2 Definition *A Hilbert C*-system $\{\mathcal{F}, \mathcal{G}, \alpha\}$ is called a* **Hilbert extension** *of a C*-algebra $\mathcal{A} \subset \mathcal{F}$ if $\mathcal{A}$ is the fixed point algebra of $\mathcal{G}$. Two Hilbert extensions $\{\mathcal{F}_i, \mathcal{G}, \alpha^i\}$, $i = 1, 2$ of $\mathcal{A}$ (w.r.t. the same group $\mathcal{G}$) are called* **$\mathcal{A}$-module isomorphic** *if there is an isomorphism $\tau : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\tau(A) = A$ for $A \in \mathcal{A}$, and τ intertwines the group actions, i.e. $\tau \circ \alpha^1_g = \alpha^2_g \circ \tau$, $g \in \mathcal{G}$.*

2.3 Remark (i) For a Hilbert C*-system $\{\mathcal{F},\mathcal{G},\alpha\}$ one has $\operatorname{spec}\alpha_{\mathcal{G}} = \widehat{\mathcal{G}}$ and the morphism $\alpha : \mathcal{G} \to \operatorname{Aut}\mathcal{F}$ is necessarily faithful. So, since $\mathcal{G}$ is compact and $\operatorname{Aut}\mathcal{F}$ is Hausdorff w.r.t. the topology of pointwise norm-convergence, α is a homeomorphism of $\mathcal{G}$ onto its image. Thus $\mathcal{G}$ and $\alpha_{\mathcal{G}}$ are isomorphic as topological groups.

(ii) Group automorphisms of $\mathcal{G}$ lead to $\mathcal{A}$-module isomorphic Hilbert extensions of $\mathcal{A}$, i.e. if $\{\mathcal{F},\mathcal{G},\alpha\}$ is a Hilbert extension of $\mathcal{A}$ and ξ an automorphism of $\mathcal{G}$, then the Hilbert extensions $\{\mathcal{F},\mathcal{G},\alpha\}$ and $\{\mathcal{F},\mathcal{G},\alpha\circ\xi\}$ are $\mathcal{A}$-module isomorphic.

Therefore, the Hilbert C*-system $\{\mathcal{F},\mathcal{G},\alpha\}$ depends, up to $\mathcal{A}$-module isomorphisms, only on $\alpha_{\mathcal{G}}$, which is isomorphic to $\mathcal{G}$. In other words, up to $\mathcal{A}$-module isomorphy we may identify $\mathcal{G}$ and $\alpha_{\mathcal{G}} \subset \operatorname{Aut}\mathcal{F}$ neglecting the action α which has no relevance from this point of view. Therefore in the following, unless it is otherwise specified, we use the notation $\{\mathcal{F},\mathcal{G}\}$ for a Hilbert extension of $\mathcal{A}$, where $\mathcal{G} \subset \operatorname{Aut}\mathcal{F}$.

(iii) As mentioned above, Hilbert C*-systems arise in DHR-superselection theory (cf. [10, 2]). Mathematically, there are constructions by means of tensor products $\mathcal{B}$ of Cuntz algebras $\mathcal{O}_{\mathcal{H}_u}$, $\mathcal{B} = \otimes_{u\in\mathrm{Ob}\,\mathcal{R}}\mathcal{O}_{\mathcal{H}_u}$, where $\mathcal{R}$ is a category whose objects u are finite-dimensional continuous unitary representations of a compact group $\mathcal{G}$ on Hilbert spaces $\mathcal{H}_u$ with $\dim\mathcal{H}_u > 1$ and whose arrows are the corresponding intertwining operators (cf. [16, Section 7]). In these examples the center $\mathcal{Z}$ of the fixed point algebra $\mathcal{A}$ is trivial.

Further examples in the context of the CAR-algebra with an abelian group $\mathcal{G} = \mathbb{T}$ and nontrivial center $\mathcal{Z}$ are given in [4]. In Section 6 we construct a family of examples of minimal and regular Hilbert C*-systems for nonabelian groups and with nontrivial $\mathcal{Z}$.

2.4 Remark A Hilbert C*-system is a very highly structured object;- below we list some important properties (for details, consult [2, 10]):

(i) Given two $\mathcal{G}$-invariant algebraic Hilbert spaces $\mathcal{H},\mathcal{K} \subset \mathcal{F}$, then $\operatorname{span}(\mathcal{H}\cdot\mathcal{K})$ is also a $\mathcal{G}$-invariant algebraic Hilbert space which we will briefly denote by $\mathcal{H}\cdot\mathcal{K}$. It carries the tensor product of the representations of $\mathcal{G}$ carried by $\mathcal{H}$ and $\mathcal{K}$.

(ii) Let $\mathcal{H},\mathcal{K}$ as before but not necessarily of support $\mathbb{1}$: There is a natural isometric embedding of $\mathcal{L}(\mathcal{H},\mathcal{K})$ into $\mathcal{F}$ given by

$$\mathcal{L}(\mathcal{H},\mathcal{K}) \ni T \to \mathcal{J}(T) := \sum_{j,k} t_{jk}\Psi_j\Phi_k^*, \quad t_{jk} \in \mathbb{C},$$

where $\{\Phi_k\}_k$ resp. $\{\Psi_j\}_j$ are orthonormal basis of $\mathcal{H}$ resp. $\mathcal{K}$ and where

$$T(\Phi_k) = \sum_j t_{jk}\Psi_j,$$

i.e. $(t_{j,k})$ is the matrix of T w.r.t. these orthonormal basis. One has

$$T(\Phi) = \mathcal{J}(T)\cdot\Phi, \quad \Phi \in \mathcal{H}.$$

For simplicity of notation we will often put $\widehat{T} := \mathcal{J}(T)$. Moreover, we have $\widehat{T} \in \mathcal{A}$ iff $T \in \mathcal{L}_{\mathcal{G}}(\mathcal{H},\mathcal{K})$, where $\mathcal{L}_{\mathcal{G}}(\mathcal{H},\mathcal{K})$ denotes the linear subspace of $\mathcal{L}(\mathcal{H},\mathcal{K})$ consisting of all intertwining operators of the representations of $\mathcal{G}$ on $\mathcal{H}$ and $\mathcal{K}$ (cf. [10, p. 222]).

(iii) Generally for a Hilbert C*-system, the assignment $D \to \mathcal{H}_D$ is not unique. If $U \in \mathcal{A}$ is unitary then also $U\mathcal{H}_D \subset \Pi_D\mathcal{F}$ is an $\mathcal{G}$-invariant algebraic Hilbert space carrying a representation in D. Note that each $\mathcal{G}$-invariant algebraic Hilbert space $\mathcal{K}$ which carries a representation of D is of this form, i.e. there is a unitary $V \in \mathcal{A}$ such that $\mathcal{K} = V\mathcal{H}_D$.

(iv) There is a useful partial order on the $\mathcal{G}$-invariant algebraic Hilbert spaces. We define $\mathcal{H} < \mathcal{K}$ to mean that there is an orthoprojection E on $\mathcal{K}$ such that $E\mathcal{K}$ is invariant w.r.t. $\mathcal{G}$ and the representation $\mathcal{G}\restriction\mathcal{H}$ is unitarily equivalent to $\mathcal{G}\restriction E\mathcal{K}$.

Note that $\mathcal{H} < \mathcal{K}$ iff there is an isometry $V \in \mathcal{A}$ such that $VV^* =: E$ is a projection of $\mathcal{K}$, i.e. $V\mathcal{H} = E\mathcal{K}$ (use (ii)).

(v) Given a Hilbert C*-system $\{\mathcal{F},\, \mathcal{G}\}$ a useful *-subalgebra of $\mathcal{F}$ is

$$\mathcal{F}_{\text{fin}} := \Big\{ F \in \mathcal{F} \ \mid\ \Pi_D F \neq 0 \quad \text{for only finitely many } D \in \widehat{\mathcal{G}} \Big\}$$

which is dense in $\mathcal{F}$ w.r.t. the C*-norm (cf. [33]).

(vi) The spectral projections satisfy:

$$\begin{aligned} \Pi_{D_1}\Pi_{D_2} &= \Pi_{D_2}\Pi_{D_1} = \delta_{D_1 D_2}\Pi_{D_1} \\ \|\Pi_D\| &\leq d(D)^{3/2}\ , \qquad d(D) := \dim(\mathcal{H}_D)\ , \\ \Pi_D\mathcal{F} &= \operatorname{span}(\mathcal{A}\mathcal{H}_D)\ , \\ \Pi_D(AFB) &= A \cdot \Pi_D(F) \cdot B, \quad A, B \in \mathcal{A},\, F \in \mathcal{F}\ , \\ \mathcal{A} &= \Pi_\iota \mathcal{F}\ , \end{aligned}$$

where $\iota \in \widehat{\mathcal{G}}$ denotes the trivial representation of $\mathcal{G}$.

(vii) In $\mathcal{F}$ there is an $\mathcal{A}$-scalar product given by $\langle F,\, G\rangle_{\mathcal{A}} := \Pi_\iota F G^*$, w.r.t. which the spectral projections are symmetric, i.e. $\langle \Pi_D F,\, G\rangle_{\mathcal{A}} = \langle F,\, \Pi_D G\rangle_{\mathcal{A}}$ for all $F,\, G \in \mathcal{F},\, D \in \widehat{\mathcal{G}}$. Using the $\mathcal{A}$-scalar product one can define a norm on $\mathcal{F}$, called the $\mathcal{A}$-norm

$$|F|_{\mathcal{A}} := \|\langle F, F\rangle\|^{1/2}, \quad f \in \mathcal{F}.$$

Note that $|F|_{\mathcal{A}} \leq \|F\|$ and that $\mathcal{F}$ in general is not closed w.r.t. the $\mathcal{A}$-norm.

The following result confirms the importance and naturalness of the previously defined norm $|\cdot|_{\mathcal{A}}$ in the context of Hilbert C*-systems. This norm plays also a fundamental role in the so-called inverse superselection theory which reconstructs the Hilbert C*-system from the data $\mathcal{A}$ and a suitable family of endomorphisms of $\mathcal{A}$ (cf. [3, 2, 7]).

2.5 Proposition *Let $\{\mathcal{F},\, \mathcal{G}\}$ be a Hilbert C*-system, then for each $F \in \mathcal{F}$ we have*

$$F = \sum_{D \in \widehat{\mathcal{G}}} \Pi_D F \tag{2}$$

where the sum on the right hand side is convergent w.r.t. the $\mathcal{A}$-norm and we have Parseval's equation:

$$\langle F, F\rangle_{\mathcal{A}} = \sum_{D \in \widehat{\mathcal{G}}} \langle \Pi_D F, \Pi_D F\rangle_{\mathcal{A}}\ . \tag{3}$$

Proof: Let $\Gamma \subset \widehat{\mathcal{G}}$, $\operatorname{card}\Gamma < \infty$. The set $\{\Gamma\}$ of all such subsets of $\widehat{\mathcal{G}}$ is a directed net. The assertion (2) means

$$\sum_{D \in \widehat{\mathcal{G}}} \Pi_D F := \lim_{\Gamma \to \widehat{\mathcal{G}}} F_\Gamma,$$

where

$$F_\Gamma := \sum_{D \in \Gamma} \Pi_D F,$$

and "lim" means convergence w.r.t. the $\mathcal{A}$-norm. On the other hand, if Γ is fixed, we put

$$G_\Gamma = G_\Gamma(C_D,\ D \in \Gamma) := \sum_{D\in\Gamma} C_D, \quad C_D \in \Pi_D \mathcal{F}.$$

Then $G_\Gamma \in \mathcal{F}_{\rm fin}$. By a simple calculation one obtains

$$\langle F - G_\Gamma, F - G_\Gamma\rangle_{\mathcal{A}} = \langle F,\, F\rangle_{\mathcal{A}} - \sum_{D\in\Gamma} \langle \Pi_D F, \Pi_D F\rangle_{\mathcal{A}} + \sum_{D\in\Gamma} \langle \Pi_D F - C_D, \Pi_D F - C_D\rangle_{\mathcal{A}}.$$

Since

$$\sum_{D\in\Gamma} \langle \Pi_D F - C_D, \Pi_D F - C_D\rangle_{\mathcal{A}} \geq 0$$

we obtain

$$\begin{aligned} \langle F - G_\Gamma, F - G_\Gamma\rangle_{\mathcal{A}} &\geq \langle F,\, F\rangle_{\mathcal{A}} - \sum_{D\in\Gamma} \langle \Pi_D F,\, \Pi_D F\rangle_{\mathcal{A}} \qquad (4)\\ &= \Big\langle F - \sum_{D\in\Gamma} \Pi_D F,\, F - \sum_{D\in\Gamma} \Pi_D F \Big\rangle_{\mathcal{A}} \geq 0. \end{aligned}$$

Therefore

$$\langle F,\, F\rangle_{\mathcal{A}} \geq \sum_{D\in\Gamma} \langle \Pi_D F, \Pi_D F\rangle_{\mathcal{A}}$$

Since $\|X\| \geq |X|_{\mathcal{A}}$ for all $X \in \mathcal{F}$ we have

$$\|F - G_\Gamma\| \geq |F - G_\Gamma|_{\mathcal{A}} \geq |F - F_\Gamma|_{\mathcal{A}}.$$

According to Shiga's theorem (see [33]) the left hand side can be chosen arbitrary small for suitable Γ and suitable coefficients C_D. Hence $|F - F_\Gamma|_{\mathcal{A}} \to 0$ for $\Gamma \to \widehat{\mathcal{G}}$ follows. This is (2) and this implies

$$\lim_{\Gamma\to\widehat{\mathcal{G}}} \|\langle F, F\rangle_{\mathcal{A}} - \sum_{D\in\Gamma} \langle \Pi_D F, \Pi_D F\rangle_{\mathcal{A}}\| = 0,$$

which proves (3). ∎

Note that (2) does not in general converge w.r.t. the C*-norm $\|\cdot\|$.

2.6 Corollary *(i) Each $F \in \mathcal{F}$ is uniquely determined by its projections $\Pi_D F$, $D \in \widehat{\mathcal{G}}$, i.e. $F = 0$ iff $\Pi_D F = 0$ for all $D \in \widehat{\mathcal{G}}$.*
(ii) We have that $|\Pi_D|_{\mathcal{A}} = 1$ for all $D \in \widehat{\mathcal{G}}$, where $|\cdot|_{\mathcal{A}}$ denotes the operator norm of Π_D w.r.t. the norm $|\cdot|_{\mathcal{A}}$ in $\mathcal{F}$.

2.7 Proposition *$\mathcal{F}$ is simple iff $\mathcal{A}$ is simple.*

Proof: Let $\mathcal{D} \lhd \mathcal{A}$ be a nontrivial closed 2-sided ideal and consider

$$\mathcal{E}_r := \mathrm{clo}_{\,\|\cdot\|}\ \mathrm{span}\,\{\mathcal{D}\,\mathcal{H}_D \mid D \in \widehat{\mathcal{G}}\}\,.$$

$\mathcal{E}_r$ is a closed right ideal in $\mathcal{F}$: Indeed, recall that

$$\mathcal{F} := \mathrm{clo}_{\,\|\cdot\|}\ \mathrm{span}\,\{\mathcal{A}\,\mathcal{H}_D \mid D \in \widehat{\mathcal{G}}\}$$

and take $A \in \mathcal{A}$. Then

$$\mathcal{D}\,\mathcal{H}_D\ A\mathcal{H}_{D'} = \underbrace{\mathcal{D}\rho_D(A)}_{\in\mathcal{D}}\ \mathcal{H}_D\,\mathcal{H}_{D'} \subset \mathcal{E}_r\,, \quad D, D' \in \widehat{\mathcal{G}}\,,$$

where the latter inclusion follows from the fact that the tensor product $\mathcal{H}_D\,\mathcal{H}_{D'}$ can be decomposed in terms of irreducible algebraic Hilbert spaces. This shows that

$$\operatorname{span}\{\mathcal{D}\,\mathcal{H}_D \mid D \in \widehat{\mathcal{G}}\} \cdot \operatorname{span}\{\mathcal{A}\,\mathcal{H}_{D'} \mid D' \in \widehat{\mathcal{G}}\} \subset \mathcal{E}_r\,.$$

Take now $\{F_n\}_n \subset \operatorname{span}\{\mathcal{A}\,\mathcal{H}_{D'} \mid D' \in \widehat{\mathcal{G}}\}$ such that $F_n \to F \in \mathcal{F}$. Then for any $E_0 \in \operatorname{span}\{\mathcal{D}\,\mathcal{H}_D \mid D \in \widehat{\mathcal{G}}\}$ we have $E_0 F \in \mathcal{E}_r$, because $\mathcal{E}_r \ni E_0 F_n \to E_0 F$. Similarly one can show that $EF \in \mathcal{E}_r$ for all $E \in \mathcal{E}_r$, $F \in \mathcal{F}$, hence $\mathcal{E}_r$ is a closed right ideal in $\mathcal{F}$. This implies that $\mathcal{E} := \mathcal{E}_r \cap \mathcal{E}_r^* \lhd \mathcal{F}$ is a nonzero closed 2-sided ideal in $\mathcal{F}$, which is proper since $\mathbb{1} \notin \mathcal{E}$.

To show the reverse implication, let $\mathcal{E} \lhd \mathcal{F}$ be a nontrivial closed 2-sided ideal and put $\mathcal{D} := \mathcal{E} \cap \mathcal{A}$, which is a closed 2-sided ideal in $\mathcal{A}$. We show next that $\mathcal{D} = \{0\}$ implies $\mathcal{E} = \{0\}$ contradicting the nontriviality assumption of the ideal $\mathcal{E}$: take $U_0\Psi_0 \in \mathcal{E} \cap (\mathcal{A}\mathcal{H}_{D_0})$ with U_0 a unitary in $\mathcal{A}$. Then $U_0\Psi_0\Psi_0^* \in \mathcal{E} \cap \mathcal{A} = \mathcal{D} = \{0\}$, hence $\Psi_0\Psi_0^* = 0$ and $\Psi_0 = 0$. This show that $\mathcal{D}$ is nonzero. Finally, $\mathcal{D}$ is proper because $\mathbb{1} \notin \mathcal{D}$. ■

3 Two natural examples of C*-categories associated with a Hilbert C*-system

In the following we introduce two important examples of C*-categories that naturally appear in the context of Hilbert C*-systems. For convenience of the reader we recall next the definition of this type of categories (see [19, 30]).

A category $\mathfrak{T}$ with objects α, β etc. is called a *C*-category* if each arrow space (α, β) is a complex Banach space where the bilinear composition of arrows R, S satisfies $\|R \circ S\| \leq \|R\|\,\|S\|$ and there is an antilinear involutive contravariant functor ${}^*\colon \mathfrak{T} \to \mathfrak{T}$ such that the analogue of the C*-property is satisfied, i.e. if $R \in (\alpha, \beta)$, $R^* \in (\beta, \alpha)$, then $\|R^* \circ R\| = \|R\|^2$. The category $\mathfrak{T}$ is said to be a *tensor* C*-category if there is an associative functor $\circ\colon\ \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ with unit ι and commuting with *. Further given two arrows $R \in (\alpha, \beta)$, $R' \in (\alpha', \beta')$ there is a tensor product of arrows denoted by $R \times R' \in (\alpha\alpha', \beta\beta')$. The mapping $R, R' \to R \times R'$ is associative and bilinear and we have (denoting by 1_ι the identity of ι)

$$1_\iota \times R = R \times 1_\iota = R\,, \quad (R \times R')^* = R^* \times R'^*\,, \tag{5}$$

as well as the interchange law

$$(S \circ R) \times (S' \circ R') = S \times S' \circ R \times R'\,,$$

whenever the left hand side is defined. (We adopt as in [19] the convention of evaluating $\times$ before $\circ$.) It is immediate to check from the previous properties that (ι, ι) is an abelian unital C*-algebra.

We may now generalize the notion of irreducible object introduced in [6, Section 5] (see also [7]) to the present general situation.

3.1 Definition *Let $\mathfrak{T}$ be a strict tensor C*-category. Then $\rho \in \operatorname{Ob}\mathfrak{T}$ is called* **irreducible** *if*

$$(\rho, \rho) = 1_\rho \times (\iota, \iota)\,.$$

We denote the set of all irreducible objects in $\mathfrak{T}$ by $\mathbf{Irr}\,\mathfrak{T}$.

