Proof of the Parr Formula for the superheating field

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Abstract
In [5], in order to prove the De Gennes Formula [16], C. Bolley and B. Helffer have obtained an upper bound for the superheating field $h^{sh}(\kappa)$ in a semi-infinite film in the weak-κ limit. Precisely, they have proved that $\kappa\left(h^{sh}(\kappa)\right)^2 \leq 2^{-\frac{3}{2}} + \mathcal{O}(\kappa^{\frac{1}{2}})$. In this paper, we improve this result and get the upper bound

$$\kappa\left(h^{sh}(\kappa)\right)^2 \leq 2^{-\frac{3}{2}} + \frac{15}{32}\kappa + \mathcal{O}(\kappa^{1+\rho}), \quad \rho > 0.$$

Combining this result with the lower bound for $h^{sh}(\kappa)$ obtained in [11], we deduce the Parr Formula [19].

1. Introduction

The states of a superconducting material in an exterior magnetic field are described by the Ginzburg-Landau theory which introduces a functional ε depending in particular on a complex wave function and on the inner magnetic potential A. When the Landau parameter κ is small and when the width of the film d is large, (in the sense that κd is large), an approximation consists in reducing the problem to the study of local minima of the functional

$$\varepsilon_\infty(f,A) = \int_0^{+\infty} \left[\frac{1}{2}(1-f(x)^2)^2 + \kappa^{-2} f'(x)^2 + f(x)^2 A(x)^2 + A'(x)^2\right] dx \ + 2\, h\, A(0)\ , \quad (1.1)$$

defined for $(f,A) \in E_\infty = \{(f,A)\ ;\ (1-f) \in H^1(]0,+\infty[),\ A \in H^1(]0,+\infty[)\}$. The corresponding Ginzburg-Landau equations expressing the necessary conditions for minima are then

$$(GL)_\infty \quad \left\{ \begin{array}{llll} (a) & -\kappa^{-2} f'' - f + f^3 + f\ A^2 & = 0 & \text{on } [0,+\infty[, \\ (b) & -A'' + Af^2 & = 0 & \text{on } [0,+\infty[, \end{array} \right. \quad (1.2)$$

with the boundary conditions

$$f'(0) = 0, \quad A'(0) = h. \quad (1.3)$$

The problem $(GL)_\infty$ is called the half-space model and was studied in [17] and [18] where numerical solutions are given.
We consider the set $\mathcal{H}_\infty \subset I\!\!R^+$ of the h's such that there exist solutions of the $(G.L.)$ system with $f > 0$. We know that $\mathcal{H}_\infty$ is a bounded interval $[0, h^+)$ (see [3], Proposition 2.1) and we then introduce the following definition:

Definition 1.1 *The superheating field $h^{sh}(\kappa)$ is defined as the supremum of the interval $\mathcal{H}_\infty$.*

In the weak-κ limit, P. G. de Gennes [16] has proposed the formula

$$\lim_{\kappa\to 0} \kappa \left(h^{sh}(\kappa)\right)^2 = 2^{-\frac{3}{2}}. \tag{1.4}$$

In [3] and [5], C. Bolley and B. Helffer have proved rigorously (1.4).
H. Parr [19] has proposed on the basis of some heuristic computations, the more general formula

$$\kappa \left(h^{sh}(\kappa)\right)^2 = 2^{-\frac{3}{2}} + \frac{15}{32}\kappa + o(\kappa). \tag{1.5}$$

Let us recall that we have proved in [11] the following theorem:

Theorem 1.2 *There exist $\kappa_0 > 0$ and C such that, for all $\kappa \leq \kappa_0$, we have the inequality*

$$\kappa(h^{sh}(\kappa))^2 \geq 2^{-\frac{3}{2}} + \frac{15}{32}\kappa + C\kappa^2. \tag{1.6}$$

In this paper, to complete the proof of the Parr Formula, we prove the following theorem:

Theorem 1.3 *There exist κ_0 and $\rho > 0$, such that, for all pairs (f, A) solution of $(GL)_\infty$, for all $\kappa \leq \kappa_0$, we have the inequality*

$$\kappa A'(0)^2 = \kappa h^2 \leq 2^{-\frac{3}{2}} + \frac{15}{32}\kappa + \mathcal{O}(\kappa^{1+\rho}). \tag{1.7}$$

Remark 1.4 *In [2], C. Bolley and the author have proved the existence and the uniqueness of a solution of $(GL)_\infty$ when the value of $f(0) \in]0,1]$ is given. This permits to introduce the map $t \mapsto \sigma(t)$ which has the property that if (f, A) is a solution of the (G.L.) equations such that $f(0) = f_0$, then $A'(0) = \sigma(f(0))$. Theorem 1.3 permits to obtain an upper bound for the maximum of σ on $]0,1]$.*

In [5], to get an upper bound for $A'(0)$ and to prove the de Gennes Formula (1.4), C. Bolley and B. Helffer have proved the following estimates, for any pairs (f, A) solutions of $(GL)_\infty$:

$$A'(0)^2 \leq \sqrt{2}[(1 - f(0)^2)f(0)^2\kappa^{-1} + 5A'(0)f(0)^{-1}], \tag{1.8}$$

$$\kappa A'(0)^2 \geq \sqrt{2}(1 - f(0)^2)f(0)^2. \tag{1.9}$$

In [8] (see Proposition 3), in order to prove that $\lim_{f(0)\to 0} \sigma(f(0)) = \frac{1}{\sqrt{2}}$, these authors have proved the following estimate.

Proposition 1.5 *There exist κ_0 and C such that, for all $\kappa \leq \kappa_0$ and any solution (f, A) of $(GL)_\infty$, we have*

$$A'(0)^2 = h^2 \leq \frac{1}{2} + C\frac{f(0)}{\kappa}. \tag{1.10}$$

From (1.8), taking the maximum of $(1-f(0)^2)f(0)^2$ on the interval $[0,1]$, and from (1.10), they have deduced the upperbound

$$\kappa h^2 \leq 2^{-\frac{3}{2}} + \mathcal{O}(\kappa^{\frac{1}{2}}), \tag{1.11}$$

for all $h \in \mathcal{H}_\infty$ and for κ small enough.

Remark 1.6 *From Proposition 1.5, we deduce that Inequality (1.7) is true for any pairs (f,A) solutions of $(GL)_\infty$ such that $f(0) \leq \dfrac{1}{10C}$. In all the following sections, we will restrict us to the pairs (f,A) such that $f(0) \geq \dfrac{1}{10C}$.*

The plan of this paper is the following. In Section 2, we first recall some estimates for the functions f, A, f' and A' obtained in [3] and [5]. Then, we improve estimates obtained for A and A'. In Section 3, we analyze the estimate for $A'(0)=h$ obtained by C. Bolley and B. Helffer in [5]. We get a better estimate for f' in the interval $[0,\kappa^{-\rho}]$, $\rho>0$. In Section 4, using a principle maximum, we improve the upper bound for $A'(0)$ given in (1.8) and get Theorem 1.3. We deduce the Parr Formula (1.5).

2. Properties of solutions of $(GL)_\infty$

2.1. General properties of the Ginzburg-Landau equations

Let us recall some properties on the functions f, A, f' and A' obtained in [3] and [4].

Proposition 2.1 *Let (f,A) be a solution of $(GL)_\infty$.*
1. *f is increasing on $\mathbb{R}^+$ and we have*

$$0 \leq f(x) \leq \tanh\left(\frac{\kappa x}{\sqrt{2}} + x_0\right), \quad \tanh(x_0) = f(0). \tag{2.1}$$

2. *A is strictly increasing on $[0,+\infty[$ and we have*

$$0 \leq A'(x) \leq h, \quad \forall\, x \in [0,+\infty[. \tag{2.2}$$

3. *f' satisfies the inequalities*

$$0 \leq f'(x) \leq \frac{\kappa}{\sqrt{2}}, \quad \forall\, x \in [0,+\infty[. \tag{2.3}$$

4. *The pair (f,A) satisfies the following energy conservation*

$$\kappa^{-2} f'(x)^2 + A'(x)^2 = A(x)^2 f(x)^2 + \frac{(1-f(x)^2)^2}{2}, \quad \forall\, x \in [0,+\infty[. \tag{2.4}$$

5. *The pair (f,A) satisfies the inequalities*

$$0 < -A\,f \leq A', \quad \forall\, x \in [0,+\infty[. \tag{2.5}$$

Remark 2.2 *From Proposition 2.1 (see (2.1)), we deduce that in a region $[0,\kappa^{-\rho}]$, $\rho>0$, we have*

$$f(0) \leq f(x) \leq f(0) + \mathcal{O}(\kappa^{1-\rho}). \tag{2.6}$$

2.2. New estimates for f', A and A'

In the next sections, we use two useful versions of the maximum principle (cf. [5] and [7]).

