The Shannon-McMillan Theorem for Ergodic Quantum Lattice Systems

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Abstract We formulate and prove a quantum Shannon-McMillan theorem. The theorem demonstrates the significance of the von Neumann entropy for translation invariant ergodic quantum spin systems on $\mathbb{Z}^{\nu}$-lattices: the entropy gives the logarithm of the essential number of eigenvectors of the system on large boxes. The one-dimensional case covers quantum information sources and is basic for coding theorems.

1. Introduction

The classical Shannon-McMillan theorem states that, for an ergodic shift system over a finite alphabet (spin chain, data stream, discrete stochastic process...) the distribution on a box of large volume v is essentially carried by a 'typical set' of approximate size $\exp(vh)$ with h being the Kolmogorov-Sinai dynamical entropy of the system, which in our case is identical to the mean (base e) Shannon entropy. Each element of this typical set has a probability of order $\exp(-vh)$. The latter fact is usually called *asymptotic equipartition property* (AEP). In order to simplify the arguments throughout this introduction we confine ourselves to the one-dimensional case. Thus instead of large boxes we consider n-blocks with n large. In classical information theory the one-dimensional lattice systems

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model information sources. The AEP is of major importance for source coding (c.f. [20]). It implies the possibility of lossless transmission for an n-block message, using typically only $nh/\log 2$ bits. The fact that on the typical set the probabilities of all elements are of the same magnitude implies that better compression is (again typically) impossible.

Considering an ergodic quantum spin lattice system (again, for simplicity, one-dimensional), we prove that the n-block state is essentially carried by a typical subspace (of the entire Hilbert space) of dimension close to $\exp(ns)$. Each one-dimensional projector into that subspace has an expected value of order $\exp(-ns)$. Here s is the mean von Neumann entropy of the system. The reduction of this statement to the commutative setting is exactly the 'classical' theorem.

Several attempts have been made to establish a quantum version of the Shannon-McMillan theorem for stationary quantum systems. Due to the fact that in quantum theory there exist several entropy notions (cf. [1]), depending on e.g. how the measuring process is incorporated, there is a number of different approaches. The paper [13] of King and Leśniewski concerns the classical stochastic process obtained from an ergodic quantum source by measurements of the individual components and derives an asymptotic dimension of a relevant subspace in terms of the classical mean entropy. This relevant subspace however is in general not minimal, and consequently not optimal for compression purposes. In particular, the AEP is not fulfilled for this subspace. For the Bernoulli case (of independent components) the King/Leśniewski result is optimal and consistent with the earlier result by Jozsa and Schumacher [12].

In the paper [7] Hiai and Petz, using a considerably stronger condition of complete ergodicity derived upper and lower bounds (in terms of the von Neumann mean entropy) for the asymptotic dimension of a minimal relevant subspace. In view of our results it turns out that the upper bound is sharp. However the results in [7] concern the more general situation of estimating minimal projectors with respect to a reference state in terms of relative entropy. For the special cases of ergodic Gibbs quantum lattice systems, [9], respectively for ergodic algebraic states, [8], Hiai and Petz could prove the Shannon-McMillan theorem. For the Gibbs situation they even established the convergence in the strong operator topology in the corresponding GNS representation.

For further special classes of quantum systems, namely for a lattice version of free bosons or fermions, Johnson and Suhov present a proof of a quantum Shannon-McMillan theorem, [11]. The authors use occupation numbers and their relation to the eigenvalues of Gibbs ensemble density matrices to obtain a sequence of independent (not identically distributed) random variables. It turns out that this classical process has a mean entropy which is precisely the mean von Neumann entropy of the corresponding quantum system.

Also in the context of quantum statistical mechanics Datta and Suhov [6] prove a Shannon-McMillan theorem for some weakly non-stationary sequence of local quantum Gibbs states in the neighborhood of a classical system. The authors use the quantum AEP to derive a reliable compression/decompression scheme which compresses the quantum state according to the von Neumann entropy rate.

Neshveyev and Størmer considered the CNT-entropy in the context of asymptotically abelian C^*-dynamical systems with locality in order to derive bounds for the asymptotic dimension of a typical subspace, [15]. They obtained a Shannon-McMillan result for the special case of tracial states which covers the classical result. For our quantum lattice systems however this essentially reduces to the situation of Bernoulli states, where the individual components are even supposed to be in the trace state.

Let us discuss, in an informal way, the main ideas behind the proof of the quantum Shannon-McMillan theorem presented here.

At the beginning, we follow mainly the approach of Hiai and Petz [8]. It is well known that one can find a classical subsystem (maximal abelian subalgebra of the entire non-commutative algebra of observables) of the n-block quantum system with Shannon entropy equal to the von Neumann entropy S_n of the full n-block quantum state. Starting from this abelian n-block algebra we construct an abelian quasilocal algebra. It is a subalgebra of the entire quasilocal algebra of the infinite quantum lattice system. Restricting the given quantum state to this classical infinite lattice system we obtain a stochastic process. The k-th component of this process is related to the k-th n-block of the original system. This classical process is ergodic if one assumes for instance *complete ergodicity* of the original system as Hiai and Petz do in their paper. We know from classical Shannon-McMillan that the probability distribution of a large k-block of this classical system (corresponding to a kn-block of the quantum system) has an essential support of size $e^{k\cdot\widetilde{h}}$ (and entropy close to $k \cdot \widetilde{h} \leq k \cdot S_n$). Here $\widetilde{h}$ denotes the mean entropy of the classical system, which is in general smaller than S_n, due to possible correlations. For the quantum context this translates into the existence of an essentially carrying subspace of dimension $e^{k\cdot\widetilde{h}}$ for the kn-block. With s being the mean von Neumann entropy, we arrive (for large n) at an estimate $kn \cdot s \ \sim \ k \cdot S_n \geq k \cdot \widetilde{h} \geq kn \cdot s$, and infer the existence of an (at most) $e^{k\cdot\widetilde{h}} \ \sim \ e^{kn\cdot s}$ -dimensional essentially carrying subspace ('typical subspace') of a kn-block of the quantum system. If we would take into account that the quantum system might have a still smaller ('better' from the point of view of resources needed) typical subspace (compared to what is delivered by the constructed classically restricted system) we would have arrived at the result that $e^{n\cdot s}$ is an *asymptotic upper bound* for the dimension of a typical subspace of the quantum system. So far, we followed the ideas of Hiai and Petz [7], where under the assumption of complete ergodicity estimates for the typical dimension were derived.

In fact, we can extend this result obtaining not only upper and lower bounds but a limit assertion for the dimension of a typical subspace. Instead of complete ergodicity we require only ergodicity, thus establishing a quantum Shannon-McMillan theorem.

Beyond technicalities, this can be accomplished by

a) observing that we *do not* have to take into account the possibility that the quantum system behaves 'better' than the classical one (see some lines above): The very fact that the asymptotic von Neumann entropy of a large n-block is $n \cdot s$ excludes the possibility of an essentially carrying subspace of dimension significantly smaller than $e^{n \cdot s}$. In fact, the contribution to the entropy due to this subspace would be smaller than $n \cdot s$. The non-typical part yields only a contribution of the order $o(n)$. This is due to a uniform integrability argument (based upon Lemma 3.3) and

b) the condition of complete ergodicity can be avoided, since a quantum analog of the following result concerning classical systems can be shown (Theorem 3.1): if a dynamical system is ergodic with respect to the transformation T, then, though it may be non-ergodic with respect to T^n (for a given $n \geq 2$), all of its T^n-ergodic components have the same entropy, and the number of ergodic components is at most n.

Note that the absence of complete ergodicity therefore reflects some intrinsic periodicity of the system. Periodicities naturally appear in a large variety of stationary (quantum) stochastic processes.

Finally let us mention that the techniques used here to derive the quantum Shannon-McMillan theorem can be modified to prove a quantum version of Breiman's strengthening (in the sense of pointwise convergence, [5]) of the classical result. With the notion of an individual trajectory being problematic in the quantum case, in the forthcoming paper [3] we give a reformulation of Breiman's theorem, which has an immediate translation into the quantum situation.

2. The Main Results

Before we state our main theorems we shortly present the mathematical description for the physical model of a ν-dimensional quantum spin lattice system, at the same time fixing notations.

