ABSENCE OF QUANTUM STATES CORRESPONDING TO UNSTABLE CLASSICAL CHANNELS: HOMOGENEOUS POTENTIALS OF DEGREE ZERO

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Abstract. We develop a general theory of absence of quantum states corresponding to unstable classical channels. A principal example treated in detail is the following: Consider a real-valued potential V on $\mathbf{R}^n$, $n \geq 2$, which is smooth outside zero and homogeneous of degree zero. Suppose that the restriction of V to the unit sphere S^{n-1} is a Morse function. We prove that there are no L^2–solutions to the Schrödinger equation $i\partial_t\phi = (-2^{-1}\Delta + V)\phi$ which asymptotically in time are concentrated near local maxima or saddle points of $V_{|S^{n-1}}$. Consequently all states concentrate asymptotically in time near the local minima. Short-range perturbations are included.

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1. Introduction

The purpose of this paper is to show in a class of models that there are no quantum states corresponding to unstable classical channels. We consider the following general situation: Suppose $h(x,\xi)$ is a real classical Hamiltonian in $C^\infty((\mathbf{R}^n\setminus\{0\})\times\mathbf{R}^n)$, $n\geq 2$, satisfying

$$x\cdot\nabla_x h(x,\xi)=0$$

in a neighborhood of a point $(\omega_0,\xi_0)\in S^{n-1}\times\mathbf{R}^n$. Suppose in addition that this neighborhood is conic in the x–variable and that the orbit $(0,\infty)\ni t\to(x(t),\xi(t))=(tk_0\omega_0,\xi_0)$ with $k_0>0$ is a solution to Hamilton's equations

$$\frac{dx}{dt}=\nabla_\xi h(x,\xi),\quad \frac{d\xi}{dt}=-\nabla_x h(x,\xi),$$

or equivalently,

$$\nabla_x h(\omega_0,\xi_0)=0,\quad \nabla_\xi h(\omega_0,\xi_0)=k_0\omega_0. \tag{1.1}$$

We consider situations in which for each energy E near $E_0=h(\omega_0,\xi_0)$ there is a (typically unique) $(\omega(E),\xi(E))\in S^{n-1}\times\mathbf{R}^n$ near (ω_0,ξ_0) depending smoothly on E such that the above structure persists, namely

$$h(\omega(E),\xi(E))=E, \tag{1.2}$$
$$\nabla_x h(\omega(E),\xi(E))=0, \tag{1.3}$$
$$\nabla_\xi h(\omega(E),\xi(E))=k(E)\omega(E). \tag{1.4}$$

Although we shall not elaborate here, we remark that one may easily derive a criterion for (1.2)–(1.4) using the implicit function theorem.

Let us restrict attention to the constant energy surface $h(x,\xi)=E$ and to values of $(\hat{x},\xi,E)$ close to $(\omega(E),\xi(E),E_0)$. (Here and henceforth $\hat{x}=|x|^{-1}x$.) Introduce a change of variables

$$\begin{aligned}&x=x_n(\omega(E)+u),\quad \xi=\xi(E)+\eta+\mu\omega(E);\\ &u\cdot\omega(E)=\eta\cdot\omega(E)=0.\end{aligned}\tag{1.5}$$

This amounts to considering coordinates $(u,x_n,\eta,\mu)\in\mathbf{R}^{n-1}\times\mathbf{R}\times\mathbf{R}^{n-1}\times\mathbf{R}$. We can solve the equation $h(\omega(E)+u,\xi(E)+\eta+\mu\omega(E))=E$ for μ using the implicit function theorem, because

$$\partial_\mu h(\omega(E),\xi(E)+\mu\omega(E))_{|\mu=0}=k(E)>0$$

for E near E_0. We obtain $\mu=-g(u,\eta,E)$ where g is smooth in a neighborhood of $(0,0,E_0)$ and $g(0,0,E_0)=0$. After introducing the "new time" $\tau=\ln x_n(t)=\ln(x(t)\cdot\omega(E))$ Hamilton's equations reduce to

$$u+\frac{du}{d\tau}=\nabla_\eta g(u,\eta,E),\quad \frac{d\eta}{d\tau}=-\nabla_u g(u,\eta,E). \tag{1.6}$$

(See [A2, p. 243].) After linearization of these equations around the fixed point $(u, \eta) = (0, 0)$ we obtain with $w = (u, \eta)$

$$\begin{aligned} &\frac{dw}{d\tau} = B(E)w; \quad B(E) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A(E) - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \\ &A(E) = \begin{pmatrix} g_{u,u} & g_{u,\eta} \\ g_{\eta,u} & g_{\eta,\eta} \end{pmatrix}. \end{aligned} \tag{1.7}$$

Here the real symmetric matrix $A(E)$ of second order derivatives is evaluated at $(0, 0, E)$. We assume all eigenvalues of $B(E)$ have nonzero real part (the hyperbolic case). These eigenvalues are easily proved to come in quadruples, $\lambda, -1 - \lambda$, and their complex conjugates (if λ is not real). If all eigenvalues of $B(E)$ have negative real part then this corresponds to a stable channel. We prefer the word channel because in the case considered $x_n(t)$ grows linearly in time. If at least one of the eigenvalues of $B(E)$ has a positive real part then the usual stable/unstable manifold theorem shows that there are always classical orbits (on the stable manifold) for which $(\hat{x}(t), \xi(t)) \to (\omega(E), \xi(E))$ for $t \to \infty$ (throughout this paper we use the convention $t \to \infty$ to mean $t \to +\infty$). In this situation the question is, do there exist quantum states whose propagation is governed by a self-adjoint quantization H of $h(x, \xi)$ on $L^2(\mathbf{R}^n)$ (possibly with the singularity at $x = 0$ removed) which exhibit this behavior? We will answer this question in the negative.

To be precise, let us first fix a (small) neighborhood $\mathcal{U}_0 \subseteq (\mathbf{R}^n \backslash \{0\}) \times \mathbf{R}^n$ of $(k(E_0)\omega_0, \xi_0)$. Then we consider a small open neighborhood I_0 of E_0 and states of the form $\psi = f(H)\psi$ with $f \in C_0^\infty(I_0)$ such that:

$$\begin{aligned} &\text{For all } g_1, g_2 \in C_0^\infty(\mathbf{R}^n) \\ &\|\{g_1(t^{-1}x) - g_1(k(H)\omega(H))1_{I_0}(H)\}\psi(t)\| \to 0 \text{ for } t \to \infty, \\ &\|\{g_2(p) - g_2(\xi(H))1_{I_0}(H)\}\psi(t)\| \to 0 \text{ for } t \to \infty; \\ &\psi(t) = e^{-itH}\psi, \quad p = -i\nabla_x, \end{aligned} \tag{1.8}$$

while

$$\begin{aligned} &\int_1^\infty t^{-1} \left\|a^w(t^{-1}x, p)\psi(t)\right\|^2 dt < \infty \text{ for all } a \in C_0^\infty(\mathcal{U}_0 \setminus \gamma(I_0)); \\ &\gamma(I_0) = \{(k(E)\omega(E), \xi(E)) \mid E \in I_0\}. \end{aligned} \tag{1.9}$$

(Here a^w signifies Weyl quantization, and 1_{I_0} is the characteristic function of I_0.)

Notice that by (1.8), at least intuitively, for all such symbols a

$$\left\|a^w(t^{-1}x, p)\psi(t)\right\| \to 0 \text{ for } t \to \infty, \tag{1.10}$$

so that (1.9) appears as a weak additonal assumption.

The states ψ obeying the above conditions (with fixed I_0) form a subspace whose closure, say $\mathcal{H}_0$, is H–reducing.

We show the following (main) result.

Theorem 1.1. *Suppose $B(E_0)$ has an eigenvalue with a positive real part. Then under a certain assumption concerning possible resonances (and other technical conditions, see* (H1)–(H8) *in Section 2) there exists a sufficiently small open neighborhood I_0 of E_0 such that*

$$\mathcal{H}_0 = \{0\} \tag{1.11}$$

A symbol satisfying the conditions (1.3) and (1.4) was studied by Guillemin and Schaeffer [GS]. In their paper the roles of x and ξ are reversed and their h is homogeneous of degree one in ξ. There is only one half-line of points in question rather than a one parameter family of half-lines (their critical set of points is at zero energy). Under the condition of no resonances they obtain a conjugation of H to a simpler normal form from which they draw conclusions about propagation of singularities for an equation of the form $H\psi = \phi$.