Lemma 2.3 *Let $d \in \mathbb{R}^+ \cup \{+\infty\}$. Let C be a bounded function on $]0, d[$ such that*

$$C(x) \geq 0, \quad \forall\, x \in]0, d[,$$

and let $u \in C^2([0, d])$ be a function such that

$$\begin{cases} -u'' + Cu \leq 0 \quad on \ ]0, d[, \\ u'(0) \geq 0. \end{cases} \tag{2.7}$$

If $d \in \mathbb{R}^+$, we assume that $u(d) \leq 0$, and if $d = +\infty$, $u' \to 0$ when $x \to +\infty$. Then, in these two cases,

$$u \leq 0 \quad on \ ]0, d[.$$

Let us recall that, from Remark 2.2, we assume that

$$f(0) \geq \frac{1}{10C}. \tag{2.8}$$

Some inequalities on the functions f', A and A' are not optimal. In this section, we get a better control on A and A'. Let us recall that in [3], C. Bolley and B. Helffer have obtained the following control for A

$$h \exp(-x) \leq -A \leq \frac{h}{f(0)} \exp(-f(0)x), \quad \forall\, x \in \mathbb{R}^+. \tag{2.9}$$

The comparison with the behavior of the formal solution given in [12] suggests that the lower bound for $-A$ is not optimal. We get a better lower bound for $-A$ in the following proposition:

Proposition 2.4 *Let $\rho \in]0, \frac{1}{2}[$. There exist κ_0 and $C > 0$ such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (2.8), the function A satisfies the inequality*

$$\forall\, x \in [0, \kappa^{-\rho}], \quad -A \geq \frac{h}{f(0)}(1 - C\kappa^{1-2\rho}) \exp -(f(0)x). \tag{2.10}$$

Proof

We set:

$$v_2(x) := \tanh(\frac{\kappa x}{\sqrt{2}} + x_0), \quad \tanh(x_0) = f(0). \tag{2.11}$$

From (2.1), we get

$$f \leq v_2. \tag{2.12}$$

Let us consider the unique solution in $H^2(]0, +\infty[)$ of the problem

$$\begin{cases} -W'' + v_2^2 W = 0, \\ W'(0) = h. \end{cases} \tag{2.13}$$

From (2.13), and applying Lemma 2.3 with $d=+\infty$, $C=v_2^2$ and $u=W$, we get $W\le 0$ on $I\!R^+$. From $(1.2)_b$, (2.12) and (2.13), we get $-(A-W)''+f^2(A-W)\le 0$. Applying again Lemma 2.3 with $d=+\infty$, $C=f^2$ and $u=A-W$, we get

$$A\le W. \tag{2.14}$$

We set

$$\phi(x):=\int_0^x (v_2+c_1\kappa)dt, \tag{2.15}$$

where c_1 is a strictly positive parameter, which will be determined later. We consider

$$z(x)=z(0)\exp\left(-\phi(x)\right), \tag{2.16}$$

where $z(0)$ is determined by the condition $z'(0)=h$. We get $z'(0)=-(v_2(0)+c_1\kappa)z(0)=h$, hence, from (2.11)

$$z(0)=-\frac{h}{f(0)+c_1\kappa}. \tag{2.17}$$

We have the equality

$$-(W-z)''+v_2^2(W-z)=z(0)\exp(-\phi(x))\big(-\tfrac{\kappa}{\sqrt2}\tfrac{1}{\cosh^2(\frac{\kappa x}{\sqrt2}+x_0)}+2v_2c_1\kappa+c_1^2\kappa^2\big). \tag{2.18}$$

From the definition of the function v_2, we have the inequality

$$-\tfrac{1}{\sqrt2}\cosh^{-2}(\tfrac{\kappa x}{\sqrt2}+x_0)+2v_2c_1+c_1^2\kappa\ge 2c_1\tanh(x_0)-\tfrac{1}{\sqrt2}\cosh^{-2}(x_0). \tag{2.19}$$

We choose c_1 such that

$$c_1>\frac{1}{2\sqrt2}\left(\sinh(x_0)\cosh(x_0)\right)^{-1}. \tag{2.20}$$

From (2.17), we have $z(0)<0$ for κ small enough. From (2.18), (2.19) and (2.20), we get

$$-(W-z)''+v_2^2(W-z)<0.$$

Applying Lemma 2.3 on $[0,+\infty[$ with $C=v_2^2$ and $u=W-z$, we deduce that $W-u\le 0$. From (2.16) and (2.17), it results that

$$-W\ge -z=\frac{h}{f(0)+c_1\kappa}\exp(-\phi(x)). \tag{2.21}$$

From (2.15), there exists $d_x\in]0,1[$ such that we have the equality

$$\phi(x)=\phi'(0)x+\frac{\phi''(d_xx)}{2}x^2.$$

According to (2.11) and (2.15), we deduce that

$$\phi(x)=(f(0)+c_1\kappa)x+\frac{1}{\sqrt2}\cosh^{-2}(\frac{d_x\kappa x}{\sqrt2}+x_0)\kappa x^2.$$

It results that there exists $\rho > 0$ such that

$$\forall\, x \in\ [0, \kappa^{-\rho}],\ \ \phi(x) \leq (f(0) + c_1\kappa)x + \frac{\kappa^{1-2\rho}}{\sqrt{2}}. \tag{2.22}$$

Finally, from (2.14), (2.21) and (2.22), we get the inequality

$$-A \geq -W \geq \frac{h}{f(0) + c_1\kappa} \exp\left(-f(0)x - c_1\kappa x - \frac{\kappa^{1-2\rho}}{\sqrt{2}}\right). \tag{2.23}$$

We deduce the existence of some $\hat{C} > 0$, such that, for κ small enough

$$-A \geq \frac{h}{f(0)}(1 - c_1\frac{\kappa}{f(0)}) \exp(-f(0)x) \exp(-\hat{C}\kappa^{1-2\rho}). \tag{2.24}$$

From (2.8) and (2.24), we deduce that there exist $C > 0$ and κ_0, such that, for all $\kappa \in]0, \kappa_0]$, for all $x \in [0, \kappa^{-\rho}]$, we have the inequality

$$-A \geq \frac{h}{f(0)}(1 - C\kappa^{1-2\rho}) \exp\left(-f(0)x\right). \tag{2.25}$$

The proof of Proposition 2.4 follows.

We have also to get a lower bound and an upper bound for A'. In [3], C. Bolley and B. Helffer get in Proposition 2.4 the following estimate for A':

$$h \exp(-x) \leq A'(x) \leq \frac{h}{f(0)} \exp(-f(0)x), \quad \forall\, x \in I\!R^+. \tag{2.26}$$

The construction of a formal solution in [12] suggests that these inequalities are not optimal. We get a better control for A' in the following proposition:

Proposition 2.5 *Let $\rho \in [0, \frac{1}{4}]$. There exists κ_0 such that, for all $\kappa \leq \kappa_0$ and for all (f, A) solutions of $(GL)_\infty$ satisfying (2.8), we have the following estimate for A':*

$$A' = \left(h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho})\right) \exp(-f(0)x), \quad \forall\, x \in [0, \kappa^{-\rho}]. \tag{2.27}$$

Proof
From $(1.2)_b$, the function A' satisfies

$$-(A')'' + f^2 A' = -2f'fA, \quad A''(0) = A(0)f(0)^2. \tag{2.28}$$

From (2.1), (2.3) and (2.9), we get the inequality

$$-2f'fA \leq \frac{2\kappa h}{\sqrt{2}f(0)} \exp(-f(0)x), \quad \forall\, x \in I\!R^+. \tag{2.29}$$

We can compare A' with the unique solution in $H^2(I\!R^+)$ of

$$\begin{cases} -y'' + f(0)^2 y = \dfrac{\sqrt{2}\kappa h}{f(0)} \exp(-f(0)x), \\ y'(0) = A(0)f(0)^2, \end{cases} \tag{2.30}$$

given by

$$y(x)=\left(-A(0)f(0)+\frac{\sqrt{2}\kappa h(1+f(0)x)}{2f(0)^3}\right)\exp(-f(0)x). \tag{2.31}$$

According to (2.28), (2.29) and (2.30), we get $-(A'-y)''+f^2A'-f(0)^2y\leq 0$. Applying the principle maximum (see Lemma 2.3) with $d=+\infty$, $C(x)=f(0)^2$ and $u=A'-y$, we get $A'\leq y$ on $I\!R^+$. From (2.5) at the point $x=0$, it results that $-A(0)f(0)\leq h$. From (1.11), we have $h=\mathcal{O}(\kappa^{-\frac{1}{2}})$. As $\frac{1}{10C}\leq f(0)\leq 1$, for $x\in[0,\kappa^{-\rho}]$, we deduce the estimate

$$\frac{\sqrt{2}\kappa h(1+f(0)x)}{2f(0)^3}=\mathcal{O}(\kappa^{\frac{1}{2}-\rho}).$$

As $A'\leq y$, and from (2.31), it results that

$$A'\leq\left(h+\mathcal{O}(\kappa^{\frac{1}{2}-\rho})\right)\exp(-f(0)x). \tag{2.32}$$

To get a lower bound for A', we proceed as in Proposition 2.4. To get a lower bound, we compare A' solution of (2.28) with the unique solution in $H^2(]0,+\infty[$ of

$$\begin{cases} -W''+v_2^2W=0\\ W'(0)=A(0)f(0)^2,\end{cases} \tag{2.33}$$

where v_2 is defined in (2.11). From Lemma 2.3 with $u=-W$ and $C=v_2^2$, let us remark that we have $W\geq 0$ on $I\!R^+$. Applying again this lemma with $d=+\infty$, $C=f^2$ and $u=W-A'$, we deduce $W\leq A'$. Now, we compare W with the function z defined in (2.16) and satisfying $z'(0)=A(0)f(0)^2$. We have $z(0)=-\dfrac{A(0)f(0)^2}{f(0)+c_1\kappa}$, and as $A(0)<0$, we have $z(0)>0$. Moreover, we have the equality (cf. (2.18))

$$-(z-W)''+v_2^2(z-W)=-z(0)\exp(-\phi(x))\Big(-\tfrac{\kappa}{\sqrt{2}}\tfrac{1}{\cosh^2(\frac{\kappa x}{\sqrt{2}}+x_0)}+2v_2c_1\kappa+c_1^2\kappa^2\Big). \tag{2.34}$$

We take c_1 as in (2.20). As $z(0)>0$, we get

$$-(z-W)''+v_2^2(z-W)<0.$$

Applying Lemma 2.3 with $d=+\infty$, $u=z-W$ and $C=v_2^2$, we get

$$z(x)=-\frac{A(0)f(0)^2}{f(0)+c_1\kappa}\exp(-\phi(x))\leq W(x).$$

Following Proposition 2.4, for $x\in[0,\kappa^{-\rho}]$, we get

$$-\frac{A(0)f(0)^2}{f(0)+c_1\kappa}\exp\left(-f(0)x-c_1\kappa-\frac{\kappa^{1-2\rho}}{\sqrt{2}}\right)\leq z(x).$$

It results that

$$-A(0)f(0)(1-C\kappa^{1-2\rho})\exp(-f(0)x)\leq A'. \tag{2.35}$$

From (2.10) at the point $x=0$, we get

$$h(1-C\kappa^{1-2\rho})\leq -A(0)f(0).$$

From (2.35), it results that

$$h(1-C\kappa^{1-2\rho})\exp(-f(0)x)\leq A'. \tag{2.36}$$

According to (2.32) and (2.36), the proof of Proposition 2.5 follows.