The standard mathematical formalism is introduced in detail e.g. in [4] and [21]. The ν-dimensional infinitely extended lattice corresponds to the group $\mathbb{Z}^\nu$. To each $\mathbf{x} \in \mathbb{Z}^\nu$ there is associated an algebra $\mathcal{A}_\mathbf{x}$ of observables for a spin located at site $\mathbf{x}$. It is given by $\mathcal{A}_\mathbf{x} = \tau(\mathbf{x})\mathcal{A}$, where $\tau(\mathbf{x})$ is an isomorphism and $\mathcal{A}$ is a finite dimensional C^*-algebra with identity. The local algebra $\mathcal{A}_\Lambda$ of observables for the finite subset $\Lambda \subset \mathbb{Z}^\nu$ is given by $\mathcal{A}_\Lambda := \bigotimes_{\mathbf{x}\in\Lambda} \mathcal{A}_\mathbf{x}$. The infinite lattice system is constructed from the finite subsets $\Lambda \subset \mathbb{Z}^\nu$. It is described by the quasilocal C^*-algebra $\mathcal{A}^\infty$, which is defined as the operator norm closure of $\bigcup_{\Lambda\subset\mathbb{Z}^\nu} \mathcal{A}_\Lambda$.
A state of the infinite spin system is given by a normed positive functional Ψ on $\mathcal{A}^\infty$. It corresponds one-to-one to a consistent family of states

$\{\Psi^{(\Lambda)}\}_{\Lambda\subset\mathbb{Z}^\nu}$, where each $\Psi^{(\Lambda)}$ is the restriction of Ψ to the finite dimensional subalgebra $\mathcal{A}_\Lambda$ of $\mathcal{A}^\infty$ and consistency means that $\Psi^{(\Lambda)} = \Psi^{(\Lambda')} \upharpoonright \mathcal{A}_\Lambda$ for $\Lambda \subset \Lambda'$. This one-to-one correspondence reflects the fact, that the state of the entire spin lattice system is assumed to be determined by the expectation values of all observables on finite subsystems Λ. Actually, it is sufficient to consider boxes only. For each $\Psi^{(\Lambda)}$ there exists a unique density operator $D_\Lambda \in \mathcal{A}_\Lambda$, such that $\Psi^{(\Lambda)}(a) = \mathrm{tr}_\Lambda D_\Lambda a,\ a \in \mathcal{A}_\Lambda$ and tr_Λ is the trace on $\mathcal{A}_\Lambda$. By $\mathcal{S}(\mathcal{A}^\infty)$ we denote the state space of $\mathcal{A}^\infty$.

Every $\mathbf{x} \in \mathbb{Z}^\nu$ defines a translation of the lattice and induces an automorphism $T(\mathbf{x})$ on $\mathcal{A}^\infty$, which is a canonical extension of the isomorphisms for finite $\Lambda \subset \mathbb{Z}^\nu$:

$$\begin{aligned} T(\mathbf{x}) : \mathcal{A}_\Lambda &\longrightarrow \mathcal{A}_{\Lambda+\mathbf{x}} \\ a &\longmapsto \left(\bigotimes_{\mathbf{z}\in\Lambda} T_{\mathbf{z}}(\mathbf{x})\right) a, \end{aligned}$$

where $T_{\mathbf{z}}(\mathbf{x}) := \tau(\mathbf{x})\tau^{-1}(\mathbf{z})$. Then $\{T(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^\nu}$ is an action of the translation group $\mathbb{Z}^\nu$ by automorphisms on $\mathcal{A}^\infty$.

Let G be any subgroup of $\mathbb{Z}^\nu$ and denote by $\mathcal{T}(\mathcal{A}^\infty, G)$ the set of states, which are invariant under the translations associated with G:

$$\mathcal{T}(\mathcal{A}^\infty, G) := \{\Psi \in \mathcal{S}(\mathcal{A}^\infty) | \Psi \circ T(\mathbf{x}) = \Psi,\ \forall \mathbf{x} \in G\}.$$

We will be concerned in particular with the space of $\mathbb{Z}^\nu$-invariant, i.e. stationary states. For simplicity we introduce the abbreviation $\mathcal{T}(\mathcal{A}^\infty) = \mathcal{T}(\mathcal{A}^\infty, \mathbb{Z}^\nu)$. Clearly, $\mathcal{T}(\mathcal{A}^\infty) \subset \mathcal{T}(\mathcal{A}^\infty, G)$ for any proper subgroup G of $\mathbb{Z}^\nu$.

For $\mathbf{n} = (n_1, \ldots, n_\nu) \in \mathbb{N}^\nu$ we denote by $\Lambda(\mathbf{n})$ the box in $\mathbb{Z}^\nu$ determined by

$$\Lambda(\mathbf{n}) := \{(x_1, \ldots, x_\nu) \in \mathbb{Z}^\nu |\ x_i \in \{0, \ldots, n_i - 1\},\ i \in \{1, \ldots, \nu\}\},$$

and for $n \in \mathbb{N}$ the hypercube $\Lambda(n)$ is given by

$$\Lambda(n) := \{\mathbf{x} \in \mathbb{Z}^\nu |\ \mathbf{x} \in \{0, \ldots, n-1\}^\nu\}. \tag{2.1}$$

In the following we simplify notations by defining $\mathcal{A}^{(\mathbf{n})} := \mathcal{A}_{\Lambda(\mathbf{n})}$ and $\Psi^{(\mathbf{n})} := \Psi^{(\Lambda(\mathbf{n}))}$, for $\mathbf{n} \in \mathbb{N}^\nu$ (respectively for $n \in \mathbb{N}$).

Obviously, because of the translation invariance any $\Psi \in \mathcal{T}(\mathcal{A}^\infty)$ is uniquely defined by the family $\{\Psi^{(\mathbf{n})}\}_{\mathbf{n}\in\mathbb{N}^\nu}$.

The von Neumann entropy of $\Psi^{(\Lambda)}$ (cf. [17]) is defined by:

$$S(\Psi^{(\Lambda)}) := -\mathrm{tr}_\Lambda D_\Lambda \log D_\Lambda,$$

where tr_Λ denotes the trace on $\mathcal{A}_\Lambda$. For $\mathrm{tr}_{\Lambda(\mathbf{n})}$ we will write $\mathrm{tr}_{\mathbf{n}}$.

It is well known that for every $\Psi \in \mathcal{T}(\mathcal{A}^\infty)$ the limit

$$s(\Psi) := \lim_{\Lambda(\mathbf{n})\nearrow\mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{n}))} S(\Psi^{(\mathbf{n})})$$

exists. We call it the mean (von Neumann) entropy. Let $l \in \mathbb{N}$ and consider the subgroup $G_l := l \cdot \mathbb{Z}^\nu$. For a G_l-invariant state Ψ we define the mean entropy with respect to G_l by

$$s(\Psi, G_l) := \lim_{\Lambda(\mathbf{n}) \nearrow \mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{n}))} S(\Psi^{(l \cdot \mathbf{n})}). \tag{2.2}$$

Observe that if the state Ψ is $\mathbb{Z}^\nu$-invariant then we have the relation $s(\Psi, G_l) = l^\nu \cdot s(\Psi)$. Further note that in the commutative case the mean entropy s coincides with the Kolmogorov-Sinai dynamical entropy h.
$\mathcal{T}(\mathcal{A}^\infty, G)$ is a convex, (weak-*) compact subset of $\mathcal{S}(\mathcal{A}^\infty)$ for any subgroup G of $\mathbb{Z}^\nu$. We denote by $\partial_{\text{ex}}\mathcal{T}(\mathcal{A}^\infty, G)$ the set of extreme points of $\mathcal{T}(\mathcal{A}^\infty, G)$. We refer to elements of $\partial_{\text{ex}}\mathcal{T}(\mathcal{A}^\infty, G)$ as G-ergodic states. The elements of $\partial_{\text{ex}}\mathcal{T}(\mathcal{A}^\infty)$ are called ergodic states.

Now we formulate our main result as a generalization of the Shannon-McMillan theorem to the case of non-commutative quasilocal C^*-algebras $\mathcal{A}^\infty$ constructed from $\mathcal{A}$, a finite dimensional C^*-algebra with identity.

Theorem 2.1 (Quantum Shannon-McMillan Theorem). *Let Ψ be an ergodic state on $\mathcal{A}^\infty$ with mean entropy $s(\Psi)$. Then for all $\delta > 0$ there is an $\mathbf{N}_\delta \in \mathbb{N}^\nu$ such that for all $\mathbf{n} \in \mathbb{N}^\nu$ with $\Lambda(\mathbf{n}) \supseteq \Lambda(\mathbf{N}_\delta)$ there exists an orthogonal projector $p_\mathbf{n}(\delta) \in \mathcal{A}^{(\mathbf{n})}$ such that*

1. *$\Psi^{(\mathbf{n})}(p_\mathbf{n}(\delta)) \geq 1 - \delta$,*
2. *for all minimal projectors $0 \neq p \in \mathcal{A}^{(\mathbf{n})}$ with $p \leq p_\mathbf{n}(\delta)$*
$$e^{-\#(\Lambda(\mathbf{n}))(s(\Psi)+\delta)} < \Psi^{(\mathbf{n})}(p) < e^{-\#(\Lambda(\mathbf{n}))(s(\Psi)-\delta)},$$
3. *$e^{\#(\Lambda(\mathbf{n}))(s(\Psi)-\delta)} < \mathrm{tr}_\mathbf{n}(p_\mathbf{n}(\delta)) < e^{\#(\Lambda(\mathbf{n}))(s(\Psi)+\delta)}$.*

Remark: *In complete analogy to the classical case the above theorem, in particular item 2., expresses the AEP in a quantum context. As will be seen in the proof of this theorem the typical subspace given by the projector $p_\mathbf{n}(\delta)$ can be chosen as the linear hull of the eigenvectors of $\Psi^{(\mathbf{n})}$ which have eigenvalues of order $e^{-\#(\Lambda(\mathbf{n}))s(\Psi)}$.*

In quantum information theory spin systems on one-dimensional lattices model quantum information sources. In such a setting essentially carrying subspaces of minimal dimension are particularly important for formulating and proving coding or compression theorems. As subspaces of probability close to 1 they are the relevant subspaces in the sense that the expectation values of any observables restricted to these subspaces are almost equal to the corresponding ones on the entire space. On the other hand the minimal dimension allows an economical use of resources (qubits) needed for quantum information storage and transmission. Define for $\varepsilon \in (0, 1)$ and $\mathbf{n} \in \mathbb{N}^\nu$

$$\beta_{\varepsilon,\mathbf{n}}(\Psi) := \min\{\log(\mathrm{tr}_\mathbf{n}\, q) |\, q \in \mathcal{A}^{(\mathbf{n})} \text{ projector},\ \Psi^{(\mathbf{n})}(q) \geq 1 - \varepsilon\},$$

Our next statement, which appears already as a conjecture in the papers [8] and [7] of Petz and Hiai, is strongly related to the previous one. In fact, it can be considered as a reformulation especially for possible applications to (quantum) data compression.