To see what our result means in a particular model, namely in the case where h is given by $h(x, \xi) = 2^{-1}\xi^2 + V(\hat{x})$ with V a Morse function on S^{n-1}, we recall from [H]: The spectrum of $H = 2^{-1}p^2 + V(\hat{x})$ is purely absolutely continuous and

$$I = \sum_{\omega_l \in \mathcal{C}_r} P_l, \tag{1.12}$$

where P_l are H–reducing orthogonal projections defined as follows: Pick any family $\{\chi_l | \omega_l \in C_r\}$ of smooth functions on S^{n-1} with $\chi_k(\omega_l) = \delta_{kl}$ (the Kronecker symbol); here C_r is the finite set of non-degenerate critical points in S^{n-1} for V. Then

$$P_l = s - \lim_{t \to \infty} e^{itH} \chi_l(\hat{x}) e^{-itH},$$

see [H] and [ACH]. Furthermore in [H] the existence of an asymptotic momentum p^+ was proved and its relationship to the above projections was shown. See also [HS2]. (There was the restriction in [H] to $n \geq 3$ but this is easily removed using the Mourre estimate [ACH, Theorem C.1].)

We notice that (1.12) has an analogue in Classical Mechanics: Any classical orbit (except for the exceptional ones that collapse at the origin) obeys $|x| \to \infty$ with $\hat{x} \to \omega_l$ for some $\omega_l \in C_r$.

Obviously the collection (1.2)–(1.4) corresponds in the potential model exactly to C_r: $(\omega(E), \xi(E)) = \left(\omega_l, \sqrt{2(E - V(\omega_l))}\omega_l\right)$ with $\omega_l \in C_r$. The assumption that the real part of one eigenvalue is positive corresponds to ω_l being either a local maximum or a saddle point of V. Moreover we have the identification

$$\mathcal{H}_0 = \operatorname{Ran}(P_l 1_{I_0}(H)). \tag{1.13}$$

Whence, upon varying I_0, Theorem 1.1 yields the following for the potential model.

Theorem 1.2. *Suppose $\omega_l \in C_r$ is the location of a local maximum or a saddle point of V. Then*

$$P_l = 0. \tag{1.14}$$

A detailed analysis of the large time asymptotic behavior of states in the range of the projections P_l which correspond to local minima was accomplished recently in [HS2]. In particular for any local minimum, $P_l \neq 0$. Moreover in this case we have (1.13) for the analogous space of that in Theorem 1.1. One may easily include in Theorem 1.2 a short-range perturbation $V_1 = O\big(|x|^{-1-\delta}\big)$, $\delta > 0$, $\partial_x^\alpha V_1 = O(|x|^{-2})$, $|\alpha| = 2$, to the Hamiltonian H, see Remarks 8.3 (1).

The result Theorem 1.1 is much more general than Theorem 1.2. In particular, as a further application, we can apply it to a problem of a quantum particle in two dimensions subject to an electromagnetic potential which is asymptotically homogeneous of degree zero in x, see [CHS2]. (For another magnetic field problem in this class see [CHS1].) Let us also give a simple example from Riemannian geometry.

Example 1.3. Consider the symbol h on $(\mathbf{R}^2 \setminus \{0\}) \times \mathbf{R}^2$

$$h = h(x, \xi) = \frac{1}{2}\big(1 + ax_2^2|x|^{-2}\big)^{-1}\xi_1^2 + \frac{1}{2}\xi_2^2;\ a > 0. \tag{1.15}$$

The family $(1, 0; \sqrt{2E}, 0)$, $E > 0$, consists of points obeying (1.2), (1.3) and (1.4). For the linearized reduced flow (1.7) we find the eigenvalues $\frac{1}{2}(-1 - \sqrt{1+4a})$ and $\frac{1}{2}(-1 + \sqrt{1+4a})$, and we conclude that the fixed points are saddle points. If a is irrational there are no resonances of any order (see Section 2 for definition), whence we may infer from Theorem 1.1 that there is no quantum channel associated to the family of fixed points in this case. Using the absence of low order resonances condition (2.6) we may in fact obtain this conclusion for $a \neq \frac{3}{4}, 2$; see Remark 2.1 for a further discussion. We have tacitly assumed that the symbol (1.15) is suitably regularized at $x = 0$ (for the quantization).

Our proof of Theorem 1.1 consists of three parts:

I) Assuming $\psi(t) = e^{-itH}\psi$ does localize in phase space as $t \to \infty$ in the region $|u| + |\eta| \leq \epsilon$ for any $\epsilon > 0$, we prove a stronger localization. Namely, for some small positive δ, the probability (assuming here that ψ is normalized) that $\psi(t)$ is localized in the region $|u| + |\eta| \geq t^{-\delta}$ goes to zero as $t \to \infty$.

II) Using I) and an iteration scheme, we construct an observable Γ which decreases "rapidly" to zero. This iteration scheme is based on one used by Poincaré (see [A1, pp. 177–180]) to obtain a change of coordinates which linearizes (1.6). The fact that if one eigenvalue of $B(E)$ has a positive real part then another has real part < -1 is relevant here. Our observable Γ is in first approximation roughly a quantization of a component of w in (1.7) which decays as $\exp(\lambda\tau)$ with $\operatorname{Re}\lambda < -1$.

III) Using Mourre theory we prove an uncertainty principle lemma for two self-adjoint operators P and Q satisfying $i[P,Q] \geq cI, \quad c > 0$, and some technical conditions. The lemma states that if $\delta_1 < \delta_2$ and g_1 and g_2 are two bounded compactly supported functions then

$$\lim_{t\to\infty} \left\| g_1\big(t^{-\delta_1}Q\big) g_2\big(t^{\delta_2}P\big) \right\| = 0.$$

If ψ is normalized this bound implies that the localizations of I) and II) are incompatible.

The basic theme of our paper may be phrased as absence of certain quantum mechanical states which are present in the corresponding classical model. Notice that given any critical point $\omega_l \in C_r$ (restricting for convenience the discussion to the potential model) there are indeed classical orbits with $|x| \to \infty$ and $\hat{x} \to \omega_l$; in particular this is the case for any given local maximum or saddle point. Intuitively, Theorem 1.1 is true because the associated classical orbits occur for only a "rare" set of initial conditions as fixed by the stable manifold theorem. Alternatively, for some components of $(\hat{x}, \xi)$ the convergence to (ω_l, ξ^+) is "too fast" thus being incompatible with the uncertainty principle in Quantum Mechanics. These two different explanations are actually connected.

For another example of this theme we refer to [G2], [S1] and [S2].

We addressed the problem of Theorem 1.2 in a previous work, [HS1], where we proved (1.14) at local maxima but only had a partial result for saddle points (using a different method). In the two-dimensional case with homogeneous potential a related result concerning distributional eigenfunctions was proved in [HMV].

1.1. **A generalization.** Before getting into the details of the proof of Theorems 1.1 and 1.2, we consider possible generalizations of the homogeneity condition $x \cdot \nabla_x h(x,\xi) = 0$. We will focus our discussion on the structure of the classical mechanics of our models and leave to the reader a formulation of the quantum problem.

The above homogeneity condition is best understood as the invariance of the Hamiltonian under the flow generated by the vector field $v(x,\xi) = \sum x_i \partial/\partial x_i$, or infinitesimally

$$vh(x,\xi) = 0. \tag{1.16}$$

Our goal is thus to find invariance conditions (1.16) which will

(a) reduce the dimension of phase space by two giving an autonomous dynamical system in dimension $2n-2$ (usually not Hamiltonian)
(b) give a natural framework for discussing stability of orbits which do not lie in a compact set. It will turn out that stability is not measured using any preexisting metric in the phase space but rather using bundles of orbits of the vector field v surrounding a given orbit of the Hamiltonian vector field, v_h.

The vector field $v(x,\xi) = \sum x_i \partial/\partial x_i$ does not generate a symplectic flow but does satisfy a crucial property. Namely $\mathcal{L}_v\omega = \omega$ where $\mathcal{L}_v$ is the Lie derivative in direction v and ω is the symplectic form. It will turn out (see Lemma 1.4) that a geometric condition such as this (although more restrictive than necessary) will guarantee that v is a suitable vector field.

We will require v to satisfy certain conditions relative to v_h, where v_h is a Hamiltonian vector field on a symplectic manifold (M,ω) with Hamiltonian h:

(1) In a neighborhood of a point $x_0 \in M$, the local flow $\phi_t^v(\cdot)$ generated by v exists for all $t \in (-\epsilon, \infty)$ for some $\epsilon > 0$ and there exists a surface S containing x_0, transverse to v, and a diffeomorphism $\sigma : B \to S$, where B is a ball in $\mathbf{R}^{2n-1}$ centered at 0, such that the map
$$B \times (-\epsilon, \infty) \ni (w,t) \to \phi_t^v(\sigma(w))$$
is a diffeomorphism onto its image, $\mathcal{K}_0$. We also assume v and v_h are parallel (and nonzero) along the positive orbit of v originating at x_0.