3.1 The category $\mathcal{T}_\mathcal{G}$ of all $\mathcal{G}$-invariant algebraic Hilbert spaces

The $\mathcal{G}$-invariant algebraic Hilbert spaces $\mathcal{H}$ of $\{\mathcal{F},\mathcal{G}\}$, satisfying $\operatorname{supp}\mathcal{H} = \mathbb{1}$, form the objects of a C*-category $\mathcal{T}_\mathcal{G}$ whose arrows are given by the $\mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{H},\mathcal{K})) \subset \mathcal{A}$. In this context the tensor product of objects is given by the product in $\mathcal{F}$, $\iota = \mathbb{C}\mathbb{1}$ and $(\iota,\iota) = \mathbb{C}\mathbb{1}$. $\mathcal{H}$ is irreducible iff $\mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{H},\mathcal{H})) = \mathbb{C}\mathbb{1}$ (Schur's lemma).

We will focus next on the additional structure of $\mathcal{T}_\mathcal{G}$. For this recall the partial order in $\operatorname{Ob}\mathcal{T}_\mathcal{G}$ given in Remark 2.4 (iv). If $\mathcal{K} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ is given, an object $\mathcal{H} < \mathcal{K}$ is called a *subobject* of $\mathcal{K}$. If $E \in \mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{K}))$ is an orthoprojection $0 < E < \mathbb{1}$, i.e. E is a reducing projection for the representation of $\mathcal{G}$ on $\mathcal{K}$, then the question arises whether there is an object $\mathcal{H}$ such that the representations on $\mathcal{H}$ and $E\mathcal{K}$ are unitarily equivalent. This suggests the concept of *closedness* of $\mathcal{T}_\mathcal{G}$ w.r.t. subobjects.

3.2 Definition *The category $\mathcal{T}_\mathcal{G}$ is* **closed w.r.t. subobjects** *if to each $\mathcal{K} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ and to each nontrivial orthoprojection $E \in \mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{K}))$ there is an isometry $V \in \mathcal{A}$ with $VV^* = E$. (In this case $\mathcal{H} := V^*\mathcal{K}$ is a subobject $\mathcal{H} < \mathcal{K}$ assigned to E.)*

Second, if $V, W \in \mathcal{A}$ are isometries with $VV^* + WW^* = \mathbb{1}$ and $\mathcal{H},\mathcal{K} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ then we call the algebraic Hilbert space $V\mathcal{H} + W\mathcal{K}$ of support $\mathbb{1}$ a *direct sum* of $\mathcal{H}$ and $\mathcal{K}$. It is $\mathcal{G}$-invariant and carries the direct sum of the representations on $\mathcal{H}$ and $\mathcal{K}$. Therefore we define

3.3 Definition *The category $\mathcal{T}_\mathcal{G}$ is* **closed w.r.t. direct sums** *if to each $\mathcal{H}_1,\mathcal{H}_2 \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ there is an object $\mathcal{K} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ and there are isometries $V_1, V_2 \in \mathcal{A}$ with $V_1V_1^* + V_2V_2^* = \mathbb{1}$ such that $\mathcal{K} = V_1\mathcal{H}_1 + V_2\mathcal{H}_2$.*

Since $\operatorname{Ob}\mathcal{T}_\mathcal{G}$ contains *all* $\mathcal{G}$-invariant algebraic Hilbert spaces, $\mathcal{T}_\mathcal{G}$ is always closed w.r.t. direct sums provided that $\mathcal{A}$ contains a pair V, W of isometries with $VV^* + WW^* = \mathbb{1}$. This condition for $\mathcal{A}$ we call *Property B*. That is we have

3.4 Proposition *If $\mathcal{A}$ satisfies Property B then $\mathcal{T}_\mathcal{G}$ is closed w.r.t. direct sums.*

3.5 Proposition *(i) If the category $\mathcal{T}_\mathcal{G}$ is closed w.r.t. direct sums, then it is closed w.r.t. subobjects.*

(ii) Let $\mathcal{G}$ be nonabelian. If the category $\mathcal{T}_\mathcal{G}$ is closed w.r.t. subobjects, then it is closed w.r.t. direct sums.

Proof: (i) First we assume that $\mathcal{T}_\mathcal{G}$ is closed w.r.t. direct sums. Note that in this case for any $n \in \mathbb{N}$ there are isometries $W_j \in \mathcal{A}, j = 1,2,...,n$, such that $\sum_{j=1}^n W_jW_j^* = \mathbb{1}$. Then to each finite-dimensional representation U there is a $\mathcal{G}$-invariant algebraic Hilbert space $\mathcal{K}$ such that U is realized on $\mathcal{K}$, because $U = \oplus_{D\in\widehat{\mathcal{G}}}\, m_D \cdot U_D$, $U_D \in D$, and in $\mathcal{A}$ there are isometries $W_{D,l}, l = 1,2,...,m_D$ (m_D being the multiplicity of U_D in the decomposition of U) such that $\sum_{D,l} W_{D,l}W_{D,l}^* = \mathbb{1}$. Therefore $\mathcal{K} := \sum_{D,l} W_{D,l}\mathcal{H}_D$ is an object from $\mathcal{T}_\mathcal{G}$ and carries exactly the representation U.

Now let $\mathcal{K} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ and $E \in \mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{K}))$, $0 < E < \mathbb{1}$, a reducing projection, i.e. $E\mathcal{K} \subset \mathcal{K}$ is a reducing subspace that carries a certain representation of $\mathcal{G}$. Note that $\operatorname{supp} E\mathcal{K} \neq \mathbb{1}$. Nevertheless there is an object $\mathcal{H} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ which carries this representation: choose in $E\mathcal{K}$ and $\mathcal{H}$ the (orthonormal) basis $\{\Phi_j\}_j$ of $E\mathcal{K}$, $\{\Psi_j\}_j$ of $\mathcal{H}$ in such a way that the representation matrices coincide. Put $A := \sum_j \Phi_j\Psi_j^*$. Then $A \in \mathcal{A}$, $A^*A = \mathbb{1}$ and $AA^* = E$ follows, i.e. $\mathcal{H}$ is a subobject of $\mathcal{K}$ w.r.t. E.

(ii) Now we assume that $\mathcal{T}_\mathcal{G}$ is closed w.r.t. subobjects. Then choose $\mathcal{H}_{D_1},\mathcal{H}_{D_2} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$, whose dimensions are larger than 1. Then $\mathcal{K} := \mathcal{H}_{D_1}\cdot\mathcal{H}_{D_2} \in \operatorname{Ob}\mathcal{T}_\mathcal{G}$ and it carries the reducible representation $U_{D_1} \otimes U_{D_2}$, i.e. there is a projection E, $0 < E < \mathbb{1}$, $E \in \mathcal{J}(\mathcal{L}_\mathcal{G}(\mathcal{K}))$. Then to E

and $\mathbb{1} - E$ there correspond isometries $V, W \in \mathcal{A}$ with $VV^* + WW^* = \mathbb{1}$, hence $\mathcal{T}_\mathcal{G}$ is closed w.r.t. direct sums. ■

3.6 Remark Note that if the group $\mathcal{G}$ is a compact abelian then $\widehat{\mathcal{G}}$ is a discrete abelian *group*, the character group. Pontryagin's duality theorem shows that in this case the notions of direct sums and subobjects are irrelevant for the duality theory (see also Subsection 4.1). If the compact group is non abelian the duality theory changes radically and closure under direct sums and subobjects become essential properties.

3.2 The category $\mathcal{T}$ of all canonical endomorphisms

3.7 Definition *To each $\mathcal{G}$-invariant algebraic Hilbert space $\mathcal{H} \subset \mathcal{F}$ there is assigned a corresponding inner endomorphism $\rho_\mathcal{H} \in \mathrm{End}\,\mathcal{F}$ given by*

$$\rho_\mathcal{H}(F) := \sum_{j=1}^{d(\mathcal{H})} \Phi_j F \Phi_j^* \,,$$

where $\{\Phi_j \mid j = 1, \ldots, d(\mathcal{H})\}$ is any orthonormal basis of $\mathcal{H}$. We call **canonical endomorphism** *the restriction of $\rho_\mathcal{H}$ to $\mathcal{A}$, i.e. $\rho_\mathcal{H}\restriction\mathcal{A} \in \mathrm{end}\,\mathcal{A}$.*

3.8 Remark (i) Note that the definition of the canonical endomorphisms uses terms of $\mathcal{F}$ explicitly. Therefore, the question arises whether the inner endomorphisms $\rho_\mathcal{H}$ can be characterized by intrinsic properties of their restriction to $\mathcal{A}$ (see the beginning of Section 4 below). This interplay between the inner and the canonical endomorphisms $\rho_\mathcal{H}$ resp. $\rho_\mathcal{H}\restriction\mathcal{A}$ plays an essential role in the DR-theory. Below, we omit the restriction symbol and regard the $\rho_\mathcal{H}$ also as endomorphisms of $\mathcal{A}$. We will identify the set of canonical endomorphisms of $\mathcal{A}$ as the objects of a very important category with interesting closure properties.

(ii) If the emphasis is only on the class $D \in \widehat{\mathcal{G}}$ and not on its corresponding algebraic Hilbert space $\mathcal{H}_D$, we will write ρ_D instead of $\rho_{\mathcal{H}_D}$.

(iii) Note that $\Phi A = \rho_\mathcal{H}(A)\Phi$ for all $\Phi \in \mathcal{H}$ and $A \in \mathcal{A}$.

(iv) Note that the identity endomorphism ι is assigned to $\mathcal{H} = \mathbb{C}\mathbb{1}$, i.e. $\rho_{\mathbb{C}\mathbb{1}} := \iota$.

(v) Let $\mathcal{H}, \mathcal{K}$ be as before, then $\rho_\mathcal{H} \circ \rho_\mathcal{K} = \rho_{\mathcal{H}\cdot\mathcal{K}}$.

(vi) Whilst an invariant algebraic Hilbert space uniquely determines its canonical endomorphism, in general the converse does not hold.

3.9 Proposition *Let $\mathcal{H}, \mathcal{K}$ be $\mathcal{G}$-invariant algebraic Hilbert spaces. Then: $\rho_\mathcal{H}\restriction\mathcal{A} = \rho_\mathcal{K}\restriction\mathcal{A}$ iff $\Psi^*\Phi \in \mathcal{A}' \cap \mathcal{F}$ for all $\Phi \in \mathcal{H}, \Psi \in \mathcal{K}$.*

Proof: It is straightforward to check the condition for orthonormal basis of $\mathcal{H}$ and $\mathcal{K}$. ■

3.10 Definition *Let $\{\mathcal{F}, \mathcal{G}\}$ be a Hilbert C*-system with fixed point algebra $\mathcal{A}$. The* **intertwiner space** *of canonical endomorphisms σ, τ is:*

$$(\sigma, \tau) := \{\, X \in \mathcal{A} \mid X\sigma(A) = \tau(A)X \quad \textit{for all } A \in \mathcal{A}\}$$

and this is a complex Banach space. We will say that $\sigma, \tau \in \mathrm{End}\,\mathcal{A}$ are **mutually disjoint** *if $(\sigma, \tau) = \{0\}$ when $\sigma \neq \tau$. We denote by $\mathcal{T}$ the category with objects consisting of the canonical endomorphisms $\rho_\mathcal{H}$ for $\mathcal{G}$-invariant algebraic Hilbert spaces $\mathcal{H} \subset \mathcal{F}$ with $\mathrm{supp}\,\mathcal{H} = \mathbb{1}$ and with arrows given by the intertwiner spaces.*

3.11 Remark (i) $\mathcal{T}$ is the second example of a C*-category. The product of objects is given by composition of endomorphisms (see Remark 3.8(v)) and $\iota = \mathrm{id}$. The composition of arrows is defined as follows: For $A \in (\sigma, \sigma')$, $B \in (\tau, \tau')$, we put $A \times B := A\sigma(B) \in (\sigma\tau, \sigma'\tau')$.

(ii) We have $(\iota, \iota) = \mathcal{Z} :=$ center of $\mathcal{A}$ and from Definition 3.1 we have $\rho_{\mathcal{H}} \in \mathrm{Ob}\,\mathcal{T}$ is **irreducible** if $(\rho_{\mathcal{H}}, \rho_{\mathcal{H}}) = \rho_{\mathcal{H}}(\mathcal{Z})$. Note that this corresponds precisely to the case where $\mathcal{G}$ acts irreducibly on $\mathcal{H}$ (see [7, Subsection 3.1] and [6, Section 5]). We denote the set of irreducible objects in $\mathcal{T}$ by $\mathbf{Irr}\,\mathcal{T}$.

(iii) Recall the isometry $\mathcal{J} : \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathcal{A}$ encountered in Remark 2.4(ii). We claim that its image is in fact contained in $(\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$. To see this, let $\Phi \in \mathcal{H}$, $A \in \mathcal{A}$ and $T \in \mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})$. Then putting $\widehat{T} := \mathcal{J}(T)$ we have

$$\widehat{T}\rho_{\mathcal{H}}(A)\Phi = \widehat{T}\Phi \cdot A = T(\Phi) \cdot A = \rho_{\mathcal{K}}(A)T(\Phi) = \rho_{\mathcal{K}}(A)\widehat{T} \cdot \Phi$$

hence

$$\widehat{T}\rho_{\mathcal{H}}(A) = \rho_{\mathcal{K}}(A)\widehat{T}$$

i.e. $\widehat{T} \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$ or

$$\mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \subseteq (\rho_{\mathcal{H}}, \rho_{\mathcal{K}}).$$

In general, the inclusion is proper.

(iv) Recall that $\mathcal{H} < \mathcal{K}$ iff there is an isometry $V \in \mathcal{A}$ such that $VV^* =: E$ is a projection of $\mathcal{K}$ i.e. $V\mathcal{H} = E\mathcal{K}$. In this case we have $V \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$ and $E \in (\rho_{\mathcal{K}}, \rho_{\mathcal{K}})$. Moreover E does not belong to the center of $\mathcal{A}$.

There is an important connection between the categories $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}$ in the case of a trivial relative commutant $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (see [16, Lemma 2.4]). Note that in this case $\mathcal{A}$ must have a trivial center $\mathcal{Z} = \mathbb{C}\mathbb{1}$.

3.12 Proposition *There is a faithful functor from the categories $\mathcal{T}_{\mathcal{G}}$ to $\mathcal{T}$ which is a bijection of objects. In general the functor is not full, but if the relative commutant $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ then $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}$ are isomorphic categories.*

Proof: For the objects the morphism is defined by $\mathcal{H} \to \rho_{\mathcal{H}}$, for the arrows by $\mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \ni A \to A \in (\rho_{\mathcal{H}}, \rho_{\mathcal{K}})$. For the second assertion use Proposition 3.9 and that $\Phi^* A \Psi \in \mathbb{C}\mathbb{1}$ for $\Phi \in \mathcal{K}, \Psi \in \mathcal{H}$. ■

We now want to exhibit closure properties of $\mathcal{T}$ similarly as for $\mathcal{T}_{\mathcal{G}}$.

3.13 Definition *(i) $\tau \in \mathrm{Ob}\,\mathcal{T}$ is a* **subobject** *of $\sigma \in \mathrm{Ob}\,\mathcal{T}$, denoted $\tau < \sigma$, if there there is an isometry $V \in (\tau, \sigma)$. In this case $\tau(\cdot) = V^*\sigma(\cdot)V$ and $VV^* =: E \in (\sigma, \sigma)$ follow.*

(ii) $\rho \in \mathrm{Ob}\,\mathcal{T}$ is a direct sum of $\sigma, \tau \in \mathrm{Ob}\,\mathcal{T}$, if there are isometries $V \in (\sigma, \rho)$, $W \in (\tau, \rho)$ with $VV^* + WW^* = \mathbb{1}$ such that

$$\rho(\cdot) = V\sigma(\cdot)V^* + W\tau(\cdot)W^* =: \sigma \oplus \tau.$$

3.14 Remark (i) The subobject relation $\tau < \sigma$ is again a partial order: let $\tau < \sigma$ and $\sigma < \mu$, so that there are isometries $V \in (\tau, \sigma)$ and $W \in (\sigma, \mu)$. Then $WV \in (\tau, \mu)$ is also an isometry, i.e. $\tau < \mu$.

(ii) Note that if $\tau = \rho_{\mathcal{H}}$, $\sigma = \rho_{\mathcal{K}}$ for $\mathcal{G}$-invariant algebraic Hilbert spaces $\mathcal{H}, \mathcal{K}$ satisfying $\mathcal{H} < \mathcal{K}$, then a fortiori $\tau < \sigma$. However if τ, σ are given and one only knows that there are algebraic Hilbert spaces $\mathcal{H} < \mathcal{K}$, then the transitivity property may not hold in general.

(iii) If $\sigma := \rho_{\mathcal{H}}$, $\tau := \rho_{\mathcal{K}}$ then $\rho = \rho_{\mathcal{L}}$ where $\mathcal{L} := V\mathcal{H} + W\mathcal{K}$.

(iv) A direct sum $\sigma \oplus \tau$, defined above (where a priori ρ is not necessarily an object of $\mathcal{T}$) with isometries $V, W \in \mathcal{A}$, $VV^* + WW^* = \mathbb{1}$ is only unique up to unitary equivalence, i.e. if ρ, ρ' are direct sums of σ and ρ, then there is a unitary $U \in (\rho, \rho')$.

The closedness of $\mathcal{T}$ w.r.t. direct sums is defined by the closedness of $\mathcal{T}_{\mathcal{G}}$ w.r.t. direct sums. The closedness of $\mathcal{T}$ w.r.t. subobjects is defined by the closedness of $\mathcal{T}_{\mathcal{G}}$ w.r.t. subobjects in the following sense: if

$$\rho = \rho_{\mathcal{H}} \in \operatorname{Ob} \mathcal{T} \tag{6}$$

is given, then for all $\mathcal{H}$ satisfying (6) and to each nontrivial projection $E \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}))$ there is an isometry $V_{\mathcal{H}} \in \mathcal{A}$ with $V_{\mathcal{H}} V_{\mathcal{H}}^* = E$.

This means

3.15 Proposition *If $\mathcal{A}$ satisfies Property B then $\mathcal{T}$ is closed w.r.t. direct sums and subobjects.*

Sometimes it is convenient to use the concept *complementary subobject*: If $\tau < \sigma$, then a complementary subobject $\tau' < \sigma$ is one which satisfies $\sigma = \tau \oplus \tau'$.

3.3 Permutation and conjugation structures on $\mathcal{T}_{\mathcal{G}}$: DR-categories

To complete the analysis of the categories $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}$ we will recall briefly their permutation and conjugation structure. We will consider these structures on $\mathcal{T}_{\mathcal{G}}$. In certain cases it possible to use the functor in Proposition 3.12 to transfer the corresponding structure to $\mathcal{T}$ (see [10, Subsection 10.5.2] for further details).

3.16 Proposition **(Permutation structure)** *$\mathcal{T}_{\mathcal{G}}$ has a permutation structure, i.e. a map*

$$\operatorname{Ob} \mathcal{T}_{\mathcal{G}} \times \operatorname{Ob} \mathcal{T}_{\mathcal{G}} \ni (\mathcal{H}, \mathcal{K}) \to \epsilon(\mathcal{H}, \mathcal{K}) \in (\mathcal{H}\mathcal{K}, \, \mathcal{K}\mathcal{H}) \,,$$

where $\epsilon(\mathcal{H}, \mathcal{K})$ is unitary and satisfies

(i) $\epsilon(\mathcal{H}, \mathcal{K}) \cdot \epsilon(\mathcal{K}, \mathcal{H}) = \mathbb{1}$.

(ii) $\epsilon(\mathbb{C}\mathbb{1}, \mathcal{H}) = \epsilon(\mathcal{H}, \mathbb{C}\mathbb{1}) = \mathbb{1}$.

(iii) $\epsilon(\mathcal{H}_1\mathcal{H}_2, \mathcal{H}_3) = \epsilon(\mathcal{H}_1, \mathcal{H}_3) \cdot \rho_{\mathcal{H}_1}(\epsilon(\mathcal{H}_2, \mathcal{H}_3)) \quad , \mathcal{H}_i \in \operatorname{Ob} \mathcal{T}_{\mathcal{G}}, \ i = 1, 2, 3.$

(iv) $\epsilon(\mathcal{H}', \mathcal{K}')\, A \times B = B \times A\, \epsilon(\mathcal{H}, \mathcal{K}) \quad , A \in (\mathcal{H}, \mathcal{H}'), \ B \in (\mathcal{K}, \mathcal{K}')$.