3. Estimates for $A'(0)$

3.1. Preliminaries

First, we show that Inequality (1.7) is true for some pairs (f,A) such that $f(0) \in [0, \frac{1}{\sqrt{2}} - \alpha\kappa^{\gamma}] \cup [\frac{1}{\sqrt{2}} + \alpha\kappa^{\gamma}, 1]$, $(\alpha,\gamma) \in (I\!R^+)^2$.

Lemma 3.1 *There exist $\alpha > 0$ and κ_0, such that, for all $\kappa \le \kappa_0$, for all pairs (f,A) solutions of $(GL)_\infty$ such that $f(0) \in]0, \frac{1}{\sqrt{2}} - \alpha\kappa^{\frac{1}{4}}] \cup [\frac{1}{\sqrt{2}} + \alpha\kappa^{\frac{1}{4}}, 1]$, we have*

$$\kappa(A'(0))^2 \le 2^{-\frac{3}{2}}. \tag{3.1}$$

Proof
We set

$$\phi_0(y) = \sqrt{2}y^2(1-y^2). \tag{3.2}$$

This function achieves its unique maximum on $[0,1]$ at the point $y = \frac{1}{\sqrt{2}}$, and $\phi_0(\frac{1}{\sqrt{2}}) = 2^{-\frac{3}{2}}$. We have seen in Remark 1.6 that Inequality (3.1) is satisfied for $f(0) \le \frac{1}{10C}$. If $f(0) \ge \frac{1}{10C}$, then from (2.27), we have $\dfrac{A'(0)}{f(0)} = \mathcal{O}(\kappa^{-\frac{1}{2}})$. From (1.8), with the choice $f(0) = \dfrac{1}{\sqrt{2}} + \alpha\kappa^{\frac{1}{4}}$, we get for some $\tilde{C}$

$$\kappa(A'(0))^2 \le 2^{-\frac{3}{2}} + \frac{1}{2}\phi_0''(\frac{1}{\sqrt{2}})\alpha^2\kappa^{\frac{1}{2}} + \tilde{C}\kappa^{\frac{1}{2}} + \mathcal{O}_\alpha(\kappa^{\frac{3}{4}}).$$

We choose first α such that $\frac{1}{2}\phi_0''(\frac{1}{\sqrt{2}})\alpha^2 + \tilde{C} \le -1$ and then $\kappa \le \kappa_0$ for κ_0 small enough. The proof of Lemma 3.1 follows.

From now on, α is fixed according to Lemma 3.1 and we assume that

$$f(0) \in [\frac{1}{\sqrt{2}} - \alpha\kappa^{\frac{1}{4}}, \frac{1}{\sqrt{2}} + \alpha\kappa^{\frac{1}{4}}],\ \alpha > 0. \tag{3.3}$$

From (1.9), for all pairs (f,A) solution of $(GL)_\infty$ satisfying (3.3), we have

$$\kappa h^2 \ge 2^{-\frac{3}{2}} + \mathcal{O}(\kappa^{\frac{1}{4}}). \tag{3.4}$$

From (1.11) and (3.4), we deduce that

$$\kappa h^2 = 2^{-\frac{3}{2}} + \mathcal{O}(\kappa^{\frac{1}{4}}), \tag{3.5}$$

for all pairs (f,A) solution of $(GL)_\infty$ satisfying (3.3).
In order to get Inequality (1.7), let us analyze the proof by C. Bolley and B. Helffer of the De Gennes formula [5] and improve some of their estimates. For doing this, we are guided by the analysis of the properties of the approximate solution constructed in [12]. To get an upper bound on $A'(0)$, C. Bolley and B. Helffer start by the identity

$$h^2 = -2\int_0^{+\infty} A'(t)A''(t)dt.$$

Then, from $(1.2)_{(b)}$, they get

$$h^2 = 2\int_0^{+\infty} A'(t)(-A(t)f(t)^2)dt. \tag{3.6}$$

Then, using Inequality (2.5), they obtain

$$h^2 \leq 2\int_0^{+\infty} f(t)A'(t)^2 dt. \tag{3.7}$$

We suspect that something has been lost when writing :

$$-Af \leq A'.$$

Then, using $(1.2)_{(a)}$ and the energy conservation (2.4), they get

$$h^2 \leq 2\int_0^{+\infty} f(t)A'(t)^2 dt = \int_0^{+\infty} f(x)(1-f(x)^4)dx - 6\kappa^{-2}\int_0^{+\infty} f(x)f'(x)^2 dx. \tag{3.8}$$

In order to get the control of the right-hand side of Inequality (3.8), in particular the two terms f' and $1-f^2$, they use the conservation law (2.4). If we rewrite the energy conservation in the form

$$(\kappa^{-1}f'(x) + A'(x))^2 = A(x)^2 f(x)^2 + 2\kappa^{-1}f'(x)A'(x) + \frac{1}{2}(1-f(x)^2)^2,$$

we observe that these authors have neglected the positive term $A(x)^2f(x)^2 + 2\kappa^{-1}f'(x)A'(x)$ for getting the inequality:

$$\kappa^{-1}f'(x) + A'(x) \geq \frac{1}{\sqrt{2}}(1-f(x)^2).$$

To improve Inequality (3.8), in the next sections, we get an upper bound and a lower bound for the difference $(A')^2 - A^2f^2$. Moreover, we improve the control on f' given in (2.3). This is the object of the following subsection.

3.2. Estimate for f'

In order to get an estimate on f' on the interval $[0, \kappa^{-\rho}]$, we establish the following lemma:

Lemma 3.2 *Let $\rho \in [0, \frac{1}{4}]$. There exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the following estimate*

$$A'f' = \frac{\kappa^2 h^3}{2f(0)^2}\exp(-f(0)x)\left(1 - \exp(-2f(0)x) + \mathcal{O}(\kappa^\rho)\right), \quad \forall\, x \in [0, \kappa^{-\rho}]. \tag{3.9}$$

Proof

Step 1: Estimate for $A'f'$ at the point $x = \kappa^{-\rho}$.

Let $\rho \in [0, \frac{1}{4}]$. We set $Y = A'f'$. As $f' \geq 0$ and $A' \geq 0$, we have $Y \geq 0$. In order to use Lemma 2.3 with $d = \kappa^{-\rho}$, we estimate $A'f'$ at the point $x = \kappa^{-\rho}$. From (2.1), (2.4) at the point $x = \kappa^{-\rho}$ and (2.5), we deduce that

$$f'(\kappa^{-\rho}) \leq \frac{\kappa}{\sqrt{2}}(1 - f(\kappa^{-\rho})^2) \leq \frac{\kappa}{\sqrt{2}}(1 - \tanh(x_0)^2).$$

According to (2.27), and as $\tanh(x_0) = f(0)$ with $f(0) \in [\frac{1}{\sqrt{2}} - \alpha\kappa^{\frac{1}{4}}, \frac{1}{\sqrt{2}} + \alpha\kappa^{\frac{1}{4}}]$, we deduce that there exists $\hat{C}$ such that

$$(A'f')(\kappa^{-\rho}) \leq \left(\frac{\kappa h}{2\sqrt{2}} + \hat{C}\kappa^{\frac{3}{4}}\right) \exp(-f(0)\kappa^{-\rho}). \tag{3.10}$$

To get a lower bound for $(A'f')(\kappa^{-\rho})$, let us remark that, at the point $x = \kappa^{-\rho}$, for κ small, we have

$$(A^2 f^2 - A'^2)(\kappa^{-\rho}) = \mathcal{O}(\kappa^{-1}) \exp(-f(0)\kappa^{-\rho}) = \mathcal{O}(\kappa^2).$$

From (2.4) at the point $x = \kappa^{-\rho}$, we deduce that

$$\kappa^{-2} f'(\kappa^{-\rho})^2 = \frac{(1 - f(\kappa^{-\rho})^2)^2}{2} + \mathcal{O}(\kappa^2) \geq \frac{1}{2}\left(1 - \tanh^2(\frac{\kappa^{1-\rho}}{\sqrt{2}} + x_0)\right) + \mathcal{O}(\kappa^2).$$

