STURM OSCILLATION AND COMPARISON THEOREMS

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ABSTRACT. This is a celebratory and pedagogical discussion of Sturm oscillation theory. Included is the discussion of the difference equation case via determinants and a renormalized oscillation theorem of Gesztesy, Teschl, and the author.

1. Introduction

Sturm's greatest contribution is undoubtedly the introduction and focus on Sturm-Liouville operators. But his mathematically deepest results are clearly the oscillation and comparison theorems. In [22, 23], he discussed these results for Sturm-Liouville operators. There has been speculation that in his unpublished papers he had the result also for difference equations, since shortly before his work on Sturm-Liouville operators, he was writing about zeros of polynomials, and there is a brief note referring to a never published manuscript that suggests he had a result for difference equations. Indeed, the Sturm oscillation theorems for difference equations written in terms of orthogonal polynomials are clearly related to Descartes' theorem on zeros and sign changes of coefficients.

In any event, the oscillation theorems for difference equations seem to have appeared in print only in 1898 [2], and the usual proof given these days is by linear interpolation and reduction to the ODE result. One of our purposes here is to make propaganda for the approach via determinants and orthogonal polynomials (see Section 2). Our discussion in Section 3 and 4 is more standard ODE theory [3] — put here to have a brief pedagogical discussion in one place. Section 5 makes propaganda for what I regard as some interesting ideas of Gesztesy, Teschl, and me [8]. Section 6 has three applications to illustrate the scope of applicability.

Date: October 15, 2003.

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Our purpose here is celebratory and pedagogical, so we make simplifying assumptions, such as only discussing bounded and continuous perturbations. Standard modern techniques allow one to discuss much more general perturbations, but this is not the place to make that precise. And we look at Schrödinger operators, rather than the more general Sturm-Liouville operators.

We study the ODE

$$Hu = -\frac{d^2u}{dx^2} + Vu = Eu \tag{1.1}$$

typically on $[0, a]$ with $u(0) = u(a) = 0$ boundary conditions or on $[0, \infty)$ with $u(0) = 0$ boundary conditions. The discrete analog is

$$(hu)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = Eu \tag{1.2}$$

for $n = 1, 2, \dots$ with $u_0 \equiv 0$.

It is a pleasure to thank W. Amrein for the invitation to give this talk and for organizing an interesting conference, Y. Last and G. Kilai for the hospitality of Hebrew University where this paper was written, and F. Gesztesy for useful comments.

2. Determinants, Orthogonal Polynomials, and Sturm Theory for Difference Equations

Given a sequence of parameters $a_1, a_2, \dots$ and b_1, b_2 for the difference equation (1.2), we look at the fundamental solution, $u_n(E)$, defined recursively by $u_1(E) = 1$ and

$$a_n u_{n+1}(E) + (b_n - E)u_n(E) + a_{n-1} u_{n-1}(E) = 0 \tag{2.1}$$

with $u_0 \equiv 0$, so

$$u_{n+1}(E) = a_n^{-1}(E - b_n)u_n(E) - a_n^{-1} a_{n-1} u_{n-1}(E) \tag{2.2}$$

Clearly, (2.2) implies, by induction, that u_{n+1} is a polynomial of degree n with leading term $(a_n \dots a_1)^{-1} E^n$. Thus, we define for $n = 0, 1, 2, \dots$

$$p_n(E) = u_{n+1}(E) \qquad P_n(E) = (a_1 \dots a_n) p_n(E) \tag{2.3}$$

Then (2.1) becomes

$$a_{n+1} p_{n+1}(E) + (b_{n+1} - E)p_n(E) + a_n p_{n-1}(E) = 0 \tag{2.4}$$

for $n = 0, 1, 2, \dots$. One also sees that

$$EP_n(E) = P_{n+1}(E) + b_{n+1}(E)P_n(E) + a_n^2 P_{n-1}(E) \tag{2.5}$$

We will eventually see p_n are orthonormal polynomials for a suitable measure on $\mathbb{R}$ and the P_n are what are known as monic orthogonal polynomials.

Let J_n be the finite $n \times n$ matrix

$$J_n = \begin{pmatrix} b_1 & a_1 & 0 & & & \\ a_1 & b_2 & a_2 & & & \\ 0 & a_2 & b_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & b_{n-1} & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix}$$

Proposition 2.1. *The eigenvalues of J_n are precisely the zeros of $p_n(E)$. We have*

$$P_n(E) = \det(E - J_n) \tag{2.6}$$

Proof. Let $\varphi(E)$ be the vector $\varphi_j(E) = p_{j-1}(E)$, $j = 1, \dots, n$. Then (2.1) implies

$$(J_n - E)\varphi(E) = -a_n p_n(E)\delta_n \tag{2.7}$$

where δ_n is the vector $(0, 0, \dots, 0, 1)$. Thus every zero of p_n is an eigenvalue of J_n. Conversely, if $\tilde{\varphi}$ is an eigenvector of J_n, then both $\tilde{\varphi}_j$ and φ_j solve (2.2), so $\tilde{\varphi}_j = \tilde{\varphi}_1 \varphi_j(E)$. This implies that E is an eigenvalue only if $p_n(E)$ is zero and that eigenvalues are simple.