Tensor categories that have a map $\epsilon(\cdot, \cdot)$ satisfying the properties (i)-(iv) adapted from above are called *symmetric* (cf. [19, p. 160]). The map $\epsilon(\cdot, \cdot)$ is also called a *permutator* or *symmetry*.

3.17 Proposition **(Conjugation structure)** *$\mathcal{T}_{\mathcal{G}}$ has a conjugation structure, i.e. to each $\mathcal{H} \in \operatorname{Ob} \mathcal{T}_{\mathcal{G}}$ there is a conjugated algebraic Hilbert space $\overline{\mathcal{H}} \in \operatorname{Ob} \mathcal{T}_{\mathcal{G}}$ carrying the corresponding conjugated representation and there are* conjugates $R_{\mathcal{H}} \in \mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathbb{C}\mathbb{1}, \overline{\mathcal{H}}\mathcal{H}))$, $S_{\mathcal{H}} = \epsilon(\overline{\mathcal{H}}, \mathcal{H})\, R_{\mathcal{H}}$ *such that*

$$S_{\mathcal{H}}^*\, \rho_{\mathcal{H}}(R_{\mathcal{H}}) = \mathbb{1} \quad \text{and} \quad R_{\mathcal{H}}^*\, \rho_{\overline{\mathcal{H}}}(S_{\mathcal{H}}) = \mathbb{1} \,.$$

3.18 Remark Permutators ϵ and conjugates can be given explicitly using orthonormal basis. Let $\{\Phi_i\}_i$,$\{\Psi_k\}_k$ and $\{\overline{\Phi}_i\}_i$ be orthonormal basis of the algebraic Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and $\overline{\mathcal{H}}$, respectively. Then

$$\begin{aligned} \epsilon(\mathcal{H},\mathcal{K}) &= \sum_{i,k} \Psi_k \, \Phi_i \, \Psi_k^* \, \Phi_i^* \\ R_{\mathcal{H}} &= \sum_i \overline{\Phi}_i \, \Phi_i \, . \end{aligned}$$

Moreover

$$R_\sigma^* R_\sigma = \sum_{j=1}^{\dim \mathcal{H}_\sigma} \Phi_j^* \Phi_j = \dim \mathcal{H}_\sigma \cdot \mathbb{1}. \tag{7}$$

We can now sum up the rich structure of the category $\mathcal{T}_\mathcal{G}$ in the notion of a (Doplicher/Roberts) DR-category (cf. [19]).

3.19 Definition *An (abstract) tensor C*-category with $(\iota,\iota) = \mathbb{C}\mathbb{1}$, closed w.r.t. direct sums and subobjects, equipped with a permutation and a conjugation structure is called an (abstract)* **DR-category***.*

If the fixed point algebra $\mathcal{A}$ of $\{\mathcal{F},\, \mathcal{G}\}$ satisfies Property B then $\mathcal{T}_\mathcal{G}$ is an example of a DR-category. In the case where the objects are finite dimensional Hilbert spaces also the name *Tannaka-Krein* category is sometimes used. Moreover if $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (which implies $\mathcal{Z} = \mathbb{C}\mathbb{1}$), the category $\mathcal{T}$ of canonical endomorphisms is another example of a DR-category (cf. Proposition 3.12).

3.20 Remark In the context of the DR-theory we can associate with any $\rho \in \mathrm{Irr}\,\mathcal{T}$ a unique element $D \in \widehat{\mathcal{G}}$, where $\mathcal{G}$ is the group associated with the DR-category $\mathcal{T}$.

4 Minimal and regular Hilbert C*-systems

The DR-theorem associates with a DR-category $\mathcal{T}$ an essentially unique compact group $\mathcal{G}$ [19, Theorem 6.1]. In the context of Hilbert C*-systems we have a bijective correspondence between

$$\{\mathcal{A},\mathcal{T}\} \quad \text{and} \quad \{\mathcal{F},\, \mathcal{G}\}\,,$$

where $\mathcal{A}$ is a unital C*-algebra with trivial center $\mathcal{Z} = \mathcal{A}' \cap \mathcal{A} = \mathbb{C}\mathbb{1}$ (and satisfying Property B) and $\mathcal{T}$ is a DR-category realized as unital endomorphisms of $\mathcal{A}$. $\{\mathcal{F},\, \mathcal{G}\}$ is a Hilbert extension of $\mathcal{A}$ having trivial relative commutant, i.e. $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$ (see [19, 20, 18, 3]). This correspondence is connected with the second part of Proposition 3.12 which requires a trivial center of $\mathcal{A}$. The DR-theorem says that in the case of Hilbert extensions of $\mathcal{A}$ with trivial relative commutant, the category $\mathcal{T}$ of all canonical endomorphisms can be indeed characterized intrinsically by their abstract algebraic properties as endomorphisms of $\mathcal{A}$ and a corresponding bijection can be established.

In this subsection we want to extend such a bijective correspondence to C*-algebras $\mathcal{A}$ with nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ and satisfying Property B. A first step in this direction is given in [7]. In this context and due to Proposition 3.12 one has to face the problem that the category $\mathcal{T}_\mathcal{G}$ and $\mathcal{T}$ *can not* be isomorphic anymore, since now we have

$$\mathbb{C}\mathbb{1} \subset \mathcal{Z} \subseteq \mathcal{A}' \cap \mathcal{F}\,.$$

We will investigate in the following the class of Hilbert extensions $\{\mathcal{F},\,\mathcal{G}\}$ with compact group $\mathcal{G}$ and where the relative commutant satisfies the following *minimality* condition

$$\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}\,. \tag{8}$$

In items (i)-(iv) of the introduction we gave several motivations that justify this choice. Therefore we define

4.1 Definition *A Hilbert C*-system $\{\mathcal{F},\mathcal{G}\}$ is called* **minimal** *if the condition*

$$\mathcal{A}'\cap\mathcal{F}=\mathcal{Z} \tag{9}$$

is satisfied.

4.2 Remark (i) The adjective minimal comes from the property of the relative commutant. Note that one always has $\mathcal{Z}\subseteq\mathcal{A}'\cap\mathcal{F}$. In the context of the DR-theory one has also minimal Hilbert C*-systems, because there $\mathcal{Z}=\mathbb{C}\mathbb{1}$ and $\mathcal{A}'\cap\mathcal{F}=\mathbb{C}\mathbb{1}$.

(ii) Let $\{\mathcal{F},\mathcal{G}\}$ be a C*-dynamical system with fixed point algebra $\mathcal{A}$ having trivial center $\mathcal{Z}(\mathcal{A})=\mathbb{C}\mathbb{1}$ and relative commutant satisfying

$$\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}(\mathcal{F})\quad\text{with}\quad\mathcal{Z}(\mathcal{F})=\mathcal{F}'\cap\mathcal{F}\,.$$

Then $\{\mathcal{F},\mathcal{G}\}$ can be obtained by inducing up from an essentially unique C*-dynamical system $\{\mathcal{F}_0,\mathcal{G}_0\}$, where $\mathcal{G}_0$ is a closed subgroup of $\mathcal{G}$, the fixed point algebra coincides with $\mathcal{A}$ and the relative commutant is trivial, i.e. $\mathcal{A}'\cap\mathcal{F}_0=\mathbb{C}\mathbb{1}$ (cf. [15, Theorem 1]). For a generalization of this result in the case where $\mathcal{Z}(\mathcal{A})$ is nontrivial and the corresponding relative commutant satisfies

$$\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}(\mathcal{A})\vee\mathcal{Z}(\mathcal{F})$$

see [36].

4.3 Proposition *Let $\{\mathcal{F},\,\mathcal{G}\}$ be a given Hilbert C*-system. Then $\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}$ iff $(\rho_D,\rho_{D'})=\{0\}$ for $D\neq D'$, i.e. iff the set $\{\rho_D\mid D\in\widehat{\mathcal{G}}\}$ is mutually disjoint.*

Proof: First note that $F\in\mathcal{A}'\cap\mathcal{F}$ iff $\Pi_D F\in(\rho_D,\iota)\mathcal{H}_D$ for all $D\in\widehat{\mathcal{G}}$. Therefore $\mathcal{A}'\cap\mathcal{F}=\mathcal{Z}$ iff $(\rho_D,\iota)=0$ for all $D\neq\iota$. But if this is true then also $(\rho_D,\rho_{D'})=0$ follows for all $D\neq D'$ (see e.g. [10, p. 193]). ■

Observe that in any Hilbert C*-system, for each $\tau\in\mathrm{Ob}\,\mathcal{T}$ the space $\mathfrak{H}_\tau:=\mathcal{H}_\tau\mathcal{Z}$, (where $\mathcal{H}_\tau$ is a $\mathcal{G}$-invariant algebraic Hilbert space) is a $\mathcal{G}$-invariant free right Hilbert $\mathcal{Z}$-module with inner product given as usual by

$$\langle H_1,H_2\rangle:=H_1^*H_2\in\mathcal{Z}\,,\quad H_1,H_2\in\mathfrak{H}_\tau$$

(cf. [6, Section 3]).

4.4 Proposition *Let $\{\mathcal{F},\,\mathcal{G}\}$ be a given minimal Hilbert C*-system, where the fixed point algebra $\mathcal{A}$ has center $\mathcal{Z}$. Then the category $\mathcal{T}$ of all canonical endomorphisms of $\{\mathcal{F},\,\mathcal{G}\}$ is isomorphic to the category $\mathcal{M}_{\mathcal{G}}$ of all free right Hilbert $\mathcal{Z}$-modules $\mathfrak{H}=\mathcal{H}\mathcal{Z}$, $\mathcal{H}\in\mathrm{Ob}\,\mathcal{T}_{\mathcal{G}}$, with arrows given by the corresponding $\mathcal{G}$-invariant module morphisms $\mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1,\mathfrak{H}_2;\mathcal{G})$.*

The bijection of objects is given by $\rho_{\mathcal{H}}\leftrightarrow\mathfrak{H}=\mathcal{H}\mathcal{Z}$ which satisfies the conditions

$$\begin{aligned}\rho_{\mathcal{H}}=(\mathrm{Ad}\,V)\circ\rho_1+(\mathrm{Ad}\,W)\circ\rho_2\quad&\longleftrightarrow\quad\mathfrak{H}=V\mathfrak{H}_1+W\mathfrak{H}_2\\ \rho_1\circ\rho_2\quad&\longleftrightarrow\quad\mathfrak{H}_1\cdot\mathfrak{H}_2\,,\end{aligned}$$

*where the latter product is the inner tensor product of the the Hilbert $\mathcal{Z}$-modules w.r.t. the natural *-homomorphism $\mathcal{Z} \to \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_2)$. The bijection on arrows is defined by*

$$\mathcal{J}\colon\ \mathcal{L}_{\mathcal{Z}}(\mathfrak{H}_1, \mathfrak{H}_2; \mathcal{G}) \to (\rho_1, \rho_2) \quad \text{with} \quad \mathcal{J}(T) := \sum_{j,k} \Psi_j Z_{j,k} \Phi_k^* =: \widehat{T} \,.$$

Here $\{\Psi_j\}_j$, $\{\Phi_k\}_k$ are orthonormal basis of $\mathfrak{H}_2, \mathfrak{H}_1$, respectively, and $(Z_{j,k})_{j,k}$ is the matrix of the right $\mathcal{Z}$-linear operator T from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ which intertwines the $\mathcal{G}$-action.

Proof: Note first that the minimality condition and Proposition 3.9 guarantee that the bijection on objects is independent of the choice of the algebraic Hilbert spaces within $\mathfrak{H}$, provided these define the same canonical endomorphism. The rest of the poof is in Proposition 3.1 and Section 4 of [6]. ■

The following result recalls the useful decomposition for a general Hilbert $\mathcal{Z}$-module $\mathfrak{H} \in \mathrm{Ob}\,\mathcal{M}_{\mathcal{G}}$ in terms of irreducible ones $\mathfrak{H}_D = \mathcal{H}_D\mathcal{Z}$, $D \in \widehat{\mathcal{G}}$. From Proposition 4.4 one has equivalently a decomposition of endomorphism $\rho_{\mathcal{H}} \in \mathrm{Ob}\,\mathcal{T}$ in terms of the corresponding irreducibles $\rho_D \in \mathrm{Irr}\,\mathcal{T}$.

4.5 Proposition *Let $\mathfrak{H} \in \mathrm{Ob}\,\mathcal{M}_{\mathcal{G}}$ be a $\mathcal{G}$-invariant free Hilbert $\mathcal{Z}$-module in $\mathcal{F}$. Then $\mathfrak{H}$ can be decomposed into the following orthogonal direct sum:*

$$\mathfrak{H} = \mathop{\oplus}_{D} (\rho_D,\, \rho_{\mathcal{H}})\, \mathfrak{H}_D \,.$$

If $\{W_{D,l}\}_{l=1}^{m(D)}$ denotes an orthonormal basis of $(\rho_D,\, \rho_{\mathcal{H}})$, where $m(D)$ is the multiplicity of $D \in \widehat{\mathcal{G}}$ in the decomposition of $U_{\mathcal{H}}$, then the isotypical projection can be written as

$$P_D := \sum_{l=1}^{m(D)} W_{D,l}\, W_{D,l}^* \,.$$

The canonical endomorphism associated with $\mathcal{H}$ is given by

$$\rho_{\mathcal{H}}(A) = \sum_{D,l,i} W_{D,l} \cdot \phi_{D,i} \cdot A \cdot \phi_{D,i}^* \cdot W_{D,l}^* = \sum_{D,l} W_{D,l} \cdot \rho_D(A) \cdot W_{D,l}^* \,.$$

The Proposition 4.4 above shows that the canonical endomorphisms uniquely fix the corresponding $\mathcal{Z}$-modules, but not the choice of the generating algebraic Hilbert spaces. The assumption of the minimality condition (9) is crucial here. From the point of view of the $\mathcal{Z}$-modules it is natural to consider next the following property of Hilbert C*-systems: the existence of a special choice of algebraic Hilbert spaces within the modules that define the canonical endomorphisms and which is compatible with products.

4.6 Definition *A Hilbert C*-system $\{\mathcal{F},\, \mathcal{G}\}$ is called* **regular** *if there is an assignment*

$$\mathrm{Ob}\,\mathcal{T} \ni \sigma \to \mathcal{H}_\sigma \in \mathrm{Ob}\,\mathcal{T}_{\mathcal{G}}$$

such that $\sigma = \rho_{\mathcal{H}_\sigma}$, i.e. σ is the canonical endomorphism of the algebraic Hilbert space $\mathcal{H}_\sigma$, and which is compatible with products:

$$\sigma \circ \tau \to \mathcal{H}_\sigma \cdot \mathcal{H}_\tau \,.$$

4.7 Remark (i) In a minimal Hilbert C*-system regularity means that there is a "generating" Hilbert space $\mathcal{H}_\tau \subset \mathfrak{H}_\tau$ for each τ (with $\mathfrak{H}_\tau = \mathcal{H}_\tau\mathcal{Z}$) such that the compatibility relation for products stated in Definition 4.6 holds. If a Hilbert C*-system is minimal and $\mathcal{Z} = \mathbb{C}\mathbb{1}$ then it is necessarily regular.

(ii) Note that the minimality condition and the compactness of the group imply that the Hilbert modules associated with objects in $\mathcal{T}$ are free. From the point of view of crossed products by endomorphisms considered in [37] this corresponds to a special case (cf. [37, Example 4.1]). Nevertheless, even in this particular situation with a compact group, there are cases where still one can not associate a symmetry to the larger category $\mathcal{T}$. In the context of Hilbert C*-modules this means that the left action *does not* coincide with the corresponding right action (in contrast with the situation considered in [37, Section 4]). The existence of a symmetry is related to the nontriviality of the chain group homomorphism to be introduced in the next section (see also Proposition 5.9, Remark 7.3 (i) and Section 8).

4.8 Lemma *Let $\{\mathcal{F},\, \mathcal{G}\}$ be a minimal and regular Hilbert C*-system. For $\sigma, \tau \in \mathrm{Ob}\,\mathcal{T}$ and $\rho, \rho' \in \mathrm{Irr}\,\mathcal{T}$ put*

$$(\sigma,\,\tau)_{\mathbb{C}} := \mathcal{J}\left(\mathcal{L}_{\mathcal{G}}(\mathcal{H}_\sigma, \mathcal{H}_\tau)\right) \leq (\sigma, \tau)\ , \quad (\rho,\,\tau)_{\mathbb{C}} := \mathcal{J}\left(\mathcal{L}_{\mathcal{G}}(\mathcal{H}_\rho, \mathcal{H}_\tau)\right) \leq (\rho,\,\tau)\ ,\ \text{etc.},$$

(cf. Remark 3.11 (iii)). If $\{W_{\tau,\,\rho,\,k}\}_k \subset (\rho,\,\tau)_{\mathbb{C}}$, $\{W_{\sigma,\,\rho',\,k'}\}_{k'} \subset (\rho',\,\sigma)_{\mathbb{C}}$ are orthonormal basis, then

$$\{W_{\tau,\,\rho,\,k} \cdot W^*_{\sigma,\,\rho,\,k'}\}_{\rho,k,k'}$$

is an orthonormal basis of $(\sigma,\,\tau)_{\mathbb{C}}$, where the ρ's are those irreducibles appearing in the decomposition of σ and *τ (cf. Proposition 4.5).*

Proof: From the orthonormality relations of $\{W_{\tau,\,\rho,\,k}\}_k$ and $\{W_{\sigma,\,\rho',\,k'}\}_{k'}$ and from the disjointness relation for irreducible objects in Proposition 4.3, it follows directly that $\{W_{\tau,\,\rho,\,k} \cdot W^*_{\sigma,\,\rho,\,k'}\}_{\rho,k,k'}$ is an orthonormal system in $(\sigma,\,\tau)_{\mathbb{C}}$. For any $A \in (\sigma,\,\tau)_{\mathbb{C}}$ put

$$\lambda_{\rho,k,k'} := W^*_{\tau,\,\rho,\,k}\, A\, W_{\sigma,\,\rho,\,k'} \in (\rho, \rho)_{\mathbb{C}} = \mathbb{C}\mathbb{1}\,,$$

where the last equation follows from the fact that the algebraic Hilbert space corresponding to ρ carries an irreducible representation. Then it is immediate to verify that

$$A = \sum_{\rho,k,k'} \lambda_{\rho,k,k'}\, W_{\tau,\,\rho,\,k}\, W^*_{\sigma,\,\rho,\,k'}\ ,$$

hence $\{W_{\tau,\,\rho,\,k} \cdot W^*_{\sigma,\,\rho,\,k'}\}_{\rho,k,k'}$ is an orthonormal basis in $(\sigma,\,\tau)_{\mathbb{C}}$. ■

4.9 Theorem *Let $\{\mathcal{F},\, \mathcal{G}\}$ be a minimal and regular Hilbert C*-system. Then $\mathcal{T}$ contains a DR-subcategory $\mathcal{T}_{\mathbb{C}}$ with the same objects, $\mathrm{Ob}\,\mathcal{T}_{\mathbb{C}} = \mathrm{Ob}\,\mathcal{T}$, and arrows $(\sigma,\,\tau)_{\mathbb{C}} \leq (\sigma, \tau)$ such that:*

$$(\sigma, \tau) = (\sigma, \tau)_{\mathbb{C}}\, \sigma(\mathcal{Z}) = \tau(\mathcal{Z})\, (\sigma, \tau)_{\mathbb{C}}\, \sigma(\mathcal{Z})\ . \tag{10}$$

Proof: Let $\mathcal{H}, \mathcal{K}$ are $\mathcal{G}$-invariant algebraic Hilbert spaces. Recall that the isometry $\mathcal{J} : \mathcal{L}(\mathcal{H}, \mathcal{K}) \to \mathcal{F}$ has the property

$$\mathcal{J}(\mathcal{L}_{\mathcal{G}}(\mathcal{H}, \mathcal{K})) \subseteq (\rho_{\mathcal{H}}, \rho_{\mathcal{K}}),$$

(cf. Remark 3.11 (iii)). Now let $\sigma \to \mathcal{H}_\sigma$ be the assignment given in Definition 4.6 for regular Hilbert C*-systems, and put

$$(\sigma,\,\tau)_{\mathbb{C}} := \mathcal{J}\left(\mathcal{L}_{\mathcal{G}}(\mathcal{H}_\sigma, \mathcal{H}_\tau)\right)\ .$$

Then the definitions of the symmetry $\epsilon(\cdot,\cdot)$, the conjugates $\overline{\sigma}$ and their intertwiners R_σ, S_σ are as follows (cf. Remark 3.18):

$$\epsilon(\sigma, \tau) := \sum_{j,k} \Psi_k \Phi_j \Psi^*_k \Phi^*_j,$$

where $\{\Phi_j\}_j$ resp. $\{\Psi_k\}_k$ are orthonormal basis of $\mathcal{H}_\sigma$ resp. $\mathcal{H}_\tau$. We also define: $\overline{\sigma} := \rho_{\mathcal{K}}$, where $\mathcal{K}$ carries the representation of $\mathcal{G}$ which is conjugated to the representation on $\mathcal{H}_\sigma$. Recall that all finite-dimensional representations of $\mathcal{G}$ can be realized by some $\mathcal{G}$-invariant algebraic Hilbert space. Moreover, $\mathcal{K}$ is chosen as the "distinguished" Hilbert space according to the assumption of regularity. Let $\{\Omega_j\}_j$ be an orthonormal basis of $\mathcal{K}$ and put

$$R_\sigma := \sum_j \Omega_j \Phi_j, \qquad S_\sigma := \epsilon(\overline{\sigma}, \sigma) R_\sigma .$$

With these choices it is easy to verify that $\mathcal{T}_\mathbb{C}$ is indeed a DR-subcategory of $\mathcal{T}$ (cf. Definition 3.19).