Hence, from (3.3), we get

$$f'(\kappa^{-\rho}) \geq \frac{\kappa}{2\sqrt{2}}\left(1 + \mathcal{O}(\kappa^{\frac{3}{4}})\right).$$

According to (2.27), it results that there exists $\tilde{C}$ such that

$$\left(\frac{\kappa h}{2\sqrt{2}} + \tilde{C}\kappa^{\frac{3}{4}}\right) \exp(-f(0)\kappa^{-\rho}) \leq (A'f')(\kappa^{-\rho}) \tag{3.11}$$

Finally, from (3.10) and (3.11), we get

$$\left(\frac{\kappa h}{2\sqrt{2}} + \tilde{C}\kappa^{\frac{3}{4}})\right) \exp(-f(0)\kappa^{-\rho}) \leq (A'f')(\kappa^{-\rho}) \leq \left(\frac{\kappa h}{2\sqrt{2}} + \hat{C}\kappa^{\frac{3}{4}}\right) \exp(-f(0)\kappa^{-\rho}). \tag{3.12}$$

Step 2: Upper bound for Y.
We have

$$Y' = A''f' + A'f''.$$

Using the Ginzburg-Landau equations (1.2), we get

$$Y' = Af^2 f' + \kappa^2 A'(-f + f^3 + A^2 f).$$

We observe that $Y'(0) = h\, f''(0)$. Differentiating once more and using (1.2), we deduce that the function Y is the unique solution in $H^2(I\!R^+)$ of the problem

$$\begin{cases} Y'' - f^2 Y \\ = 2ff'^2 A + 2\kappa^2 A f^2(-f + f^3 + A^2 f) + \kappa^2 A'(-f' + 3f'f^2) + 2\kappa^2 A'^2 Af + 2\kappa^2 A' A^2 f' \\ Y'(0) = h\, f''(0). \end{cases}$$

Hence, we get

$$\begin{cases} -Y'' + G\, Y = -2ff'^2 A - 2\kappa^2 A f^2(-f + f^3) - 2\kappa^2 A^3 f^3 - 2\kappa^2 A'^2 Af, \\ Y'(0) = h\, f''(0), \end{cases} \tag{3.13}$$

where

$$G := f^2 + \kappa^2(-1 + 3f^2 + A^2). \tag{3.14}$$

According to (2.6) and (2.9), for $x \in [0, \kappa^{-\rho}]$, we get

$$f(0)^2 \leq G(x) \leq f(0)^2 + \mathcal{O}(\kappa^{1-\rho}). \tag{3.15}$$

In order to get an upper bound for Y, we look for an upper bound of the right-hand side of (3.13). According to (2.1), (2.2) and (2.3), we get

$$2f(f')^2 A \leq 2ff'^2 A - 2\kappa^2 A f^2 (f - f^3).$$

As $h = \mathcal{O}(\kappa^{-\frac{1}{2}})$ (see (1.11)), and from (2.3), (2.9),(2.27) and (3.3), there exists C_1 such that, for all $x \in [0, \kappa^{-\rho}]$, we get

$$2f(f')^2 A \geq C_1 \kappa^{\frac{3}{2}} \exp(-f(0)x).$$

From (3.13), we deduce that, for $x \in [0, \kappa^{-\rho}]$, we have

$$-Y'' + GY \leq -2\kappa^2 A f(A^2 f^2 + A'^2) - C_1 \kappa^{\frac{3}{2}} \exp(-f(0)x).$$

Now, let us analyze the main term $-2\kappa^2 A f(A^2 f^2 + A'^2)$. According to (2.6), (2.9), and Proposition 2.5, we get

$$\begin{aligned} &-2\kappa^2 A f(A^2 f^2 + A'^2) \\ &\leq \left(2\kappa^2 \frac{h^3}{f(0)^3} (f(0) + \mathcal{O}(\kappa^{1-\rho}))^3 + 2\kappa^2 \frac{h}{f(0)} \left(f(0) + \mathcal{O}(\kappa^{1-\rho})\right) (h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho}))^2\right) \exp(-3f(0)x). \end{aligned}$$

It results that there exists C_2 such that, for $x \in [0, \kappa^{-\rho}]$, we get

$$-2\kappa^2 A f(A^2 f^2 + A'^2) \leq \left(4\kappa^2 h^3 + C_2 \kappa^{1-\rho}\right) \exp(-3f(0)x).$$

Then, to get an upper bound for Y, we compare this function with the solution in $H^2([0, \kappa^{-\rho}])$ of

$$-y_1{}'' + f(0)^2 y_1 = \left(4\kappa^2 h^3 + C_2 \kappa^{1-\rho}\right) \exp(-3f(0)x) - C_1 \kappa^{\frac{3}{2}} \exp(-f(0)x), \tag{3.16}$$

satisfying

$$y_1'(0) = Y'(0), \quad y_1(\kappa^{-\rho}) = \left(\frac{\kappa h}{2\sqrt{2}} + \hat{C}\kappa^{\frac{3}{4}}\right) \exp(-f(0)\kappa^{-\rho}). \tag{3.17}$$

From (3.15) and as $Y \geq 0$, we have

$$-(Y - y_1)'' + f(0)^2 (Y - y_1) \leq -(Y - y_1)'' + GY - f(0)^2 y_1 \leq 0.$$

From (3.12) and (3.17), we have $(Y - y_1)'(0) = 0$ and $(Y - y_1)(\kappa^{-\rho}) \leq 0$. Applying Lemma 2.3 with $d = \kappa^{-\rho}$, $C = f(0)^2$ and $u = Y - y_1$, for $x \in [0, \kappa^{-\rho}]$, we get $Y \leq y_1$ on $[0, \kappa^{-\rho}]$. The solution of (3.16) is given by

$$\begin{aligned} &y_1(x) = \tilde{C}_1(\kappa) \exp(-f(0)x) + \tilde{C}_2(\kappa) \exp(f(0)x) \\ &-\frac{1}{8f(0)^2} \exp(-3f(0)x) \left(4\kappa^2 h^3 + C_2 \kappa^{1-\rho}\right) + \exp(-f(0)x) \left(-\frac{C_1}{4f(0)^2} \kappa^{\frac{3}{2}} (1 + 2f(0)x)\right). \end{aligned} \tag{3.18}$$

For $x \in [0, \kappa^{-\rho}]$, $(\rho \leq \frac{1}{4})$, we have $\kappa^{\frac{3}{2}}(1+2f(0)x) = \mathcal{O}(\kappa)$. Hence, from (3.17) and (3.18), we get

$$\begin{aligned} & y_1(\kappa^{-\rho}) = \tilde{C}_2(\kappa)\exp(f(0)\kappa^{-\rho}) + \left(\tilde{C}_1(\kappa) + \mathcal{O}(\kappa)\right)\exp(-f(0)\kappa^{-\rho}) \\ & = \left(\tfrac{\kappa h}{2\sqrt{2}} + \hat{C}\kappa^{\frac{3}{4}}\right)\exp(-f(0)\kappa^{-\rho}). \end{aligned} \tag{3.19}$$

As $\kappa h = \mathcal{O}(\kappa^{\frac{1}{2}})$, from (3.19), it results that

$$\tilde{C}_2(\kappa) = \left(-\tilde{C}_1(\kappa) + \mathcal{O}(\kappa^{\frac{1}{2}})\right)\exp(-2f(0)\kappa^{-\rho}). \tag{3.20}$$

From (3.18), we have

$$y_1'(0) = f(0)\tilde{C}_2(\kappa) - f(0)\tilde{C}_1(\kappa) + \frac{1}{8f(0)}(12\kappa^2h^3 - 2C_1\kappa^{\frac{3}{2}} + 3C_2\kappa^{1-\rho}).$$

From (3.20) and the condition $y_1'(0) = Y'(0) = h\,f''(0)$, it results that

$$\tilde{C}_1(\kappa) = -\frac{hf''(0)}{f(0)} + \frac{3}{2}\frac{\kappa^2h^3}{f(0)^2} + \mathcal{O}(\kappa^{1-\rho}).$$

From (1.2), (2.9) and (2.10), we get

$$f''(0) = \frac{\kappa^2h^2}{f(0)} + \mathcal{O}(\kappa^2). \tag{3.21}$$

Hence, we get

$$\tilde{C}_1(\kappa) = \frac{\kappa^2h^3}{2f(0)^2} + \mathcal{O}(\kappa^{1-\rho}). \tag{3.22}$$

According to (3.18) and (3.20), for $x \in [0, \kappa^{-\rho}]$, we get

$$\begin{aligned} & y_1(x) = -\frac{\kappa^2h^3}{2f(0)^2}\exp(-3f(0)x) \\ & + \exp(-f(0)x)[\tilde{C}_1(\kappa) + \left(-\tilde{C}_1(\kappa) + \mathcal{O}(\kappa^{\frac{1}{2}})\right)\exp\left(2f(0)(x-\kappa^{-\rho})\right)]. \end{aligned} \tag{3.23}$$

From (3.22), we have $\tilde{C}_1(\kappa) = \mathcal{O}(\kappa^{\frac{1}{2}})$. Let us remark that, for $x \in [0, \frac{\kappa^{-\rho}}{2}]$, for all $n \in I\!N$, we have

$$\left(-\tilde{C}_1(\kappa) + \mathcal{O}(\kappa^{\frac{1}{2}})\right)\exp\left(2f(0)(x-\kappa^{-\rho})\right) = \mathcal{O}(\kappa^n).$$