Since J_n is real symmetric and eigenvalues are simple, $p_n(E)$ has n distinct eigenvalues $E_j^{(n)}$, $j = 1, \dots, n$ with $E_{j-1}^{(n)} < E_j^{(n)}$. Thus, since p_n and P_n have the same zeros,

$$P_n(E) = \prod_{j=1}^{n} (E - E_j^{(n)}) = \det(E - J_n)$$

□

Proposition 2.2. (i) *The eigenvalues of J_n and J_{n+1} strictly interlace, that is,*

$$E_1^{(n+1)} < E_1^{(n)} < E_2^{(n+1)} < \cdots < E_n^{(n)} < E_{n+1}^{(n+1)} \tag{2.8}$$

(ii) *The zeros of $p_n(E)$ are simple, all real, and strictly interlace those of $p_{n+1}(E)$.*

Proof. (i) J_n is obtained from J_{n+1} by restricting the quadratic form $u \to \langle u, J_{n+1}u\rangle$ to $\mathbb{C}^n$, a subspace. It follows that $E_1^{(n+1)} = \min_{u, \|u\|=1} \langle u, J_{n+1}u\rangle \leq \min_{u \in \mathbb{C}^n, \|u\|=1} \langle u, J_{n+1}u\rangle = E_1^{(n)}$. More generally, using that min-max principle

$$E_j^{(n+1)} = \max_{\varphi_1, \dots, \varphi_{j-1}} \min_{\substack{\|u\|=1 \\ u \perp \varphi_1, \dots, \varphi_{j-1}}} \langle u, J_{n+1}u\rangle$$

one sees that

$$E_j^{(n)} \geq E_j^{(n+1)}$$

By replacing min's with max's,

$$E_j^{(n)} \leq E_{j+1}^{(n+1)}$$

All that remains is to show that equality is impossible. If $E_0 \equiv E_j^{(n)} = E_j^{(n+1)}$ or $E_0 \equiv E_j^{(n)} = E_j^{(n+1)}$, then $p_{n+1}(E_0) = p_n(E_0) = 0$. By (2.4), this implies $p_{n-1}(E_0) = 0$ so, by induction, $p_0(E) = 0$. But $p_0 \equiv 1$. Thus equality is impossible.

(ii) Given (2.6), a restatement of what we have proven about the eigenvalues of J_n. □

Here is our first version of Sturm oscillation theorems:

Theorem 2.3. *Suppose E_0 is not an eigenvalue of J_k for $k = 1, 2, \dots, n$. Then*

$$\#(j \mid E_j^{(n)} > E_0) = \#\{\ell = 1, \dots, n \mid \operatorname{sgn}(P_{\ell-1}(E_0)) \neq \operatorname{sgn}(P_\ell(E_0))\} \tag{2.9}$$

$$\#(j \mid E_j^{(n)} < E_0) = \#\{\ell = 1, \dots, n \mid \operatorname{sgn}(P_{\ell-1}(E_0) = \operatorname{sgn}(P_\ell(E_0))\} \tag{2.10}$$

Proof. (2.9) clearly implies (2.10) since the sum of both sides of the equalities is n. Thus we need only prove (2.9).

Suppose that $E_1^{(\ell)} < \dots < E_k^{(\ell)} < E_0 < E_{k+1}^{(\ell)} < E_n^{(\ell)}$. By eigenvalue interlacing, $J_{\ell+1}$ has k eigenvalues in $(-\infty, E_k^{(\ell)})$ and $n-k$ eigenvalues in $(E_{k+1}^{(\ell)}, \infty)$. The question is whether the eigenvalue in $(E_k^{(\ell)}, E_{k+1}^{(\ell)})$ lies above E_0 or below. Since $\operatorname{sgn}\det(E - J^{(\ell+1)}) = (-1)^{\#(j \mid E_j^{(\ell)} > E_0)}$, and similarly for $J_{\ell+1}$, and there is at most one extra eigenvalue above E_0, we see

$$\operatorname{sgn} P_\ell(E_0) = \operatorname{sgn} P_{\ell+1}(E_0) \Leftrightarrow \#(j \mid E_j^{(\ell)} > E_0) = \#(j \mid E_j^{(\ell+1)} > E_0)$$

$$\operatorname{sgn} P_\ell(E_0) = \operatorname{sgn} P_{\ell+1}(E_0) \Leftrightarrow \#j(\mid E_j^{(\ell)} > E_0) + 1 = \#(j \mid E_j^{(\ell+1)} > E_0)$$

(2.9) follows from this by induction. □

We want to extend this in two ways. First, we can allow $P_k(z_0) = 0$ for some $k < n$. In that case, by eigenvalue interlacing, it is easy to see J_{k+1} has one more eigenvalue than J_{k-1} in (E_0, ∞) and also in $(-\infty, E_0)$, so $\operatorname{sgn}(P_{k-1}(z_0)) = -\operatorname{sgn}(P_{k+1}(z_0))$ (also evident from (2.5) and $P_k(z_0) = 0$). Thus we need to be sure to count the change of sign from $< 0, 0$ to $> 0, a$ as only a simple change of sign. We therefore have

Proposition 2.4. (2.9) *and* (2.10) *remain true so long as* $P_n(E_0) \neq 0$ *so long as we define* $\operatorname{sgn}(0) = 1$. *If* $P_n(E_0) = 0$, *they remain true so long as* $\ell = n$ *is dropped from the right side.*

One can summarize this result as follows: For $x \in [0, n]$, define $y(x)$ by linear interpolation, that is,

$$x = [x] + (x) \Rightarrow y(x) = P_{[x]} + (x)(P_{[x]+1} - P_{[x]})$$

Then the number of eigenvalues of J_n above E is the number of zeros of $y(x, E)$ in $[0, n)$. If we do the same for $\tilde{y}$ with $P_{[x]}$ replaced by $(-1)^{[x]} P_{[x]}$, then the number of eigenvalues below E is the number of zeros of $\tilde{y}$ in $[0, n)$. Some proofs (see [5]) of oscillation theory for difference equations use y and mimic the continuum proof of the next section.