Proposition 2.1. *Let Ψ be an ergodic state on $\mathcal{A}^\infty$ with mean entropy $s(\Psi)$. Then for every $\varepsilon \in (0,1)$*

$$\lim_{\Lambda(\mathbf{n})\nearrow\mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{n}))}\beta_{\varepsilon,\mathbf{n}}(\Psi) = s(\Psi). \tag{2.3}$$

In the proof of the quantum Shannon-McMillan theorem this proposition is an intermediate result.

3. Proofs of the Main Results

A basic tool for the proof of the Shannon-McMillan theorem under the general assumption of ergodicity is the structural assertion Theorem 3.1. It is used to circumvent the complete ergodicity assumption made by Hiai and Petz. The substitution of the quantum system by a classical approximation on large boxes leads to an ergodicity problem for these classical approximations. Theorem 3.1 combined with the subsequent lemma allow to control not only the mean (per site limit) entropies of the ergodic components (with respect to the subshift generated by a large box), but also to cope with the obstacle that some of these components might have an atypical entropy on this large but finite box. Using these prerequisites, we prove Lemma 3.2 which is the extension of the Hiai/Petz upper bound result to ergodic states. Finally, from the simple probabilistic argument expressed in Lemma 3.3 we infer that the upper bound is really a limit.

Theorem 3.1. *Let $\Psi \in \partial_{ex}\mathcal{T}(\mathcal{A}^\infty)$. Then for every subgroup $G_l := l \cdot \mathbb{Z}^\nu$, with $l > 1$ an integer, there exists a $\mathbf{k}(l) \in \mathbb{N}^\nu$ and a unique convex decomposition of Ψ into G_l-ergodic states $\Psi_{\mathbf{x}}$:*

$$\Psi = \frac{1}{\#(\Lambda(\mathbf{k}(l)))} \sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} \Psi_{\mathbf{x}}. \tag{3.1}$$

The G_l-ergodic decomposition (3.1) has the following properties:

1. *$k_j(l) \le l$ and $k_j(l)|l$ for all $j \in \{1,\dots,\nu\}$*
2. *$\{\Psi_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} = \{\Psi_0 \circ T(-\mathbf{x})\}_{\mathbf{x}\in\Lambda(\mathbf{k}(l))}$*
3. *For every G_l-ergodic state $\Psi_{\mathbf{x}}$ in the convex decomposition (3.1) of Ψ the mean entropy with respect to G_l, $s(\Psi_{\mathbf{x}}, G_l)$, is equal to the mean entropy $s(\Psi, G_l)$, i.e.*

$$s(\Psi_{\mathbf{x}}, G_l) = s(\Psi, G_l) \tag{3.2}$$

for all $\mathbf{x} \in \Lambda(\mathbf{k}(l))$.

Proof of Theorem 3.1: Let $(\mathcal{H}_\Psi, \pi_\Psi, \Omega_\Psi, U_\Psi)$ be the GNS representation of the C^*-dynamical system $(\mathcal{A}^\infty, \Psi, \mathbb{Z}^\nu)$. U_Ψ is the unitary representation of $\mathbb{Z}^\nu$ on $\mathcal{H}_\Psi$. It satisfies for every $\mathbf{x} \in \mathbb{Z}^\nu$:

$$U_\Psi(\mathbf{x})\Omega_\Psi = \Omega_\Psi, \tag{3.3}$$

$$U_\Psi(\mathbf{x})\pi_\Psi(a)U_\Psi^*(\mathbf{x}) = \pi_\Psi(T(\mathbf{x})a), \qquad \forall a \in \mathcal{A}^\infty. \tag{3.4}$$

Define

$$\mathcal{N}_{\Psi,G_l} := \pi_\Psi(\mathcal{A}^\infty) \cup U_\Psi(G_l), \tag{3.5}$$

$$\mathcal{P}_{\Psi,G_l} := \{P \in \mathcal{N}'_{\Psi,G_l} | \ P = P^* = P^2\}. \tag{3.6}$$

By $'$ we denote the commutant. Observe that $\mathcal{N}_{\Psi,G_l}$ is selfadjoint. Thus $\mathcal{N}'_{\Psi,G_l}$ (as the commutant of a selfadjoint set) is a von Neumann algebra. Further it is a known result that $\mathcal{N}'_{\Psi,G_l}$ is abelian, (cf. Proposition 4.3.7. in [4] or Lemma IV.3.4 in [10]).

Consider some $l > 1$ such that $\Psi \notin \partial_{\text{ex}}\mathcal{T}(\mathcal{A}^\infty, G_l)$. (If there is no such l the statement of the theorem is trivial.) Then

$$\mathcal{P}_{\Psi,G_l} \backslash \{0, \mathbf{1}\} \neq \emptyset. \tag{3.7}$$

In fact, $\Psi \notin \partial_{\text{ex}}\mathcal{T}(\mathcal{A}^\infty, G_l)$ is equivalent to the reducibility of $\mathcal{N}_{\Psi,G_l}$, (cf. Theorem 4.3.17 in [4]). This means that there is a non-trivial closed subspace of $\mathcal{H}_\Psi$ invariant under the action of $\pi_\Psi(\mathcal{A}^\infty)$ and $U_\Psi(G_l)$. Let P be the projection on this subspace and $P^\perp = \mathbf{1} - P$. Then $P, P^\perp \notin \{0, \mathbf{1}\}$ and of course P and $P^\perp$ are contained in $\mathcal{N}'_{\Psi,G_l}$. Thus (3.7) is clear.

Let I be a countable index set. An implication of the $\mathbb{Z}^\nu$-ergodicity of the G_l-invariant Ψ is the following:

$$\{Q_i\}_{i \in I} \text{ orthogonal } \mathbf{1} - \text{decomposition in } \mathcal{N}'_{\Psi,G_l} \Longrightarrow |I| \le l^\nu. \tag{3.8}$$

To see (3.8) observe at first that for any $Q \in \mathcal{P}_{\Psi,G_l} \backslash \{0\}$ the projector $U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})$, $\mathbf{x} \in \Lambda(l)$, belongs to the abelian algebra $\mathcal{N}'_{\Psi,G_l}$, namely

$$\begin{aligned}\pi_\Psi(a)U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x}) &= U_\Psi(\mathbf{x})\pi_\Psi(T(-\mathbf{x})a)QU_\Psi^*(\mathbf{x}) \quad (\text{by } (3.4)) \\ &= U_\Psi(\mathbf{x})Q\pi_\Psi(T(-\mathbf{x})a)U_\Psi^*(\mathbf{x}) \\ &= U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})\pi_\Psi(a) \quad (\text{by } (3.4))\end{aligned}$$

holds for every $a \in \mathcal{A}^\infty$ and $[U_\Psi(\mathbf{y}), U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})] = 0$ is obvious by $[U_\Psi(\mathbf{y}), U_\Psi(\mathbf{x})] = 0$ for all $\mathbf{y} \in G_l$ and $\mathbf{x} \in \mathbb{Z}^\nu$. Thus $\{U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})\}_{\mathbf{x} \in \Lambda(l)}$ is a family of mutually commuting projectors. The Gelfand isomorphism represents the projectors $U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})$ as continuous characteristic functions 1_{Q_x} on some compact (totally disconnected) space. Define $\bar{Q} := \bigvee_{\mathbf{x} \in \Lambda(l)} U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})$, which has the representation as $\bigvee_{\mathbf{x} \in \Lambda(l)} 1_{Q_\mathbf{x}} = 1_{\bigcup_{\mathbf{x} \in \Lambda(l)} Q_\mathbf{x}}$. Note that $\bar{Q}$ is invariant under the action of

$U_\Psi(\mathbb{Z}^\nu)$. From the $\mathbb{Z}^\nu$-ergodicity of Ψ we deduce that $\bar{Q} = \mathbf{1}$. If we translate back the finite subadditivity of probability measures to the expectation values of the projectors $U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})$ we obtain:

$$\begin{aligned} 1 = \langle \Omega_\Psi, \bar{Q}\Omega_\Psi\rangle &\leq \sum_{\mathbf{x}\in\Lambda(l)} \langle \Omega_\Psi, U_\Psi(\mathbf{x})QU_\Psi^*(\mathbf{x})\Omega_\Psi\rangle \\ &= l^\nu \cdot \langle \Omega_\Psi, Q\Omega_\Psi\rangle \quad \text{(by (3.3))}. \end{aligned}$$

Thus (3.8) is clear.