(2) There are smooth functions β and γ such that
$$[v, v_h] = \beta v_h + \gamma v \text{ in } \mathcal{K}_0.$$

(3) $vh = 0$ in $\mathcal{K}_0$.

Condition (1) allows us to assume (after a change of coordinates) that $\mathcal{K}_0 = B \times (-\epsilon, \infty)$, $x_0 = (0,0)$, and $v = (0,\dots,0,1)$ in $\mathcal{K}_0$. With the notation $x_\perp = (x_1, \dots, x_{2n-1})$ for $x \in \mathbf{R}^{2n}$, condition (2) implies
$$(v_h)_\perp(x) = k(x)(v_h)_\perp(x_\perp, 0)$$
for some positive $k(\cdot)$ so that introducing the new time variable τ with $d\tau/dt = k(x(t))$ the first $2n-1$ of Hamilton's equations become
$$\frac{dx_\perp}{d\tau} = (v_h)_\perp(x_\perp, 0).$$
As long as $dh(x_0) \neq 0$, using condition (3) we can eliminate one more variable using energy conservation, $h(x) = h(x_\perp, 0) = E$. For example if $\partial h/\partial x_{2n-1} \neq 0$ we obtain $x_{2n-1} = g(u,E)$ with $u = (x_1, \dots, x_{2n-2})$. Here we assume (u,E) is in a neighborhood of $(0,E_0)$, $E_0 = h(x_0) = h(0)$. We obtain
$$\frac{du}{d\tau} = f(u,E), \tag{1.17}$$
where $f(u,E) = \big((v_h)_1(u,g(u,E)), \dots, (v_h)_{2n-2}(u,g(u,E))\big)$. The orbit of v_h along v corresponds to $u = 0$, $E = E_0$ (in which case $f(0,E_0) = 0$). If $\det(\partial f_i/\partial u_j(0,E_0)) \neq 0$ there will be a smooth family of fixed points of (1.17), $u = u(E)$, in a neighborhood of E_0 (with $u(E_0) = 0$). This situation is entirely analogous to the case

$v(x,\xi) = \sum x_i \partial/\partial x_i$ discussed above and we can define stability of orbits in M in terms of the stability of the fixed points $u(E)$.

If a proof of absence of channels is contemplated along the lines carried out in this paper, it is necessary that low order resonances do not occur at more than a discrete set of energies. In particular, the equations (1.17) should not have a Hamiltonian structure.

The only place where the Hamiltonian nature of the equations appeared above was where we used conservation of energy. To bring in the symplectic form ω we introduce what turns out to be a more restrictive but more geometric condition (see Remark 1.5 for an interpretation):

Lemma 1.4. *Fix an open set $U \subseteq M$.*

(a) *Suppose $\mathcal{L}_v\omega = \alpha\omega$ in U for some $\alpha \in C^\infty(U)$. Suppose in addition that $vh = 0$ in U. Then $[v, v_h] = -\alpha v_h$ in U.*

(b) *Suppose v is nonzero in U and for any smooth function h on U satisfying $vh = 0$ in a neighborhood of a point of U, v satisfies $[v, v_h] = -\alpha v_h$ in this neighborhood. Then $\mathcal{L}_v\omega = \alpha\omega$ in U.*

Proof. We shall use the general relations $dh(w) = \omega(v_h, w)$, $[\mathcal{L}_v, i_w] = i_{[v,w]}$ and $[\mathcal{L}_w, d] = 0$. Here i_w represents interior product with w (see for example [C, p. 84] or [A2, p. 198]).

For (a) we compute in U

$$i_{[v,v_h]}\omega = [\mathcal{L}_v, i_{v_h}]\omega = \mathcal{L}_v dh - i_{v_h}\alpha\omega = d\mathcal{L}_v h - i_{\alpha v_h}\omega = i_{-\alpha v_h}\omega.$$

Since ω is non-degenerate we conclude (a).

As for (b) we use the same computation to conclude that

$$i_{v_h}(-\mathcal{L}_v\omega + \alpha\omega) = 0$$

in open subsets where $vh = 0$. Since v is nonzero there are sufficiently many choices of h to conclude from this that indeed $\mathcal{L}_v\omega = \alpha\omega$. □

Remark 1.5. By integrating the condition of Lemma 1.4 (a), $\mathcal{L}_v\omega = \alpha\omega$, we obtain

$$(\phi_t^v)^*\omega = \exp\Big(\int_0^t \alpha \circ \phi_s^v ds\Big)\omega.$$

In particular if $\mathcal{L}_v\omega = \alpha\omega$ holds in M and ϕ_t^v is a global flow we see that the diffeomorphisms ϕ_t^v preserve the family of Lagrangian manifolds.

Conversely one may readily prove that if ϕ_t^v is a global flow and the diffeomorphisms ϕ_t^v preserve the family of Lagrangian manifolds, then indeed $\mathcal{L}_v\omega = \alpha\omega$ for some smooth α.

Example 1.6. We give another simple example from Riemannian geometry. Consider the symbol h on $(\mathbf{R}^2 \setminus \{0\}) \times \mathbf{R}^2$

$$h = h(x,\xi) = \frac{1}{2}\big(x_1^2 + ax_2^2\big)^{-1}\xi^2;\ a > 0. \tag{1.18}$$

Let $v(x,\xi) = \sum(x_i\partial/\partial x_i + \xi_i\partial/\partial\xi_i)$. Then the vector field v and the Hamitonian vector field v_h fulfill the condition (1)–(3) along the positive orbit of v originating at $(1+2E)^{-1/2}(1,0;\sqrt{2E},0)$, $E > 0$. Here we have taken the S in condition (1) to be a subset of the unit-sphere S^3. Notice also that $(\phi_t^v)^*\omega = \exp(2t)\omega$, and therefore $\mathcal{L}_v\omega = 2\omega$. After linearizing the reduced flow (1.17) we find the eigenvalues $-\sqrt{2E}(\sqrt{a}+1)$ and $\sqrt{2E}(\sqrt{a}-1)$, and we conclude that the family of fixed points consists of sinks for $a \in (0,1)$ and saddle points for $a > 1$. If $\sqrt{a}$ is irrational there are no resonances (of any order). The symbol (1.18) needs to be regularized at $x = 0$ (for the quantization).

This paper is organized as follows: In Section 2 we elaborate on all technical conditions needed for Theorem 1.1 and give a more detailed outline of its proof, cf. the steps I)–III) indicated above. In Section 3 we have collected a few technical preliminaries. In Section 4 we prove the $t^{-\delta}$ –localization, cf. step I), while the localization of Γ is given in Section 5. Finally, Section 6 is devoted to the Mourre theory for this observable. We complete the proof of Theorem 1.1 in Section 7 and give a few missing details of the proof of Theorem 1.2 in Section 8.

2. Technical conditions and outline of proof

We fix $(\omega_0, \xi_0) \in S^{n-1} \times \mathbf{R}^n$ and a small open neighborhood I_0 of $E_0 = h(r\omega_0, \xi_0)$ as in Section 1. We shall elaborate on conditions on the real-valued symbol $h(x, \xi)$, see (H1)–(H8) below. For convenience we remove a possible singularity at $x = 0$ caused by the imposed (local) homogeneity assumption of Section 1. This may be done as follows. Let $\mathcal{N}_0$ be as small open neighborhood of (ω_0, ξ_0). We shall now and henceforth assume that for some $r_0 > 0$

$$\begin{aligned} &h(x,\xi) = h(r_0\hat{x},\xi) \ \textit{in}\ \mathcal{C}_0 := \{(x,\xi) \mid (\hat{x},\xi) \in \mathcal{N}_0, |x| > r_0\},\\ &h \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n). \end{aligned} \tag{H1}$$

Notice that this modification intuitively is irrelevant for the issue of Theorem 1.1 (which concerns states propagating linearly in time in configuration space).

We assume that for some $r, l \geq 0$

$$h \in S\Big(\langle\xi\rangle^r \langle x\rangle^l, g_0\Big);\ \ g_0 = \langle x\rangle^{-2} dx^2 + d\xi^2,\ \ \langle x\rangle = \big(1+|x|^2\big)^{1/2}, \tag{H2}$$

and that

$$H = h^w(x,p)\ \textit{is essentially self–adjoint on}\ C_0^\infty(\mathbf{R}^n). \tag{H3}$$

(See Section 3 for notation.)

Remark. There is some freedom in choosing a global condition like (H2), for example it suffices to have (H2) with g_0 replaced by $\langle x\rangle^{-2\delta_1}dx^2 + \langle x\rangle^{2\delta_2}d\xi^2$ with $0 \leq \delta_2 < \delta_1 \leq 1$.