The second extension involves infinite Jacobi matrices. In discussing eigenvalues of an infinite J, domain issues arise if J is not bounded (if the moment problem is not determinate, these are complicated issues; see Simon [21]). Thus, let us suppose

$$\sup_n \, (|a_n| + |b_n|) < \infty \tag{2.11}$$

If J is bounded, the quadratic form of J_n is a restriction of J to $\mathbb{C}^n$. As in the argument about eigenvalues interlacing, one shows that if J has only $N_0 < \infty$ eigenvalues in (E_0, ∞), then J_n has at most N_0 eigenvalues there. Put differently, if $E_1^{(\infty)} > E_2^{(\infty)} > \cdots$ are the eigenvalues of J, $E_j^{(\infty)} \geq E_j^{(n)}$. Thus, if $N_n(E) = \#$ of eigenvalues of J_n in (E, ∞) and N_∞ the dimension of $\operatorname{Ran} P_{(E,\infty)}(J)$, the spectral projection

$$N_n(E) \leq N_{n+1}(E) \leq \cdots \leq N_\infty(E) \tag{2.12}$$

On the other hand, suppose we can find an orthonormal set $\{\varphi_j\}_{j=1}^N$ with $M_{jk}^{(\infty)} = \langle \varphi_j, J\varphi_k \rangle = e_j \delta_{jk}$ and $\min(e_j) = e_0 > E_0$. If $M_{jk}^{(n)} = \langle \varphi_j, J_n \varphi_k \rangle$, $M^{(n)} \to M^{(\infty)}$, so for n large, $M^{(n)} \geq \min(e_j) + \frac{1}{2}(e_0 - E_0) > E_0$. Thus $N_n(E_0) \geq N$ for n large. It follows that $\lim N_n \geq N_\infty$, that is, we have shown that $N_\infty(E_0) = \lim_{n\to\infty} N_n(E_0)$. Thus,

Theorem 2.5. *Let* J *be an infinite Jacobi matrix with* (2.11). *Then* (*with* $\operatorname{sgn}(0) = 1$) *we have*

$$N_\infty(E_0) = \#\{\ell = 1, 2, \dots \mid \operatorname{sgn}(P_{\ell-1}(E_0)) \neq \operatorname{sgn}(P_\ell(E_0))\} \tag{2.13}$$

$$\dim P_{(-\infty,E_0)}(J) = \#\{\ell = 1, 2, \dots \mid \operatorname{sgn}(P_{\ell-1}(E_0)) = \operatorname{sgn}(P_\ell(E_0))\} \tag{2.14}$$

Corollary 2.6. *$a_- \leq J \leq a_+$ if and only if for all ℓ,*

$$P_\ell(a_+) > 0 \qquad \text{and} \qquad (-1)^\ell P_\ell(a_-) > 0 \tag{2.15}$$

While on the subject of determinants and Jacobi matrices, I would be remiss if I did not make two further remarks.

Given (2.6), (2.5) is an interesting relation among determinants, and you should not be surprised it has a determinantal proof. The matrix J_{n+1} has b_{n+1} and a_n in its bottom row. The minor of $E - b_{n+1}$ in $E - J_{n+1}$ is clearly $\det(E - J_n)$. A little thought shows the minor of $-a_n$ is $-a_n \det(E - J_{n-1})$. Thus

$$\det(E - J_{n+1}) = (E - b_{n+1}) \det(E - J_n) - a_n^2 \det(E - J_{n-1}) \tag{2.16}$$

which is just (2.5).

Secondly, one can look at determinants where we peel off the top and left rather than the right and bottom. Let $J^{(1)}, J^{(2)}$ be the Jacobi matrices obtained from J by removing the first row and column, the first two, ... Making the J-dependence of $P_n(\,\cdot\,)$ explicit, Cramer's rule implies

$$(z - J_n)^{-1}_{11} = \frac{P_{n-1}(z, J^{(1)})}{P_n(z, J)} \tag{2.17}$$

In the OP literature, $a_1^{-1} p_n(z, J^{(1)})$ are called the second kind polynomials.