Combining the results (3.7) and (3.8) we get the existence of an orthogonal $\mathbf{1}$-decomposition $\{P_i\}_{i=0}^{n_l-1}$ in $n_l \leq l^\nu$ minimal projectors $P_i \in \mathcal{P}_{\Psi,G_l}\setminus\{0,\mathbf{1}\}$. Here we use the standard definition of minimality:

$$\begin{aligned} P \text{ minimal projector in } \mathcal{N}'_{\Psi,G_l} \quad &:\Longleftrightarrow \quad 0 \neq P \in \mathcal{P}_{\Psi,G_l} \text{ and} \\ &\qquad Q \leq P \;\Rightarrow Q = P \quad \forall Q \in \mathcal{P}_{\Psi,G_l}\setminus\{0\} \end{aligned}$$

The abelianness of $\mathcal{N}'_{\Psi,G_l}$ implies the uniqueness of the orthogonal $\mathbf{1}$-decomposition $\{P_i\}_{i=0}^{n_l-1}$. Further it follows that $\{P_i\}_{i=0}^{n_l-1}$ is a generating subset for $\mathcal{P}_{\Psi,G_l}$ in the following sense:

$$Q \in \mathcal{P}_{\Psi,G_l} \implies \exists\{P_{i_j}\}_{j=0}^{s\leq n_l-1} \subset \mathcal{P}_{\Psi,G_l} \text{ such that } Q = \sum_{j=0}^{s} P_{i_j}. \quad (3.9)$$

Define $p_i := \langle \Omega_\Psi, P_i\Omega_\Psi\rangle$ and order the minimal projectors P_i such that

$$p_0 \leq p_i, \qquad \forall i \in \{1,\ldots,n_l-1\}. \quad (3.10)$$

Let

$$G(P_0) := \{\mathbf{x}\in\mathbb{Z}^\nu \,|\, U(\mathbf{x})P_0U^*(\mathbf{x}) = P_0\}.$$

Note that $G(P_0)$ is a subgroup of $\mathbb{Z}^\nu$ and contains G_l, since $P_0 \in \mathcal{P}_{\Psi,G_l}$. This leads to the representation

$$G(P_0) = \bigoplus_{j=1}^{\nu} k_j(l)\mathbb{Z}, \quad \text{with } k_j(l)|l \quad \text{for all } j\in\{1,\ldots,\nu\},$$

where the integers $k_j(l)$ are given by

$$k_j(l) := \min\{x_j \,|\, x_j \text{ is the j-th component of } \mathbf{x}\in G(P_0) \text{ and } x_j > 0\}.$$

For P_0, as an element of $\mathcal{P}_{\Psi,G_l}$, $\{U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})\}_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} \subseteq \mathcal{P}_{\Psi,G_l}$ for $\mathbf{k}(l) = (k_1(l),\ldots,k_\nu(l))$. Thus by (3.9) each $U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})$, $\mathbf{x}\in\Lambda(\mathbf{k}(l))$, can be represented as a sum of minimal projectors. But then by linearity of the expectation values and the assumed ordering (3.10) each $U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})$ must be a minimal projector for $\mathbf{x}\in\Lambda(\mathbf{k}(l))$. Otherwise there would be a contradiction to $\langle\Omega_\Psi, U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})\Omega_\Psi\rangle = p_0$. Consequently $\{U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})\}_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} \subseteq \{P_i\}_{i=0}^{n_l-1}$. Consider $\bar{P}_0 =$

$\sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})$. Obviously $\bar{P}_0$ is invariant under the action of $U_\Psi(\mathbb{Z}^\nu)$ and because of the $\mathbb{Z}^\nu$-ergodicity of Ψ

$$\bar{P}_0 = \mathbf{1}.$$

It follows by the uniqueness of the orthogonal $\mathbf{1}$-decomposition

$$\{U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x})\}_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} = \{P_j\}_{j=0}^{n_l-1}.$$

Obviously $n_l = \#(\Lambda(\mathbf{k}(l)))$ and for each P_i, $i \in \{0, \ldots, n_l - 1\}$, there is only one $\mathbf{x} \in \Lambda(\mathbf{k}(l))$ such that

$$P_i = U_\Psi(\mathbf{x})P_0U_\Psi^*(\mathbf{x}) =: P_\mathbf{x}. \tag{3.11}$$

It follows $p_i = p_0$ for all $i \in \{0, \ldots, n_l - 1\}$ and hence

$$p_i = \frac{1}{n_l} = \frac{1}{\#(\Lambda(\mathbf{k}(l)))}, \qquad i \in \{0, \ldots, n_l - 1\}.$$

Finally, set for every $\mathbf{x} \in \Lambda(\mathbf{k}(l))$

$$\Psi_\mathbf{x}(a) := \#(\Lambda(\mathbf{k}(l)))\langle\Omega_\Psi, P_\mathbf{x}\pi_\Psi(a)\Omega_\Psi\rangle, \qquad a \in \mathcal{A}^\infty.$$

From (3.11), (3.3) and (3.4) we get

$$\begin{aligned}\Psi_\mathbf{x}(a) &= \#(\Lambda(\mathbf{k}(l)))\langle\Omega_\Psi, P_\mathbf{x}\pi_\Psi(a)\Omega_\Psi\rangle \\ &= \#(\Lambda(\mathbf{k}(l)))\langle\Omega_\Psi, P_0\pi_\Psi(T(-\mathbf{x})a)\Omega_\Psi\rangle \\ &= \Psi_0(T(-\mathbf{x})a), \qquad a \in \mathcal{A}^\infty,\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{\#(\Lambda(\mathbf{k}(l)))}\sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))}\langle\Omega_\Psi, P_\mathbf{x}\pi_\Psi(a)\Omega_\Psi\rangle &= \langle\Omega_\Psi, (\sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))} P_\mathbf{x})\pi_\Psi(a)\Omega_\Psi\rangle \\ &= \Psi(a).\end{aligned}$$

Thus we arrive at the convex decomposition of Ψ:

$$\Psi = \frac{1}{\#(\Lambda(\mathbf{k}(l)))}\sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))}\Psi_0 \circ T(-\mathbf{x}).$$

By construction this is a G_l-ergodic decomposition of Ψ. It remains to prove the fact that the mean entropies with respect to the lattice G_l are the same for all G_l-ergodic components $\Psi_\mathbf{x}$.

Proof of item 3.: It is a well known result that the quantum mean entropy with respect to a given lattice G_l is affine on the convex set of G_l-invariant states, (cf. prop. 7.2.3 in [21]). Thus to prove (3.2) it is sufficient to show:

$$s(\Psi_\mathbf{x}, G_l) = s(\Psi_0, G_l), \qquad \forall\mathbf{x} \in \Lambda(\mathbf{k}(l)).$$

By the definition of the mean entropy this is equivalent to the statement

$$|S(\Psi_{\mathbf{x}}^{(l\mathbf{n})}) - S(\Psi_0^{(l\mathbf{n})})| = o(|\mathbf{n}|) \qquad \text{as } \mathbf{n} \to \infty. \tag{3.12}$$