We assume

$$(1.2)\text{–}(1.4)\ \textit{for}\ E \in I_0. \tag{H4}$$

We define $\omega_n(E) = \omega(E)$, and shrinking I_0 if necessary we pick smooth functions

$$\omega_1(E), \dots, \omega_{n-1}(E) \in S^{n-1}$$

such that $\omega_1(E), \dots, \omega_n(E)$ are mutually orthogonal. We define, cf. (1.5), $x_j = x \cdot \omega_j(E)$ for $j \le n$, $u_j = x_j/x_n$ and $\eta_j = (\xi - \xi(E)) \cdot \omega_j(E)$ for $j \le n-1$ and $\mu = (\xi - \xi(E)) \cdot \omega_n(E)$. Let $w = (u, \eta) = (u_1, \dots, u_{n-1}, \eta_1, \dots, \eta_{n-1})$.

As for the matrix $B(E)$ of (1.7) in these coordinates we need the condition:

$$\begin{aligned}&\textit{The real part of all eigenvalues of } B(E) \textit{ is nonzero}\\ &\textit{for } E \in I_0. \end{aligned} \tag{H5}$$

Let us order the eigenvalues as $\beta_1^s(E), \dots, \beta_{n^s}^s(E), \beta_1^u(E), \dots, \beta_{n^u}^u(E)$ where $\mathrm{Re}\,\big(\beta_j^s(E)\big) < 0$ ($\beta_j^s(E)$ are the stable ones) and $\mathrm{Re}\,\big(\beta_j^u(E)\big) > 0$ ($\beta_j^u(E)$ are the unstable ones). Let $\beta(E)$ refer to the $\mathbf{C}^{2n-2}$ –vector of eigenvalues $(\beta_1^s(E), \dots, \beta_{n^u}^u(E))$ counted with multiplicity.

We are interested in the case

$$n^u = n^u(E) \ge 1. \tag{H6}$$

Let $V^s(E)$ and $V^u(E)$ be the sum of the generalized eigenspaces of $B(E)$ correponding to stable and unstable eigenvalues, respectively. Then we have the decomposition

$$\mathbf{C}^{2n-2} = V^s(E) \oplus V^u(E).$$

Using basis vectors respecting this structure we can find a smooth $M_{2n-2}(\mathbf{C})$ –valued function $T(E)$ such that

$$T(E)^{-1} B(E) T(E) = \mathrm{diag}\,(B^s(E), B^u(E)). \tag{2.1}$$

We may assume the following at $E = E_0$: Corresponding to the decomposition into generalized eigenspaces

$$\begin{aligned}&\mathbf{C}^{2n-2} = V^s \oplus V^u = V_1^s \oplus \cdots \oplus V_{n^s}^s \oplus V_1^u \oplus \cdots \oplus V_{n^u}^u,\\ &T(E_0)^{-1} B(E_0) T(E_0) = \mathrm{diag}\,(B_1^s, \dots, B_{n^u}^u),\end{aligned}$$

where for all entries $N_j^{\#} := B_j^{\#} - \beta_j^{\#}(E_0) I_{\dim\left(V_j^{\#}\right)}$ is strictly lower triangular. Given any $\epsilon > 0$ we may assume (by rescaling the basis vectors) that

$$\|N_j^{\#}\| \le \epsilon. \tag{2.2}$$

We introduce a vector of new variables $\gamma = (\gamma^s, \gamma^u) = (\gamma_1, \dots, \gamma_{2n-2})$

$$\gamma = \gamma(w(E), E) = T(E)^{-1} w(E), \tag{2.3}$$

where γ^s and γ^u are the vectors of coordinates of the part of $w(E)$ in $V^s(E)$ and $V^u(E)$, respectively.

We shall make the assumption (using "*tr*" to denote transposed):

There exists a smooth eigenvector $v(E)$ of $B(E)^{tr}$ in $E \in I_0$, such that $\operatorname{Re}(\lambda(E)) < -1$ *for the corresponding eigenvalue $\lambda(E)$.* (H7)

See Remark 2.3 below for an alternative condition.

The ordering of the eigenvalues may be chosen such that

$$\beta_1^s(E) = \lambda(E). \tag{2.4}$$

It may also be assumed that $v(E)$ is the first row of $T(E)^{-1}$. Clearly by (2.4) $\beta_1^s(E)$ is smooth for $E \in I_0$.

We call E_0 a resonance of order $m \in \{2, 3, \dots\}$ for an eigenvalue $\beta_j^{\#}(E_0)$ if for some $\alpha = (\alpha_1, \dots \alpha_{2n-2}) \in (\mathbf{N} \cup \{0\})^{2n-2}$ with $|\alpha| = m$,

$$\beta_j^{\#}(E_0) = \beta(E_0) \cdot \alpha. \tag{2.5}$$

We assume that

$$E_0 \textit{ is not a resonance of order } \leq m_0 \textit{ for } \beta_1^s(E_0). \tag{H8}$$

Here m_0 may be extracted from the bulk of the paper; the condition

$$m_0 > \max\left(4, \frac{1 + \operatorname{Re}(\beta_1^s(E_0))}{-\operatorname{Re}(\beta_1^s(E_0))}, \dots, \frac{1 + \operatorname{Re}(\beta_{n^s}^s(E_0))}{-\operatorname{Re}(\beta_{n^s}^s(E_0))}\right) \tag{2.6}$$

suffices.

Remark 2.1. The theorem proved with (H8) does not exclude cases where there are low order resonances as long as they constitute a discrete set. This is used in the proof of Theorem 1.2 in Section 8. For the exceptional values $a = \frac{3}{4}$ and $a = 2$ of Example 1.3 there are resonances of order 5 and 4, respectively. For these values of a *all* positive energies are resonances, and consequently our theorem is not applicable.

We shall build a (classical) observable Γ from the first coordinate $\gamma_1 = \gamma_1(w(E), E) = v(E) \cdot w(E)$ of $\gamma^s = \gamma^s(w(E), E)$

$$\Gamma = \gamma_1(w(E), E) + O\big(|\gamma(w(E), E)|^2\big). \tag{2.7}$$

In the study of an analogous quantum observable we consider in detail the case where for some $1 \leq l \leq n-1$

$$\partial_{\eta_l} \gamma_1(w, E_0)_{|w=0} \neq 0. \tag{2.8}$$

We notice that if (2.8) is not true then for some $1 \leq l \leq n-1$

$$\partial_{u_l} \gamma_1(w, E_0)_{|w=0} \neq 0. \tag{2.9}$$

The construction of the quantum Γ in the case of (2.8) and an elaboration of its decay properties will be given in Section 5. A Mourre esimate is given in Section 6, and we complete the proof of Theorem 1.1 in this case in Section 7. We refer the reader to Remarks 5.3, 6.3 and 7.2 for the modifications needed for showing Theorem 1.1 in the case of (2.9).

2.1. **Outline of proof of Theorem 1.1.** Consider a classical orbit with $(\hat{x}(t), \xi(t)) \to (\omega(E), \xi(E))$ for $t \to \infty$ (and E nearby E_0). How do we prove the bound $|u| + |\eta| \leq Ct^{-\delta}$ for some positive δ?

We consider the observables

$$q^s = |\gamma^s|^2, \;\; q^u = |\gamma^u|^2, \;\; q^- = q^u - q^s, \;\; q^+ = q^u + q^s = |\gamma|^2. \tag{2.10}$$

Using (1.6) and (2.1) we compute

$$\frac{d}{dt}\gamma = \frac{\partial_\mu h}{x_n}\big\{(B^s(E)\gamma^s, B^u(E)\gamma^u) + O\big(q^+\big)\big\}. \tag{2.11}$$

For $\epsilon > 0$ small enough in (2.2) the equation (2.11) leads to

$$\frac{d}{dt}q^- = 2\,\mathrm{Re}\left\langle \gamma^u, \frac{d}{dt}\gamma^u \right\rangle_{\mathbf{C}^{n^u}} - 2\,\mathrm{Re}\left\langle \gamma^s, \frac{d}{dt}\gamma^s \right\rangle_{\mathbf{C}^{n^s}} \geq \delta^- t^{-1} q^+ \tag{2.12}$$

for some positive δ^- (which may be chosen independent of E close enough to E_0) and for all $t \geq t^-$ (with t^- large enough).

In particular q^- is increasing and hence

$$q^- \leq 0; \quad t \geq t^-. \tag{2.13}$$

Using (2.11), (2.13) and the Cauchy-Schwarz inequality we compute

$$\frac{d}{dt}q^s = 2\,\mathrm{Re}\left\langle \gamma^s, \frac{d}{dt}\gamma^s \right\rangle_{\mathbf{C}^{n^s}} \leq -2\delta^s t^{-1} q^s \tag{2.14}$$

for some positive δ^s and all $t \geq t^s$.