It remains to show Eq. (10). The inclusion $(\sigma,\tau) \supseteq (\sigma,\tau)_\mathbb{C}\, \sigma(\mathcal{Z})$ follows immediately from the fact that (σ,τ) is a right $\sigma(\mathcal{Z})$-module. To show the reverse inclusion let $\{W_{\tau,\rho,k}\}_k \subset (\rho,\tau)_\mathbb{C}$, $\{W_{\sigma,\rho',k'}\}_{k'} \subset (\rho',\sigma)_\mathbb{C}$ be orthonormal basis as in Lemma 4.8. Take $A \in (\sigma,\tau)$ and define

$$\rho(Z_{\rho,k,k'}) := W^*_{\tau,\rho,k}\, A\, W_{\sigma,\rho,k'} \in (\rho,\rho) = \rho(\mathcal{Z}) .$$

Then

$$\begin{aligned} A &= \sum_{\rho,k,k'} W_{\tau,\rho,k}\, \rho(Z_{\rho,k,k'})\, W^*_{\sigma,\rho,k'} \\ &= \sum_{\rho,k,k'} W_{\tau,\rho,k}\, W^*_{\sigma,\rho,k'}\, \sigma(Z_{\rho,k,k'}) \in (\sigma,\tau)_\mathbb{C}\, \sigma(\mathcal{Z}) , \end{aligned}$$

and the proof of Eq. (10) is completed. ■

4.10 Remark (i) Recall that the category $\mathcal{T}_\mathcal{G}$ introduced in Subsection 3.1 is an example of a DR-category (recall Remark 3.11) and in fact it plays the role of the subcategory $\mathcal{T}_\mathbb{C}$ as a subcategory of the in general larger category $\mathcal{T}$ of canonical endomorphisms of the minimal and regular Hilbert C*-system.

(ii) The assumptions of the previous theorem imply that each basis of $(\sigma,\tau)_\mathbb{C}$ is simultaneously a module basis of (σ,τ) modulo $\sigma(\mathcal{Z})$ as a right module, i.e. the module (σ,τ) is free.

(iii) For simplicity we will sometimes call a DR-subcategory $\mathcal{T}_\mathbb{C}$ of $\mathcal{T}$ satisfying the properties of Theorem 4.9 **admissible**.

(iv) Note that the properties (P.2)-(P.4) (with the exception of P.2.6) in [7, Section 2] are contained in the assumption that $\mathcal{T}_\mathbb{C}$ is a DR-subcategory of $\mathcal{T}$ (cf. Definition 3.19).

Let $\mathrm{Ob}\,\mathcal{T} \ni \rho \to V_\rho \in (\rho,\rho)$ be a choice of unitaries such that

$$V_{\rho\circ\sigma} = V_\rho \times V_\sigma . \tag{11}$$

Note that (11) implies that $V_\iota = \mathbb{1}$, because $V_\sigma = V_{\iota\sigma} = V_\iota \times V_\sigma = V_\iota V_\sigma$. This choice allows one to define from the subcategory $\mathcal{T}_\mathbb{C}$ of $\mathcal{T}$ another subcategory $\mathcal{T}'_\mathbb{C}$ of $\mathcal{T}$ with the same properties as $\mathcal{T}_\mathbb{C}$. To do this, put

$$(\rho,\sigma)'_\mathbb{C} := V_\sigma (\rho,\sigma)_\mathbb{C} V^*_\rho \subset (\rho,\sigma) \tag{12}$$

and for the corresponding permutation structure $\epsilon'(\rho,\sigma)$ for $\mathcal{T}'_\mathbb{C}$ one takes:

$$\epsilon'(\rho,\sigma) := (V_\sigma \times V_\rho) \cdot \epsilon(\rho,\sigma) \cdot (V_\rho \times V_\sigma)^* . \tag{13}$$

It is easy to check that ϵ' defines a permutation structure (cf. Proposition 3.16). The corresponding conjugates R'_ρ are defined by

$$R'_\rho := V_{\overline{\rho}\rho} R_\rho, \qquad S'_\rho := \epsilon'(\rho,\overline{\rho}) R'_\rho \tag{14}$$

(recall also Proposition 3.17). Then it is straightforward to verify that the new subcategory $\mathcal{T}'_{\mathbb{C}}$ also satisfies the assumptions of Theorem 4.9.

This suggests the following definition of equivalence between subcategories.

4.11 Definition *Two admissible DR-subcategories $\mathcal{T}_{\mathbb{C}}$ and $\mathcal{T}'_{\mathbb{C}}$ of $\mathcal{T}$ are called* **equivalent** *if there is an assignment*

$$\mathrm{Ob}\,\mathcal{T} \ni \rho \to V_\rho \in (\rho,\rho), \quad \text{with } V_\rho \text{ unitary satisfying} \quad V_{\rho\circ\sigma} = V_\rho \times V_\sigma,$$

and such that the equations (12), (13) and (14) hold.

The converse of Theorem 4.9 is also true:

4.12 Theorem *Let $\mathcal{T}$ be a tensor C*-category of unital endomorphisms of $\mathcal{A}$ and let $\mathcal{T}_{\mathbb{C}}$ be an admissible DR-subcategory. Then there is a minimal and regular Hilbert extension $\{\mathcal{F},\,\mathcal{G}\}$ of $\mathcal{A}$, such that $\mathcal{T}$ is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F},\,\mathcal{G}\}$. Moreover, if $\mathcal{T}_{\mathbb{C}}$, $\mathcal{T}'_{\mathbb{C}}$ are two admissible DR-subcategories of $\mathcal{T}$, then the corresponding Hilbert extensions are $\mathcal{A}$-module isomorphic iff $\mathcal{T}_{\mathbb{C}}$ is equivalent to $\mathcal{T}'_{\mathbb{C}}$.*

Proof: This is proven in [7]. ■

The previous result says that for minimal and regular Hilbert C*-systems there is an intrinsic characterization of the category of all canonical endomorphisms in terms of $\mathcal{A}$ only. Moreover, up to $\mathcal{A}$-module isomorphisms, there is a bijection between minimal and regular Hilbert extensions and tensor C*-categories $\mathcal{T}$ of unital endomorphisms of $\mathcal{A}$ with admissible DR-subcategories.

Note that Theorem 4.12 is an immediate generalization of the DR-theorem (mentioned at the beginning of this section) for the case that $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$, i.e. if $\mathcal{Z} = \mathbb{C}\mathbb{1}$ then $\mathcal{T}$ itself is admissible (hence a DR-category) and the corresponding Hilbert extensions have trivial relative commutant.

4.1 Hilbert C*-systems with abelian groups

In the case that $\mathcal{G}$ is abelian the preceding structure simplifies radically (see Remark 3.6). Specifically, $\widehat{\mathcal{G}}$ is a discrete abelian group (the character group), each $\mathcal{H}_D$, $D \in \widehat{\mathcal{G}}$, is one-dimensional with a generating unitary U_D, hence the canonical endomorphisms $\rho_{\mathcal{H}_D}$ (denoted briefly by ρ_D, see Remark 3.8(ii)) are in fact automorphisms (necessarily outer on $\mathcal{A}$). Since $\rho_{D_1} \circ \rho_{D_2} = \rho_{D_1 D_2}$ in this case the set Γ of all canonical endomorphisms $\rho_{\mathcal{H}_D}$ is a group with the property

$$\widehat{\mathcal{G}} \cong \Gamma/\mathrm{int}\,\mathcal{A}\,.$$

Therefore, it is not necessary to take into account direct sums or subobjects in this case and one can drop Property B as an assumption on $\mathcal{A}$ in this subsection.

4.13 Proposition *Let $\{\mathcal{F},\,\mathcal{G}\}$ be a Hilbert C*-system with $\mathcal{G}$ abelian. Then $\mathcal{Z} = \mathbb{C}\mathbb{1}$ implies that $\{\mathcal{F},\,\mathcal{G}\}$ is minimal and regular.*

Proof: Let $D, D' \in \widehat{\mathcal{G}}$ and $D \neq D'$. Assume $(\rho_D, \rho_{D'}) \neq 0$. Since $\rho_D, \rho_{D'}$ are automorphisms of $\mathcal{A}$ and $\mathcal{Z} = \mathbb{C}\mathbb{1}$ one obtains that ρ_D and $\rho_{D'}$ are unitarily equivalent, i.e. there is a unitary $V \in \mathcal{A}$ such that $\rho_{D'} = \mathrm{ad}\,V \circ \rho_D$. Let $\rho_D := \mathrm{ad}\,U_D \restriction \mathcal{A}$, $\rho_{D'} := \mathrm{ad}\,U_{D'} \restriction \mathcal{A}$. Then $U_{D'} A U_{D'}^{-1} = V U_D A (V U_D)^{-1}$ for all $A \in \mathcal{A}$ or $U_{D'} \in \mathcal{A} U_D = \Pi_D \mathcal{F}$ contradicting $D \neq D'$. Applying Proposition 4.3 one obtains $\mathcal{A}' \cap \mathcal{F} = \mathbb{C}\mathbb{1}$, i.e. $\{\mathcal{F},\,\mathcal{G}\}$ is minimal, hence necessarily regular. ■

4.14 Remark If $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ this conclusion is no longer true. In this case $X \in (\rho_D, \rho_{D'})$ implies only $X^* X = X X^* \in \mathcal{Z}$, i.e. X is "normal", but a construction of a unitary in $(\rho_D, \rho_{D'})$ is not possible, in general.

In the case $\mathcal{Z} = \mathbb{C}\mathbb{1}$ the permutators ϵ (restricted to $\widehat{\mathcal{G}} \times \widehat{\mathcal{G}}$) are elements of the second cohomology group $H^2(\widehat{\mathcal{G}})$ and

$$U_{D_1} \cdot U_{D_2} = \omega(D_1, D_2) U_{D_1 \circ D_2},$$

where

$$\epsilon(D_1, D_2) = \frac{\omega(D_1, D_2)}{\omega(D_2, D_1)}$$

and ω is a corresponding 2-cocycle. The field algebra $\mathcal{F}$ is just the ω-twisted discrete crossed product of $\mathcal{A}$ with $\widehat{\mathcal{G}}$ (see e.g. [2, p. 86 ff.] for details). For the case $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$ see [8]. (The minimal case is not specially mentioned there.)

5 Minimal Hilbert C*-systems for nonabelian groups

Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal Hilbert C*-system with $\mathcal{G}$ nonabelian and such that the fixed point algebra $\mathcal{A}$ satisfies Property B. Recall from Remark 3.11 (i) that $\rho \in \operatorname{Irr} \mathcal{T}$ if $(\rho, \rho) = \rho(\mathcal{Z})$. In this case one has trivially that $\mathcal{Z} \subseteq \rho(\mathcal{Z})$. We will next show that for irreducible endomorphisms the previous inclusion is actually an equality.

5.1 Proposition *For any $\rho \in \operatorname{Irr} \mathcal{T}$ we have $\rho(\mathcal{Z}) = \mathcal{Z}$ and $\rho{\restriction}\mathcal{Z}$ is an automorphism of $\mathcal{Z}$, i.e. $\rho \in \operatorname{aut} \mathcal{Z}$.*

Proof: Choose an irreducible endomorphisms $\rho \in \operatorname{Irr} \mathcal{T}$ and recall that in this case the conjugated endomorphism is also irreducible, i.e. $(\overline{\rho}, \overline{\rho}) = \overline{\rho}(\mathcal{Z})$. From Proposition 3.17 we have that the conjugates R_ρ and $S_\rho := \epsilon(\overline{\rho}, \rho) R_\rho$ satisfy the relations

$$R_\rho^* \overline{\rho}(S_\rho) = \mathbb{1} \quad \text{and} \quad S_\rho^* \rho(R_\rho) = \mathbb{1} \ ,$$

and we can define in terms of these the following two vector space isomorphisms (see e.g. [29]):

$$(\overline{\rho}, \overline{\rho}) \ni Y \to Y^* R_\rho \in (\iota, \overline{\rho}\rho) \quad \text{and} \quad (\iota, \overline{\rho}\rho) \ni X \to S_\rho^* \rho(Y) \in (\rho, \rho).$$

Composing these isomorphisms in the case of an irreducible pair $\rho, \overline{\rho}$ we obtain

$$(\overline{\rho}, \overline{\rho}) \ni \overline{\rho}(Z) \to S_\rho^* \rho(\overline{\rho}(Z)^* R_\rho) = Z^* \in (\rho, \rho)$$

i.e.

$$(\overline{\rho}, \overline{\rho}) \ni \overline{\rho}(Z) \to Z^* \in (\rho, \rho)$$

is a vector space isomorphism from $(\overline{\rho}, \overline{\rho})$ onto (ρ, ρ). Therefore $\rho(\mathcal{Z}) = \mathcal{Z}$ and $\rho{\restriction}\mathcal{Z}$ is an automorphism of $\mathcal{Z}$. ■

From the previous proposition it follows immediately:

5.2 Corollary *(i) Let $\lambda = \rho_1 \circ \rho_2 \circ \ldots \circ \rho_n$, where $\rho_i \in \operatorname{Irr} \mathcal{T}$, $i = 1, \ldots, n$. Then $\lambda(\mathcal{Z}) = \mathcal{Z}$ and $\lambda{\restriction}\mathcal{Z} \in \operatorname{aut} \mathcal{Z}$.*

(ii) For any unitary $U \in \mathcal{A}$ we have $(\operatorname{ad} U \circ \rho){\restriction}\mathcal{Z} = \rho{\restriction}\mathcal{Z}$, $\rho \in \operatorname{Irr} \mathcal{T}$.

From the previous result we can now introduce the following automorphism on $\mathcal{Z}$ which only depends on the class $D \in \widehat{\mathcal{G}}$:

$$\widehat{\mathcal{G}} \ni D \to \alpha_D := \rho_D{\restriction}\mathcal{Z} \in \operatorname{aut} \mathcal{Z} \,. \tag{15}$$

We will introduce next an equivalence relation[1] in $\widehat{\mathcal{G}}$ which, roughly speaking, relates elements $D, D' \in \widehat{\mathcal{G}}$ if there is a "chain of tensor products" of elements in $\widehat{\mathcal{G}}$ containing D and D'. This equivalence relation appears naturally when considering the action of irreducible canonical endomorphisms on $\mathcal{Z}$ (see Theorem 5.6 and Remark 5.7 below).

To make the preceding notion precise recall the algebraic structure of $\widehat{\mathcal{G}}$: denote by $\times$ the natural operation on subsets of $\widehat{\mathcal{G}}$ associated with the decomposition of the tensor products of irreducible representations: for $D_1, D_2 \in \widehat{\mathcal{G}}$ the set $D_1 \times D_2$ contains the corresponding classes that appear in the decomposition of $U_{D_1} \otimes U_{D_2}$, where $U_{D_i} \in D_i$, $i = 1, 2$, is any representant of the corresponding class. That means that if

$$U_{D_1} \otimes U_{D_2} = \mathop{\oplus}_{k=1}^{s} m_k \, U_{D'_k} \quad (m_k \equiv \text{multiplicities}\,),$$

then $D_1 \times D_2 = \{D'_1, \ldots, D'_s\}$. For $\Gamma, \Gamma_1, \Gamma_2 \subset \widehat{\mathcal{G}}$ put

$$\Gamma_1 \times \Gamma_2 = \cup\{D_1 \times D_2 \mid D_i \in \Gamma_i,\, i = 1, 2\} \quad \text{and} \quad D \times \Gamma = \{D\} \times \Gamma\,.$$

Further if $\overline{D} \in \widehat{\mathcal{G}}$ denotes the conjugate class to $D \in \widehat{\mathcal{G}}$ denote by $\overline{P} = \{\overline{D} \mid D \in P\}$. Recall in particular that if $D \in D_0 \times D_1$, $D' \in D'_0 \times D'_1$, then $D \times D' \subset D_0 \times D_1 \times D'_0 \times D'_1$ or that $1 \in \Gamma \times \overline{\Gamma}$ (cf. [28, Definition 27.35] for further details).

We can now make precise the previous idea:

5.3 Definition *$D, D' \in \widehat{\mathcal{G}}$ are called equivalent, $D \approx D'$, if there exist $D_1, \ldots, D_n \in \widehat{\mathcal{G}}$ such that*

$$D, D' \in D_1 \times \ldots \times D_n\,.$$

5.4 Remark (i) It is easy to check that the relation $\approx$ is an equivalence relation in $\widehat{\mathcal{G}}$. Indeed, reflexivity and symmetry are trivial. To show transitivity let $D \approx D'$ as above and $D' \approx D''$, i.e. $D', D'' \in D'_1 \times \ldots \times D'_k$ for some $D'_1, \ldots, D'_k \in \widehat{\mathcal{G}}$. We will show that D, D'' is contained in the larger chain

$$\Gamma := D_1 \times \ldots \times D_n \times \overline{D_1 \times \ldots \times D_n} \times D'_1 \times \ldots \times D'_k\,.$$

Indeed, $D'' \in \Gamma$, because $1 \in D_1 \times \ldots \times D_n \times \overline{D_1 \times \ldots \times D_n}$ and $D'' \in D'_1 \times \ldots \times D'_k$. Further, since $\overline{D'} \in \overline{D_1 \times \ldots \times D_n}$ and $D' \in D'_1 \times \ldots \times D'_k$ we also have that $1 \in \overline{D_1 \times \ldots \times D_n} \times D'_1 \times \ldots \times D'_k$, and therefore $D \in \Gamma$ (cf. [28, Theorem 27.38]).

Denote by square brackets $[\cdot]$ the corresponding chain equivalence classes and by $\mathfrak{C}(\mathcal{G})$ the set of these equivalence classes, i.e.

$$\mathfrak{C}(\mathcal{G}) := \{[D] \mid D \in \widehat{\mathcal{G}}\}\,.$$

(ii) Note that *any* pair $D, D' \in D_0 \times D_1$ satisfies by definition $D \approx D'$. Therefore for $D_0, D_1 \in \widehat{\mathcal{G}}$ we have that $D_0 \times D_1$ also specifies an element of $\mathfrak{C}(\mathcal{G})$ and we can simply put

$$[D_0 \times D_1] := [D]\,,$$

where D is any element in $D_0 \times D_1$.

(iii) It is also possible to formulate the previous equivalence relation entirely in terms of characters.

[1] We would like to acknowledge an observation of P.A. Zito concerning the equivalence relation that served to simplify the presentation below.

We will define on $\mathfrak{C}(\mathcal{G})$ an abelian product $\boxtimes$ (see Eq. (16) below) so that $(\mathfrak{C}(\mathcal{G}),\boxtimes)$ becomes an abelian group which for simplicity we call **chain group**. Moreover the chain group can be related to the character group of the center $\mathcal{C}$ of $\mathcal{G}$. For this recall also the notion of conjugacy class of a representation (cf. [23]): let $D\in\widehat{\mathcal{G}}$ and U_D any representant in D. By Schur's Lemma we have

$$U_D\restriction\mathcal{C}=\Psi_D\cdot\mathbb{1}\,,$$

and it can be easily seen that Ψ_D is a character on the center $\mathcal{C}$ of $\mathcal{G}$ which only depends on D, i.e. $\Psi_D\in\widehat{\mathcal{C}}$.