This can be seen as follows: In view of the definition of $\Psi_{\mathbf{x}}^{(l\mathbf{n})}$ we have $S(\Psi_{\mathbf{x}}^{(l\mathbf{n})}) = S(\Psi_{\mathbf{x}}^{(\Lambda(l\mathbf{n}))}) = S(\Psi_0^{(\Lambda(l\mathbf{n})-\mathbf{x})})$. We introduce the box $\tilde{\Lambda}$ being concentric with $\Lambda(l\mathbf{n})$, with all edges enlarged by l on both directions, i.e. an l-neighborhood of $\Lambda(l\mathbf{n})$. The two expressions $S(\Psi_{\mathbf{x}}^{(l\mathbf{n})})$ and $S(\Psi_0^{(l\mathbf{n})})$ are von Neumann entropies of the restrictions of $\Psi_0^{(\widetilde{\Lambda})}$ to the smaller sets $\Lambda(l\mathbf{n})$ and $\Lambda(l\mathbf{n}) - \mathbf{x}$, respectively. On the other hand we consider the box $\widehat{\Lambda}$ being concentric with $\Lambda(l\mathbf{n})$ with all edges shortened by l at both sides. $S(\Psi_0^{(\widehat{\Lambda})})$ is the von Neumann entropy of $\Psi_0^{(\Lambda(l\mathbf{n}))}$ and $\Psi_{\mathbf{x}}^{(\Lambda(l\mathbf{n}))}$ after their restriction to the set $\widehat{\Lambda}$. $S(\Psi_{\mathbf{x}}^{(\Lambda(l\mathbf{n}))})$ and $S(\Psi_0^{(\Lambda(l\mathbf{n}))})$ can be estimated simultaneously using the subadditivity of the von Neumann entropy

$$S(\Psi_0^{(\widetilde{\Lambda})}) - \log \mathrm{tr}_{\tilde{\Lambda}\setminus\Lambda(l\mathbf{n})}\mathbf{1} \le S(\Psi_{\natural}^{(l\mathbf{n})}) \le S(\Psi_0^{(\widehat{\Lambda})}) + \log \mathrm{tr}_{\Lambda(l\mathbf{n})\setminus\widehat{\Lambda}}\mathbf{1},$$

where $\natural \in \{\mathbf{x}, 0\}$. Thus (3.12) is immediate. $\square$

In order to simplify our notation in the next lemma we introduce some abbreviations. We choose a positive integer l and consider the decomposition of $\Psi \in \partial_{ex}\mathcal{T}(\mathcal{A}^\infty)$ into states $\Psi_{\mathbf{x}}$ being ergodic with respect to the action of G_l, i.e. $\Psi = \frac{1}{\#(\Lambda(\mathbf{k}(l)))}\sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l))}\Psi_{\mathbf{x}}$. Then we set

$$s := s(\Psi, \mathbb{Z}^\nu) = s(\Psi),$$

i.e. the mean entropy of the state Ψ computed with respect to $\mathbb{Z}^\nu$. Moreover we set

$$s_{\mathbf{x}}^{(l)} := \frac{1}{\#(\Lambda(l))}S(\Psi_{\mathbf{x}}^{(\Lambda(l))}) \quad \text{and} \quad s^{(l)} := \frac{1}{\#(\Lambda(l))}S(\Psi^{(\Lambda(l))}).$$

From the previous lemma we know that

$$s(\Psi_{\mathbf{x}}, G_l) = s(\Psi, G_l) = l^\nu \cdot s(\Psi), \qquad \forall \mathbf{x} \in \Lambda(\mathbf{k}(l)). \tag{3.13}$$

For $\eta > 0$ let us introduce the following set

$$A_{l,\eta} := \{\mathbf{x} \in \Lambda(\mathbf{k}(l)) |\ s_{\mathbf{x}}^{(l)} \ge s + \eta\}. \tag{3.14}$$

By $A_{l,\eta}^c$ we denote its complement. The following lemma states that the density of G_l-ergodic components of Ψ which have too large entropy on the box of side length l vanishes asymptotically in l.

Lemma 3.1. *If Ψ is a $\mathbb{Z}^\nu$-ergodic state on $\mathcal{A}^\infty$, then*

$$\lim_{l\to\infty}\frac{\#A_{l,\eta}}{\#\Lambda(\mathbf{k}(l))}=0$$

holds for every $\eta>0$.

Proof of Lemma 3.1: We suppose on the contrary that there is some $\eta_0>0$ such that $\limsup_l \frac{\#A_{l,\eta_0}}{\#\Lambda(\mathbf{k}(l))}=a>0$. Then there exists a subsequence (l_j) with the property

$$\lim_{j\to\infty}\frac{\#A_{l_j,\eta_0}}{\#\Lambda(\mathbf{k}(l_j))}=a.$$

By the concavity of the von Neumann entropy we obtain

$$\begin{aligned}
\#\Lambda(\mathbf{k}(l_j))\cdot s^{(l_j)} &\geq \sum_{\mathbf{x}\in\Lambda(\mathbf{k}(l_j))} s_{\mathbf{x}}^{(l_j)}\\
&= \sum_{\mathbf{x}\in A_{l_j,\eta_0}} s_{\mathbf{x}}^{(l_j)} + \sum_{\mathbf{x}\in A^c_{l_j,\eta_0}} s_{\mathbf{x}}^{(l_j)}\\
&\geq \#A_{l_j,\eta_0}\cdot(s+\eta_0)+\#A^c_{l_j,\eta_0}\cdot\min_{\mathbf{x}\in A^c_{l_j,\eta_0}} s_{\mathbf{x}}^{(l_j)}.
\end{aligned}$$

Here we made use of (3.14) at the last step. Using that for the mean entropy holds

$$s(\Psi_{\mathbf{x}},G_l)=\lim_{\Lambda(\mathbf{m})\nearrow\mathbb{N}^\nu}\frac{1}{\#\Lambda(\mathbf{m})}S(\Psi_{\mathbf{x}}^{(l\mathbf{m})})=\inf_{\Lambda(\mathbf{m})}\frac{1}{\#\Lambda(\mathbf{m})}S(\Psi_{\mathbf{x}}^{(l\mathbf{m})})$$

we obtain a further estimation for the second term on the right hand side:

$$\begin{aligned}
\#A^c_{l_j,\eta_0}\cdot\min_{\mathbf{x}\in A^c_{l_j,\eta_0}} s_{\mathbf{x}}^{(l_j)} &\geq \#A^c_{l_j,\eta_0}\cdot\min_{\mathbf{x}\in A^c_{l_j,\eta_0}}\frac{1}{l_j^\nu}s(\Psi_{\mathbf{x}},G_{l_j})\\
&= \#A^c_{l_j,\eta_0}\cdot s(\Psi) \qquad \text{(by(3.13))}.
\end{aligned}$$

After dividing $\#\Lambda(\mathbf{k}(l_j))\cdot s^{(l_j)}\geq\#A_{l_j,\eta_0}\cdot(s+\eta_0)+\#A^c_{l_j,\eta_0}\cdot s(\Psi)$ by $\#\Lambda(\mathbf{k}(l_j))$ and taking limits we arrive at the following contradictory inequality:

$$s\geq a(s+\eta_0)+(1-a)s=s+a\eta_0>s.$$

So, $a=0$. $\Box$

Lemma 3.2. *Let Ψ be an ergodic state on $\mathcal{A}^\infty$. Then for every $\varepsilon\in(0,1)$*

$$\limsup_{\Lambda(\mathbf{n})\nearrow\mathbb{N}^\nu}\frac{1}{\#\Lambda(\mathbf{n})}\beta_{\varepsilon,\mathbf{n}}(\Psi)\leq s(\Psi).$$

Proof of Lemma 3.2: We fix $\varepsilon > 0$ and choose arbitrary $\eta, \delta > 0$. Consider the G_l-ergodic decomposition

$$\Psi = \frac{1}{\# \Lambda(\mathbf{k}(l))} \sum_{\mathbf{x} \in \Lambda(\mathbf{k}(l))} \Psi_x$$

of Ψ for integers $l \geq 1$. By Lemma 3.1 there is an integer $L \geq 1$ such that for any $l \geq L$

$$\frac{\varepsilon}{2} \geq \frac{1}{\# \Lambda(\mathbf{k}(l))} \# A_{l,\eta} \geq 0$$

holds, where $A_{l,\eta}$ is defined by (3.14). This inequality implies

$$\frac{1}{\# \Lambda(\mathbf{k}(l))} \# A^c_{l,\eta} \cdot (1 - \frac{\varepsilon}{2}) \geq 1 - \varepsilon. \tag{3.15}$$

On the other hand by

$$S(\Psi^{(\mathbf{n})}) = \inf\{S(\Psi^{(\mathbf{n})} \upharpoonright \mathcal{B}) | \, \mathcal{B} \text{ maximal abelian } C^* - \text{subalgebra of } \mathcal{A}^{(\mathbf{n})}\}$$

(cf. Theorem 11.9 in [16] and use the one-to-one correspondence between maximal abelian $*$-subalgebras and orthogonal $\mathbf{1}$-decompositions into minimal projectors contained in $\mathcal{A}^{(\mathbf{n})}$) there exist maximal abelian C^*-subalgebras $\mathcal{B}_{\mathbf{x}}$ of $\mathcal{A}_{\Lambda(l)}$ with the property

$$\frac{1}{\# \Lambda(l)} S(\Psi^{(\Lambda(l))}_{\mathbf{x}} \upharpoonright \mathcal{B}_{\mathbf{x}}) < s(\Psi) + \eta, \qquad \forall \, \mathbf{x} \in A^c_{l,\eta}. \tag{3.16}$$