Integrating (2.14) yields

$$q^s \leq C^s t^{-2\delta^s}, \quad t \geq t^s. \tag{2.15}$$

Finally from (2.13) and (2.15) we conclude that $q^+ \leq 2C^s t^{-2\delta^s}$ and therefore that

$$|\gamma| \leq Ct^{-\delta}; \quad \delta \leq \delta^s. \tag{2.16}$$

Remarks 2.2.

1. (1) We may choose the positive δ in (2.16) as close to the (optimal) exponent $\min\left(\mathrm{Re}\,(-\beta^s_1(E_0)), \dots, \mathrm{Re}\,(-\beta^s_{n^s}(E_0))\right)$ as we wish (provided E is taken close enough to E_0).
2. (2) Although not needed, one may easily prove using similar differential inequalities that indeed $q^u = O\big((q^s)^2\big)$ in complete agreement with the stable manifold theorem.

2.2. **Classical Γ.** We shall for each $m \in \{1, \dots, m_0\}$ construct a $\gamma^{(m)}$ of the form (2.7) such that

$$\frac{d}{dt}\gamma^{(m)} = \frac{\partial_\mu h}{x_n}\beta^s_1\big\{\gamma^{(m)} + O\big(|\gamma|^{m+1}\big)\big\}; \quad \beta^s_1 = \beta^s_1(E). \tag{2.17}$$

Specifically we shall require

$$\gamma^{(1)} = \gamma_1, \text{ and } \gamma^{(m)} = \gamma_1 + \sum_{2 \leq |\alpha| \leq m} c_\alpha \gamma^\alpha; \quad m \geq 2, \tag{2.18}$$

with $\gamma^\alpha = \gamma_1^{\alpha_1} \cdots \gamma_{2n-2}^{\alpha_{2n-2}}$. (It will follow from the construction below that the coefficients $c_\alpha = c_\alpha(E)$ will be smooth; this will be important for "quantizing" the symbol.)

We proceed inductively. Clearly by (2.11) we have (2.17) for $m = 1$. Now suppose we have constructed a function $\gamma^{(m-1)} = \sum_{|\alpha| \le m-1} c_\alpha \gamma^\alpha$ obeying

$$\frac{d}{dt}\gamma^{(m-1)} = \frac{\partial_\mu h}{x_n}\beta_1^s\Big(\gamma^{(m-1)} + \sum_{|\alpha|=m} d_\alpha \gamma^\alpha + O\big(|\gamma|^{m+1}\big)\Big),$$

then we add to $\gamma^{(m-1)}$ a function of the form $\sum_{|\alpha|=m} c_\alpha \gamma^\alpha$ and we need to solve

$$\frac{d}{dt}\sum_{|\alpha|=m} c_\alpha \gamma^\alpha = \frac{\partial_\mu h}{x_n}\beta_1^s \sum_{|\alpha|=m}(c_\alpha - d_\alpha)\gamma^\alpha + O\big(|\gamma|^{m+1}\big). \tag{2.19}$$

For that we compute the derivative using again (2.11). Let us denote by β_{ij} the ij'th entry of the matrix $\operatorname{diag}\big(B^s(E)^{tr}, B^u(E)^{tr}\big)$. Then (2.19) reduces to solving

$$\sum_{|\tilde\alpha|=m}\sum_{i,j}\tilde\alpha_i \beta_{ij} c_{\tilde\alpha}\gamma^{\tilde\alpha - e_i + e_j} = \beta_1^s \sum_{|\alpha|=m}(c_\alpha - d_\alpha)\gamma^\alpha, \tag{2.20}$$

which in turn reduces to solving the system of algebraic equations

$$\sum_{i,j}(\alpha_i + 1 - \delta_{ij})\beta_{ij} c_{\alpha+e_i-e_j} = \beta_1^s(c_\alpha - d_\alpha); \quad |\alpha| = m. \tag{2.21}$$

Here e_i and e_j denote canonical basis vectors in $\mathbf{R}^{2n-2}$ and δ_{ij} is the Kronecker symbol.

Clearly (2.21) amounts to showing that β_1^s is not an eigenvalue of the linear map $\tilde B$ on $\mathbf{C}^{\tilde n}$ with

$$\tilde n = \#\big\{\alpha \in (\mathbf{N}\cup\{0\})^{2n-2} \big| \quad |\alpha| = m\big\} = \frac{(m+2n-3)!}{(2n-3)!m!}$$

given by

$$\mathbf{C}^{\tilde n} \ni c = (c_\alpha)_\alpha \to \big(\tilde B c\big)_\alpha = \Big(\sum_{i,j}(\alpha_i + 1 - \delta_{ij})\beta_{ij} c_{\alpha+e_i-e_j}\Big)_\alpha \in \mathbf{C}^{\tilde n}.$$

Since $\beta_{ij} = \beta_{ij}(E)$ depends continuously on $E \in I_0$ we only need to show that

$$\tilde B(E_0) - \beta_1^s(E_0)I \text{ is invertible.} \tag{2.22}$$

By the condition (H8) indeed (2.22) holds since $m \le m_0$ and the spectrum

$$\sigma\Big(\tilde B(E_0)\Big) = \{\beta(E_0)\cdot\alpha| \quad |\alpha| = m\}.$$

The latter is obvious if diag $\big(B^s(E_0)^{tr}, B^u(E_0)^{tr}\big)$ is diagonal. In general this may be seen by a perturbation argument, see [N, p. 37].

Finally we define

$$\Gamma = \gamma^{(m_0)}.$$

If we have m_0 so large that $\delta(m_0+1) > -\beta_1^s(E)$ where δ is given as in (2.16) we infer by integrating (2.17) (since $\lim_{t\to\infty} t\frac{\partial_\mu h}{x_n} = 1$) that

$$\Gamma = \gamma_1 + O\big(|\gamma|^2\big) = O\Big(t^{\beta_1^s(E)+\epsilon'}\Big); \quad \epsilon' > 0. \tag{2.23}$$

Remark 2.3. We could have used a different observable constructed by a similar iteration using as $\gamma^{(1)}$ a component of γ corresponding to an eigenvector with eigenvalue $\lambda(E)$ having $\mathrm{Re}\,(\lambda(E)) > 0$. We would again need smoothness of the eigenvector and a non-resonance condition for $\lambda(E_0)$, cf. (H7) and (H8). The analogous observable $\gamma^{(m)}$ decreases as $t^{-\delta(m+1)}$ with no upper bound on m (assuming E_0 is not a resonance of any order). But as we will see below, the correspondence between classical and quantum behavior is not so precise as to allow a similar statement in Quantum Mechanics. Thus it does not much matter which of these observables is used.

2.3. **Quantum** Γ**.** To get a statement like (2.23) in Quantum Mechanics we need to quantize the classical symbol $\gamma^{(m)} = \gamma^{(m)}(x,\xi)$. We choose a quantization that takes into account localizations of the states $\psi = f(H)\psi$ obeying (1.8) and (1.9). We fix $m = m_0$ depending on an analogue of the classical bound (2.16), cf. the classical case discussed above. Without going into details, in the case of (2.8) this operator takes the form

$$\Gamma = \Gamma(t) = (p - \xi(E_0))\cdot\omega_l(E_0) + B_1(t); \quad B_1(t) \text{ bounded.}$$

Strictly speaking to get this expression we first make the modification of the classical Γ of dividing by the constant $c_l = \partial_{\eta_l}\gamma_1(w,E_0)_{|w=0}$ and then taking the real part; we shall not discuss the case of (2.9) here. We show the following analogue of (2.23):

Given $\sigma > 0$ we have for some Γ of this form the strong localization

$$\big\|1_{[t^{\sigma-1},\infty)}(|\Gamma|)e^{-itH}\psi\big\| \to 0 \text{ for } t\to\infty. \tag{2.24}$$

We notice that (2.24) is a weaker bound than (2.23); to control various commutators we need to have σ positive. On the other hand it may appear somewhat surprising that such localization result can be proved at all for $\sigma < 2^{-1}$. According to folklore wisdom there is usually a strong connection for pseudodifferential operators between the functional calculus and the pseudodifferential calculus, see for example [DG, Appendix D]. In our case one might think that (2.24) is equivalent to a statement like

$$e^{-itH}\psi \approx a_t^w(x,p)e^{-itH}\psi \text{ for } t\to\infty,$$

where the symbol $a_t = h\big(t^{1-\sigma}\,\mathrm{Re}\,\big(c_l^{-1}\gamma^{(m_0)}\big)\big)$ for suitable $h \in C_0^\infty(\mathbf{R})$ and $\gamma^{(m_0)}$ given by the classical symbol (possibly modified by cut-offs) discussed above. However for $\sigma < 2^{-1}$ such symbols a_t do not fit into any standard (parameter-dependent) pseudodifferential calculus which essentially would require the uniform bounds $\partial_\xi^\beta \partial_x^\alpha a_t = O\big(t^{\delta_2|\beta|-\delta_1|\alpha|}\big)$ with $\delta_2 < \delta_1$. As a consequence we shall base our proof of (2.24) on a functional calculus approach. Using a differential *equality* related to (2.17) we can indeed bound certain quantum errors in a calculus even for $\sigma < 2^{-1}$. It is important that we can take σ small; see the next subsection. Somewhat related problems were studied in [G1] and [CHS1].