The analog of (2.16) is

$$P_n(z, J) = (z - b_1) P_{n-1}(z, J^{(1)}) - a_1^2 P_{n-2}(z, J^{(2)})$$

which, by (2.17), becomes

$$[(z - J)^{-1}_{11}]^{-1} = \frac{1}{(z - b_1) - a_1^2 (z - J^{(1)}_{n-1})^{-1}_{11}} \tag{2.18}$$

In particular, since $d\gamma$ is the spectral measure of δ_1, J, we have

$$(z - J)^{-1}_{11} = \int \frac{d\gamma(x)}{z - x} \equiv -m(z, J) \tag{2.19}$$

and (2.18) becomes in the limit with $(z - J^{(1)})^{-1}_{11} \to -m(z, J^{(1)})$

$$m(z; J) = \frac{1}{b_1 - z - a_1^2 m(z; J^{(1)})} \tag{2.20}$$

(2.18) leads to a finite continued fraction expansion of $(z - J_n)^{-1}_{11}$ due to Jacobi, and (2.20) to the Stieltjes continued fraction. Sturm's celebrated paper on zeros of polynomials is essentially also a continued fraction expansion. It would be interesting to know how much Sturm and Jacobi knew of each other's work. Jacobi visited Paris in 1829 (see James [10]), but I have no idea if he and Sturm met at that time.

3. Sturm Theory of the Real Line

We will suppose V is a bounded function $[0,\infty)$. We are interested in solutions of

$$-u'' + Vu = Eu \tag{3.1}$$

for E real.

Theorem 3.1 (Sturm Comparison Theorem)**.** *For $j = 1, 2$, let u_j be not identically zero and solve $-u_j'' + Vu_j = E_j u_j$. Suppose $a < b$, $u_1(a) = u_1(b) = 0$ and $E_2 > E_1$. Then u_2 has a zero in (a, b). If $E_2 = E_1$ and $u_2(a) \neq 0$, then u_2 has a zero in (a, b).*

Proof. Define the Wronskian

$$W(x) = u_1'(x)u_2(x) - u_1(x)u_2'(x) \tag{3.2}$$

Then

$$W'(x) = (E_2 - E_1)u_1(x)u_2(x) \tag{3.3}$$

Without loss, suppose a and b are successive zeros of u_1. By changing signs of u if need be, we can suppose $u_1 > 0$ on (a, b) and $u_2 > 0$ on $(a, a + \varepsilon)$ for some ε. Thus $W(a) = u_1'(a)u_2(a) \geq 0$ (and, in case $E_1 = E_2$ and $u_2(a) \neq 0$, $W(a) > 0$). If u_2 is nonvanishing in (a, b), then $u_2 \geq 0$ there, so $W(b) > 0$ (if $E_2 > E_1$, $(E_2 - E_1)\int_a^b u_1 u_2\,dx > 0$, and if $E_2 = E_1$ but $u_2(a) \neq 0$, $W(a) > 0$). Since $W(b) = u_1'(b)u_2(b)$ with $u_1'(b) < 0$ and $u_2(b) \geq 0$, this is impossible. Thus we have the result by contradiction. □

Corollary 3.2. *Let $u(x, E)$ be the solution of (3.1) with $u(0, E) = 0$, $u'(0, E) = 1$. Let $N(a, E)$ be the number of zeros of $u(x, E)$ in $(0, a)$. Then, if $E_2 > E_1$, we have $N(a, E_2) \geq N(a, E_1)$ for all a.*

Proof. If $n = N(a, E_1)$ and $0 < x_1 < \cdots < x_n < a$ are the zeros of $u(x, E_1)$, then, by the theorem, $u(x, E_2)$ has zeros in $(0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n)$. □

This gives us the first version of the Sturm oscillation theorem:

Theorem 3.3. *Let $E_0 < E_1 < \cdots$ be the eigenvalues of $H \equiv -\frac{d^2}{dx^2} + V(x)$ on $L^2(0, a)$ with boundary conditions $u(0) = u(a) = 0$. Then $u(x, E_n)$ has exactly n zeros in $(0, a)$.*

Proof. If $u_k \equiv u(\,\cdot\,, E_k)$ has m zeros $x_1 < x_2 < \cdots x_m$ in $(0, a)$, then for any $E > E_k$, $u(\,\cdot\,, E)$ has zeros in $(0, x), \dots, (x_{m-1}, x_m), (x_m, a)$ and so, u_{k+1} has at least $m + 1$ zeros. It follows by induction that u_n has at least n zeros, that is, $m \geq n$.

Suppose u_n has m zeros $x_1 < \cdots < x_m$ in $(0, a)$. Let $v_0, \dots, v_m$ be the function u_n restricted successively to $(0, x_1), (x_1, x_2), \dots, (x_m, a)$.

The v's are continuous and piecewise C^1 with $v_\ell(0) = v_\ell(a) = 0$. Thus they lie in the quadratic form domain of H (see [16, 17] for discussions of quadratic forms) and

$$\begin{aligned} \langle v_j, Hv_k\rangle &= \int_0^a v_j' v_k' + \int_0^a V v_j v_k \\ &= \delta_{jk} E \int_0^a v_j^2\, dx \end{aligned} \tag{3.4}$$

since if $j = k$, we can integrate by parts and use $-u'' + Vu = Eu$.

It follows that for any v in the span of v_j's, $\langle v, Hv\rangle = E\|v\|^2$, so by the variational principle, H has at least $m+1$ eigenvalues in $(-\infty, E_n)$, that is, $n+1 \geq m+1$. $\square$

Remark. The second half of this argument is due to Courant-Hilbert [4].