We fix an $l \geq L$ and consider the abelian quasi-local C^*-algebras $\mathcal{B}^\infty_{\mathbf{x}}$, constructed with $\mathcal{B}_{\mathbf{x}}$, as C^*-subalgebras of $\mathcal{A}^\infty$ and set

$$m_{\mathbf{x}} := \Psi_{\mathbf{x}} \upharpoonright \mathcal{B}^\infty_{\mathbf{x}} \text{ and } m^{(\mathbf{n})}_{\mathbf{x}} := \Psi_{\mathbf{x}} \upharpoonright \mathcal{B}^{(\mathbf{n})}_{\mathbf{x}}$$

for $\mathbf{x} \in A^c_{l,\eta}$ and $\mathbf{n} \in \mathbb{N}^\nu$. The states $m_{\mathbf{x}}$ are G_l-ergodic since they are restrictions of G_l-ergodic states $\Psi_{\mathbf{x}}$ on a quasi-local algebra. This easily follows from Theorem 4.3.17. in [4]. Moreover, by the Gelfand isomorphism and Riesz representation theorem, we can identify the states $m_{\mathbf{x}}$ with probability measures on corresponding (compact) maximal ideal spaces of $\mathcal{B}^\infty_{\mathbf{x}}$. By commutativity and finite dimensionality of the algebras $\mathcal{B}_{\mathbf{x}}$ these compact spaces can be represented as $B^{\mathbb{Z}^\nu}_{\mathbf{x}}$ with finite sets $B_{\mathbf{x}}$ for all $\mathbf{x} \in A^c_{l,\eta}$. By the Shannon-McMillan-Breiman theorem (cf. [18], [14])

$$\lim_{\Lambda(\mathbf{n}) \nearrow \mathbb{N}^\nu} -\frac{1}{\# \Lambda(\mathbf{n})} \log m^{(\mathbf{n})}_{\mathbf{x}}(\omega_{\mathbf{n}}) = h_{\mathbf{x}} \tag{3.17}$$

$m_{\mathbf{x}}$-almost surely and in $L^1(m_{\mathbf{x}})$ for all $\mathbf{x} \in A^c_{l,\eta}$, where $h_{\mathbf{x}}$ denotes the Kolmogorov-Sinai entropy of $m_{\mathbf{x}}$, and $\omega_{\mathbf{n}} \in {B_{\mathbf{x}}}^{\Lambda(\mathbf{n})}$ are the components of $\omega \in B^{\mathbb{Z}^\nu}_{\mathbf{x}}$ corresponding to the box $\Lambda(\mathbf{n})$. Actually, as we shall see, we need the theorem cited above only in its weaker form (convergence in probability) known as Shannon-McMillan theorem. For each n and $\mathbf{x} \in A^c_{l,\eta}$ let

$$
\begin{aligned}
C^{(\mathbf{n})}_{\mathbf{x}} &:= \{\omega_{\mathbf{n}} \in B^{(\mathbf{n})}_{\mathbf{x}} | \, | - \frac{1}{\#\Lambda(\mathbf{n})} \log m^{(\mathbf{n})}_{\mathbf{x}}(\omega_{\mathbf{n}}) - h_{\mathbf{x}}| < \delta\} \\
&= \{\omega_{\mathbf{n}} \in B^{(\mathbf{n})}_{\mathbf{x}} | \, e^{-\#\Lambda(\mathbf{n})\cdot(h_{\mathbf{x}}+\delta)} < m^{(\mathbf{n})}_{\mathbf{x}}(\omega_{\mathbf{n}}) < e^{-\#\Lambda(\mathbf{n})\cdot(h_{\mathbf{x}}-\delta)}\}.
\end{aligned}
$$

Since lower bounds on the probability imply upper bounds on the cardinality we obtain

$$
\#C^{(\mathbf{n})}_{\mathbf{x}} = \mathrm{tr}_{\mathbf{n}}\left(p^{(\mathbf{n})}_{\mathbf{x}}\right) \leq e^{\#\Lambda(\mathbf{n})\cdot(h_{\mathbf{x}}+\delta)} \leq e^{\#\Lambda(\mathbf{n})\cdot(l^\nu(s(\Psi)+\eta)+\delta)} \tag{3.18}
$$

where $p^{(\mathbf{n})}_{\mathbf{x}}$ is the projector in $\mathcal{B}^{(\mathbf{n})}_{\mathbf{x}}$ corresponding to the function $1_{C^{(\mathbf{n})}_{\mathbf{x}}}$. In the last inequality we have used that $h_{\mathbf{x}} \leq S(\Psi^{(\Lambda(l))}_{\mathbf{x}} \upharpoonright \mathcal{B}_{\mathbf{x}}) < l^\nu(s(\Psi)+\eta)$ for all $\mathbf{x} \in A^c_{l,\eta}$ by

$$
h_{\mathbf{x}} = \lim_{\Lambda(\mathbf{n}) \nearrow \mathbb{N}^\nu} \frac{1}{\#\Lambda(\mathbf{n})} H(m^{(\mathbf{n})}_{\mathbf{x}}) = \inf_{\Lambda(\mathbf{n})} \frac{1}{\#\Lambda(\mathbf{n})} H(m^{(\mathbf{n})}_{\mathbf{x}}),
$$

(cf. [21]), and by (3.16). Here H denotes the Shannon entropy.
From (3.17) it follows that there is an $N \in \mathbb{N}$ (depending on l) such that for all $\mathbf{n} \in \mathbb{N}^\nu$ with $\Lambda(\mathbf{n}) \supset \Lambda(N)$

$$
m^{(\mathbf{n})}_{\mathbf{x}}(C^{(\mathbf{n})}_{\mathbf{x}}) \geq 1 - \frac{\varepsilon}{2}, \qquad \forall\, \mathbf{x} \in A^c_{l,\eta}. \tag{3.19}
$$

For each $\mathbf{y} \in \mathbb{N}^\nu$ with $y_i \geq Nl$ let $y_i = n_i l + j_i$, where $n_i \geq N$ and $0 \leq j_i < l$. We set

$$
q := \bigvee_{\mathbf{x} \in A^c_{l,\eta}} p^{(\mathbf{n})}_{\mathbf{x}}.
$$

and denote by $q_{\mathbf{y}}$ the embedding of q in $\mathcal{A}^{(\mathbf{y})}$. By (3.19) and (3.15) we obtain

$$
\begin{aligned}
\Psi^{(\mathbf{y})}(q_{\mathbf{y}}) &= \frac{1}{\#\Lambda(\mathbf{k}(l))} \sum_{\mathbf{x} \in \Lambda(\mathbf{k}(l))} \Psi^{(\mathbf{y})}_{\mathbf{x}}(q_{\mathbf{y}}) \\
&\geq \frac{1}{\#\Lambda(\mathbf{k}(l))} \#A^c_{l,\eta} \cdot (1 - \frac{\varepsilon}{2}) \geq (1 - \varepsilon).
\end{aligned}
$$

Thus the condition in the definition of $\beta_{\varepsilon,\mathbf{y}}(\Psi)$ is satisfied. Moreover by (3.18)

$$
\begin{aligned}
\beta_{\varepsilon,\mathbf{y}}(\Psi) &\leq \log \mathrm{tr}_{\mathbf{y}}(q_{\mathbf{y}}) \\
&\leq \log \sum_{\mathbf{x} \in A^c_{l,\eta}} e^{\#\Lambda(\mathbf{n})\cdot(h_{\mathbf{x}}+\delta)} + \mathrm{tr}_{\Lambda(\mathbf{y}) \setminus \Lambda(l\mathbf{n})} \mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \log \# A^c_{l,\eta} \cdot e^{\# \Lambda(\mathbf{n})(l^\nu(s(\Psi)+\eta)+\delta)}) + \mathrm{tr}_{\Lambda(\mathbf{y})\setminus\Lambda(l\mathbf{n})}\mathbf{1} \\
&\leq \log \# A^c_{l,\eta} + \#\Lambda(\mathbf{n})(l^\nu(s(\Psi)+\eta)+\delta) \\
&\quad + \mathrm{tr}_{\Lambda(\mathbf{y})\setminus\Lambda(l\mathbf{n})}\mathbf{1} \\
&\leq \log \# A^c_{l,\eta} + \#(\Lambda(\mathbf{y}))(s(\Psi)+\eta+\delta) \\
&\quad + \mathrm{tr}_{\Lambda(\mathbf{y})\setminus\Lambda(l\mathbf{n})}\mathbf{1}.
\end{aligned}
$$

We can conclude from this that

$$\limsup_{\Lambda(\mathbf{y})\nearrow\mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{y}))}\beta_{\varepsilon,\mathbf{y}}(\Psi) \leq s(\Psi)+\eta+\delta,$$

because $\# A^c_{l,\eta}$ does not depend on $\mathbf{n}$ and $\Lambda(\mathbf{y}) \nearrow \mathbb{N}^\nu$ if and only if $\Lambda(\mathbf{n}) \nearrow \mathbb{N}^\nu$. This leads to

$$\limsup_{\Lambda(\mathbf{y})\nearrow\mathbb{N}^\nu} \frac{1}{\#\Lambda(\mathbf{y})}\beta_{\varepsilon,\mathbf{y}}(\Psi) \leq s(\Psi),$$

since η, $\delta > 0$ were chosen arbitrarily. $\square$

Let $\nu \in \mathbb{N}$. For $\mathbf{n} = (n_1,\ldots,n_\nu) \in \mathbb{N}^\nu$ we define $|\mathbf{n}| := \prod_{i=1}^{\nu} n_i$ and write $\mathbf{n}\to\infty$ alternatively for $\Lambda(\mathbf{n}) \nearrow \mathbb{N}^\nu$. Further we introduce the notation

$$\mathbf{n} \geq \mathbf{m} \;:\Longleftrightarrow\; n_i \geq m_i, \quad \forall i \in \{1,\ldots,\nu\}.$$