5.5 Theorem *Let $\mathcal{G}$ be a compact nonabelian group and denote its center by $\mathcal{C}$.*

(i) The set $\mathfrak{C}(\mathcal{G})$ becomes an abelian group w.r.t. the following multiplication: for $D_0,D_1\in\widehat{\mathcal{G}}$ put

$$[D_0]\boxtimes[D_1]:=[D_0\times D_1] \tag{16}$$

(recall Remark 5.4 (ii)).

(ii) The conjugacy classes Ψ_D depend on the chain equivalence class $[D]$ and there is an epimorphism $\eta:\mathfrak{C}(\mathcal{G})\to\widehat{\mathcal{C}}$ given by

$$\eta([D]):=\Psi_{[D]}\,,$$

where $\Psi_{[D]}$ is the conjugacy class associated with $[D]$.

Proof: (i) Recall first that by Remark 5.4 (ii) the r.h.s. of (16) is well defined. We need to verify next that the l.h.s. of (16) is independent of the representants $D_0,D_1\in\widehat{\mathcal{G}}$: suppose that $D_0\approx D_0'$ as well as $D_1\approx D_1'$ and we will show that any $D\in D_0\times D_1$ is related to any $D'\in D_0'\times D_1'$, i.e. $D\approx D'$, so that they specify the same equivalence class. Let

$$D_i,D_i'\in D_{i1}\times\ldots\times D_{ik_i}\,,\quad i=0,1\,,$$

for some $D_{i1},\ldots,D_{ik_i}\in\widehat{\mathcal{G}}$. Then

$$D\in D_0\times D_1\subset D_{01}\times\ldots\times D_{0k_0}\times D_{11}\times\ldots\times D_{1k_1}\supset D_0'\times D_1'\ni D'\,.$$

This shows that $D\approx D'$ and the product $\boxtimes$ is well defined.

The neutral element of $\mathfrak{C}(\mathcal{G})$ is given by the class generated by the trivial representation 1. The inverse element to $[D]$ is given by $[\overline{D}]$, because $1\in D\times\overline{D}$. Finally the commutativity of the product is guaranteed by the equation $D_0\times D_1=D_1\times D_0$ (see [28, Theorem 27.38] for details).

(ii) To show that the mapping η is well defined note that for the tensor product of *irreducible* representations U_{D_i}, $i=1,\ldots,n$, we have

$$U_{D_1}\otimes\ldots\otimes U_{D_n}\restriction\mathcal{C}=\Psi_{D_1}\cdot\ldots\cdot\Psi_{D_n}\mathbb{1}\,. \tag{17}$$

Even more any irreducible appearing in the decomposition of $U_{D_1}\otimes\ldots\otimes U_{D_n}$ defines the same conjugacy class $\Psi_{D_1}\cdot\ldots\cdot\Psi_{D_n}$. This shows that if $D\approx D'$, then $\Psi_D=\Psi_{D'}$ and that the η is a group homomorphism. Finally Theorem 27.46 in [28] implies surjectivity. ■

We will leave till the end of this section the computation of chain groups associated with several finite and compact Lie groups. We will now concentrate on the relation of the chain group $\mathfrak{C}(\mathcal{G})$, associated with the group $\mathcal{G}$ of a Hilbert C*-system $\{\mathcal{F},\,\mathcal{G}\}$, with the irreducible canonical endomorphisms restricted to $\mathcal{Z}$. In particular recall the automorphisms on $\mathcal{Z}$ given in Eq. (15) by $\alpha_D:=\rho_D\restriction\mathcal{Z}\in\operatorname{aut}\mathcal{Z}$ which are associated with any class $D\in\widehat{\mathcal{G}}$.

5.6 Theorem *(i) Let $D,D'\in\widehat{\mathcal{G}}$ be equivalent, i.e. $D\approx D'$. Then $\alpha_D=\alpha_{D'}$ and we can associate the automorphism $\alpha_{[D]}\in\operatorname{aut}\mathcal{Z}$ with the chain group element $[D]\in\mathfrak{C}(\mathcal{G})$.*

(ii) There is a natural group homomorphism between the chain group and the automorphism group generated by the irreducible endomorphisms restricted to $\mathcal{Z}$ (cf. Proposition 5.1):

$$\mathfrak{C}(\mathcal{G}) \ni [D] \mapsto \alpha_{[D]} \in \mathrm{aut}\, \mathcal{Z} \,. \tag{18}$$

Proof: (i) First we show that if $\lambda = \rho_1 \circ \ldots \circ \rho_n$ is a finite product of irreducibles whose dimensions are larger than one, then $\lambda \restriction \mathcal{Z} = \alpha_D$ for all $D \in \widehat{\mathcal{G}}$ appearing in the decomposition of λ: for this recall from Proposition 4.5 that λ can be decomposed as

$$\lambda(\cdot) = \sum_{D,j} W_{D,j}\; \rho_D(\cdot)\; W_{D,j}^* \,, \tag{19}$$

where the ρ_D's, $D \in \widehat{\mathcal{G}}$, are irreducible and $\{W_{D,j}\}_j$ is an orthonormal basis of (ρ_D, λ). Then by Corollary 5.2 (i) and Proposition 5.1 we have $\lambda(Z) \in \mathcal{Z}$ for all $Z \in \mathcal{Z}$ as well as

$$\lambda(Z) = \sum_{D,j} W_{D,j}\; \alpha_D(Z)\; W_{D,j}^* = \sum_D \alpha_D(Z) \sum_j W_{D,j} W_{D,j}^* = \sum_D \alpha_D(Z) E_D \,,$$

where

$$E_D := \sum_j W_{D,j} W_{D,j}^* \tag{20}$$

is the so-called isotypical projection w.r.t. $D \in \widehat{\mathcal{G}}$. Therefore $\sum_D \alpha_D(Z) E_D = \lambda(Z)$ or $\sum_D (\alpha_D(Z) - \lambda(Z)) E_D = 0$ and this implies $(\alpha_D(Z) - \lambda(Z)) E_D = 0$ for all D appearing in the decomposition of λ. Using (20) and the orhtogonality relations of the $W_{D,j}$'s we obtain finally $\alpha_D = \lambda \restriction \mathcal{Z}$, hence all irreducibles ρ_D occurring in the decomposition of λ coincide on $\mathcal{Z}$.

Second, let $D \approx D'$, i.e. $D, D' \in D_1 \times \ldots \times D_n$ for some $D_1, \ldots, D_n \in \widehat{\mathcal{G}}$ (cf. Definition 5.3). For the endomorphisms this implies that ρ_D and $\rho_{D'}$ appear in the decomposition of $\rho_{D_1} \circ \ldots \circ \rho_{D_n}$ The arguments of the first part of the proof show that

$$\alpha_D = \alpha_{D'} \,.$$

(ii) The homomorphism property follows immediately from the arguments of the proof of part (i) since for any $D_1, D_2 \in \widehat{\mathcal{G}}$ we have

$$\alpha_{[D_1]} \circ \alpha_{[D_2]} = \alpha_{\,[D_1 \times D_2]} = \alpha_{[D_1] \boxtimes [D_2]}$$

(recall Theorem 5.5 (i)). ■

5.7 Remark Note that the chain group and in particular Theorem 5.6 (i) completes the picture of the action of the irreducible canonical endomorphisms on the center $\mathcal{Z}$ of the fixed point algebra $\mathcal{A}$ (recall also Eq. (15)). Indeed, we may summarize this action by means of the following diagram

$$\begin{array}{ccccc} \widehat{\mathcal{G}} & \rightarrow & \mathfrak{C}(\mathcal{G}) & \rightarrow & \mathrm{aut}\, \mathcal{Z} \\ D & \mapsto & [D] & \mapsto & \alpha_{[D]} \end{array}$$

5.8 Theorem *Let $\lambda \in \mathrm{Ob}\, \mathcal{T}$. Then its action on $\mathcal{Z}$ can be described by means of the following formula*

$$\lambda(Z) = \sum_{[D]} \alpha_{[D]}(Z) \cdot \Big(\sum_{D' \,\in\, [D]} E_{D'} \Big) \,, \quad Z \in \mathcal{Z} \,,$$

where $E_{D'}$ is the isotypical projection w.r.t. $D' \in \widehat{\mathcal{G}}$.

Proof: First note that for a general $\lambda \in \mathrm{Ob}\,\mathcal{T}$ the equation (19) is still valid (cf. Proposition 4.5). From this we have for $Z \in \mathcal{Z}$

$$\lambda(Z) = \sum_D \alpha_D(Z) \sum_j W_{D,j} W_{D,j}^* = \sum_D \alpha_D(Z) E_D = \sum_{[D]} \alpha_{[D]}(Z) \cdot \Big(\sum_{D' \in [D]} E_{D'} \Big) ,$$

where for the last equation we have used Theorem 5.6 (i). ■

Next we show how a nontrivial chain group homomorphism (18) acts as an obstruction to the existence of a symmetry associated with the larger category $\mathcal{T}$.

Let $\mathcal{H}, \widetilde{\mathcal{H}} \in \mathrm{Irr}\,\mathcal{T}_{\mathcal{G}}$ be irreducible algebraic Hilbert spaces and $\mathfrak{H} = \mathcal{H}\mathcal{Z}, \widetilde{\mathfrak{H}} = \widetilde{\mathcal{H}}\mathcal{Z} \in \mathrm{Irr}\,\mathcal{M}_{\mathcal{G}}$ the corresponding free $\mathcal{Z}$-modules. By Proposition 4.4 we associate with them the irreducible endomorphisms $\rho, \widetilde{\rho} \in \mathrm{Irr}\,\mathcal{T}$ and denote the automorphisms of their restriction to $\mathcal{Z}$ by

$$\alpha := \rho \restriction \mathcal{Z} \quad \text{and} \quad \widetilde{\alpha} := \widetilde{\rho} \restriction \mathcal{Z} .$$

If $\{\Phi_i\}_i$ and $\{\widetilde{\Phi}_j\}_i$ are orthonormal basis of $\mathcal{H}$ resp. $\widetilde{\mathcal{H}}$, then

$$\Psi_i := \sum_{i'} \Phi_{i'}\, Z_{i'i} \quad \text{and} \quad \widetilde{\Psi}_j := \sum_{j'} \widetilde{\Phi}_{j'}\, \widetilde{Z}_{j'j} \quad Z_{i'i}, \widetilde{Z}_{j'j} \in \mathcal{Z} , \tag{21}$$

are arbitrary orthonormal basis of the corresponding modules $\mathfrak{H}$ resp. $\widetilde{\mathfrak{H}}$, where

$$\mathfrak{Z} := (Z_{i'i})_{i',i} \quad \text{and} \quad \widetilde{\mathfrak{Z}} := (\widetilde{Z}_{j'j})_{j',j} \in \mathrm{Mat}\,(\mathcal{Z}) \quad \text{satisfy} \quad \mathfrak{Z}^*\mathfrak{Z} = \mathfrak{Z}\mathfrak{Z}^* = \mathbb{1} = \widetilde{\mathfrak{Z}}^*\widetilde{\mathfrak{Z}} = \widetilde{\mathfrak{Z}}\widetilde{\mathfrak{Z}}^* .$$

5.9 Proposition *With the previous notation define*

$$\epsilon(\mathcal{H}, \widetilde{\mathcal{H}}) := \sum_{i,j} \widetilde{\Phi}_j\, \Phi_i\, \widetilde{\Phi}_j^*\, \Phi_i^* \quad \text{and} \quad \epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) := \sum_{i,j} \widetilde{\Psi}_j\, \Psi_i\, \widetilde{\Psi}_j^*\, \Psi_i^* .$$

Then we have

$$\epsilon(\mathcal{H}, \widetilde{\mathcal{H}}) = \epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) \quad \text{iff} \quad \widetilde{\alpha}\,(\mathfrak{Z})\, \mathfrak{Z}^* = \mathbb{1} \quad \text{and} \quad \widetilde{\mathfrak{Z}}\, \alpha(\widetilde{\mathfrak{Z}}^*) = \mathbb{1} ,$$

where $\widetilde{\alpha}(\mathfrak{Z}) := (\widetilde{\alpha}(Z_{i'i}))_{i',i} \in \mathrm{Mat}\,(\mathcal{Z})$ *(similarly for* $\alpha(\widetilde{\mathfrak{Z}}^*)$*).*

Proof: Using Eq. (21) we have

$$\begin{aligned}
\epsilon(\mathfrak{H}, \widetilde{\mathfrak{H}}) &= \sum_{i,j} \widetilde{\Psi}_j\, \Psi_i\, \widetilde{\Psi}_j^*\, \Psi_i^* = \sum_{i,j,i',j',i'',j''} \widetilde{\Phi}_{j'}\, \widetilde{Z}_{j'j}\, \Phi_{i'}\, Z_{i'i} \widetilde{Z}_{j''j}^* \widetilde{\Phi}_{j''}^*\, Z_{i''i}^*\, \Phi_{i''}^* \\
&= \sum_{i,j,i',j',i'',j''} \widetilde{\Phi}_{j'}\, \Phi_{i'}\, \alpha^{-1}(\widetilde{Z}_{j'j})\, Z_{i'i}\, \widetilde{Z}_{j''j}^*\, \widetilde{\alpha}^{-1}(Z_{i''i}^*)\, \widetilde{\Phi}_{j''}^*\, \Phi_{i''}^* \\
&\overset{!}{=} \sum_{i,j} \widetilde{\Phi}_j\, \Phi_i\, \widetilde{\Phi}_j^*\, \Phi_i^* = \epsilon(\mathcal{H}, \widetilde{\mathcal{H}})
\end{aligned}$$

Multiplying the previous equation from the left with $\Phi_{i_0'}^* \widetilde{\Phi}_{j_0'}^*$ and from the right with $\Phi_{i_0} \widetilde{\Phi}_{j_0}$ we obtain

$$\sum_{i,j} \alpha^{-1}(\widetilde{Z}_{j_0'j})\, \widetilde{Z}_{j_0j}^*\, Z_{i_0'i}\, \widetilde{\alpha}^{-1}(Z_{i_0i}^*) = \delta_{j_0'j_0}\, \delta_{i_0'i_0} .$$

That means

$$\sum_j \widetilde{Z}_{j_0'j}\, \alpha(\widetilde{Z}_{j_0j}^*) = \delta_{j_0'j_0} \quad \text{and} \quad \sum_i \widetilde{\alpha}(Z_{i_0'i})\, Z_{i_0i}^* = \delta_{i_0'i_0} .$$

and the proof is concluded. ■

5.1 Examples of chain groups for some finite and compact Lie groups

We will give next several examples of chain groups associated with nonabelian finite and compact Lie groups. We will also show that in all the examples considered the chain group is isomorphic to the character group of the center. Note also that if the group is abelian, then one can identify the chain group with the corresponding character group.

If $\mathcal{G}$ is the group we will denote its center by $\mathcal{C}(\mathcal{G})$ and the corresponding chain group by $\mathfrak{C}(\mathcal{G})$.

Compact Lie groups: We begin with the case $\mathcal{G} = \mathrm{SU}(2)$. Denote by

$$l \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\} = \widehat{\mathrm{SU}(2)}$$

the class specified by the usual representation $T^{(l)}$ of SU(2) on the space of complex polynomials of degree $\leq 2l$ which has dimension $2l+1$. Then the decomposition theory for the tensor products $T^{(l)} \otimes T^{(l')}$ (cf. [28, Theorem 29.26]) gives

$$l \times l' = \left\{\, |l-l'|, |l-l'|+1, \ldots, l+l' \,\right\}\,, \quad l, l' \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\}\,.$$

This decomposition structure implies that

$$l \approx l' \quad \text{iff} \quad l, l' \ \text{are both integers or both half-integers}\,.$$

We can finally conclude that

$$\mathfrak{C}(\mathrm{SU}(2)) = \left\{[0], [\tfrac{1}{2}]\right\} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathrm{SU}(2))}\,.$$

Using Brauer-Weyl theory it can be shown that in general for $\mathcal{G} = \mathrm{SU}(N)$ we still have an isomorphism between the chain groups and the character groups of the corresponding centers.[2]

Similarly one can proceed in other examples. Using well-known results on the decomposition of the tensor product of irreducible representations (see e.g. [28, Section 29]) we list the following further examples of chain groups.

(i) If $\mathcal{G} = \mathrm{U}(2)$ we have that its dual is given by the following labels:

$$\widehat{\mathrm{U}(2)} = \left\{\, (m,l) \mid l \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\}\,,\ m \in \mathbb{Z} \quad \text{and} \quad m+2l \ \text{even} \,\right\}\,.$$

Then we compute

$$\mathfrak{C}\,(\mathrm{U}(2)) = \left\{\ [(m_+,0)]\ ,\ [(m_-,\tfrac{1}{2})]\ \mid m_+/m_- \ \text{is even/odd} \right\} \cong \mathbb{Z} \cong \widehat{\mathcal{C}(\mathrm{U}(2))}\,.$$

(ii) If $\mathcal{G} = \mathrm{O}(3)$ we have

$$\widehat{\mathrm{O}(3)} = \left\{\, (0,l)\,,\ (1,l) \mid l \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\}\right\}$$

and

$$\mathfrak{C}\,(\mathrm{O}(3)) = \{\ [(0,0)]\ ,\ [(1,0)]\ \} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathrm{O}(3))}\,.$$

(iii) For $\mathcal{G} = \mathrm{SO}(3)$ recall that

$$\widehat{\mathrm{SO}(3)} = \{0, 1, 2, \ldots\}\,.$$

In this case the corresponding center as well as the chain group are trivial:

$$\mathfrak{C}(\mathrm{SO}(3)) = \{[0]\} \cong \widehat{\mathcal{C}(\mathrm{SO}(3))}\,.$$

[2]Christoph Schweigert, private communication.

Finite groups: We consider first the family of dihedral groups $\mathcal{G} = \mathbb{D}_{2m}$, $m \geq 2$. The group $\mathbb{D}_{2m}$ has order $2m$ and is generated by two elements a, b that satisfy the relations

$$a^m = b^2 = e \quad \text{and} \quad bab = a^{m-1} \, .$$

We consider first the case where $m = 2l$, $l \in \mathbb{N}$, is even. Then the center of $\mathbb{D}_{2m}$ is $\mathcal{C}(\mathbb{D}_{2m}) = \{e, a^l\} \cong \mathbb{Z}_2$ and its dual is given by

$$\widehat{\mathbb{D}_{2m}} = \{1, \chi_1, \chi_2, \chi_3\} \cup \{D_k \mid k = 1, \dots, \tfrac{m-2}{2}\} \, ,$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{\frac{m-2}{2}}$ are 2-dimensional representations. From the results concerning the decomposition of tensor products of irreducible representations stated in [28, §27.62 (d)] we conclude:

$$\mathfrak{C}(\mathbb{D}_{2m}) = \{\, [1] \, , \, [D_1] \,\} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathbb{D}_{2m})} \, .$$

To check the details of the previous example $\mathbb{D}_{2m}$, $m = 2l$, $l \in \mathbb{N}$, it is useful to distinguish further between the cases l even or odd. Indeed, if l is even, then

$$1 \approx \chi_1 \approx \chi_2 \approx \chi_3 \approx D_k \, , \; k \text{ even} \quad \text{and} \quad D_1 \approx D_{k'} \, , \; k' \text{ odd} \; .$$

If l is odd, then the corresponding chain classes have slightly different representants

$$1 \approx \chi_1 \approx D_k \, , \; k \text{ even} \quad \text{and} \quad \chi_2 \approx \chi_3 \approx D_1 \approx D_{k'} \, , \; k' \text{ odd} \, .$$

Similarly we can use the results in [28, Section 27] to compute the following family of examples:

(iv) $\mathcal{G} = \mathbb{D}_{2m}$ with m odd. Then the center is trivial, $\mathcal{C}(\mathbb{D}_{2m}) = \{e\}$, and

$$\widehat{\mathbb{D}_{2m}} = \{1, \chi_1\} \cup \{D_k \mid k = 1, \dots, \tfrac{m-1}{2}\} \, ,$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{\frac{m-1}{2}}$ are 2-dimensional representations. As before we compute

$$\mathfrak{C}(\mathbb{D}_{2m}) = \{\, [1] \,\} \cong \widehat{\mathcal{C}(\mathbb{D}_{2m})} \, .$$

(v) Let $\mathcal{G} = \mathbb{Q}_{4m}$ be the generalized quaternion groups which is a group of order $4m$ generated by two elements a, b that satisfy the relations $a^{2m} = b^4 = e$, $b^2 = a^m$ and $bab^{-1} = a^{2m-1}$. Its center is given by $\mathcal{C}(\mathbb{Q}_{4m}) = \{e, a^m\} \cong \mathbb{Z}_2$ and

$$\widehat{\mathbb{Q}_{4m}} = \{1, \chi_1, \chi_2, \chi_3\} \cup \{D_k \mid k = 1, \dots, m-1\} \, ,$$

where $\{1, \chi_1, \chi_2, \chi_3\}$ are 1-dimensional representations and $\{D_k\}_{k=1}^{m-1}$ are 2-dimensional representations. Distinguishing again between the cases m even or odd we obtain using [28, §27.62 (e)]

$$\mathfrak{C}(\mathbb{Q}_{4m}) = \{\, [1] \, , \, [D_1] \,\} \cong \mathbb{Z}_2 \cong \widehat{\mathcal{C}(\mathbb{Q}_{4m})} \, .$$

(vi) We conclude this list of examples mentioning the cases of the permutation groups $\mathbb{P}_3, \mathbb{P}_4$ and the the alternating group $\mathbb{A}_4$ which have trivial center. It is straightforward to verify that the corresponding chain groups are also trivial.