If we combine this result with Corollary 3.2, we immediately have:

Theorem 3.4 (Sturm Oscillation Theorem)**.** *The number of eigenvalues of H strictly below E is exactly the number of zeros of $u(x,E)$ in $(0,a)$.*

As in the discrete case, if H_a is $-\frac{d^2}{dx^2} + V(x)$ on $[0,a]$ with $u(0) = u(a) = 0$ boundary conditions and H_∞ is the operator on $L^2(0,\infty)$ with $u(0) = 0$ boundary conditions, and if $N_a(E) = \dim P_{(-\infty,E)}(H_a)$, then $N_a(E) \to N_\infty(E)$, so

Theorem 3.5. *The number of eigenvalues of H_∞ strictly below E, more generally $\dim P_{(-\infty,E)}(H)$, is exactly the number of zeros of $u(x,E)$ in $(0,\infty)$.*

There is another distinct approach, essentially Sturm's approach in [22], to Sturm theory on the real line that we should mention. Consider zeros of $u(x,E)$, that is, solutions of

$$u(x(E),E) = 0 \tag{3.5}$$

u is a C^1 function of x and E, and if $u(x_0,E) = 0$, then $u'(x_0,E_0) \neq 0$ (since u obeys a second-order ODE). Thus, by the implicit function theorem, for E near E_0, there is a unique solution, $x(E)$, of (3.4) near x_0, and it obeys

$$\left.\frac{dx}{dE}\right|_{E_0} = -\left.\frac{\partial u/\partial E}{\partial u/\partial x}\right|_{x=x_0, E=E_0} \tag{3.6}$$

Now, $v \equiv \partial u/\partial E$ obeys the equation

$$-v'' + Vv = Ev + u \tag{3.7}$$

by taking the derivative of $-u'' + Vu = Eu$. Multiply (3.7) by u and integrate by parts from 0 to x_0. Since $v(0) = 0$, there is no boundary term at 0, but there is at x_0, and we find

$$v(x_0)u'(x_0) = \int_0^{x_0} |u(x)|^2 \, dx$$

Thus (3.6) becomes

$$\frac{dx_0}{dE} = -|u'(x_0, E)|^{-2} \int_0^{x_0} |u(x, E)|^2 \, dx < 0 \tag{3.8}$$

Thus, as E increases, zeros of u move towards zero. This immediately implies the comparison theorem. Moreover, starting with u_n, the $(n+1)$-st eigenfunction at energy E_n, if it has m zeros in $(0, a)$ as E decreases from E_n to a value, E' below $-\|V\|_\infty$ (where $u(x, E') > 0$ has no zeros in $(0, \infty)$), the m zeros move out continuously, and so $u(a, E) = 0$ exactly m times, that is, $m = n$. This proves the oscillation theorem.

4. Rotation Numbers and Oscillations

Take the solution $u(x, E)$ of the last section and look at the point

$$\pi(x, E) = \begin{pmatrix} u'(x, E) \\ u(x, E) \end{pmatrix}$$

in $\mathbb{R}^2$. π is never zero since u and u' have no common zeros. At most points in $\mathbb{R}^2$, the argument of π, that is, the angle π makes with $\binom{1}{0}$, can increase or decrease. u can wander around and around. But not at points where $u = 0$. If $u' > 0$ at such a point, π moves from the lower right quadrant to the upper right, and similarly, if $u' < 0$, it moves from the upper left to lower left. Thus, since π starts at $\binom{1}{0}$, we see

Theorem 4.1. *If $u(x, E)$ has m zeros in $(0, a)$, then $\operatorname{Arg} \pi(a, E)$ (defined by continuity and $\operatorname{Arg} \pi(0, E) = 0$) lies in $(m\frac{\pi}{2}, (m+1)\frac{\pi}{2}]$.*

If u and v are two solutions of $-u'' + Vu = Eu$ with $u(0) = 0$, $v(0) \neq 0$, we can look at

$$\tilde{\pi}(x, E) = \begin{pmatrix} u \\ v \end{pmatrix}$$

$\tilde{\pi}$ is never zero since u and v are linear independent. $W(x) = u'v - v'u$ is a constant, say c. $c \neq 0$ since u and v are linear independent. Suppose $c > 0$. Then if $u(x_0) = 0$, $u'(x_0) = c/v(x_0)$ has the same sign as $v(x_0)$.

So the above argument applies (if $c < 0$, there is winding in the (u, v)-plane in the opposite direction). Rather than look at $\tilde{\pi}$, we can look at $\varphi = u + iv$. Then $u'v - vu' = \operatorname{Im}(\bar{\varphi}\varphi')$. Thus we have

Theorem 4.2. *Let $\varphi(x, E)$ obey $-\varphi'' + V\varphi = E\varphi$ and be complex-valued with*

$$\operatorname{Im}(\bar{\varphi}(0)\varphi'(0)) > 0 \tag{4.1}$$

Suppose $\operatorname{Re}\varphi(0) = 0$. *Then, if* $\operatorname{Re}\varphi$ *has m zeros in $(0, a)$, then* $\operatorname{Arg}(\varphi(a))$ *is in $(m\frac{\pi}{2}, (m+1)\frac{\pi}{2}]$.*

The ideas of this section are the basis of the relation of rotation numbers and density of states used by Johnson-Moser [12] (see also [11]). We will use them as the starting point of the next section.