From (3.22) and (3.23), for $x \in [0, \frac{\kappa^{-\rho}}{2}]$, it results that

$$y_1(x) = -\frac{\kappa^2h^3}{2f(0)^2}\exp(-3f(0)x) + \left(\frac{\kappa^2h^3}{2f(0)^2} + \mathcal{O}(\kappa^{1-2\rho})\right)\exp(-f(0)x).$$

As $Y \leq y_1$, it results the inequality

$$Y(x) \leq \frac{\kappa^2h^3}{2f(0)^2}\exp(-f(0)x)\left(1 - \exp(-2f(0)x) + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho})\right), \quad \forall\, x \in [0, \frac{\kappa^{-\rho}}{2}]. \tag{3.24}$$

Taking κ_0 smaller, we get (3.9) on $[0,\kappa^{-\rho}]$.
Step 3: Lower bound for Y.
According to (2.1) and (2.3), and as $A'\geq 0$ and $A<0$, for $x\in I\!R^+$, we get

$$2ff'^2A+2\kappa^2Af^2(-f+f^3)\leq 2\kappa^2Af^2(-f+f^3).$$

From (2.1) and (2.9), there exists C_3 such that, for $x\in[0,\kappa^{-\rho}]$, we have

$$-2\kappa^2Af^2(f-f^3)\leq C_3\kappa^{\frac{3}{2}}\exp(-f(0)x).$$

It results that for $x\in[0,\kappa^{-\rho}]$, we obtain the inequality

$$2ff'^2A+2\kappa^2Af^2(-f+f^3)\leq C_3\kappa^{\frac{3}{2}}\exp(-f(0)x).$$

From (2.10) and (2.27), it results that there exists C_4 such that

$$-2\kappa^2Af(A^2f^2+A'^2)\geq\left(4\kappa^2h^3+C_4\kappa^{1-\rho}\right)\exp(-3f(0)x),\quad\forall\, x\,\in[0,\kappa^{-\rho}].$$

We want compare Y with the solution in $H^2([0,\kappa^{-\rho}])$ of

$$-y_2''+(f(0)^2+C_5\kappa^{1-\rho})y_2=(4\kappa^2h^3+C_4\kappa^{1-\rho})\exp(-3f(0)x)-C_3\kappa^{\frac{3}{2}}\exp(-f(0)x)\quad(3.25)$$

satisfying the conditions

$$y_2'(0)=Y'(0),\quad y_2(\kappa^{-\rho})=\left(\frac{\kappa h}{2\sqrt{2}}+\tilde{C}\kappa^{\frac{3}{4}}\right)\exp(-f(0)\kappa^{-\rho}).\quad(3.26)$$

According to (3.12), (3.25) and (3.26), Lemma 2.3 with $d=\kappa^{-\rho}$, $C=f(0)^2+C_5\kappa^{1-\rho}$ and $u=y_2-Y$, we get $y_2\leq Y$ on $[0,\kappa^{-\rho}]$. We have

$$\begin{aligned}y_2(x)&=\hat{C}_1(\kappa)\exp\left(-\sqrt{f(0)^2+C_5\kappa^{1-\rho}}x\right)+\hat{C}_2(\kappa)\exp\left(\sqrt{f(0)^2+C_5\kappa^{1-\rho}}x\right)\\&-\frac{4\kappa^2h^3\exp(-2f(0)x)+8\frac{C_3}{C_5}\kappa^{\frac{1}{2}+\rho}f(0)^2-C_3\kappa^{\frac{3}{2}}+C_4\kappa^{1-\rho}\exp(-2f(0)x)}{8f(0)^2-C_5\kappa^{1-\rho}}\exp(-f(0)x).\end{aligned}\quad(3.27)$$

From (3.5), we have $\kappa^2h^3=\mathcal{O}(\kappa^{\frac{1}{2}})$, hence

$$\frac{4\kappa^2h^3\exp(-2f(0)\kappa^{-\rho})+8\frac{C_3}{C_5}\kappa^{\frac{1}{2}+\rho}f(0)^2-C_3\kappa^{\frac{3}{2}}+C_4\kappa^{1-\rho}\exp(-2f(0)\kappa^{-\rho})}{-8f(0)^2+C_5\kappa^{1-\rho}}=\mathcal{O}(\kappa^{\frac{1}{2}}).$$

From (3.25) and (3.27) at the point $x=\kappa^{-\rho}$, it results that

$$\begin{aligned}&\hat{C}_2(\kappa)=\\&-\hat{C}_1(\kappa)\exp(-2\sqrt{f(0)^2+C_5\kappa^{1-\rho}}\kappa^{-\rho})+\mathcal{O}(\kappa^{\frac{1}{2}})\exp\left(-\left(\sqrt{f(0)^2+C_5\kappa^{1-\rho}}+f(0)\right)\kappa^{-\rho}\right).\end{aligned}\quad(3.28)$$

From (3.27), we get

$$\begin{aligned}y_2'(0)=&-\frac{8\frac{C_3}{C_5}\kappa^{\frac{1}{2}+\rho}f(0)^3+12\kappa^2h^3f(0)+3\kappa^{1-\rho}C_4f(0)-f(0)C_3\kappa^{\frac{3}{2}}}{(-8f(0)^2+C_5\kappa^{1-\rho})}\\&-\hat{C}_1(\kappa)\sqrt{f(0)^2+C_5\kappa^{1-\rho}}+\hat{C}_2(\kappa)\sqrt{f(0)^2+C_5\kappa^{1-\rho}}.\end{aligned}$$

The condition $y_2'(0) = hf''(0)$, (3.21) and (3.28) lead to

$$-\hat{C}_1(\kappa)\sqrt{f(0)^2 + C_5\kappa^{1-\rho}} + \frac{3\kappa^2 h^3}{2f(0)} + \mathcal{O}(\kappa^{\frac{1}{2}+\rho}) = hf''(0),$$

hence

$$\hat{C}_1(\kappa) = -\frac{hf''(0)}{f(0)} + \frac{3\kappa^2 h^3}{f(0)^2} + \mathcal{O}(\kappa^{\frac{1}{2}+\rho}).$$

Following the step 1, on $[0, \frac{\kappa^{-\rho}}{2}]$, we get the inequality

$$Y(x) \geq \frac{3\kappa^2 h^3}{2f(0)^2}\exp(-3f(0)x) + \left(\frac{\kappa^2 h^3}{2f(0)^2} - \frac{hf''(0)}{f(0)^2} + \mathcal{O}(\kappa^{\frac{1}{2}+\rho})\right)\exp(-f(0)x),$$

hence, from (3.21)

$$Y(x) \geq \frac{\kappa^2 h^3}{2f(0)^2}\exp(-f(0)x)\left(1 - \exp(-2f(0)x) + \mathcal{O}(\kappa^\rho)\right). \tag{3.29}$$

From (3.24) and (3.29), we deduce that there exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) satisfying (3.3), Estimate (3.9) is true on $[0, \kappa^{-\rho}]$. The proof of Lemma 3.2 follows.

From Lemma 3.2, we deduce an estimate for f' in the following proposition:

Proposition 3.3 *Let $\rho \in [0, \frac{1}{8}]$. There exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the estimate*

$$f'(x) = \kappa^2 h^2\left(1 - \exp(-\sqrt{2}x) + \mathcal{O}(\kappa^\rho)\right), \quad \forall\, x \in [0, \kappa^{-\rho}]. \tag{3.30}$$

Proof

According to Proposition 2.5 and Lemma 3.2, we get

$$f'(x) = \frac{\kappa^2 h^3}{2f(0)^2}\left(1 - \exp(-2f(0)x) + \mathcal{O}(\kappa^\rho)\right).\left(\frac{1}{h} + \mathcal{O}(\kappa^{\frac{3}{2}-2\rho})\right).$$

For $f(0) = \frac{1}{\sqrt{2}} + \mathcal{O}(\kappa^{\frac{1}{4}})$, and for $x \in [0, \kappa^{-\rho}]$, we have

$$\exp(-2f(0)x) = \exp(-\sqrt{2}x) + \mathcal{O}(\kappa^{\frac{1}{4}-\rho}).$$

As $\rho \in [0, \frac{1}{8}]$, Estimate (3.30) follows and this achieved the proof of Proposition 3.3.

Remark 3.4 *Let us recall that in [12] (see also [15]), we have obtained an asymptotic matched solution of $(GL)_\infty$ denoted by $(f^{vd,(n)}, A^{vd,(n)})$. For $n = 1$, at the superheating field, we have shown that*

$$f^{vd,(1)}(x) = \tanh(\frac{\kappa x}{\sqrt{2}} + x_0) + \frac{\kappa}{4}\exp(-\sqrt{2}x), \tag{3.31}$$

where $\tanh(x_0) = \frac{1}{\sqrt{2}}$. It results that, for $x \in [0, \kappa^{-\rho}]$, we have

$$(f^{vd,(1)})'(x) = \frac{\sqrt{2}}{4}\kappa\left(1 - \exp(-\sqrt{2}x)\right) + \mathcal{O}(\kappa^{2-\rho}). \tag{3.32}$$

According to (1.9) and (1.10) with $f(0) = \dfrac{1}{\sqrt{2}}$, the main term of Estimate (3.30) is the one of Estimate (3.32). Therefore, we suspect that Estimate (3.30) is optimal.

3.3. Estimate for $A' + Af$

To get an estimate for $A' + Af$, we use Lemma 2.3 and estimates for A', A, f' and f obtained in Propositions 2.4 and 2.5.