5.10 Conjecture In all the examples mentioned before we have shown that the chain group $\mathfrak{C}(\mathcal{G})$ is isomorphic to the character group $\widehat{\mathcal{C}(\mathcal{G})}$ of the center of $\mathcal{G}$. It is therefore natural to conjecture that the epimorphism considered in Theorem 5.5 (ii) is actually an isomorphism for any compact group.

At the same time the previous examples also show the difficulty of proving this statement in such general terms. By definition of the chain product in (16) it is clear that one needs to know explicit formulas for the decomposition of tensor products of irreducible representations in order to compute the chain group. For a general compact group this is of course a very difficult task (see e.g. the notes to §29 in [28]).

Concerning this question the groups $\mathbb{D}_{8l}$ and $\mathbb{Q}_{8l}$, $l \in \mathbb{N}$, are particularly interesting. Indeed, these groups are non-isomorphic but have isomorphic duals (cf. [28, §27.62 (f)]) and therefore isomorphic chain groups. However, since the centers of $\mathbb{D}_{8l}$ and $\mathbb{Q}_{8l}$ are isomorphic we can still verify the isomorphism between the corresponding chain groups and character groups of the center.

6 A family of examples of admissible subcategories

In this section we will give a family of examples of pairs of categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathbb{C}}$ is admissible (recall Remark 4.10 (iii)).

Let $\mathcal{A}_{\mathbb{C}}$ be a unital C*-algebra with trivial center, $\mathcal{Z}(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}\mathbb{1}$, and satisfying Property B. Denote by $\mathfrak{Z}$ a unital abelian C*-algebra and define

$$\mathcal{A} := \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}\,,$$

which is again a C*-algebra with unit $\mathbb{1} \otimes \mathbb{1}$ and center $\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \mathbb{1} \otimes \mathfrak{Z}$. Let $\mathcal{T}_{\mathrm{DR}}$ be a DR-category (recall Definition 3.19) realized as endomorphisms of $\mathcal{A}_{\mathbb{C}}$. The objects of $\mathcal{T}_{\mathrm{DR}}$ are denoted by ρ, σ etc. and the corresponding arrows by (ρ, σ). Let $\mathcal{G}$ be the compact group associated with $\mathcal{T}_{\mathrm{DR}}$ and denote by $\mathfrak{C}$ its corresponding chain group. We consider also a fixed group homomorphism (recall Theorem 5.6)

$$\mathfrak{C} \ni [D] \mapsto \alpha_{[D]} \in \operatorname{aut} \mathcal{Z}\,. \tag{22}$$

We can now start defining the C*-category $\mathcal{T}$ realized as endomorphism of the larger algebra $\mathcal{A}$ with nontrivial center $\mathcal{Z}$. To identify the new objects we proceed in two steps: first we extend irreducible endomorphisms in $\mathcal{T}_{\mathrm{DR}}$ to endomorphisms of $\mathcal{A}$. Second, we use the decomposition result in Proposition 4.5 to extend general objects in $\mathcal{T}_{\mathrm{DR}}$ to endomorphisms of $\mathcal{A}$. The extended endomorphisms of the larger algebra $\mathcal{A}$ are interpreted as new objects of the category $\mathcal{T}$.

(a) If ρ is irreducible, $\rho \in \operatorname{Irr} \mathcal{T}_{\mathrm{DR}}$, we define

$$\widetilde{\rho} := \rho \otimes \alpha_{[D]} \in \operatorname{end} \mathcal{A}\,,$$

where $D \in \widehat{\mathcal{G}}$ is the corresponding class associated with $\rho \in \operatorname{Irr} \mathcal{T}_{\mathrm{DR}}$ (cf. Remark 3.20).

(b) Let $\tau \in \operatorname{Ob} \mathcal{T}_{\mathrm{DR}}$. According to Proposition 4.5, the endomorphism τ can be decomposed in terms of irreducible objects as

$$\tau(\cdot) = \sum_{\rho,\, l} W_{\rho,\, l}\, \rho(\cdot)\, W^*_{\rho,\, l}\,,$$

where $\{W_{\rho,l}\}_l$ denotes an orthonormal basis of (ρ,τ) and $\rho \in \mathrm{Irr}\,\mathcal{T}_{\mathrm{DR}}$. We assign to τ the following endomorphism of $\mathcal{A}$

$$\begin{aligned} \widetilde{\tau}(\cdot) &:= \sum_{\rho,l} (W_{\rho,l} \otimes \mathbb{1})\, \widetilde{\rho}(\cdot)\, (W_{\rho,l} \otimes \mathbb{1})^* \\ &= \sum_{\rho,l} (W_{\rho,l}\, \rho(\cdot)\, W^*_{\rho,l}) \otimes \alpha_{[D]}(\cdot)\,, \end{aligned} \tag{23}$$

where for the second equation we have used the previous item (a).

(c) The arrows in $\mathcal{T}$ are defined as usual

$$(\widetilde{\rho}, \widetilde{\tau}) := \{A \in \mathcal{A} \mid A\,\widetilde{\rho}(X) = \widetilde{\tau}(X)\,A\,, \quad X \in \mathcal{A}\}\,.$$

6.1 Proposition *Let $\mathcal{T}$ be the C^*-category defined by means of (a),(b) and (c) above. Then the objects $\widetilde{\rho}, \widetilde{\rho}_1, \widetilde{\rho}_2$ defined in part (a) satisfy:*

(i) Irreducibility: $\widetilde{\rho} \in \mathrm{Irr}\,\mathcal{T}$, i.e. $(\widetilde{\rho}, \widetilde{\rho}) = \widetilde{\rho}(\mathcal{Z}) = \mathcal{Z}$.

(ii) Pairwise disjointness: $(\widetilde{\rho}_1, \widetilde{\rho}_2) = \{0\}$ if $\widetilde{\rho}_1 \neq \widetilde{\rho}_2$.

Proof: For the proof it is convenient to apply Gelfand's theorem: any $Z \in \mathfrak{Z}$ can be identified with a continuous function over the compact space $\mathrm{spec}\,(\mathfrak{Z})$, $Z(\cdot) \in C(\mathrm{spec}\,\mathfrak{Z})$, and

$$\mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z} \cong C(\mathrm{spec}\,\mathfrak{Z} \to \mathcal{A}_{\mathbb{C}})\,.$$

In particular, we will need below that any elementary tensor $A = A_0 \otimes Z \in \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$ can be expressed as the function $\mathrm{spec}\,\mathfrak{Z} \ni \mu \to A(\mu) = Z(\mu) A_0$.

(i) Let $\widetilde{\rho} := \rho \otimes \alpha_{[D]} \in \mathrm{Ob}\,\mathcal{T}$ with $D \in \widehat{\mathcal{G}}$ associated with $\rho \in \mathrm{Irr}\,\mathcal{T}_{\mathrm{DR}}$ by means of the DR-Theorem (cf. Remark 3.20). It is clear that $\widetilde{\rho}(\mathcal{Z}) \subseteq (\widetilde{\rho}, \widetilde{\rho})$, since

$$\widetilde{\rho}\,(\mathcal{Z}) = \mathbb{1} \otimes \alpha_{[D]}(\mathfrak{Z}) = \mathbb{1} \otimes \mathfrak{Z} = \mathcal{Z} \subseteq (\widetilde{\rho}, \widetilde{\rho})\,.$$

For the converse inclusion let $A \in (\widetilde{\rho}, \widetilde{\rho})$. In particular, this implies

$$A\,\widetilde{\rho}\,(X \otimes \mathbb{1}) = \widetilde{\rho}\,(X \otimes \mathbb{1})\,A\,, \quad X \in \mathcal{A}_{\mathbb{C}}\,.$$

The previous equation can be rewritten using Gelfand's theorem in terms of functions over $\mathrm{spec}\,\mathfrak{Z}$ as

$$A(\mu)\, \rho\,(X) = \rho\,(X)\, A(\mu)\,, \quad X \in \mathcal{A}_{\mathbb{C}}\,,\ \mu \in \mathrm{spec}\,\mathfrak{Z}\,.$$

Since $\rho \in \mathrm{Irr}\,\mathcal{T}_{\mathrm{DR}}$, i.e. $(\rho,\rho) = \mathbb{C}\mathbb{1}$, we conclude that $A(\mu) = \lambda(\mu)\,\mathbb{1}$, where λ is a continuous scalar function on $\mathrm{spec}\,\mathfrak{Z}$. Applying once more Gelfand's theorem we have

$$A \in \mathbb{1} \otimes \mathfrak{Z} = \mathbb{1} \otimes \alpha_{[D]}(\mathfrak{Z}) = \widetilde{\rho}\,(\mathcal{Z})$$

and we have shown that $\widetilde{\rho} \in \mathrm{Irr}\,\mathcal{T}$.

(ii) Let $\widetilde{\rho}_i = \rho_i \otimes \alpha_{[D_i]}$, $\rho_i \in \mathrm{Irr}\,\mathcal{T}_{\mathrm{DR}}$, $i = 1,2$, and $\widetilde{\rho}_1 \neq \widetilde{\rho}_2$. To show the disjointness relation $(\widetilde{\rho}_1, \widetilde{\rho}_2) = \{0\}$ choose $A \in (\widetilde{\rho}_1, \widetilde{\rho}_2) \subset \mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$. The intertwiner element A must satisfy in particular

$$A\,(\rho_1 \otimes \alpha_{[D_1]})\,(X \otimes \mathbb{1}) = (\rho_2 \otimes \alpha_{[D_2]})\,(X \otimes \mathbb{1})\,A\,, \quad X \in \mathcal{A}_{\mathbb{C}}\,,$$

which in terms of functions over $\mathrm{spec}\,\mathfrak{Z}$ means

$$A(\mu)\, \rho_1\,(X) = \rho_2\,(X)\, A(\mu)\,, \quad X \in \mathcal{A}_{\mathbb{C}}\,,\ \mu \in \mathrm{spec}\,\mathfrak{Z}\,.$$

Since $\rho_1 \neq \rho_2$ are irreducible in $\mathcal{T}_{\rm DR}$ we have $(\rho_1, \rho_2) = \{0\}$, hence $A(\mu) = 0$, $\mu \in \mathrm{spec}\,\mathfrak{Z}$, and we conclude that $A = 0$. ∎

We can now state the main result of this section, namely the specification of a family of examples satisfying the hypothesis of Theorem 4.12. Note that from the construction prescription in (a) and (b) above, there is a bijective correspondence between $\mathrm{Ob}\,\mathcal{T}_{\rm DR}$ (which are realized as endomorphisms of $\mathcal{A}_{\mathbb{C}}$) and $\mathrm{Ob}\,\mathcal{T}$ (which are realized as endomorphisms of the larger algebra $\mathcal{A} = \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z}$).

6.2 Theorem *Let $\mathcal{A}_{\mathbb{C}}$ be a unital C*-algebra with trivial center and satisfying Property B. Denote by $\mathfrak{Z}$ a unital abelian C*-algebra and define*

$$\mathcal{A} := \mathcal{A}_{\mathbb{C}} \otimes \mathfrak{Z} \quad \text{so that} \quad \mathcal{Z} := \mathcal{Z}(\mathcal{A}) = \mathbb{1} \otimes \mathfrak{Z} \,.$$

If $\mathcal{T}_{DR}$ is a DR-category (cf. Definition 3.19) realized as endomorphisms of $\mathcal{A}_{\mathbb{C}}$, let $\mathcal{T}$ be the C-category specified by (a),(b),(c) above. Define the C*-category $\mathcal{T}_{\mathbb{C}}$ as follows:* $\mathrm{Ob}\,\mathcal{T}_{\mathbb{C}} := \mathrm{Ob}\,\mathcal{T}$ *and*

$$(\,\widetilde{\sigma}, \widetilde{\tau}\,)_{\mathbb{C}} := (\sigma, \tau) \otimes \mathbb{1} \subset \mathcal{A} \,,$$

where σ, τ are the objects in $\mathcal{T}_{DR}$ corresponding to $\widetilde{\sigma}, \widetilde{\tau} \in \mathrm{Ob}\,\mathcal{T}$. Then $\mathcal{T}_{\mathbb{C}}$ is an admissible subcategory of $\mathcal{T}$, i.e. the arrows satisfy the equation

$$(\widetilde{\sigma}, \widetilde{\tau}) = (\widetilde{\sigma}, \widetilde{\tau})_{\mathbb{C}}\, \widetilde{\sigma}(\mathcal{Z}) \,, \tag{24}$$

(cf. Theorem 4.9).

Proof: We need to show Eq. (24), that means that we can generate the "larger" arrows set of $\mathcal{T}$ with the "smaller" arrow set of $\mathcal{T}_{\mathbb{C}}$ and the center $\mathcal{Z}$. From the decomposition result in Proposition 4.5 it is sufficient to prove the special case

$$(\widetilde{\rho}, \widetilde{\tau}) = (\widetilde{\rho}, \widetilde{\tau})_{\mathbb{C}}\, \widetilde{\rho}(\mathcal{Z}) \,, \quad \widetilde{\rho} \in \mathrm{Irr}\,\mathcal{T} \,,\, \widetilde{\tau} \in \mathrm{Ob}\,\mathcal{T} \,.$$

First we show $(\widetilde{\rho}, \widetilde{\tau}) \supseteq (\widetilde{\rho}, \widetilde{\tau})_{\mathbb{C}}\, \widetilde{\rho}(\mathcal{Z})$. For this, take an orthonormal basis $\{W_{\rho,l}\}_{l=1}^{n} \subset (\rho, \tau)$, where $\rho \in \mathrm{Irr}\,\mathcal{T}_{\rm DR}$, $\tau \in \mathrm{Ob}\,\mathcal{T}_{\rm DR}$. It is enough to show that for any $l = 1, \ldots, n$ and any $Z_0 \in \mathfrak{Z}$ the following equation holds:

$$\Big((W_{\rho,\,l} \otimes \mathbb{1})\ \widetilde{\rho}\,(\mathbb{1} \otimes Z_0)\Big)\widetilde{\rho}\,(X \otimes Z) = \widetilde{\tau}\,(X \otimes Z)\Big((W_{\rho,\,l} \otimes \mathbb{1})\ \widetilde{\rho}\,(\mathbb{1} \otimes Z_0)\Big) \,, \quad X \in \mathcal{A}_{\mathbb{C}} \,,\ Z \in \mathfrak{Z} \,.$$

Using the definition of irreducible $\widetilde{\rho}$ in part (a) above we can rewrite the last equation as

$$W_{\rho,\,l}\, \rho\,(X) \otimes \alpha_{[D]}(Z_0 Z) = \widetilde{\tau}\,(X \otimes Z)\,(W_{\rho,l} \otimes \alpha_{[D]}(Z_0)) \,, \quad X \in \mathcal{A}_{\mathbb{C}} \,,\ Z \in \mathfrak{Z} \,, \tag{25}$$

where $D \in \widehat{\mathcal{G}}$ corresponds to $\rho \in \mathrm{Irr}\,\mathcal{T}_{\rm DR}$ according to DR-Theorem. We consider now the expression $\widetilde{\tau}\,(X \otimes Z)$ separately and use Eq. (23) to obtain

$$\widetilde{\tau}\,(X \otimes Z) = \sum_{\rho',\, l'} (W_{\rho',\, l'}\ \rho'(X)\, W^*_{\rho',\, l'}) \otimes \alpha_{[D']}(Z) \,,$$

where $\{W_{\rho',l'}\}_{l'} \subset (\rho', \tau)$ is an orthonormal basis and $D' \in \widehat{\mathcal{G}}$ corresponds to $\rho' \in \mathrm{Irr}\,\mathcal{T}_{\rm DR}$. Inserting this in the r.h.s. of Eq. (25) and using the orthogonality relations $W^*_{\rho',l'} W_{\rho,l} = \delta_{\rho\rho'}\delta_{ll'}$ we obtain

$$\begin{aligned}
\widetilde{\tau}\,(X \otimes Z)\,(W_{\rho,l} \otimes \alpha_{[D]}(Z_0)) &= \sum_{\rho',\, l'} (W_{\rho',\, l'}\ \rho'\,(X)\, W^*_{\rho',\, l'} W_{\rho,\, l}) \otimes (\alpha_{[D']}(Z)\alpha_{[D]}(Z_0)) \\
&= W_{\rho,\, l}\, \rho\,(X) \otimes \alpha_{[D]}(Z_0 Z)
\end{aligned}$$

which coincides with the l.h.s. of Eq. (25). This concludes the proof of the inclusion $(\widetilde{\rho},\widetilde{\tau}) \supseteq (\widetilde{\rho},\widetilde{\tau})_{\mathbb{C}}\, \widetilde{\rho}(\mathcal{Z})$.

To show the reverse inclusion choose $A \in (\widetilde{\rho},\widetilde{\tau})$ so that

$$\begin{aligned} A\,\widetilde{\rho}\,(X\otimes Z) &= \widetilde{\tau}\,(X\otimes Z)\,A\,, \quad X\in\mathcal{A}_{\mathbb{C}}\,,\ Z\in\mathfrak{Z} \\ A\,(\rho\,(X)\otimes\alpha_{[D]}(Z)) &= \Big(\sum_{\rho',\,l'}(W_{\rho',\,l'}\,\rho'\,(X)\,W^*_{\rho',\,l'})\otimes\alpha_{[D']}(Z)\Big)\,A\,, \end{aligned}$$

where $\{W_{\rho',l'}\}_{l'} \subset (\rho',\tau)$ is an orthonormal basis as before. Multiplying the previous equation with $W^*_{\rho,l}\otimes\mathbb{1}$ from the left we get

$$((W^*_{\rho,l}\otimes\mathbb{1})A)\;(\rho\,(X)\otimes\alpha_{[D]}(Z)) = (\rho\,(X)\otimes\alpha_{[D]}(Z))\,((W^*_{\rho,\,l}\otimes\mathbb{1})\,A)\,,\quad X\in\mathcal{A}_{\mathbb{C}}\,,\ Z\in\mathfrak{Z}\,,$$

and this shows that

$$(W^*_{\rho,l}\otimes\mathbb{1})A\in(\widetilde{\rho},\widetilde{\rho}) = \widetilde{\rho}(\mathbb{1}\otimes\mathfrak{Z})\,,$$

where for the last equation we have used Proposition 6.1 (i). Therefore, for any $W_{\rho,l}$ there is a $Z_{\rho,l}\in\mathfrak{Z}$ such that $(W^*_{\rho,l}\otimes\mathbb{1})A = \mathbb{1}\otimes\alpha_{[D]}(Z_{\rho,l})$. Multiplying this relation from the left by $W_{\rho,l}\otimes\mathbb{1}$ and summing up w.r.t. l we obtain

$$\sum_l (E_\rho\otimes\mathbb{1})\,A = \sum_l W_{\rho,l}\otimes\alpha_{[D]}(Z_{\rho,l})\,,$$

where $E_\rho := \sum_l W_{\rho,l}\,W^*_{\rho,l}\in(\tau,\tau)$ is the isotypical projection w.r.t. $\rho\in\mathrm{Irr}\,\mathcal{T}_{\mathrm{DR}}$. To conclude the proof recall the disjointness relation in Proposition 6.1 (ii) which implies

$$(W^*_{\rho',l'}\otimes\mathbb{1})A\in(\widetilde{\rho},\widetilde{\rho'}) = \{0\}\quad\text{for all}\quad \widetilde{\rho}\neq\widetilde{\rho'}\,,$$

hence $(E_{\rho'}\otimes\mathbb{1})\,A = 0$ for all $\widetilde{\rho}\neq\widetilde{\rho'}$. Now from the property $\sum_\rho E_\rho = \mathbb{1}$ of the isotypical projections we obtain

$$A = \sum_l W_{\rho,l}\otimes\alpha_{[D]}(Z_{\rho,l}) = \sum_l (W_{\rho,l}\otimes\mathbb{1})\,(\mathbb{1}\otimes\alpha_{[D]}(Z_{\rho,l}))\in(\widetilde{\rho},\widetilde{\tau})_{\mathbb{C}}\,\widetilde{\rho}(\mathcal{Z})$$

and the proof is concluded. ■

We can now apply Theorem 4.12 to the pair of categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ constructed in this section to obtain the following result:

6.3 Proposition *Let $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ be the pair of C*-categories constructed in Theorem 6.2, where $\mathcal{T}_{\mathbb{C}}$ is an admissible subcategory of $\mathcal{T}$. Then there exists an essentially unique minimal and regular Hilbert extension $\{\mathcal{F},\,\mathcal{G}\}$ of $\mathcal{A}$.*

6.4 Remark Note that construction of the inclusion of C*-categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$ in Theorem 6.2 depends crucially on the choice of the chain group homomorphism in Eq. (22). Therefore different choices of this homomorphism will produce different minimal and regular Hilbert extensions.