5. Renormalized Oscillation Theory

Consider $H = -\frac{d^2}{dx^2} + V$ on $[0, \infty)$ with $u(0) = 0$ boundary conditions where, as usual, for simplicity, we suppose that V is bounded. By Theorem 3.5, $\dim P_{(-\infty,E)}(H)$ is the number of zeros of $u(x, E)$ in $(0, \infty)$. If we want to know $\dim P_{[E_1,E_2)}(H)$, we can just subtract the number of zeros of $u(x, E_1)$ on $(0, \infty)$ from those of $u(x, E_2)$. At least, if $\dim P_{(-\infty,E_2)}(H)$ is finite, one can count just by subtracting. But if $\dim P_{(-\infty,E_1)}(H) = \infty$ while $\dim P_{[E_1,E_2)}$ is finite, both $u(x, E_2)$ and $u(x, E_1)$ have infinitely many zeros, and so subtraction requires regularization.

One might hope that

$$\dim P_{[E_1,E_2)}(H) = \lim_{a\to\infty} (N(E_2, a) - N(E_1, a)) \tag{5.1}$$

where $N(E, a)$ is the number of zeros of $u(x, E)$ in $(0, a)$. This is an approach of Hartmann [9]. (5.1) cannot literally be true since $N(E_2, a) - N(E_1, a)$ is an integer which clearly keeps changing when one passes through a zero of $u(x, E_2)$ that is not also a zero of $u(x, E_1)$. One can show that for a large, the absolute value of difference of the two sides of (5.1) is at most one, but it is not obvious when one has reached the asymptotic region.

Instead, we will describe an approach of Gesztesy, Simon, and Teschl [8]; see Schmidt [19] for further discussion. Here it is for the half-line (the theorem is true in much greater generality than V bounded and there are whole-line results).

Theorem 5.1. *Let V be bounded and let $H = -\frac{d^2}{dx^2} + V(x)$ on $[0, \infty)$ with $u(0) = 0$ boundary conditions. Fix $E_1 < E_2$. Let*

$$W(x) = u(x, E_1)u'(x, E_2) - u'(x, E_1)u(x, E_2) \tag{5.2}$$

and let N be the number of zeros of W in $(0, \infty)$. Then

$$\dim P_{(E_1,E_2)}(H) = N \tag{5.3}$$

The rest of this section will sketch the proof of this theorem under the assumption that $\dim P_{(-\infty,E_2)}(H) = \infty$. This will allow a simplification of the argument and covers cases of greatest interest. Following [8], we will prove this in three steps:

(1) Prove the result in a finite interval $[0, a]$ in case $u(a, E_2) = 0$.
(2) Prove $\dim P_{(E_1,E_2)}(H) \leq N$ by limits from (1) when $\dim P_{(-\infty,E_2)}(H) = \infty$.
(3) Prove $\dim P_{(E_1,E_2)}(H) \geq N$ by a variational argument.

Step 1. We use the rotation number picture of the last section. Define the Prüfer angle $\theta(x, E)$ by

$$\tan(\theta(x, E)) = \frac{u(x, E)}{u'(x, E)} \tag{5.4}$$

with $\theta(0, E) = 0$ and θ continuous at points, x_0, where $u'(x_0, E) = 0$. Using $\frac{d}{dy} \tan y = 1 + \tan^2 y$, we get

$$\frac{d\theta}{dx} = \frac{(u')^2 - uu''}{u^2 + (u')^2} \tag{5.5}$$

Let θ_1, θ_2 be the Prüfer angles for $u_1(x) \equiv u(x, E_1)$ and $u_2(x) \equiv u(x, E_2)$. Suppose $W(x_0) = 0$. This happens if and only if $u(x_0, E)/u'(x_0, E_1) = u(x_0, E_2)/u'(x_0, E_2)$, that is, $\theta_2 = \theta_1 + k\pi$ with $k \in \mathbb{Z}$. If it happens, we can multiply u_2 by a constant so $u_1(x_0) = u_2(x_0)$, $u_1'(x_0) = u_2'(x_0)$. Once we do that, (5.5) says

$$\frac{d}{dx}(\theta_2 - \theta_1) = \frac{(E_2 - E_1)u_1^2(x_0)}{u_1'(x_0)^2 + u_1^2(x_0)} > 0$$

Thus

$$\theta_1 = \theta_2 \mod \pi \Rightarrow \theta_2' > \theta_1' \tag{5.6}$$

Think of θ_2 as a hare and θ_1 as a tortoise running around a track of length π. There are two rules in their race. They can each run in either direction, except they can only pass the starting gate going forward (i.e., $\theta_j = 0 \mod \pi \Rightarrow \theta_j' > 0$), and the hare can pass the tortoise, not vice-versa (i.e., (5.6) holds).