Lemma 3.5 *Let $\rho \in [0, \frac{1}{8}]$. There exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the following estimate*

$$A' + Af = \frac{1}{2}\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2 h^3 \exp(-\sqrt{2}x) - \frac{1}{8h} + 3\kappa^2 h^3 + \mathcal{O}(\kappa^{\frac{1}{2}+\rho})\right), \quad \forall\, x \in [0, \kappa^{-\rho}]. \quad (3.33)$$

Proof .

Step 1: Upper bound for $A' + Af$

We introduce the function $Z := A' + Af$. From (2.5), this function is positive. At the point $x = \kappa^{-\rho}$, from (2.9), (2.10) and (2.27), we deduce that there exists $\hat{C}$ and $\tilde{C}$ such that

$$\hat{C}\kappa^{\frac{1}{2}-2\rho}\exp(-f(0)\kappa^{-\rho}) \leq Z(\kappa^{-\rho}) = (A' + Af)(\kappa^{-\rho}) \leq \tilde{C}\kappa^{\frac{1}{2}-2\rho}\exp(-f(0)\kappa^{-\rho}). \quad (3.34)$$

From $(1.2)_b$, we have

$$Z' = Af^2 + A'f + Af'.$$

Differentiating more, and using the $(G.L.)$ equations, we get

$$Z'' = A'f^2 + 2Af'f + Af^3 + 2A'f' + \kappa^2 A(-f + f^3 + A^2 f).$$

It results that the function Z satisfies the equation

$$-Z'' + (f^2 + 2f')Z = -\kappa^2 A(-f + f^3 + A^2 f). \quad (3.35)$$

From $(1.2)_b$, we have $Z'(0) = f(0)(h + A(0)f(0)) \geq 0$. From (2.4) at the point $x = 0$, we get

$$h + A(0)f(0) = \frac{(1 - f(0)^2)^2}{2(h - A(0)f(0))}.$$

From (2.5) and (2.10) at the point $x = 0$, we get

$$h(2 + \mathcal{O}(\kappa^{\frac{3}{2}-2\rho})) \leq h - A(0)f(0) \leq 2h.$$

It results that

$$\frac{f(0)(1 - f(0)^2)^2}{4h} \leq Z'(0) = f(0)\left(h + A(0)f(0)\right) \leq \frac{f(0)(1 - f(0)^2)^2}{4h(1 + \mathcal{O}(\kappa^{\frac{3}{2}-2\rho}))}.$$

Hence, as $f(0) = \dfrac{1}{\sqrt{2}} + \mathcal{O}(\kappa^{\frac{1}{4}})$, we get

$$Z'(0) = f(0)\left(h + A(0)f(0)\right) = \frac{1}{16\sqrt{2}h}\left(1 + \mathcal{O}(\kappa^{\frac{1}{4}})\right). \quad (3.36)$$

Now, we can estimate the term $-\kappa^2 A(-f + f^3 + A^2 f)$. First, for $x \in \mathbb{R}^+$, as $A < 0$ and $0 \leq f \leq 1$, we have

$$-\kappa^2 A(-f + f^3) < 0.$$

According to (2.6), (2.9) and using $f(0)=\frac{1}{\sqrt{2}}+\mathcal{O}(\kappa^{\frac14})$, we deduce that there exists C_6 such that, for $x\in[0,\kappa^{-\rho}]$, we get the estimate

$$-\kappa^2A^3f\le\left(2\kappa^2h^3+C_6\kappa^{\frac34}\right)\exp(-3f(0)x).$$

To get an upper bound for Z, as $f^2+2f'\ge f(0)^2$, we compare this function with the solution in $H^2([0,\kappa^{-\rho}])$ of the equation

$$-V''+f(0)^2V=\left(2\kappa^2h^3+C_6\kappa^{\frac34}\right)\exp(-3f(0)x),\tag{3.37}$$

and satisfying

$$V'(0)=Z'(0),\quad V(\kappa^{-\rho})=\tilde{C}\kappa^{\frac12-2\rho}\exp(-f(0)\kappa^{-\rho}).\tag{3.38}$$

From (3.34), (3.35), (3.37), (3.38) and as $Z\ge 0$, we get on $[0,\kappa^{-\rho}]$

$$-(Z-V)''+f(0)^2(Z-V)\le 0,$$

and

$$(Z-V)'(0)=0,\quad (Z-V)(\kappa^{-\rho})\le 0.$$

We apply Lemma 2.3 with $d=\kappa^{-\rho}$, $C=f(0)^2$ and $u=Z-V$ and get $Z\le V$.
For $x\in[0,\kappa^{-\rho}]$, the solution of (3.37) is given by

$$\begin{aligned}&V(x)\\&=\tilde{C}_3(\kappa)\exp(-f(0)x)+\tilde{C}_4(\kappa)\exp(f(0)x)-\left(\frac{\kappa^2h^3}{4f(0)^2}+\frac{C_6}{8f(0)^2}\kappa^{\frac34}\right)\exp(-3f(0)x).\end{aligned}\tag{3.39}$$

From (3.38) and (3.39), we get

$$\tilde{C}_4(\kappa)=\left(-\tilde{C}_3(\kappa)+\mathcal{O}(\kappa^{\frac12-2\rho})\right)\exp(-2f(0)\kappa^{-\rho}).\tag{3.40}$$

From (3.39), we deduce that

$$V'(0)=-f(0)\tilde{C}_3(\kappa)+f(0)\tilde{C}_4(\kappa)+\frac{3}{4f(0)}\kappa^2h^3+\frac{3C_6}{8f(0)}\kappa^{\frac34}.$$

From (3.40) and taking account the condition $V'(0)=Z'(0)$ (see (3.36)) and replacing $f(0)$ with $\frac{1}{\sqrt{2}}+\mathcal{O}(\kappa^{\frac14})$, we get

$$\tilde{C}_3(\kappa)=-\frac{1}{16h}\left(1+\mathcal{O}(\kappa^{\frac14})\right)+\frac{3\kappa^2h^3}{2}.\tag{3.41}$$

We have $\exp(-f(0)x)=\exp(-\frac{1}{\sqrt{2}}x)\left(1+\mathcal{O}(\kappa^{\frac14-\rho})\right)$ on $[0,\kappa^{-\rho}]$. From (3.39), (3.40) and (3.41), we get the estimate, for $x\in[0,\frac{\kappa^{-\rho}}{2}]$

$$V(x)=\frac12\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2h^3\exp(-\sqrt{2}x)-\frac{1}{8h}+3\kappa^2h^3+\mathcal{O}(\kappa^{\frac34-\rho})\right).\tag{3.42}$$

As $Z\le V$ on $[0,\kappa^{-\rho}]$, we deduce that there exists κ_0 such that, for all $\kappa\le\kappa_0$, we get

$$Z(x)\le\frac12\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2h^3\exp(-\sqrt{2}x)-\frac{1}{8h}+3\kappa^2h^3+\mathcal{O}(\kappa^{\frac34-\rho})\right),\quad\forall\, x\in[0,\kappa^{-\rho}].\tag{3.43}$$

Step 2: Lower bound for $A' + Af$
First, from (2.10), we remark that there exists C_7 such that, for $x \in [0, \kappa^{-\rho}]$,

$$-\kappa^2 A(-f + f^3) \geq C_7 \kappa^{\frac{3}{2}} \exp(-f(0)x).$$

According to (2.3) and (2.6), we deduce that there exists C_8 such that $f^2 + 2f' \leq (f(0)^2 + C_8\kappa^{1-\rho})$. To get a lower bound, we compare Z with the solution of

$$\begin{cases} -W'' + (f(0)^2 + C_8\kappa^{1-\rho})W = \left(2\kappa^2 h^3 + C_6\kappa^{\frac{3}{4}}\right)\exp(-3f(0)x) + C_7\kappa^{\frac{3}{2}}\exp(-f(0)x), \\ W'(0) = Z'(0), \quad W(\kappa^{-\rho}) = \hat{C}\kappa^{\frac{1}{2}-2\rho}\exp(-f(0)\kappa^{-\rho}). \end{cases} \tag{3.44}$$

Following the step 1, we apply Lemma (2.3) with $d = \kappa^{-\rho}$, $C = f(0)^2 + C_8\kappa^{1-\rho}$ and $u = W - Z$, and we get $W \leq Z$ on $[0, \kappa^{-\rho}]$. The function W is equal to

$$\begin{aligned} W(x) &= \hat{C}_3(\kappa)\exp(-\sqrt{f(0)^2 + C_8\kappa^{1-\rho}}x) + \hat{C}_4(\kappa)\exp(\sqrt{f(0)^2 + C_8\kappa^{1-\rho}}x) \\ &- \frac{2\kappa^2 h^3 \exp(-2f(0)x) - 8\frac{C_7}{C_8}\kappa^{\frac{1}{2}+\rho}f(0)^2 + C_7\kappa^{\frac{3}{2}} + C_6\kappa^{1-\rho}\exp(-2f(0)x)}{8f(0)^2 - C_8\kappa^{1-\rho}}\exp(-f(0)x). \end{aligned} \tag{3.45}$$

From (3.45) at the point $x = \kappa^{-\rho}$ and from (3.44), it results that

$$\begin{aligned} &\hat{C}_4(\kappa) \\ &= -\hat{C}_3(\kappa)\exp(-2\sqrt{f(0) + C_8\kappa^{1-\rho}}\kappa^{-\rho}) + \mathcal{O}(\kappa^{\frac{1}{2}})\exp\left(-(\sqrt{f(0) + C_8\kappa^{1-\rho}} + f(0))\kappa^{-\rho}\right). \end{aligned} \tag{3.46}$$