Remark. Although suppressed in the above discussion it is *important* from a calculus point of view that the localization similar to the classical bound (2.16) used to define Γ and proving (2.24) is "somewhat weak" (to be specified later); in particular it must be weaker than that used (and being of the same type) in the uncertainty principle argument of the next subsection, see (2.25).

2.4. **Implementing the uncertainty principle.** The last step in our proof of Theorem 1.1 is the decisive one; here Quantum Mechanics enters crucially. We show that a localization similar to the classical bound (2.16) and (2.24) are incompatible unless $\psi = 0$. First fix $\delta > 0$ in agreement with (2.16). More precisely we need the localization

$$e^{-itH}\psi \approx h_2\big(\bar{A}\big)e^{-itH}\psi \to 0 \text{ for } t \to \infty, \tag{2.25}$$

for some $h_2 \in C_0^\infty(\mathbf{R})$ and some operator of the form

$$\bar{A} = t^{\delta-1}x_l + B_2(t); \quad B_2(t) = O\big(t^\delta\big), \quad x_l = x \cdot \omega_l(E_0).$$

Then fix any $\sigma \in (0, \delta)$ and introduce with Γ as in (2.24) the operator $\bar{H} = t^{1-\delta}\Gamma$.

We prove a global Mourre estimate

$$i\big[\bar{H}, \bar{A}\big] \geq 2^{-1}I. \tag{2.26}$$

Abstract Mourre theory and (2.26) lead to the bound

$$\big\|h_2\big(\bar{A}\big)h_1\big(t^{\delta-\sigma}\bar{H}\big)\big\| \leq Ct^{(\sigma-\delta)/2}, \tag{2.27}$$

valid for all $h_1, h_2 \in C_0^\infty(\mathbf{R})$.

Finally picking localization functions in agreement with (2.25) and (2.24) we conclude from (2.27) that

$$e^{-itH}\psi \approx h_2\big(\bar{A}\big)h_1\big(t^{\delta-\sigma}\bar{H}\big)e^{-itH}\psi \to 0 \text{ for } t \to \infty,$$

completing the proof.

3. Preliminaries

We use the notation $\Psi(m, g)$ for the space of operators given by quantizing symbols in the symbol class $S(m, g)$ as defined by [Hö, (18.4.6)]. For the weight functions m and metrics g relevant for this paper it does not matter here whether "quantize" refers to Weyl or Kohn-Nirenberg quantization. For $a \in S(m,g)$ we use the notation $a^w(x,p)$ to denote the Weyl quantization of a. We refer the reader to [DG, Appendix D] and [Hö, Chapter 18] for a detailed account of the calculus of pseudodifferential operators. We shall deal with various kinds of parameter-dependent symbols. In one case the parameter is time $t \geq 1$ and for that we introduce the following shorthand notation.

Definition 3.1. *A family $\{a_t | t \geq 1\}$ of symbols in $S(m,g)$ is said to be uniform in $S(m,g)$ if for all semi-norms $|| \cdot ||_k$ on $S(m,g)$ (cf. [Hö, (18.4.6)]) $\sup_t \|a_t\|_k < \infty$. In this case we write $a_t \in S_{unif}(m,g)$ and $a_t^w(x,p) \in \Psi_{unif}(m,g)$.*

Given this uniformity property various bounds from the calculus of pseudodifferential operators are uniform in the parameter (by continuity properties of the calculus).

We shall also deal with parameter-dependent metrics. Specifically we shall consider for $0 \leq \delta_2 < \delta_1 \leq 1$ and $t \geq 1$

$$g_t = g_t^{\delta_1,\delta_2} = t^{-2\delta_1}dx^2 + t^{2\delta_2}d\xi^2. \tag{3.1}$$

Similarly to Definition 3.1 we shall write (for given $l \in \mathbf{R}$), $a_t \in S_{unif}\big(t^l, g_t\big)$ and $a_t^w(x,p) \in \Psi_{unif}\big(t^l, g_t\big)$ meaning that for all (time-dependent) semi-norms $\sup_t \|a_t\|_{t,k} < \infty$. Also in this case various bounds from the calculus of pseudodifferential operators will be uniform in the parameter. Some extensions of this idea will be used without further comments.

One may verify that (1.10) follows from (1.8) by applying a partition of unity to the f of any state $\psi = f(H)\psi$ of (1.8) to decompose it as $f = \sum f_i$ and by noticing that (1.8) remains valid for the sharper localized states $\psi \to \psi_i = f_i(H)\psi$. (Notice that if $\operatorname{supp}(f_i)$ is located near E_i this leads to $t^{-1}x \approx k(E_i)\omega(E_i)$ and $p \approx \xi(E_i)$ along $\psi_i(t)$.) The latter follows readily upon commutation and applying Lemma 3.2 stated below. The same argument shows that indeed $\mathcal{H}_0$ is H–reducing. (This property may also be verified without appealing to Lemma 3.2.)

Pick non-negative $g_1, \tilde g_1, \tilde{\tilde g}_1 \in C_0^\infty(\mathbf{R}^n)$ such that $g_1 = 1$ in a (small) neighborhood of $k(E_0)\omega_0$, $\tilde g_1 = 1$ in a neighborhood of $\operatorname{supp}(g_1)$ and $\tilde{\tilde g}_1 = 1$ in a neighborhood of $\operatorname{supp}(\tilde g_1)$. Similarly, pick non-negative $g_2, \tilde g_2, \tilde{\tilde g}_2 \in C_0^\infty(\mathbf{R}^n)$ such that $g_2 = 1$ in a neighborhood of ξ_0, $\tilde g_2 = 1$ in a neighborhood of $\operatorname{supp}(g_2)$ and $\tilde{\tilde g}_2 = 1$ in a neighborhood of $\operatorname{supp}(\tilde g_2)$. We suppose $\operatorname{supp}\big(\tilde{\tilde g}_1\big) \times \operatorname{supp}\big(\tilde{\tilde g}_2\big) \subseteq \mathcal{U}_0$ (with $\mathcal{U}_0$ given as in (1.9)), and in fact that the supports are so small that for some $t_0 \geq 1$ the symbol

$$\begin{aligned} h_t(x,\xi) :&= h(x,\xi)\tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi) \\ &= h(r_0\hat x,\xi)\tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi); \quad t \geq t_0, \end{aligned} \tag{3.2}$$

cf. (H1). By the assumption (H2) we then have

$$h_t \in S_{unif}(1, g_0) \cap S_{unif}\big(1, g_t^{1,0}\big). \tag{3.3}$$

Lemma 3.2. *For all $f \in C_0^\infty(\mathbf{R})$ the family*

$$f(h_t^w(x,p)) \in \Psi_{unif}(1, g_0) \cap \Psi_{unif}\big(1, g_t^{1,0}\big) \tag{3.4}$$

and

$$\big\|g_1\big(t^{-1}x\big)g_2(p)\{f(h_t^w(x,p)) - f(H)\}\big\| = O\big(t^{-\infty}\big). \tag{3.5}$$

Proof. As for (3.4) we may proceed as in the proofs of [DG, Propositions D.4.7 and D.11.2]. (One verifies the Beals criterion using the representation (3.10) given below and the calculus of pseudodifferential operators.)

For (3.5) we let $B = h_t^w(x,p)$ and $G = h^w(x,p) - h_t^w(x,p)$. By (3.10)

$$f(h_t^w(x,p)) - f(H) = \frac{1}{\pi}\int_{\mathbf{C}} \big(\bar\partial\tilde f\big)(z)(B-z)^{-1}G(H-z)^{-1}dudv. \tag{3.6}$$

For any large $m \in \mathbf{N}$ we may decompose

$$(B-z)^{-1}G = \sum_{k=1}^{m} ad_B^k(G)(B-z)^{-k} + (B-z)^{-1}ad_B^m(G)(B-z)^{-m}, \tag{3.7}$$

yielding (by the calculus)

$$\begin{aligned} g_1\big(t^{-1}x\big)g_2(p)(B-z)^{-1}G = \sum_{k=1}^{m} R_k(B-z)^{-k}& \\ + g_1\big(t^{-1}x\big)g_2(p)(B-z)^{-1}ad_B^m(G)(B-z)^{-m};& \\ R_k = O\big(t^{-\infty}\big).& \end{aligned} \tag{3.8}$$

By (H2), $ad_B^m(G) \in \Psi_{unif}\Big(\langle x\rangle^{l-m}, g_0\Big)$ and therefore $ad_B^m(G) = O\big(t^{l-m}\big)$, whence

$$\big\|g_1\big(t^{-1}x\big)g_2(p)(B-z)^{-1}G\big\| \le Ct^{l-m}|\operatorname{Im} z|^{-(m+1)} \tag{3.9}$$

uniformly in $z \in \operatorname{supp}\big(\tilde f\big)$.