Suppose H_a, the operator on $(0, a)$ with $u(0) = u(a) = 0$ boundary condition, has m eigenvalues below E_2 and n below E_1. Since $u(a, E_2) = 0$, $\theta_2(a) = (m + 1)\pi$, that is, at $x = a$, the hare makes exactly $m + 1$ loops of the track. At $x = a$, the tortoise has made n loops plus part, perhaps all, of an additional one. Since $\theta_2' - \theta_1' > 0$ at $x = 0$, the hare starts out ahead. Thus, the hare must overtake the tortoise exactly $m-n$ times between 0 and a (if $\theta_1(a)=(n+1)\pi$, since then $\theta_2'-\theta_1'>0$ at $x=0$, $\theta_2-(m+1)\pi<\theta_1-(n-1)\pi$, and $x=a$; so it is still true that there are exactly $m-n$ crossings). Thus

$$P_{(E_1,E_2)}(H_a)=\#\{x_0\in(0,a)\mid W(x_0)=0\} \tag{5.7}$$

Step 2. Since $\dim P_{(-\infty,E_2)}(H)=\infty$, there is, by Theorem 3.5, an infinite sequence $a_1<a_2<\cdots\to\infty$ so that $u(a_j,E_2)=0$. $H_{a_j}\to H$ in strong resolvent sense, so by a simple argument,

$$\begin{aligned}\dim P_{(E_1,E_2)}(H)&\leq\liminf\dim P_{(E_1,E_2)}(H_a)\\&=N\end{aligned} \tag{5.8}$$

with N the number of zeros of W in $(0,\infty)$. (5.8) comes from (5.7).

Step 3. Suppose $N<\infty$. Let $0<x_1<\cdots<x_N$ be the zeros of W. Define

$$\eta_j(x)=\begin{cases}u_1(x)-\gamma_j u_2(x) & 0<x\leq x_j\\ 0 & x\geq x_j\end{cases} \tag{5.9}$$

$$\tilde\eta_j(x)=\begin{cases}u_1(x)+\gamma_j u_2(x) & 0<x<x_j\\ 0 & x>x_j\end{cases} \tag{5.10}$$

where $u_j(x)=u(x,E_j)$ and γ_j is chosen by

$$\gamma_j=\begin{cases}u_1(x_j)/u_2(x_j) & \text{if } u(x_j)\neq0\\ u_1'(x_j)/u_2'(x_j) & \text{if } u(x_j)=0\end{cases} \tag{5.11}$$

Since $W(x_j)=0$, η_j is a C^1 function of compact support and piecewise C^2, and so in $D(H)$. But $\tilde\eta$ is discontinuous.

We claim that if η is in the span of $\{\eta_j\}_{j=1}^N$, then

$$\left\|\left(H-\frac{E_2+E_1}{2}\right)\eta\right\|=\frac{|E_2-E_1|}{2}\,\|\eta\| \tag{5.12}$$

Moreover, such η's are never a finite linear combination of eigenfunctions of H. Accepting these two facts, we note that since the η_j are obviously linear independent, (5.12) implies $\dim P_{(E_1,E_2)}(H)\geq N$. This, together with (5.8), proves the result.

To prove (5.12), we note that

$$\left(H-\frac{E_2+E_1}{2}\right)\eta_j=-\frac{|E_2-E_1|}{2}\,\tilde\eta_j \tag{5.13}$$

Since $\tilde\eta_j$ is not C^1 at x_j, no $\tilde\eta$ is in $D(H)$, hence no η can be in $D(H^2)$ (so we get control of $\dim P_{(E_1,E_2)}(H)$, not just $\dim P_{[E_1,E_2]}(H)$).

Next, note that since $W'(x) = (E_2 - E_1)u_2u_1$, we have if $W(x_i) = W(x_{i+1}) = 0$ that

$$\int_{x_i}^{x_{i+1}} u_1(x)u_2(x)\,dx = 0$$

for $i = 0, 1, 2, \dots, N$ where $x_0 = 0$. Thus

$$\langle \eta_i, \eta_j \rangle = \langle \tilde{\eta}_i, \tilde{\eta}_j \rangle \tag{5.14}$$

since if $i < j$, the difference of the two sides is $2(\gamma_i + \gamma_j) \int_{x_i}^{x_j} u_1(x)u_2(x) = 0$. (5.14) and (5.13) implies (5.12). That completes the proof if $N < \infty$.

If N is infinite, pick $0 < x_1 < \cdots < x_L$ successive zeros and deduce $\dim P_{(E_1,E_2)}(H) \geq L$ for all L. □

6. Some Applications

We will consider three typical applications in this section: one classical (i.e., fifty years old!), one recent to difference equations, and one of Theorem 5.1.

Application 1: Bargmann's Bound. Let u obey $-u'' + Vu = 0$ with $u(0) = 0$ so, if V is bounded, $u(x)/x$ has a finite limit as $x \downarrow 0$. Also suppose $V \leq 0$.

Define $\tilde{m} = -u'/u$ so

$$\tilde{m}' = |V| + \tilde{m}^2 \tag{6.1}$$

since $-V = |V|$. Thus $\tilde{m}$ is monotone increasing. It has a pole at each zero, $x_0 = 0$, $x_1, x_2, \dots, x_\ell, \dots$ of u. Define

$$b(x) = -\frac{xu'}{u} = x\tilde{m}(x) \tag{6.2}$$

Then $b(x)$ has limit -1 as $x \downarrow 0$ and

$$b'(x) = x|V| + \frac{(b + b^2)}{x} \tag{6.3}$$

In particular,

$$-1 \leq b \leq 0 \Rightarrow b'(x) \leq x|V| \tag{6.4}$$

By the monotonicity of $\tilde{m}$, there are unique points $0 < z_1 < x_1 < \cdots < x_{\ell-1} < z_\ell < x_\ell$ where $b_\ell = 0$, and since $b \to -\infty$ as $x \downarrow x_j$, there are last points $y_j \subset [x_{j-1}, z_j]$ where $b(y) = -1$ for $j = 2, 3, \dots, \ell$ and at $y_1 = 0$, $b(0) = -1$. Integrating b' from y_j to z_j, using (6.4), we find