As $W'(0) = Z'(0)$ and $f(0) = \dfrac{1}{\sqrt{2}} + \mathcal{O}(\kappa^{\frac{1}{4}})$, we get from (3.46) and following the step 1

$$\hat{C}_3(\kappa) = -\frac{1}{16h}\left(1 + \mathcal{O}(\kappa^{\frac{1}{4}})\right) + \frac{3\kappa^2 h^3}{2} + \mathcal{O}(\kappa^{\frac{1}{2}+\rho}). \tag{3.47}$$

On $[0, \frac{\kappa^{-\rho}}{2}]$, according to (3.45), (3.46) and (3.47), we get that W is given by

$$\frac{1}{2}\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2 h^3 \exp(-\sqrt{2}x) - \frac{1}{8h} + 3\kappa^2 h^3 + \mathcal{O}(\kappa^{\frac{1}{2}+\rho})\right).$$

Hence, as $W \leq Z$ on $[0, \kappa^{-\rho}]$, we get the estimate for $x \in [0, \frac{\kappa^{-\rho}}{2}]$

$$\frac{1}{2}\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2 h^3 \exp(-\sqrt{2}x) - \frac{1}{8h} + 3\kappa^2 h^3 + \mathcal{O}(\kappa^{\frac{1}{2}+\rho})\right) \leq Z(x). \tag{3.48}$$

Taking account (3.43) and (3.48), we deduce that there exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) satisfying (3.3), we get Estimate (3.33) on $[0, \kappa^{-\rho}]$. The proof of Lemma 3.5 follows.

Then, from Lemma 3.5, we can improve Inequality (2.5) and get the following proposition:

Proposition 3.6 *There exists κ_0 such that, for all $\kappa \leq \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the following estimate*

$$-Af = A' - B, \quad \forall\, x \in [0, \kappa^{-\rho}], \tag{3.49}$$

where B is defined by

$$B(x) := \frac{1}{2}\exp(-\frac{1}{\sqrt{2}}x)\left(-\kappa^2 h^3 \exp(-\sqrt{2}x) - \frac{1}{8h} + 3\kappa^2 h^3 + \mathcal{O}(\kappa^{\frac{1}{2}+\rho})\right). \tag{3.50}$$

3.4. Control of the function $S := \sqrt{(A')^2 - A^2 f^2}$

In all the following sections, we set

$$S^2 := (A')^2 - A^2 f^2. \tag{3.51}$$

From Proposition 3.5, we can state the following proposition:

Proposition 3.7 *Let $\rho \in [0, \frac{1}{8}]$ and S^2 be the function defined in (3.51). There exists κ_0 such that, for all $\kappa \le \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the following estimate*

$$S^2 = \frac{1}{8}\left(-8\kappa^2 h^4 \exp(-\sqrt{2}x) - 1 + 24\kappa^2 h^4 + \mathcal{O}(\kappa^\rho)\right)\exp(-\sqrt{2}x), \quad \forall\, x \in [0, \kappa^{-\rho}]. \tag{3.52}$$

Proposition 3.8 *Let $\rho \in [0, \frac{1}{8}]$ and S^2 be the function defined in (3.51). There exists κ_0 such that, for all $\kappa \le \kappa_0$, for all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have the following estimate*

$$S^2 = \frac{1}{8}\left(-8\kappa^2 h^4 \exp(-\sqrt{2}x) - 1 + 24\kappa^2 h^4 + \mathcal{O}(\kappa^\rho)\right)\exp(-\sqrt{2}x), \quad \forall\, x \in [0, \kappa^{-\rho}]. \tag{3.53}$$

Proof
From Inequalities (2.9) and (2.26), for $x \in [0, \kappa^{-\rho}]$, we get

$$A' - Af \le \left(h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho})\right)\exp(-f(0)x) + \frac{h}{f(0)}\left(f(0) + \mathcal{O}(\kappa^{1-\rho})\right)\exp(-f(0)x).$$

We deduce the upper bound

$$A' - Af \le 2\left(h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho})\right)\exp(-f(0)x). \tag{3.54}$$

Moreover, from (2.10) and (2.27), we get the lower bound

$$(h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho}))\exp(-f(0)x) + \frac{h}{f(0)} f(0)(1 - C\kappa^{1-2\rho})\exp(-f(0)x) \le A' - Af. \tag{3.55}$$

From (3.54) and (3.55), for $x \in [0, \kappa^{-\rho}]$, we deduce the estimate

$$A' - Af = 2\left(h + \mathcal{O}(\kappa^{\frac{1}{2}-2\rho})\right)\exp(-f(0)x). \tag{3.56}$$

From (3.56) and Lemma 3.5 (see (3.33)), we can write

$$\begin{aligned} S^2 &= (A' - Af).(A' + Af) \\ &= \frac{1}{8}\left(-8\kappa^2 h^4 \exp(-\sqrt{2}x) - 1 + 24\kappa^2 h^4 + \mathcal{O}(\kappa^\rho)\right)\exp(-\sqrt{2}x). \end{aligned}$$

The proof of Proposition 3.8 follows.

4. Proof of Theorem 1.7

In this section, first, using Propositions 3.3, 3.6 and 3.8, we get Inequality 1.7 for all pairs (f,A) solution of $(GL)_\infty$ such that $f(0)\in[\frac{1}{\sqrt{2}}-\alpha\kappa^{\frac{1}{4}},\frac{1}{\sqrt{2}}+\alpha\kappa^{\frac{1}{4}}]$. In the following, we use the elementary equality, for $a>0$ and $b>0$

$$\sqrt{a^2+b^2}=a+b-\frac{2ab}{a+b+\sqrt{a^2+b^2}}. \tag{4.1}$$

Step 1: New estimate for $A'(0)$
According to (3.6) and Proposition 3.6 (see (3.49)), we get the upper bound for $A'(0)^2$

$$h^2\le 2\int_0^\infty f(t)A'(t)^2dt-2\int_0^{\kappa^{-\rho}}A'(t)f(t)B(t)dt,$$

where B is defined in (3.50). From (3.8), we get

$$h^2\le -2\int_0^{\kappa^{-\rho}}A'(t)f(t)B(t)dt+\int_0^{+\infty}f(x)(1-f(x)^4)dx-6\kappa^{-2}\int_0^{+\infty}f(x)f'(x)^2dx. \tag{4.2}$$

According to (2.4) and (4.1) with the choice $a=\kappa^{-1}f'$ and $b=S$, it results the equality

$$\frac{1}{\sqrt{2}}(1-f^2)=\kappa^{-1}f'+S-T, \tag{4.3}$$

where T is defined on $I\!R^+$ by

$$T(x):=\frac{2\kappa^{-1}f'S}{S+\kappa^{-1}f'+\sqrt{S^2+\kappa^{-2}(f')^2}}. \tag{4.4}$$

From (4.3) and following [5] (see Proposition 3.1), we get

$$\int_0^\infty(1-f^4(x))f(x)dx=\frac{1}{2\sqrt{2}}\kappa^{-1}(1-f(0)^2)(3+f(0)^2)$$
$$+\sqrt{2}\int_0^\infty f(x)(1+f(x)^2).S(x)dx-\sqrt{2}\int_0^\infty f(x)(1+f(x)^2)T(x)dx,$$

and

$$\int_0^\infty f(x)f'(x)^2dx=\frac{\kappa}{4\sqrt{2}}(1-f(0)^2)^2-\kappa\int_0^\infty f(x)f'(x)S(x)dx$$
$$+\kappa\int_0^\infty f(x)f'(x)T(x)dx.$$

Hence, from (4.2) we get

$$h^2\le\kappa^{-1}\phi_0(f(0))+I_1+I_2+I_3,$$

where ϕ_0 is defined in (3.2),

$$I_1:=-2\int_0^{\kappa^{-\rho}}A'(t)f(t)B(t)dt, \tag{4.5}$$

$$I_2:=\sqrt{2}\int_0^\infty f(x)(1+f(x)^2)S(x)dx+6\kappa^{-1}\int_0^\infty f(x)f'(x)S(x)dx, \tag{4.6}$$

$$I_3 := -\sqrt{2}\int_0^\infty f(x)(1+f(x)^2)T(x)dx - 6\kappa^{-1}\int_0^\infty f(x)f'(x)T(x)dx. \tag{4.7}$$

For all $f(0)\in]0,1]$, we have $\phi_0(f(0)) \le 2^{-\frac{3}{2}}$. Then, we get

$$h^2 \le \kappa^{-1}2^{-\frac{3}{2}} + I_1 + I_2 + I_3. \tag{4.8}$$

For $\rho\in]0,\frac{1}{8}[$, we cut the integration interval in $[0,\kappa^{-\rho}]$ and $[\kappa^{-\rho},+\infty[$. As the functions f, f' are bounded and the function S admits for upper bound and lower bound an exponential polynomial in the form of $P(x)\exp(-\sqrt{2}x)$, the integral in (4.6) on the interval $[\kappa^{-\rho},+\infty[$ are equal to $\mathcal{O}(\kappa^n)$ for all $n\in I\!N$. According to (2.6), (2.27) and (3.50), we get

$$\begin{aligned}&-2\int_0^{\kappa^{-\rho}} A'(t)f(t)B(t)dt\\ &\le -(f(0)+\mathcal{O}(\kappa^{1-\rho}))\int_0^{\kappa^{-\rho}} \exp(-\sqrt{2}x)\left(\kappa^2h^4\exp(-\sqrt{2}x) - \frac{1}{8} + 3\kappa^2h^4 + \mathcal{O}(\kappa^\rho)\right)dx.\end{aligned}$$