Recall that for a probability distribution P on a finite set A the Shannon entropy is defined by

$$H(P) := -\sum_{a\in A} P(a)\log P(a).$$

Lemma 3.3. *Let $D > 0$ and $\{(A^{(\mathbf{n})}, P^{(\mathbf{n})})\}_{\mathbf{n}\in\mathbb{N}^\nu}$ be a family, where each $A^{(\mathbf{n})}$ is a finite set with $\frac{1}{|\mathbf{n}|}\log\# A^{(\mathbf{n})} \leq D$ for all $\mathbf{n}\in\mathbb{N}^\nu$ and $P^{(\mathbf{n})}$ is a probability distribution on $A^{(\mathbf{n})}$. Define*

$$\alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) := \min\{\log\#\Omega |\ \Omega\subset A^{(\mathbf{n})}, P^{(\mathbf{n})}(\Omega)\geq 1-\varepsilon\}. \quad (3.20)$$

If $\{(A^{(\mathbf{n})}, P^{(\mathbf{n})})\}_{\mathbf{n}\in\mathbb{N}^\nu}$ satisfies the following two conditions:

1. $\lim_{\mathbf{n}\to\infty}\frac{1}{|\mathbf{n}|}H(P^{(\mathbf{n})}) = h < \infty$
2. $\limsup_{\mathbf{n}\to\infty}\frac{1}{|\mathbf{n}|}\alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) \leq h, \qquad \forall\varepsilon\in(0,1)$

then for every $\varepsilon\in(0,1)$

$$\lim_{\mathbf{n}\to\infty}\frac{1}{|\mathbf{n}|}\alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) = h. \quad (3.21)$$

Note that we do not expect either $\{A^{(\mathbf{n})}\}_{\mathbf{n}\in\mathbb{N}^\nu}$ or $\{P^{(\mathbf{n})}\}_{\mathbf{n}\in\mathbb{N}^\nu}$ to fulfill any consistency conditions. We will see later on that this is the important point for why Lemma 3.3 will be useful in the non-commutative setting.

Proof of Lemma 3.3: Let $\delta > 0$ and define

$$\begin{aligned}
A_1^{(\mathbf{n})}(\delta) &:= \left\{ a \in A^{(\mathbf{n})} \mid P^{(\mathbf{n})}(a) > e^{-|\mathbf{n}|(h-\delta)} \right\}, \\
A_2^{(\mathbf{n})}(\delta) &:= \left\{ a \in A^{(\mathbf{n})} \mid e^{-|\mathbf{n}|(h+\delta)} \le P^{(\mathbf{n})}(a) \le e^{-|\mathbf{n}|(h-\delta)} \right\}, \\
A_3^{(\mathbf{n})}(\delta) &:= \left\{ a \in A^{(\mathbf{n})} \mid P^{(\mathbf{n})}(a) < e^{-|\mathbf{n}|(h+\delta)} \right\}.
\end{aligned}$$

We fix an arbitrary $\delta > 0$ and use the abbreviation $A_i^{(\mathbf{n})} = A_i^{(\mathbf{n})}(\delta)$, $i \in \{1,2,3\}$. To see that $\lim_{\mathbf{n}\to\infty} P^{(\mathbf{n})}\left(A_3^{(\mathbf{n})}\right) = 0$ assume the contrary and observe that the upper bound on the probability of elements from $A_3^{(\mathbf{n})}$ implies a lower bound on the cardinality of elements from $A_3^{(\mathbf{n})}$ needed to cover say an $\varepsilon-$fraction, $\varepsilon \in (0,1)$, of $A^{(\mathbf{n})}$ with respect to $P^{(\mathbf{n})}$. Namely one has $\min\left\{ \#C \mid C \subset A_3^{(\mathbf{n})}, P^{(\mathbf{n})}(C) > \varepsilon \right\} > \varepsilon \cdot e^{|\mathbf{n}|(h+\delta)}$ which contradicts condition 2 in the lemma. Furthermore the set $A_3^{(\mathbf{n})}$ cannot asymptotically contribute to the mean entropy h since

$$\begin{aligned}
&-\frac{1}{|\mathbf{n}|} \sum_{a \in A_3^{(\mathbf{n})}} P^{(\mathbf{n})}(a) \log P^{(\mathbf{n})}(a) \\
&\le -\frac{1}{|\mathbf{n}|} \sum_{a \in A_3^{(\mathbf{n})}} P^{(\mathbf{n})}(a) \log \frac{1}{\#A_3^{(\mathbf{n})}} P^{\mathbf{n}}(A_3^{\mathbf{n}})
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\mathbf{n}\to\infty} -\frac{1}{|\mathbf{n}|} \sum_{a \in A_3^{(\mathbf{n})}} P^{(\mathbf{n})}(a) \log \frac{1}{\#A_3^{(\mathbf{n})}} P^{(\mathbf{n})}(A_3^{(\mathbf{n})}) \\
&= \lim_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} \left(P^{(\mathbf{n})}(A_3^{(\mathbf{n})}) \log \#A_3^{(\mathbf{n})} - P^{(\mathbf{n})}(A_3^{(\mathbf{n})}) \log P^{(\mathbf{n})}(A_3^{(\mathbf{n})}) \right) = 0.
\end{aligned}$$

Here we used the fact that $\frac{\log \#A^{(\mathbf{n})}}{|\mathbf{n}|}$ stays bounded and $-\sum p_i \log p_i \le -\sum p_i \log q_i$ for finite vectors $(p_i), (q_i)$ with $\sum_i p_i = \sum_i q_i \le 1$ and $p_i, q_i \ge 0$. Since $A_3^{(\mathbf{n})}$ does not contribute to the entropy one easily concludes that $\lim_{\mathbf{n}\to\infty} P^{(\mathbf{n})}\left(A_1^{(\mathbf{n})}\right) = 0$ because otherwise $\liminf_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} H\left(P^{(\mathbf{n})}\right) < h$ would hold. Recall that $\delta > 0$ was chosen arbitrarily.

Thus

$$\lim_{\mathbf{n}\to\infty} P^{(\mathbf{n})}\left(A_2^{(\mathbf{n})}(\delta)\right) = 1, \qquad \forall \delta > 0. \tag{3.22}$$

Consequently the lemma follows since $P^{(\mathbf{n})}(\Omega) \geq 1-\varepsilon$ implies $P^{(\mathbf{n})}\left(\Omega \cap A_2^{(\mathbf{n})}(\delta)\right) \geq (1-\varepsilon)^2$ for $|\mathbf{n}|$ sufficiently large and one needs at least $(1-\varepsilon)^2 \cdot e^{|\mathbf{n}|(h-\delta)}$ elements from $A_2^{(\mathbf{n})}(\delta)$ to cover $\Omega \cap A_2^{(\mathbf{n})}(\delta)$ and δ can be chosen arbitrarily small. $\square$

Proof of Proposition 2.1: $\mathcal{A}^{(\mathbf{n})}$ as a finite dimensional C^*-algebra is isomorphic to a finite direct sum $\bigoplus_{j=1}^{M} \mathcal{B}(\mathcal{H}_j^{(\mathbf{n})})$, where each $\mathcal{H}_j^{(\mathbf{n})}$ is a Hilbert space with $\dim \mathcal{H}_j^{(\mathbf{n})} = d_j^{(\mathbf{n})} < \infty$ and any minimal projector in $\mathcal{A}^{(\mathbf{n})}$ is represented by a one-dimensional projector on $\mathcal{H}^{(\mathbf{n})} := \bigoplus_{j=1}^{M} \mathcal{H}_j^{(\mathbf{n})}$ with $\dim \mathcal{H}^{(\mathbf{n})} = \sum_{j=1}^{M} d_j^{(\mathbf{n})} =: d_{\mathbf{n}}$. Note that $\bigoplus_{j=1}^{M} \mathcal{B}(\mathcal{H}_j^{(\mathbf{n})}) \subset \mathcal{B}(\mathcal{H}^{(\mathbf{n})})$. Consider the spectral representation of the density operator $D_{\mathbf{n}}$ of $\Psi^{(\mathbf{n})}$ in $\mathcal{B}(\mathcal{H}^{(\mathbf{n})})$:

$$D_{\mathbf{n}} = \sum_{i=1}^{d_{\mathbf{n}}} \lambda_i^{(\mathbf{n})} |q_i^{(\mathbf{n})}\rangle\langle q_i^{(\mathbf{n})}|, \qquad \lambda_i^{(\mathbf{n})} \in [0,1],\ |q_i^{(\mathbf{n})}\rangle \in \mathcal{H}^{(\mathbf{n})}.$$