$$\int_{y_j}^{z_j} x|V(x)|\,dx \geq 1$$

so

$$\int_0^{x_\ell} x|V(x)|\,dx \geq \ell$$

By the oscillation theorem, if $N(V) = \dim P_{(-\infty,0)}(H)$, then

$$N(V) \leq \int_0^\infty x|V(x)|\,dx \tag{6.5}$$

This is Bargmann's bound [1]. For further discussion, see Schmidt [20].

Application 2: Denisov-Rakhmanov Theorem. Rakhmanov [14, 15] (see also [13]) proved a deep theorem about orthogonal polynomials on the unit circle that translates to

Rakhmanov's Theorem. *If J is an infinite Jacobi matrix , $d\mu = f\,dx + d\mu_{\rm s}$ and $f(x) > 0$ and $x \in [-2,2]$ and* $\operatorname{supp}(d\mu_{\rm s}) \subset [-2,2]$ *(i.e.,* $\operatorname{spec}(J) \subset [-2,2]$*), then $a_n \to 1$, $b_n \to 0$.*

From the 1990's, there was some interest in extending this to the more general result, where $\operatorname{spec}(J) \subset [-2,2]$ is replaced by $\operatorname{ess\,spec}(J) \subset [-2,2]$. By using the ideas of the proof of Rakhmanov's theorem, one can prove:

Extended Rakhmanov Theorem. *There exist $C(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$ so that if $d\mu = f\,dx + du$ and $f(x) > 0$ a.e. x in $[-2,2]$ and* $\operatorname{spec}(J) \subset [-2-\varepsilon, 2+C]$*, then*

$$\limsup(|a_n - 1| + |b_n|) \leq C(\varepsilon)$$

Here is how Denisov [6] used this to prove

Denisov-Rakhmanov Theorem. *If $d\mu = f(x)\,dx + d\mu_0$, $f(x) > 0$ a.e. $x \in [-2,2]$ and $\sigma_{\rm ess}(J) \subset [-2,2]$, then $a_n \to 1$ and $b_n \to 0$.*

His proof goes as follows. Fix ε. Since J has only finitely many eigenvalues in $[2+\varepsilon,\infty)$, $P_n(2+\varepsilon)$ has only finitely many sign changes. Similarly, $(-1)^n P_n(-2-\varepsilon)$ has only finitely many sign changes. Thus, we can find N_0 so $P_n(2+\varepsilon)$ and $(-1)^n P_n(-2-\varepsilon)$ both have fixed signs if $n > N_0$. Let $\tilde a, \tilde b$ be given by

$$\tilde a_n = a_{N_0+n} \qquad \tilde b_n = b_{N_0+n}$$

By a use of the comparison and oscillation theorems, $\tilde J$ has no eigenvalues in $(-\infty,-2-\varepsilon) \cup (2+\varepsilon,\infty)$. Thus, by the Extended Rakhmanov Theorem,

$$\limsup(|a_n - 1| + |b_n|) = \limsup(|\tilde a_n - 1| + |\tilde b_n|) \leq C(\varepsilon)$$

Since ε is arbitrary, the theorem is proven.

Application 3: Teschl's Proof of the Rofe-Beketov Theorem. Let $V_0(x)$ be periodic and continuous. Let $H_0 = -\frac{d^2}{dx^2} + V_0$ on $L^2(0,\infty)$ with $u(0)=0$ boundary condition. Then

$$\sigma_{\text{ess}}(H_0) = \bigcup_{j=1}^{\infty} [a_j, b_j]$$

with $b_j < a_{j+1}$. (In some special cases, there is only a finite union with one infinite interval.) (b_j, a_{j+1}) are called the gaps. In each gap, H_0 has either zero or one eigenvalue. Suppose $X(x) \to 0$ as $x \to \infty$, and let $H = H_0 + X$. Since $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$, H also has gaps in its spectrum. When is it true that each gap has at most finitely many eigenvalues? Teschl [24, 25] has proven that if $\int_0^\infty x|X(x)|\,dx < \infty$, then for each j, the Wronskian, $w(x)$, of $u(x, b_j)$ and $u(x, a_{j+1})$ has only finitely many zeros. He does this by showing for H_0 that $|X(x)| \to \infty$ as $x \to \infty$ and by an ODE perturbation argument, this implies $|w(x)| \to \infty$ for H. Thus, by the results of Section 5, there are finitely many eigenvalues in each gap.

It is easy to go from half-line results to whole-line results, so Teschl proves if $\int |x|\,|X(x)|\,dx < \infty$, each gap has only finitely many eigenvalues.

This result was first proven by Rofe-Beketov [18] with another simple proof in Gesztesy-Simon [7]; see that later paper for additional references. Teschl's results are stated for the discrete (Jacobi) case (and may be the first proof for the finite difference situation), but his argument translates to the one above for Schrödinger operators.

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