From (3.5) and as $e^{-\sqrt{2}\kappa^{-\rho}} = \mathcal{O}(\kappa^n)$ for κ small, it results that

$$I_1 \le -\frac{3}{32} + \mathcal{O}(\kappa^\rho). \tag{4.9}$$

Step 2: Estimate for I_2
On the other hand, using Proposition 3.8, and making the scaling $u=\exp(-\sqrt{2}x)$, we get

$$\begin{aligned}&\int_0^{\kappa^{-\rho}} S(x)dx\\ &= \tfrac{1}{4}\int_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 \frac{\sqrt{2u-u^2}}{u}du + \mathcal{O}(\kappa^\rho) = \left[\frac{1}{4}\sqrt{2u-u^2} + \frac{1}{4}\arcsin(u-1)\right]_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 + \mathcal{O}(\kappa^\rho)\end{aligned}$$

hence, for κ small

$$\int_0^{\kappa^{-\rho}} S(x)dx = \frac{\pi}{8} + \frac{1}{4} + \mathcal{O}(\kappa^\rho). \tag{4.10}$$

From (2.6), (4.10) and Proposition 3.3 (see (3.30)), it results that

$$6\kappa^{-1}\int_0^{\kappa^{-\rho}} f(x)f'(x)S(x)dx \le 6\kappa^{-1}(f(0)+\mathcal{O}(\kappa^{1-\rho}))\int_0^{\kappa^{-\rho}} \kappa^2h^2(1-\exp(-\sqrt{2}x)).S(x)dx.$$

From (3.5) and making the scaling $u=\exp(-\sqrt{2}x)$, we get

$$\begin{aligned}&\int_0^{\kappa^{-\rho}} (1-\exp(-\sqrt{2}x)).S(x)dx = \frac{1}{4}\int_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 \frac{(1-u)}{u}\sqrt{2u-u^2}du + \mathcal{O}(\kappa^\rho)\\ &= \left[(\frac{3}{8}-\frac{u}{8})\sqrt{2u-u^2} + \frac{1}{8}\arcsin(u-1)\right]_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 + \mathcal{O}(\kappa^\rho),\end{aligned}$$

hence

$$\int_0^{\kappa^{-\rho}} (1-\exp(-\sqrt{2}x)).S(x)dx = \frac{\pi}{16} + \frac{1}{4} + \mathcal{O}(\kappa^\rho).$$

From (3.3) and (3.5), we get

$$6\kappa^{-1}\int_0^{\kappa^{-\rho}} f(x)f'(x)S(x)dx \leq \frac{3}{2}(\frac{\pi}{16}+\frac{1}{4}) + \mathcal{O}(\kappa^\rho). \tag{4.11}$$

According to (2.6) and (4.10), we deduce that

$$\sqrt{2}\int_0^{\kappa^{-\rho}} f(x)(1+f(x)^2)S(x)dx \leq \sqrt{2}\left(f(0)+\mathcal{O}(\kappa^{1-\rho})\right)\left(1+(f(0)+\mathcal{O}(\kappa^{1-\rho}))^2\right)\int_0^{\kappa^{-\rho}} S(x)dx.$$

From (4.10), we get

$$\sqrt{2}\int_0^{\kappa^{-\rho}} f(x)(1+f(x)^2)S(x)dx \leq \frac{3}{2}(\frac{\pi}{8}+\frac{1}{4}) + \mathcal{O}(\kappa^\rho). \tag{4.12}$$

According to (4.6), (4.11) and (4.12), we get

$$I_2 \leq \frac{9\pi}{32}+\frac{3}{4}+\mathcal{O}(\kappa^\rho). \tag{4.13}$$

Step 3: Estimate for I_3
From Proposition 3.3 (see (3.30)) and (3.5), on the interval $[0,\kappa^{-\rho}]$, we have

$$T \geq U := \frac{B}{C}, \tag{4.14}$$

where

$$B(x) := \frac{\sqrt{2}}{2}\left(1-\exp(-\sqrt{2}x)+\mathcal{O}(\kappa^\rho)\right).S, \tag{4.15}$$

and

$$C(x) := S+\frac{\sqrt{2}}{4}\left(1-\exp(-\sqrt{2}x)+\mathcal{O}(\kappa^\rho)\right).S+\sqrt{S^2+\frac{1}{8}\left(1-\exp(-\sqrt{2}x)+\mathcal{O}(\kappa^\rho)\right)^2}. \tag{4.16}$$

As the functions f' and S are positive on $I\!R^+$, from (4.4), we deduce that $T \geq 0$. Thus, from (2.6) and (4.14), we get

$$-\sqrt{2}\int_0^{\infty} f(x)(1+f(x)^2)T(x)dx \leq -\frac{3}{2}\int_0^{\kappa^{-\rho}} U(x)dx + \mathcal{O}(\kappa^\rho). \tag{4.17}$$

Moreover, we have

$$-6\kappa^{-1}\int_0^{\infty} f(x)f'(x)T(x)dx \leq -\frac{3}{2}\int_0^{\kappa^{-\rho}} (1-\exp(-\sqrt{2}x)).U(x)dx + \mathcal{O}(\kappa^\rho). \tag{4.18}$$

Making the scaling $u=\exp(-\sqrt{2}x)$, from (3.5), (4.14), (4.15) and (4.16), we get

$$\int_0^{\kappa^{-\rho}} U(x)dx = \frac{1}{\sqrt{2}}\int_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 \frac{1}{\sqrt{2}u}\frac{\sqrt{u(2-u)}(u-1)}{u-2-\sqrt{u(2-u)}}du + \mathcal{O}(\kappa^\rho), \tag{4.19}$$

and

$$\int_0^{\kappa^{-\rho}} (1-\exp(-\sqrt{2}x)).U(x)dx = \frac{1}{\sqrt{2}}\int_{e^{-\sqrt{2}\kappa^{-\rho}}}^1 -\frac{1}{\sqrt{2}u}\frac{\sqrt{u(2-u)}(u-1)^2}{u-2-\sqrt{u(2-u)}}du + \mathcal{O}(\kappa^\rho). \tag{4.20}$$

For $u \in]0,2[$, we have

$$\int \frac{1}{\sqrt{2}u} \frac{\sqrt{u(2-u)}(u-1)}{u-2-\sqrt{u(2-u)}} du = \frac{\sqrt{2}}{4} \left(\arcsin(u-1) - u + \sqrt{2u-u^2} \right),$$

and

$$\int -\frac{1}{\sqrt{2}u} \frac{\sqrt{u(2-u)}(u-1)^2}{u-2-\sqrt{u(2-u)}} du$$
$$= \frac{1}{8} \arcsin(u-1) + \frac{\sqrt{2}}{8} \sqrt{2u-u^2}(3-u) - \frac{\sqrt{2}}{4} u(1-\frac{u}{2}).$$

From (4.19), it results that

$$\int_0^{\kappa^{-\rho}} U(x)dx = \frac{\pi}{8} + \mathcal{O}(\kappa^\rho). \tag{4.21}$$

Moreover, from (4.20), we get

$$\int_0^{\kappa^{-\rho}} (1-\exp(-\sqrt{2}x)).U(x)dx = \frac{1}{8}(1+\frac{\pi}{2}) + \mathcal{O}(\kappa^\rho). \tag{4.22}$$

From (4.17), (4.18), (4.21) and (4.22), we get

$$I_3 \leq -\frac{9\pi}{32} - \frac{3}{16} + \mathcal{O}(\kappa^\rho). \tag{4.23}$$

Step 4: Upper bound for $A'(0)$
Thus, according to (4.8), (4.9), (4.13) and (4.23), we deduce that there exists κ_0, such that, for all $\kappa \leq \kappa_0$ and all pairs (f, A) solutions of $(GL)_\infty$ satisfying (3.3), we have

$$h^2 \leq \kappa^{-1} 2^{-\frac{3}{2}} + \frac{15}{32} + \mathcal{O}(\kappa^\rho). \tag{4.24}$$

From Lemma 3.1 and (4.24), the proof of Theorem 1.3 follows.

According to (1.6) and Theorem 1.3, we deduce the Parr Formula.

Theorem 4.1 *(Parr Formula) There exists κ_0 such that, for all $\kappa_0 \leq \kappa$, we have*

$$\kappa \left(h^{sh}(\kappa) \right)^2 = 2^{-\frac{3}{2}} + \frac{15}{32}\kappa + o(\kappa).$$

5. Open problem

The challenge is now to obtain a complete expansion in powers of $\kappa^{\frac{1}{2}}$ for the superheating field. In [12], we have obtained a formal expansion for the superheating field, denoted by $h^{sh,f}(\kappa) = \kappa^{-\frac{1}{2}} \sum_{i=0}^{\infty} h_i \kappa^i$. In [14], we have proved that for all $n \in I\!N$, there exist κ_0 and C such that, for all $\kappa \leq \kappa_0$, we have the inequality

$$\kappa^{\frac{1}{2}} h^{sh}(\kappa) \geq \sum_{i=0}^{n} h_i \kappa^i + C\kappa^{n+1}.$$

What remains is the following conjecture:

Conjecture 5.1 *Let $n \geq 2$. There exist κ_0 and C such that, for all $\kappa \leq \kappa_0$, we have the inequality*

$$\kappa^{\frac{1}{2}} h^{sh}(\kappa) \leq \sum_{i=0}^{n} h_i \kappa^i + C\kappa^{n+1}.$$

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