Clearly (3.5) follows from (3.6) and (3.9). □

Remark 3.3. The statements of Lemma 3.2 extend to any smooth function f with $\frac{d^k}{d\lambda^k}f(\lambda) = O\big(\lambda^{m-k}\big)$ (for fixed $m \in \mathbf{R}$); in particular Lemma 3.2 holds for $f(\lambda) = \lambda$.

Definition 3.4. *Let $\mathcal{F}_+$ denote the largest set of $F = F_+ \in C^\infty(\mathbf{R})$, such that $0 \le F \le 1$, $F' \ge 0, F' \in C_0^\infty\big(\big(\frac12, \frac34\big)\big)$, $F\big(\frac12\big) = 0$, $F\big(\frac34\big) = 1$ and $\sqrt{1-F}$, $\sqrt{F}$, $\sqrt{F'} \in C^\infty$, which is stable under the maps $F \to F^m$ and $F \to 1-(1-F)^m$; $m \in \mathbf{N}$. Let $\mathcal{F}_-$ denote the set of functions $F_- = 1 - F_+$ where $F_+ \in \mathcal{F}_+$.*

We shall in Section 5 use a modification of the abstract calculus [D, Lemma A.3 (b)], see also [DG, Appendix C], [G1, Appendix] or [Mø].

Lemma 3.5. *Suppose $\bar{H}$ and B are self-adjoint operators on a complex Hilbert space $\mathcal{H}$, and that $\{B(t) \mid t > t_0\}$ is a family of self-adjoint operators on $\mathcal{H}$ with the common domain $\mathcal{D}(B(t)) = \mathcal{D}(B)$. Suppose that $\bar{H}$ is bounded, that the commutator form $i\big[\bar{H}, B(t)\big]$ defined on $\mathcal{D}(B)$ is a symmetric operator with same (operator) domain $\mathcal{D}(B)$ and that the $\mathcal{B}(\mathcal{H})$-valued function $B(t)(B-i)^{-1}$ is continuously differentiable. Then*

(A) For any given $F \in C_0^\infty(\mathbf{R})$ we let $\tilde{F} \in C_0^\infty(\mathbf{C})$ denote an almost analytic extension. In particular

$$F(B(t)) = \frac{1}{\pi}\int_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(z)(B(t)-z)^{-1}dudv, \quad z = u + iv. \tag{3.10}$$

The $\mathcal{B}(\mathcal{H})$-valued function $F(B(t))$ is continuously differentiable, and introducing the Heisenberg derivative $\mathbf{D} = \frac{d}{dt} + i\big[\bar{H}, \cdot\big]$, the form

$$\frac{d}{dt}F(B(t)) + i\big[\bar{H}, F(B(t))\big]$$

is given by the bounded operator

$$\mathbf{D}F(B(t)) = -\frac{1}{\pi}\int_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(z)(B(t)-z)^{-1}(\mathbf{D}B(t))(B(t)-z)^{-1}dudv. \tag{3.11}$$

In particular if $\mathbf{D}B(t)$ is bounded then for any $\epsilon > 0$ (with $\langle z\rangle = (1+|z|^2)^{\frac{1}{2}}$)

$$\|\mathbf{D}F(B(t))\| \leq C_\epsilon \sup_{z\in\mathbf{C}} \Big(\langle z\rangle^{\epsilon+2}|\operatorname{Im} z|^{-2}\big|\big(\bar{\partial}\tilde{F}\big)(z)\big|\Big)||\mathbf{D}B(t)||. \tag{3.12}$$

(B) Suppose in addition that we can split $\mathbf{D}B(t) = D(t) + D_r(t)$, where $D(t)$ and $D_r(t)$ are symmetric operators on $\mathcal{D}(B)$ and that the form $i^k ad^k_{B(t)}(D(t)) = i\big[i^{k-1}ad^{k-1}_{B(t)}(D(t)), B(t)\big]$ for $k = 1$ defined on $\mathcal{D}(B)$ is a symmetric operator on $\mathcal{D}(B)$; $ad^0_{B(t)}(D(t)) = D(t)$. (No assumption is made for the form when $k = 2$.) Then the contribution from $D(t)$ to (3.11) can be written as

$$\begin{aligned} -\frac{1}{\pi}\int_{\mathbf{C}} &\big(\bar{\partial}\tilde{F}\big)(z)(B(t)-z)^{-1}D(t)(B(t)-z)^{-1}dudv \\ &= \frac{1}{2}(F'(B(t))D(t) + D(t)F'(B(t))) + R_1(t); \qquad (3.13)\\ R_1(t) = \frac{1}{2\pi}\int_{\mathbf{C}} &\big(\bar{\partial}\tilde{F}\big)(z)(B(t)-z)^{-2} \\ &\cdot ad^2_{B(t)}(D(t))(B(t)-z)^{-2}dudv. \end{aligned}$$

For all $f \in C_0^\infty(\mathbf{R})$

$$\tfrac{1}{2}\big(f^2(B(t))D(t) + D(t)f^2(B(t))\big)$$
$$= f(B(t))D(t)f(B(t)) + R_2(t); \tag{3.14}$$
$$R_2(t) = \frac{1}{2\pi^2}\int_{\mathbf{C}}\int_{\mathbf{C}} \big(\bar\partial\tilde f\big)(z_2)\big(\bar\partial\tilde f\big)(z_1)(B(t)-z_2)^{-1}(B(t)-z_1)^{-1}$$
$$ad^2_{B(t)}(D(t))(B(t)-z_1)^{-1}(B(t)-z_2)^{-1}du_1dv_1du_2dv_2.$$

(C) Suppose in addition to previous assumptions that for all $t > t_0$ *the form* $i[D(t), B(t)]$ *extends from* $\mathcal{D}(B)$ *to a bounded self-adjoint operator. Similarly suppose the operator* $D_r(t)$ *extends to a bounded self-adjoint operator. Then for all* $F \in \mathcal{F}_+$ *the* $\mathcal{B}(\mathcal{H})$*-valued function* $F(B(t))(B-i)^{-1}$ *is continuously differentiable, and there is an almost analytic extension with*

$$\big|\big(\bar\partial\tilde F\big)(z)\big| \le C_k\langle z\rangle^{-1-k}|\operatorname{Im} z|^k;\ \in \mathbf{N}, \tag{3.15}$$

yielding the representation

$$\mathbf{D}F(B(t)) = F'^{\frac12}(B(t))D(t)F'^{\frac12}(B(t)) + R_1(t) + R_2(t) + R_3(t), \tag{3.16}$$

where $R_1(t)$ *is given by (3.13),* $R_2(t)$ *by (3.14) with* $f = \sqrt{F'}$ *and* $R_3(t)$ *is the contribution from* $D_r(t)$ *to (3.11).*

Remarks 3.6.

(1) The left hand side of (3.16) is initially defined as a form on $\mathcal{D}(B)$ while the terms on the right hand side are bounded operators. We shall use the stated representation formulas for bounding these operators in an application in the proof of Proposition 5.1; this will be in the spirit of (3.12) although somewhat more sophisticated.
(2) There are versions of Lemma 3.5 without the assumption $\bar H$ be bounded; they are not needed in this paper.

4. $t^{-\delta}$–LOCALIZATION

Let $\psi = f(H)\psi$ be any state obeying (1.8) and (1.9) with f supported in a very small neighborhood of E_0 (in agreement with the smallness of the neighborhood I_0 of Theorem 1.1). Let $g_1, \tilde g_1, g_2, \tilde g_2 \in C_0^\infty(\mathbf{R}^n)$ be given as in (3.2) and (3.5). In particular we have

$$g_1(k(E)\omega(E))f(E) = f(E),\ g_2(\xi(E))f(E) = f(E).$$

Consider for $t, \kappa \ge 1$ symbols

$$a = a_{t,\kappa}(x,\xi) = F_+\big(\kappa q^-(x,\xi)\big)\tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi), \tag{4.1}$$

where F_+ is given as in Definition 3.4 and q^- is built from the q^- of (2.10) by writing $q^- = q^-(w(E), E)$ and substituting for E the symbol $h(r_0\hat x, \xi)$ cf. (3.2),

$$q = q^-(w(h(r_0\hat x,\xi)), h(r_0\hat x,\xi)). \tag{4.2}$$

We shall consider $\kappa \in [1, t^\nu]$ with $\nu > 0$. To have a good calculus for the symbol a we need $\nu < 1/2$. Notice that

$$a_{t,\kappa} \in S_{unif}\big(1, g_t^{1-\nu,\nu}\big). \tag{4.3}$$

Denoting by $\langle\cdot\rangle_t$ the expectation in the state $\psi(t) = e^{-itH}\psi$ we have the following localization.