7 The case of a trivial chain group homomorphism

We will assume in this section that the chain group homomorphism given in Eq. (18) is trivial. We will see that in this case the analysis of minimal Hilbert C*-systems $\{\mathcal{F},\,\mathcal{G}\}$ simplifies considerably. In fact, this assumption implies that any irreducible endomorphism acts trivially on the center $\mathcal{Z}$ and by Proposition 4.5 we finally obtain

$$\rho\restriction\mathcal{Z} = \mathrm{id}\restriction\mathcal{Z}\quad\text{for any}\ \ \rho\in\mathrm{Ob}\,\mathcal{T}\,. \tag{26}$$

For example, the chain group homomorphism is trivial if the chain group $\mathfrak{C}(\mathcal{G})$ itself is trivial (see the examples in (iii),(iv) and (vi) of Subsection 5.1). This means that any $D \in \widehat{\mathcal{G}}$ lies in the chain equivalence class of the trivial representation.

7.1 Proposition *Let $\{\mathcal{F}, \mathcal{G}\}$ be minimal Hilbert C*-system with fixed point algebra $\mathcal{A}$ and center $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$. Then the center of $\mathcal{F}$ coincides with $\mathcal{Z}$, i.e. $\mathcal{F} \cap \mathcal{F}' = \mathcal{Z}$.*

Proof: Let $\rho = \rho_{\mathcal{H}}$ be irreducible, $\rho(A) = \sum_j \Phi_j A \Phi_j^*$ with an orthonormal basis $\{\Phi_j\}_j$ of $\mathcal{H}$. Since $\rho(Z) = Z$ we get $\Phi_j Z = Z\Phi_j$ for all j. Further $\mathcal{F} = C^*(\mathcal{A}, \{\mathcal{H}\})$, where $\mathcal{H}$ runs through all irreducible Hilbert spaces. This implies $ZF = FZ$ for all $F \in \mathcal{F}$. Therefore $\mathcal{Z} \subseteq \mathcal{F}' \cap \mathcal{F} \subseteq \mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, hence $\mathcal{F}' \cap \mathcal{F} = \mathcal{Z}$ follows. ■

Next we show that in the case of a trivial chain group homomorphism one can still associate a symmetry with the larger category $\mathcal{T}$. For this purpose, recall from Proposition 4.4 that to any $\rho \in \mathrm{Ob}\,\mathcal{T}$ there exists a unique free right $\mathcal{Z}$-module $\mathfrak{H}_\rho \in \mathrm{Ob}\,\mathcal{M}_{\mathcal{G}}$.

7.2 Corollary *Each canonical endomorphism of a minimal Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ is a $\mathcal{Z}$-module endomorphism. The symmetry $\epsilon(\rho, \sigma)$, $\rho, \sigma \in \mathrm{Ob}\,\mathcal{T}$, defined for $\{\Phi_i\}_i$,$\{\Psi_j\}_j$ orthonormal basis in $\mathfrak{H}_\rho$, $\mathfrak{H}_\sigma$ by*

$$\epsilon(\rho, \sigma) := \sum_{i,j} \Psi_j \Phi_i \Psi_j^* \Phi_i^* \in (\rho\sigma, \sigma\rho)\,,$$

satisfy the corresponding properties of Proposition 3.16.

Proof: From the result $\mathcal{F}' \cap \mathcal{F} = \mathcal{Z}$ and the definition of a canonical endomorphisms we get immediately that $\rho(AZ) = \rho(A)Z = Z\rho(A)$ for any $\rho \in \mathrm{Ob}\,\mathcal{T}$ and any $Z \in \mathcal{Z}$, hence the objects in $\mathcal{T}$ are $\mathcal{Z}$-module endomorphism. In particular, this also implies $\alpha := \rho\lceil \mathcal{Z} = \mathrm{id}\lceil \mathcal{Z}$ for all $\rho \in \mathrm{Ob}\,\mathcal{T}$. Therefore by Proposition 5.9 we get that the definition of $\epsilon(\cdot, \cdot)$ is independent of the module basis chosen in the Hilbert $\mathcal{Z}$-module assigned to the objects of $\mathcal{T}$. The additional properties of $\epsilon(\cdot, \cdot)$ are then verified easily (cf. Proposition 3.16). ■

7.3 Remark (i) As already mentioned in [6, Remark 6.4] it is not possible in general to associate a symmetry $\epsilon(\cdot, \cdot)$ with the larger category of canonical endomorphisms $\mathcal{T}$. The reason is that, in general, the formula in the previous corollary is not independent of the module basis chosen (recall Proposition 5.9). Therefore the existence of a symmetry in the present context suggests that the nontriviality of the chain group homomorphism given in Eq. (18) is an obstruction to the existence of a well defined ϵ within the category $\mathcal{T}$.

(ii) The present section is also related to the notion of extention of C*-categories $\mathcal{C}$ by abelian C*-algebras $C(\Gamma)$ studied in [35]. In this reference it is shown that the DR-algebra associated with an object of the extension category $\mathcal{C}^\Gamma$ is a continuous field of DR-algebras corresponding to the initial category $\mathcal{C}$. (For the construction of DR-algebras associated with suitable C*-categories see [17].)

The previous corollary means that, in the present situation, the category $\mathcal{T}$ of all canonical endomorphisms is "almost" a DR-category (cf. Definition 3.19): in fact, there is a permutation and a conjugation structure. The only difference is that we have $(\iota, \iota) = \mathcal{Z} \supset \mathbb{C}\mathbb{1}$.

The next theorem shows that, using the central decomposition w.r.t. the common center $\mathcal{Z}$ (cf. Proposition 7.1), a minimal Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ satisfying $\rho\lceil \mathcal{Z} = \mathrm{id}\lceil \mathcal{Z}$, $\rho \in \mathrm{Ob}\,\mathcal{T}$, can be considered as a direct integral of Hilbert C*-systems $\{\mathcal{F}(\lambda), \mathcal{G}\}_\lambda$, $\lambda \in \mathrm{spec}\,\mathcal{Z}$, with trivial relative commutant and a fiber-independent compact group. Moreover, the Hilbert C*-systems

$\{\mathcal{F}(\lambda), \mathcal{G}\}$ associated with the base points λ are the ones that appear in the original DR-theory (recall the beginning of Section 4).

We assume for the rest of this section that the C*-algebras $\mathcal{A}$ and $\mathcal{F}$ are separable and faithfully represented in some Hilbert space $\mathfrak{h}$. We put $\mathcal{Z} \cong C(\Gamma)$, $\Gamma := \operatorname{spec} \mathcal{Z}$ (cf. Section 9).

7.4 Theorem *Let $\{\mathcal{F}, \mathcal{G}\}$ be a minimal Hilbert C*-system and $\mathcal{Z}$ be the common center of $\mathcal{A}$ and $\mathcal{F}$ (cf. Proposition 7.1). The fiber C*-algebras corresponding to the central decomposition w.r.t. $\mathcal{Z}$ (cf. Section 9) are denoted by $\mathcal{A}(\lambda)$, $\mathcal{F}(\lambda)$, $\lambda \in \Gamma$. Then for μ-almost every $\lambda \in \Gamma$ we have*

(i) $\mathcal{A}(\lambda) \subset \mathcal{F}(\lambda)$ and $\mathcal{A}(\lambda)$, $\lambda \in \Gamma$, satisfies Property B (cf. Subsection 3.1).

(ii) Let $\mathcal{H} \subset \mathcal{F}$ be a $\mathcal{G}$-invariant algebraic Hilbert space with support $\mathbb{1}$. Then the fiber spaces $\mathcal{H}(\lambda) \subset \mathcal{F}(\lambda)$ are again $\mathcal{G}$-invariant algebraic Hilbert spaces satisfying $\operatorname{supp} \mathcal{H}(\lambda) = \mathbb{1}$*. If $\mathfrak{H} = \mathcal{H}\mathcal{Z}$ is the free right $\mathcal{Z}$-module generated by $\mathcal{H}$, then $\mathfrak{H}(\lambda) = \mathcal{H}(\lambda)$.*

(iii) $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a Hilbert C-system. Let $\rho, \sigma \in \operatorname{Ob} \mathcal{T}$ be canonical endomorphisms and denote by $\rho_\lambda, \sigma_\lambda$ their fibre decomposition (cf. Proposition 9.3). Then $\rho_\lambda, \sigma_\lambda$ are canonical endomorphisms associated with $\{\mathcal{F}(\lambda), \mathcal{G}\}$ and the intertwiner space $(\rho_\lambda, \sigma_\lambda)$ is given by $(\rho_\lambda, \sigma_\lambda) = (\rho, \sigma)_\lambda$.*

(iv) The category $\mathcal{T}_\lambda$ of canonical endomorphisms associated with $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a DR-category, that means in particular $(\iota_\lambda, \iota_\lambda) = \mathbb{C}\mathbb{1}_\lambda$ and the Hilbert C-system $\{\mathcal{F}(\lambda), \mathcal{G}\}$ has a trivial relative commutant.*

Proof: (i) This follows from the inclusion $\mathcal{A}E(\Delta) \subset \mathcal{F}E(\Delta)$ for all Borel sets $\Delta \subset \Gamma$. The C*-algebra $\mathcal{A}$ satisfies Property B, if there exist isometries $V, W \in \mathcal{A}$ satisfying $VV^* + WW^* = \mathbb{1}$. By Theorem 9.1 it follows that the representation of V, W on the fibre spaces satisfy analogous properties.

(ii) Let $\{\Phi_i\}_i$ be an orthonormal basis of the $\mathcal{G}$-invariant algebraic Hilbert $\mathcal{H} \in \operatorname{Ob} \mathcal{T}_{\mathcal{G}}$. It transforms according to a unitary representation U of $\mathcal{G}$. Again, by Theorem 9.1 we have that, for μ-almost all $\lambda \in \Gamma$, $\{\Phi_i(\lambda)\}_i$ is an orthonormal basis of $\mathcal{H}(\lambda)$ transforming according to the *same* representation U (hence $\mathcal{H}(\lambda)$ is $\mathcal{G}$-invariant) and $\operatorname{supp} \mathcal{H}(\lambda) = \mathbb{1}$. Finally, let $\mathfrak{H} = \mathcal{H}\mathcal{Z} \subset \mathcal{F}$ be the free right $\mathcal{Z}$-module generated by $\mathcal{H}$. Any $H \in \mathfrak{H}$ can be written as $\sum_i \Phi_i Z_i$ for some $Z_i \in \mathcal{Z}$, hence its fibre component becomes

$$H(\lambda) = \sum_i \Phi_i(\lambda)\, Z_i(\lambda) \in \mathcal{H}(\lambda).$$

This shows $\mathfrak{H}(\lambda) = \mathcal{H}(\lambda)$ for μ-almost all $\lambda \in \Gamma$.

(iii) The first part follows already from (ii). Eq. (26) implies that all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$ are $\mathcal{Z}$-module endomorphisms, hence from Proposition 9.3 we have

$$\rho_\lambda(A(\lambda)) = \sum_i \Phi_i(\lambda)\, A(\lambda)\, \Phi_i(\lambda)^*\,,$$

i.e. ρ_λ is canonical w.r.t. $\{\mathcal{F}(\lambda), \mathcal{G}\}$. Finally, it is straightforward to show that $A \in (\rho, \sigma)$ iff $A(\lambda) \in (\rho_\lambda, \sigma_\lambda)$ for μ-almost all $\lambda \in \Gamma$.

(iv) The permutation and conjugation structures of $\mathcal{T}_{\mathcal{G}}$ (recall Subsection 3.3) can be "disintegrated" and the fibre components define the permutation and conjugation structures of $\mathcal{T}_\lambda$. Finally, $(\iota, \iota) = \mathcal{Z}$ implies $(\iota_\lambda, \iota_\lambda) = \mathbb{C}\mathbb{1}_\lambda$ and $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ gives $\mathcal{A}(\lambda)' \cap \mathcal{F}(\lambda) = \mathbb{C}\mathbb{1}_\lambda$ for μ-almost all $\lambda \in \Gamma$. ∎

Furthermore we have the following inverse theorem.

7.5 Theorem *Let $\mathcal{A}$ be a unital C*-algebra with center $\mathcal{Z}$ and satisfying Property B. Suppose that $\Gamma := \mathrm{spec}\,\mathcal{Z}$ is connected and let $\mathcal{T}$ be a tensor C*-category realized as unital endomorphisms of $\mathcal{A}$ and equipped with the following properties:*

(i) All $\rho \in \mathrm{Ob}\,\mathcal{T}$ are $\mathcal{Z}$-module endomorphisms.

(ii) $\mathcal{T}$ is closed w.r.t. direct sums and subobjects.

(iii) $\mathcal{T}$ is equipped with a permutation and a conjugation structure (cf. Propositions 3.16 and 3.17).

Then there is a minimal Hilbert extension $\{\mathcal{F},\mathcal{G}\}$ of $\mathcal{A}$ such that $\mathcal{T}$ is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F},\mathcal{G}\}$.

Proof: Let $\mathcal{A} = \int_\Gamma \mathcal{A}(\lambda)\mu(d\lambda)$ be the central decomposition of $\mathcal{A}$ over the direct integral $L^2(\Gamma,\mu,\mathfrak{f}_\lambda)$ (cf. Theorem 9.1) and denote by $\mathcal{T}(\lambda)$, for μ-almost all $\lambda \in \Gamma$, the C*-category whose objects ρ_λ are realized as endomorphisms of $\mathcal{A}(\lambda)$ (cf. Proposition 9.3). First we show that $\mathcal{T}(\lambda)$ is a DR-category (recall Definition 3.19): indeed, $(\iota_\lambda,\iota_\lambda) = \mathbb{C}\mathbb{1}_\lambda$ follows from the fact that that the C*-algebras $\mathcal{A}(\lambda)$ have trivial center (cf. Theorem 9.1). Note also that $\rho \in \mathrm{Irr}\,\mathcal{T}$ iff $(\rho_\lambda,\rho_\lambda) = \mathbb{C}\mathbb{1}_\lambda$ for μ-almost all $\lambda \in \Gamma$, i.e. $\rho_\lambda \in \mathrm{Irr}\,\mathcal{T}(\lambda)$. Closure under direct sums and subobjects follows from (ii). Recall that from (i), (iii) and Corollary 7.2 we have a well defined permutation structure on $\mathcal{T}$ which can be carried over to $\mathcal{T}(\lambda)$. Similarly we can "disintegrate" the conjugation structure, e.g. we have for the conjugate

$$(\iota,\overline{\rho}\rho) \ni R_\rho = \int_\Gamma R_{\rho_\lambda}\,\mu(d\lambda)$$

and R_{ρ_λ} satisfies the corresponding properties on the fibre. This shows that $\mathcal{T}(\lambda)$ is a DR-category for μ-almost all $\lambda \in \Gamma$.

Now, by the DR-theory we have on the one hand that

$$R^*_{\rho_\lambda} R_{\rho_\lambda} = d_{\rho_\lambda}\mathbb{1}_\lambda\,,$$

where $d_{\rho_\lambda} \in \mathbb{N}$. On the other hand $R^*_\rho R_\rho \in \mathcal{Z} \cong C(\Gamma)$ and therefore $\lambda \to d_{\rho_\lambda}$ is continuous on Γ. That means that $d_{\rho_\lambda} = d_\rho \in \mathbb{N}$ is constant over Γ. We use this result to analyze the Hilbert extension $\{\mathcal{F}(\lambda),\mathcal{G}(\lambda)\}$ of $\mathcal{A}(\lambda)$ which satisfies $\mathcal{A}(\lambda)' \cap \mathcal{F}(\lambda) = \mathbb{C}\mathbb{1}_\lambda$ (cf. [3])). The existence of this fibre Hilbert C*-system for μ-almost all $\lambda \in \Gamma$ is guaranteed by the DR-Theorem. For any $\rho_\lambda \in \mathrm{Irr}\,\mathcal{T}(\lambda)$ we consider a d_ρ-dimensional algebraic Hilbert space $\mathcal{H}_{\rho_\lambda} \equiv \mathcal{H}_\rho$, which is constant over Γ and generated by the an orthonormal basis $\{\Phi_{\rho,i}\}_{i=1}^{d_\rho}$, i.e.

$$\mathcal{H}_\rho = \mathrm{span}\,\{\Phi_{\rho,i} \mid i = 1,\ldots,d_\rho\}\,. \tag{27}$$

We may assume that the isometries $\Phi_{\rho,i}$ are represented on a fixed Hilbert space $\mathfrak{h}_0$. For any arbitrary $\tau_\lambda \in \mathrm{Ob}\,\mathcal{T}(\lambda)$ we associate the algebraic Hilbert space

$$\mathcal{H}_\tau(\lambda) := \bigoplus_{\rho_\lambda \in \mathrm{Irr}\,\mathcal{T}(\lambda)} (\rho_\lambda,\tau_\lambda)\,\mathcal{H}_\rho\,.$$

We have

$$\mathcal{H}_\tau(\lambda) \subset \mathcal{F}_{\mathrm{fin}}(\lambda) := \left\{ \sum_{\rho_\lambda \in \mathrm{Irr}\,\mathcal{T}(\lambda)} A_{\rho_\lambda,j}(\lambda)\,\Phi_{\rho,j} \right\} \subset \mathcal{F}(\lambda) := \mathrm{clo}_{\|\cdot\|_\lambda}\mathcal{F}_{\mathrm{fin}}(\lambda)$$

and the elements of the algebraic Hilbert space are bounded operators on $\mathfrak{f}_\lambda \otimes \mathfrak{h}_0$. Put

$$\mathcal{F} := \int_\Gamma \mathcal{F}(\lambda)\mu(d\lambda) \subset \mathcal{L}\left(L^2(\Gamma,\mu,\mathfrak{f}_\lambda \otimes \mathfrak{h}_0)\right)$$

with C*-norm given by $\|\cdot\| := \text{ess}\sup_\lambda \|\cdot\|_\lambda$.