For $\mathbf{n} = (n_1, \ldots, n_\nu) \in \mathbb{N}^\nu$ let $A^{(\mathbf{n})}$ be the finite set consisting of the eigen-projectors $q_i^{(\mathbf{n})} := |q_i^{(\mathbf{n})}\rangle\langle q_i^{(\mathbf{n})}|$ of $\Psi^{(\mathbf{n})}$, i.e.

$$A^{(\mathbf{n})} := \{q_i^{(\mathbf{n})}\}_{i=1}^{d_{\mathbf{n}}}. \tag{3.23}$$

Let $P^{(\mathbf{n})}$ be the probability distribution on $A^{(\mathbf{n})}$ given by:

$$P^{(\mathbf{n})}(q_i^{(\mathbf{n})}) := \Psi^{(\mathbf{n})}(q_i^{(\mathbf{n})}) = \lambda_i^{(\mathbf{n})}. \tag{3.24}$$

Recall that $|\mathbf{n}| = \prod_{i=1}^{\nu} n_i$. Let $D := \log(\dim \mathcal{H}^{(0)})$, then $\frac{1}{|\mathbf{n}|} \log \# A^{(\mathbf{n})} \leq D$ for all $\mathbf{n} \in \mathbb{N}^\nu$. We show that the family $\{(A^{(\mathbf{n})}, P^{(\mathbf{n})})\}_{\mathbf{n}\in\mathbb{N}^\nu}$ fulfills both conditions in Lemma 3.3 and consequently

$$\lim_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} \alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) = \lim_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} H(P^{(\mathbf{n})}), \qquad \forall \varepsilon \in (0,1). \tag{3.25}$$

It is clear that $H(P^{(\mathbf{n})}) = -\sum_{i=1}^{d_{\mathbf{n}}} \lambda_i^{(\mathbf{n})} \log \lambda_i^{(\mathbf{n})} = S(\Psi^{(\mathbf{n})})$. Thus

$$h := \lim_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} H(P^{(\mathbf{n})}) = s(\Psi). \tag{3.26}$$

Next assume the following ordering:

$$i < j \implies \lambda_i^{(\mathbf{n})} \geq \lambda_j^{(\mathbf{n})}$$

and define for $\varepsilon \in (0,1)$

$$n_{\varepsilon,\mathbf{n}} := \min\{k \in \{1,\ldots,d_{\mathbf{n}}\} | \sum_{j=1}^{k} \lambda_j^{(\mathbf{n})} \geq 1-\varepsilon\}.$$

Thus $\alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) = \log \#(\{q_i^{(\mathbf{n})}\}_{i=1}^{n_{\varepsilon,\mathbf{n}}}) = \log n_{\varepsilon,\mathbf{n}}$. We claim :

$$\alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) = \beta_{\varepsilon,\mathbf{n}}(\Psi^{(\mathbf{n})}), \qquad \forall \varepsilon \in (0,1). \tag{3.27}$$

From $\Psi^{(\mathbf{n})}(\sum_{i=1}^{n_{\varepsilon,\mathbf{n}}} q_i^{(\mathbf{n})}) \geq 1-\varepsilon$ and $\mathrm{tr}_{\mathbf{n}} \sum_{i=1}^{n_{\varepsilon,\mathbf{n}}} q_i^{(\mathbf{n})} = n_{\varepsilon,\mathbf{n}}$ it is obvious that $\beta_{\varepsilon,\mathbf{n}}(\Psi^{(\mathbf{n})}) \leq \alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})})$.
Assume $\beta_{\varepsilon,\mathbf{n}}(\Psi^{(\mathbf{n})}) < \alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})})$. Then there exists a projector $q \in \mathcal{A}^{(\mathbf{n})}$ with $\Psi^{(\mathbf{n})}(q) \geq 1-\varepsilon$ such that $m := \mathrm{tr}_{\mathbf{n}}\, q < n_{\varepsilon,\mathbf{n}}$. Let $\sum_{i=1}^{m} |q_i\rangle\langle q_i|$, where $|q_i\rangle \in \mathcal{H}^{(\mathbf{n})}$, be the spectral representation of q. For $D_{\mathbf{n}}$ as density matrix on $\mathcal{H}^{(\mathbf{n})}$ we use Ky Fan's maximum principle, [2], and obtain the contradiction

$$1-\varepsilon \leq \Psi^{(\mathbf{n})}(q) = \mathrm{tr}_{\mathbf{n}} D_{\mathbf{n}} q = \sum_{i=1}^{m} \langle q_i, D_{\mathbf{n}} q_i\rangle \leq \sum_{i=1}^{m} \lambda_i^{(\mathbf{n})} < 1-\varepsilon.$$

Ψ is ergodic. Thus we can apply Lemma 3.2:

$$\limsup_{\Lambda(\mathbf{n})\nearrow\mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{n}))} \beta_{\varepsilon,\mathbf{n}}(\Psi^{(\mathbf{n})}) \leq s(\Psi), \qquad \forall \varepsilon \in (0,1). \tag{3.28}$$

Setting (3.27) and (3.26) in (3.28) and using that $\#(\Lambda(\mathbf{n})) = |\mathbf{n}|$ we obtain

$$\limsup_{\mathbf{n}\to\infty} \frac{1}{|\mathbf{n}|} \alpha_{\varepsilon,\mathbf{n}}(P^{(\mathbf{n})}) \leq h, \qquad \forall \varepsilon \in (0,1). \tag{3.29}$$

With (3.26) and (3.29) both conditions in Lemma 3.3 are satisfied. It follows (3.25). Now we set back (3.27) and (3.26) in (3.25) and arrive at

$$\lim_{\Lambda(\mathbf{n})\nearrow\mathbb{N}^\nu} \frac{1}{\#(\Lambda(\mathbf{n}))} \beta_{\varepsilon,\mathbf{n}}(\Psi) = s(\Psi), \qquad \forall \varepsilon \in (0,1). \qquad \square$$

Proof of the Quantum Shannon-McMillan Theorem:
Fix $\delta > 0$. Adopt the family $\{(A^{(\mathbf{n})}, P^{(\mathbf{n})})\}_{\mathbf{n}\in\mathbb{N}^\nu}$ and further notations from the proof of Proposition 2.1. Choose some $\delta' < \delta$. Let $A_2^{(\mathbf{n})}(\delta')$ be the subset of $A^{(\mathbf{n})}$ defined in the proof of Lemma 3.3 with $h = s(\Psi)$, appropriate to (3.26). Let $I_{\mathbf{n}}(\delta') := \{i \in \{1,\ldots,d_{\mathbf{n}}\} | \, q_i^{\mathbf{n}} \in A_2^{(\mathbf{n})}(\delta')\}$. Set

$$p_{\mathbf{n}}(\delta) = \sum_{I_{\mathbf{n}}(\delta')} q_i^{\mathbf{n}}.$$

By (3.22) there exists an $\mathbf{N}_\delta \in \mathbb{N}^\nu$ such that $p_{\mathbf{n}}(\delta)$ is a projector with

$$\Psi^{(\mathbf{n})}(p_{\mathbf{n}}(\delta)) = P^{(\mathbf{n})}(A_2^{(\mathbf{n})}(\delta')) \geq 1 - \delta, \qquad \forall \mathbf{n} \geq \mathbf{N}_\delta.$$

Any minimal projector $0 \neq p \in \mathcal{A}^{(\mathbf{n})}$ with $p \leq p_{\mathbf{n}}(\delta)$ is represented as a one-dimensional projector $|p\rangle\langle p|$ on $\mathcal{H}^{(\mathbf{n})}$, such that $|p\rangle = \sum_{i \in I_{\mathbf{n}}(\delta')} \gamma_i |q_i^{\mathbf{n}}\rangle$ and $\sum_{i \in I_{\mathbf{n}}(\delta')} \gamma_i^2 = 1$. Hence

$$\Psi^{(\mathbf{n})}(p) = \sum_{i \in I_{\mathbf{n}}(\delta')} \gamma_i^2 \lambda_i^{(\mathbf{n})}$$

is a weighted average of the eigenvalues $\lambda_i^{(\mathbf{n})}$ corresponding to the set $A_2^{(\mathbf{n})}(\delta')$. Thus we obtain by the definition of this set

$$e^{-\#\Lambda(\mathbf{n})(s(\Psi)+\delta)} < \Psi^{(\mathbf{n})}(p) < e^{-\#\Lambda(\mathbf{n})(s(\Psi)-\delta)}. \qquad (3.30)$$

Using the linearity of $\Psi^{(\mathbf{n})}$ and applying (3.30) to the projectors $q_i^{(\mathbf{n})}$ we arrive at the following estimation

$$e^{\#\Lambda(\mathbf{n})(s(\Psi)-\delta)} < \mathrm{tr}_{\mathbf{n}} p_{\mathbf{n}}(\delta) < e^{\#\Lambda(\mathbf{n})(s(\Psi)+\delta)},$$

if $\mathbf{n}$ is large enough. We have shown all assertions of the theorem. □

4. Comment

It seems that our result can be extended to the case of discrete amenable group actions on quasilocal C^*-algebras. Limits in this situation have to be taken along tempered Følner sequences. The relevant classical theorem in this setting can be found in [14].

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