Finally we have to define a compact group action on $\mathcal{F}$. For this, recall that on each fibre λ the compact group $\mathcal{G}(\lambda)$ acts as follows: each $A(\lambda) \in \mathcal{A}(\lambda)$ is left pointwise invariant under the group action. Moreover any $\mathcal{H}_\rho$, $\rho \in \operatorname{Irr}\mathcal{T}$, carries an irreducible representation of $\mathcal{G}(\lambda)$ which does not depend on λ (cf. Eq. (27)). Therefore the action $\mathcal{G}(\lambda)$ is independent of λ, hence we put $\mathcal{G}(\lambda) \equiv \mathcal{G}$, where $\mathcal{G}$ is compact. Since $\{\mathcal{F}(\lambda), \mathcal{G}\}$ is a Hilbert C*-system it follows immediately that $\{\mathcal{F}, \mathcal{G}\}$ is also a Hilbert C*-system. We still need to show that it is also minimal. For this take for any $D \in \mathcal{G}$ a canonical endomorphism $\rho_{D,\lambda} \in \operatorname{Irr}\mathcal{T}(\lambda)$, $\lambda \in \Gamma$. Since $\rho_{D,\lambda}$ is disjoint from ι_λ for any $D \neq \iota$, i.e. $(\rho_{D,\lambda}, \iota_\lambda) = 0$, we have that the corresponding integrated endomorphism satisfies the same property $(\rho_D, \iota) = 0$ for any $D \neq \iota$. From [10, Lemma 10.1.8] we have that $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ and the proof is completed. ■

7.6 Corollary *Each minimal Hilbert C*-system satisfying* $\rho\restriction\mathcal{Z} = \mathrm{id}\restriction\mathcal{Z}$, $\rho \in \operatorname{Ob}\mathcal{T}$, *with* $\Gamma :=$ $\operatorname{spec}\mathcal{Z}$ *connected, is regular.*

Proof: First recall from the proof of the previous theorem that the fiber Hilbert C*-systems have trivial relative commutant. This means that for μ-almost all $\lambda \in \Gamma$ there is a one-to-one correspondence between the fiber endomorphisms and the generating Hilbert spaces,

$$\rho_\lambda \leftrightarrow \mathcal{H}_{\rho_\lambda} \subset \mathcal{F}(\lambda)$$

such that

$$\rho_\lambda \circ \sigma_\lambda \leftrightarrow \mathcal{H}_{\rho_\lambda} \cdot \mathcal{H}_{\sigma_\lambda}.$$

If one chooses for μ-almost all $\lambda \in \Gamma$ a fixed Hilbert space $\mathcal{H}_{\rho_\lambda}$, then these fiber spaces define a Hilbert space $\mathcal{H}_\rho \subset \mathcal{F}$ such that

$$\rho \circ \sigma \leftrightarrow \mathcal{H}_\rho \cdot \mathcal{H}_\sigma.$$

Therefore $\{\mathcal{F}, \mathcal{G}\}$ is regular. ■

7.7 Remark We will extend in this remark the inverse result stated in Theorem 7.5 to the following situation: let $\{\mathcal{A}, \mathcal{T}\}$ satisfy the assumptions of Theorem 7.5 except that now Γ is a disjoint union of (in general infinite) connected components Γ_a, $a \in \mathbb{A}$, i.e.

$$\Gamma = \dot{\cup}\,\Gamma_a\,.$$

With each Γ_a, $a \in \mathbb{A}$, we can associate a central orthoprojection P_a, which is defined by means of the following continuous function over Γ:

$$P_a(\lambda) := \begin{cases} 1 & \text{if } \lambda \in \Gamma_a \\ 0 & \text{otherwise}\,. \end{cases}$$

The projections in $\{P_a\}_{a\in\mathbb{A}}$ are mutually disjoint and satisfy $\sum_a P_a = \mathbb{1}$ (strong operator convergence; to define the previous sum in the infinite case consider a net of projections indexed by the class of all finite subsets of $\mathbb{A}$ partially ordered by inclusion $\subseteq$, cf. [25, Sections 2.5 and 2.6]). The Hilbert space $\mathfrak{h}$, on which the algebra $\mathcal{A}$ is represented, decomposes as $\mathfrak{h} = \oplus_a \mathfrak{h}_a$ and $(x_a)_{a\in\mathbb{A}} \in \oplus_{a\in\mathbb{A}}\mathfrak{h}_a$ if $\sum_a \|x_a\|^2 < \infty$. Therefore we can decompose $\mathcal{A}$ as a direct sum

$$\mathcal{A} = \oplus_{a\in\mathbb{A}}\mathcal{A}_a \quad \text{and in particular} \quad \mathcal{Z} = \oplus_{a\in\mathbb{A}}\mathcal{Z}_a$$

where $\mathcal{A}_a := \mathcal{A}\,P_a$ has center $\mathcal{Z}_a := \mathcal{Z}\,P_a$. Recall that $(A_a)_{a\in\mathbb{A}} \in \oplus_{a\in\mathbb{A}}\mathcal{A}_a$ if $\sup\{\|A_a\| \mid a \in \mathbb{A}\} < \infty$.

From property (i) we can consistently define a family of $\mathcal{Z}_a$-module endomorphisms $\mathcal{T}_a := \{\rho_a\} \subset \operatorname{end} \mathcal{A}_a$ by means of

$$\rho_a(AP_a) := \rho(AP_a) = \rho(A)P_a \in \mathcal{A}_a\,, \quad A \in \mathcal{A}\,.$$

Moreover, since $\mathcal{A}$ satisfies Property B, then $\mathcal{A}_a$ also satisfies this property on $\mathfrak{h}_a$, i.e. $\mathcal{A}_a$ contains are isometries V_a, W_a satisfying $V_a V_a^* + W_a W_a^* = P_a$. Similarly we can adapt the assumptions (i)-(iii) to the pair $\{\mathcal{A}_a, \mathcal{T}_a\}$, $a \in \mathbb{A}$. By the proof of Theorem 7.5 we can construct Hilbert C*-systems $\{\mathcal{F}_a, \mathcal{G}_a\}$, with $\mathcal{G}_a$ compact and satisfying

$$\mathcal{A}_a' \cap \mathcal{F}_a = \mathcal{Z}_a \quad \text{as well as} \quad \mathcal{F}_a' \cap \mathcal{F}_a = \mathcal{Z}_a\,, \quad a \in \mathbb{A}\,.$$

Now, in order to be able to built up from these systems a minimal Hilbert C*-system with a compact group we need to make the following additional assumption:

> **Assumption:** The compact groups $\mathcal{G}_a$ are mutually isomorphic, i.e.
>
> $$\mathcal{G}_a \cong \mathcal{G}\,, \quad \text{for some compact group}\, \mathcal{G}\,. \tag{28}$$

Under this assumption put finally $\mathcal{F} := \oplus_a \mathcal{F}_a$, where $\mathcal{G}$ acts on each component. The Hilbert C*-system $\{\mathcal{F}, \mathcal{G}\}$ satisfies

$$\mathcal{A}' \cap \mathcal{F} = \oplus_a \mathcal{Z}_a = \mathcal{Z}\,,$$

hence it is minimal. Finally, note that Corollary 7.6 can be also adapted to the present more general situation satisfying the assumption (28).

7.8 Remark The reason why we need to make the assumption (28) is that we want to reconstruct Hilbert C*-systems $\{\mathcal{F}, \mathcal{G}\}$ of the type studied in Theorem 7.4. A similar situation that considers more general groups, where e.g. the mapping $\Gamma \ni \lambda \to \mathcal{G}_\lambda$ is not constant, is studied in [37, Section 3, Example 3.1].

8 Conclusions

In the present paper we have described a generalization of the DR-duality theory of compact groups, to the case where the underlying C*-algebra $\mathcal{A}$ has a nontrivial center. The abstract characterization of minimal and regular Hilbert C*-systems with a compact group $\mathcal{G}$ is now given by the inclusion of C*-categories $\mathcal{T}_{\mathbb{C}} < \mathcal{T}$, where $\mathcal{T}_{\mathbb{C}}$ is an admissible DR-subcategory of $\mathcal{T}$, the latter category being realized as endomorphisms of $\mathcal{A}$. A crucial new entity that appears when the center $\mathcal{Z}$ of $\mathcal{A}$ is nontrivial is the chain group $\mathfrak{C}(\mathcal{G})$, which is an abelian group constructed from a suitable equivalence relation in $\widehat{\mathcal{G}}$, the dual object of $\mathcal{G}$. Our results suggest the following considerations:

- As far as the symmetry ϵ is concerned, the special case studied in Section 7 was also addressed in the context of vector bundles and crossed products by endomorphisms (see e.g. [37, Eq. (3.7) and Section 4]). In the mentioned reference, the existence of a symmetry is guaranteed by the fact that the left and right $\mathcal{Z}$-actions on $(\iota, \mathcal{E})$ coincide, where $\mathcal{E}$ is a vector bundle over the compact Hausdorff space $\operatorname{spec} \mathcal{Z}$ and $(\iota, \mathcal{E})$ denotes the natural $\mathcal{Z}$-bimodule vector bundle morphism.

  However, the situation analyzed in the present paper can not be fully compared with the case studied in [37]. In the latter paper much more general groups are considered and, in fact, many of them are not even locally compact. For this reason no decomposition theory in terms of irreducible objects is mentioned in that context. It is therefore not clear how the notion of a nontrivial chain group should be extended to the general framework of vector bundles. Recall that the notion of chain group was suggested by the decomposition theory of canonical endomorphisms and their restriction to $\mathcal{Z}$ (cf. Theorem 5.6 (ii)). The nontriviality of the chain group homomorphism Eq. (18) gives an obstruction to the existence of a symmetry associated with the larger category $\mathcal{T}$ (see Proposition 5.9 and Remark 7.3 (i)).

- In lower dimensional quantum field theory models (see e.g. [11, 31, 22]), a nontrivial center appears when one constructs the so-called universal algebra. In the case of nets of C*-algebras indexed by open intervals of S^1, the universal algebra replaces the notion of quasi-local algebra (inductive limit). (Recall that in this case the index set is not directed. See [21] or [10, Chapter 5].) Although these models do not fit completely within the frame studied in this paper (there is no DR-Theorem and a nontrivial monodromy in two dimensions) we still hope that some pieces of the analysis considered here can be also applied in that situation. E.g. the generalization of the notion of irreducible objects and the analysis of their restriction to the center $\mathcal{Z}$ that in our context led to the definition of the chain group.

9 Appendix: Decomposition of a C*-algebra w.r.t. its center

For convenience of the reader we recall the following facts: Let $\mathcal{A}$ be a unital and separable C*-algebra, $\mathcal{Z}$ its center and π a representation of $\mathcal{A}$ on a Hilbert space $\mathfrak{h}$, $\pi(\mathcal{A}) \subset \mathcal{L}(\mathfrak{h})$. According to Gelfand's theorem we have $\mathcal{Z} \cong C(\Gamma)$, $\Gamma := \operatorname{spec} \mathcal{Z}$ a compact Hausdorff space. Then $\pi\restriction\mathcal{Z}$ defines a distinguished spectral measure $E_\pi(\cdot)$ on the Borel sets $\{\Delta\} \subset \Gamma$ such that

$$\pi(Z) = \int_\Gamma Z(\lambda) E_\pi(d\lambda)\,,$$

where $Z(\cdot) \in C(\Gamma)$ is the continuous function corresponding to $Z \in \mathcal{Z}$ (see e.g. M.A. Neumark [32, p. 278]). Note that $\pi(\mathcal{Z})'' = \{E_\pi(\Delta) \mid \Delta \subset \Gamma \text{ is Borel}\}''$. Since $\pi(\mathcal{Z}) \subseteq \pi(\mathcal{A}) \cap \pi(\mathcal{A})' \subseteq \pi(\mathcal{A})'$ we get $\pi(\mathcal{A}) \subset \pi(\mathcal{A})'' \subseteq \pi(\mathcal{Z})'$ or

$$E_\pi(\Delta)\pi(A) = \pi(A)E_\pi(\Delta), \quad A \in \mathcal{A},\ \Delta \subset \Gamma,$$

i.e. each projection $E_\pi(\Delta)$ is reducing for the representation π and each Borel set Δ defines a representation of $\mathcal{A}$ on $E_\pi(\Delta)\mathfrak{h}$:

$$\pi_\Delta(A) := \pi(A)\restriction E_\pi(\Delta)\mathfrak{h}.$$

Recall that $\Delta_1 \cap \Delta_2 = \emptyset$ implies that the representations $\pi_{\Delta_1}(\mathcal{A})$ and $\pi_{\Delta_2}(\mathcal{A})$ are disjoint.

Let $\Phi : \mathfrak{h} \to \widehat{\mathfrak{h}} := L^2(\Gamma, \mu, \mathfrak{f}_\lambda)$ (notation as in [9, Chapter 4]) be the unitary spectral transformation assigned to E_π, where μ is an appropriate Borel measure on Γ and $\mathfrak{f}_\lambda$ are the fibre Hilbert spaces. The spectral representation space $\widehat{\mathfrak{h}}$ (direct integral) is denoted also as $\int_\Gamma \mathfrak{f}_\lambda\, \mu(d\lambda)$. Putting $\mathfrak{h} \ni x \to \widehat{x} := \Phi(x) \in \widehat{\mathfrak{h}}$ we have (cf. [32, §41] and [34, Theorem IV.8.25])

9.1 Theorem *The C*-algebra $ad\,\Phi \circ \pi(\mathcal{A})$ is a C*-subalgebra of the von Neumann algebra $L^\infty(\Gamma, \mu, \mathcal{L}(\mathfrak{f}_\lambda))$ of all decomposable operators on $\widehat{\mathfrak{h}}$, i.e. for μ-almost all $\lambda \in \Gamma$ there are unital C*-algebras $(\pi(\mathcal{A}))(\lambda) \subset \mathcal{L}(\mathfrak{f}_\lambda)$ such that $\pi(A)$ is represented on $\widehat{\mathfrak{h}}$ by*

$$((ad\,\Phi \circ \pi(A))\,\widehat{x})(\lambda) = (\pi(A))(\lambda)\,\widehat{x}(\lambda)$$

and

$$ess\ sup_{\lambda\in\Gamma} \|(\pi(A))(\lambda)\|_\lambda < \infty\,,$$

where $\|\cdot\|_\lambda$ denotes the operator norm in $\mathcal{L}(\mathfrak{f}_\lambda)$. Moreover, the representations

$$\mathcal{A} \ni A \to (\pi(A))(\lambda) \in \mathcal{L}(\mathfrak{f}_\lambda)$$

are measurable and have trivial center.

9.2 Remark (i) Note that

$$L^\infty(\Gamma,\mu,\mathbb{C}\mathbb{1}_\lambda) \subseteq (\mathrm{ad}\Phi\circ\pi(\mathcal{A}))'' \subseteq L^\infty(\Gamma,\mu,\mathcal{L}(\mathfrak{f}_\lambda))$$

and

$$\pi(\mathcal{Z})'' \subseteq \pi(\mathcal{A})''\cap\pi(\mathcal{A})'.$$

Whereas the centers of the C*-algebras $(\pi(\mathcal{A}))(\lambda)$ are trivial, the von Neumann hulls $(\pi(\mathcal{A}))(\lambda)''$ may have nontrivial center. In other words, if one wants to decompose $\pi(\mathcal{A})''$ into factorial representations it is necessary to use the spectrum of the center $\pi(\mathcal{A})''\cap\pi(\mathcal{A})'$.

(ii) We mention here the special case where the spectral measure E_π has homogeneous multiplicity. Then there is a unique fiber space $\mathfrak{f}$ and $\widehat{\mathfrak{h}} = L^2(\Gamma,\mu,\mathfrak{f})$. Moreover $\widehat{\mathcal{A}} := \mathrm{ad}\,\Phi\circ\pi(\mathcal{A}) \subseteq L^\infty(\Gamma,\mu,\mathcal{L}(\mathfrak{f}))$, $\widehat{\mathcal{A}}(\lambda) := (\mathrm{ad}\,\Phi\circ\pi(\mathcal{A}))(\lambda)$ is a C*-algebra on $\mathfrak{f}$ for μ-almost all $\lambda\in\Gamma$ and for $\widehat{A}(\lambda)\in\widehat{\mathcal{A}}(\lambda)$

$$\mathrm{ess\ sup}_{\lambda\in\Gamma}\|\widehat{A}(\lambda)\|_{\mathcal{L}(\mathfrak{f})} < \infty\,.$$

If we assume that all operator functions $\Gamma\ni\lambda\to\widehat{A}(\lambda)\in\mathcal{L}(\mathfrak{f})$ are continuous w.r.t. the operator norm $\|\cdot\|_{\mathcal{L}(\mathfrak{f})}$, then also $\lambda\to\|\widehat{A}(\lambda)\|_{\mathcal{L}(\mathfrak{f})}$ is continuous and $\widehat{\mathcal{A}}\subset C(\Gamma,\mathcal{L}(\mathfrak{f}))\subset L^\infty(\Gamma,\mu,\mathcal{L}(\mathfrak{f}))$. Putting

$$\widehat{\mathcal{A}}_\lambda := \mathrm{clo}\,\{\widehat{A}(\lambda)\}_{\widehat{A}\in\widehat{\mathcal{A}}} \subseteq \mathcal{L}(\mathfrak{f}),$$

then $\widehat{\mathcal{A}}_\lambda$ is a unital C*-subalgebra of $\mathcal{L}(\mathfrak{f})$, $\{\widehat{\mathcal{A}}_\lambda,\widehat{\mathcal{A}}\}$ is a *continuous field* of C*-algebras over Γ and $\widehat{\mathcal{A}}$ is simultaneously the C*-algebra defined by this field (see Dixmier [12, p. 218 ff.]). If $\widehat{\mathcal{A}}_\lambda$ is independent of λ, for example $\widehat{\mathcal{A}}_\lambda = \mathcal{L}(\mathfrak{f})$ (this is true if $\widehat{\mathcal{A}} = C(\Gamma,\mathcal{L}(\mathfrak{f})) = C(\Gamma)\otimes\mathcal{L}(\mathfrak{f})$ consists of *all* continuous operator functions on Γ), then the field is trivial (in the sense of Dixmier).

9.1 $\mathcal{Z}$-module endomorphisms

To keep notation simple we consider in this subsection that $\mathcal{A}$ is faithfully represented in a Hilbert space $\mathfrak{h}$ and omit the explicit use of the representation π. Recall that a unital endomorphism ρ of $\mathcal{A}$ is called a $\mathcal{Z}$-module endomorphism if

$$\rho(AZ) = \rho(A)Z, \quad A\in\mathcal{A},\, Z\in\mathcal{Z}. \tag{29}$$

9.3 Proposition *Let ρ be a unital $\mathcal{Z}$-module endomorphism of $\mathcal{A}$. Then for μ-almost all $\lambda\in\Gamma$ there is a measurable family of unital endomorphism $\{\rho_\lambda\}_{\lambda\in\Gamma}$ of $\{\mathcal{A}(\lambda)\}_{\lambda\in\Gamma}$ defined for μ-almost all by*

$$\rho_\lambda(A(\lambda)) := (\rho(A))(\lambda), \quad A = \int_\Gamma A(\lambda)\mu(d\lambda)\in\mathcal{A}. \tag{30}$$

Conversely, any measurable family of unital endomorphism $\{\rho_\lambda\}_{\lambda\in\Gamma}$ satisfying

$$\mathit{ess\ sup}_{\lambda\in\Gamma}\|\rho_\lambda(A(\lambda))\|_\lambda < \infty, \quad A\in\mathcal{A},$$

defines a $\mathcal{Z}$-module endomorphism.

Proof: From Eq. (29) we have that each Borel set $\Delta\subset\Gamma$ defines a unital endomorphism ρ_Δ of $\mathcal{A}\restriction E(\Delta)\mathfrak{h}$ by means of

$$\rho_\Delta(AE(\Delta)) := \rho(A)E(\Delta)\,.$$

Therefore (30) follows and it is clear that $\lambda\to\rho_\lambda$ is measurable. Moreover one has

$$\|\rho_\lambda(A(\lambda))\|_\lambda \le \|A(\lambda)\|_\lambda, \quad A(\lambda)\in\mathcal{A}(\lambda),$$

because ρ_λ is an algebraic unital endomorphism of $\mathcal{A}(\lambda)$ and therefore by Theorem 9.1 we obtain

$$\text{ess sup}_{\lambda\in\Gamma}\|\rho_\lambda(A(\lambda))\|_\lambda < \infty\,.$$

Conversely, let $\{\rho_\lambda\}_\lambda$ be a measurable family of unital endomorphisms of $\mathcal{A}(\lambda)$, defined for μ-almost all $\lambda \in \Gamma$ such that

$$\text{ess sup}_{\lambda\in\Gamma}\|\rho_\lambda(A(\lambda))\|_\lambda < \infty, \quad A \in \mathcal{A},$$

is satisfied. Then this family defines a unital endomorphism ρ of $\mathcal{A}$ which is a $\mathcal{Z}$-module endomorphism, because

$$(\rho(AZ))(\lambda) = \rho_\lambda((AZ)(\lambda)) = \rho_\lambda(A(\lambda)Z(\lambda)) = Z(\lambda)\rho_\lambda(A(\lambda)) = (\rho(A)Z)(\lambda)$$

and the proof is concluded. ∎

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