Lemma 4.1. *For all* $\nu \in (0, 2/5)$

$$\big\langle a^w_{t,t^\nu}(x,p)\big\rangle_t \to 0 \text{ for } t \to \infty. \tag{4.4}$$

Proof. We shall use a scheme of proof from [D]. Let

$$A_{t,\kappa} = L_1(t)^* a^w_{t,\kappa}(x,p) L_1(t); \quad L_1(t) = g_1\big(t^{-1}x\big) g_2(p). \tag{4.5}$$

From (1.10) and the calculus of pseudodifferential operators we immediately conclude that for fixed κ

$$\langle A_{s,\kappa}\rangle_s \to 0 \text{ for } s \to \infty,$$

yielding

$$-\langle A_{t,\kappa}\rangle_t = \int_t^\infty \langle \mathbf{D} A_{s,\kappa}\rangle_s ds, \tag{4.6}$$

where $\mathbf{D}$ refers to the Heisenberg derivative $\mathbf{D} = \frac{d}{ds} + i[H, \cdot]$. We shall show that the expectation of $\mathbf{D}A_{s,\kappa}$ is essentially positive (in agreement with (2.12)). Up to terms $O(s^{-\infty})$ we may replace $\mathbf{D}$ by $\mathbf{D}_s = \frac{d}{ds} + i[h^w_s(x,p), \cdot]$, cf. Remark 3.3. First we notice that

$$g_2(p) g_1\big(s^{-1}x\big)\big(\mathbf{D}_s a^w_{s,\kappa}(x,p)\big) g_1\big(s^{-1}x\big) g_2(p) \geq -Cs^{5\nu-3}, \tag{4.7}$$

where $C > 0$ is independent of $\kappa \in [1, t^\nu]$.

This bound follows from the calculus. The classical Poisson bracket contributes by a positive symbol when differentiating $q(x,\xi)$. The Fefferman-Phong inequality (see [Hö, Theorem 18.6.8 and Lemma 18.6. 10]) for this term yields the lower bound $O\Big(s^{\nu-1}(s^{2\nu-1})^2\Big) = O(s^{5\nu-3})$.

Hence (uniformly in κ)

$$\begin{aligned} &\mathbf{D}A_{s,\kappa} \geq \{T + T^*\} - Cs^{5\nu-3}; \\ &T = g_2(p) g_1\big(s^{-1}x\big) a^w_{s,\kappa}(x,p) \mathbf{D}_s\big(g_1\big(s^{-1}x\big) g_2(p)\big). \end{aligned}$$

For the contribution from the first term on the right hand side we invoke (1.9) after symmetrizing. We conclude that

$$\int_t^\infty \langle \mathbf{D}A_{s,\kappa}\rangle_s ds \geq o\big(t^0\big) - Ct^{5\nu-2} \text{ uniformly in } \kappa \in [1, t^\nu]. \tag{4.8}$$

Pick $\kappa = t^\nu$.

By combining (4.6) and (4.8) we infer that

$$\langle A_{t,t^\nu}\rangle_t \to 0 \text{ for } t \to \infty,$$

and therefore (4.4). □

Let q^+, q^s and q^u be given as in (2.10) upon substituting E by the symbol $h(r_0\hat{x}, \xi)$, cf. the use of q^- above. We introduce the symbols

$$a_t^1 = t^{\nu-1} q^-(x,\xi) F_+'\big(t^\nu q^-(x,\xi)\big)\tilde{g}_1\big(t^{-1}x\big)\tilde{g}_2(\xi),$$
$$a_t^2 = t^{\nu-1} q^+(x,\xi) F_+'\big(t^\nu q^-(x,\xi)\big)\tilde{g}_1\big(t^{-1}x\big)\tilde{g}_2(\xi).$$

We get the following integral estimate from the above proof employing the uniform boundedness of the family of "propagation observables" A_{t,t^ν}, cf. a standard argument of scattering theory see for example [D, Lemma A.1 (b)].

Lemma 4.2. *In the state $\psi_1(t) = L_1(t)\psi(t)$*

$$\int_1^\infty \big(|\big\langle \big(a_t^1\big)^w(x,p)\big\rangle_t| + |\big\langle \big(a_t^2\big)^w(x,p)\big\rangle_t|\big) dt < \infty.$$

Proof. We substitute $\kappa = t^\nu$ in the construction (4.5). Then up to integrable terms the left hand side of (4.7) (with $s = t$) is given by $c_t^w(x,p)$ with

$$c_t(x,\xi) = g_2(\xi)^2 g_1\big(t^{-1}x\big)^2$$
$$\big(\nu t^{\nu-1}q^-(x,\xi) + t^\nu\big\{h(x,\xi), q^-(x,\xi)\big\}\big)F_+'\big(t^\nu q^-(x,\xi)\big),$$

where $\{\cdot,\cdot\}$ signifies Poisson bracket.

We have the bounds for some $C > 0$ and all large enough t

$$C^{-1}c_t(x,\xi) \le g_2(\xi)^2 g_1\big(t^{-1}x\big)^2\big(a_t^1(x,\xi) + a_t^2(x,\xi)\big) \le C c_t(x,\xi),$$

from which we readily get the lemma by the Fefferman-Phong inequality. □

Remark 4.3. We shall not use Lemma 4.2. However the proof will be important. In particular we shall need the non-negativity of the above symbol c_t.

Let for $t, \kappa \ge 1$ and $0 < 2\delta < \min(\nu, 2\delta^s)$ with $\nu < 2/5$ and δ^s as in (2.14) (this number may be taken independent of E close to E_0, cf. Remarks 2.2 (1)),

$$b_{t,\kappa}(x,\xi) = F_+\big(\kappa^{-1}t^{2\delta}q^s(x,\xi)\big)F_-\big(t^\nu q^-(x,\xi)\big)$$
$$\tilde{g}_1\big(t^{-1}x\big)\tilde{g}_2(\xi) \in S_{unif}\big(1, g_t^{1-\nu,\nu}\big).$$

Lemma 4.4. *For all $\epsilon > 0$*

$$\big\langle b^w_{t,t^\epsilon}(x,p)\big\rangle_t \to 0 \text{ for } t \to \infty. \tag{4.9}$$

Proof. We shall use another scheme of proof from [D]. Let

$$B_{t,\kappa} = L_1(t)^* b^w_{t,\kappa}(x,p) L_1(t), \tag{4.10}$$

cf. (4.5), and write for any (large) t_0

$$\langle B_{t,\kappa}\rangle_t = \langle B_{t_0,\kappa}\rangle_{t_0} + \int_{t_0}^t \langle \mathbf{D}B_{s,\kappa}\rangle_s ds. \tag{4.11}$$

To show that the left hand side of (4.11) vanishes as $t \to \infty$ (with $\kappa = t^{\epsilon}$) we look at the integrand on the right hand side: As in the proof of Lemma 4.1 we may replace $\mathbf{D}$ by $\mathbf{D}_s$ up to a term $r_{s,\kappa}$ such that

$$\int_{t_0}^{t} r_{s,\kappa} ds \to 0 \text{ uniformly in } \kappa \geq 1 \text{ and } t \geq t_0 \text{ as } t_0 \to \infty.$$

Using (1.9) and Remark 4.3 we may estimate the integrand up to terms of this type as

$$\cdots \leq \big\langle L_1(s)^* \big(b^1_{s,\kappa}\big)^w(x,p) L_1(s)\big\rangle_s,$$

where

$$b^1_{s,\kappa}(x,\xi) = \kappa^{-1} s^{2\delta}\big(2\delta s^{-1} q^s(x,\xi) + \{h(x,\xi), q^s(x,\xi)\}\big) c_{s,\kappa}(x,\xi);$$
$$c_{s,\kappa}(x,\xi) = F'_+\big(\kappa^{-1} s^{2\delta} q^s(x,\xi)\big) F_-\big(s^{\nu} q^-(x,\xi)\big) \tilde g_1\big(s^{-1}x\big)\tilde g_2(\xi).$$

We compute, cf. (2.14), that for all large s and a large constant $C > 0$

$$\begin{aligned} &- C s^{2\delta-\nu-1} - C s^{2\delta-1} q^s(x,\xi) c_{s,\kappa}(x,\xi) \\ &\leq b^1_{s,\kappa}(x,\xi) \leq C s^{2\delta-\nu-1} - C^{-1} s^{2\delta-1} q^s(x,\xi) c_{s,\kappa}(x,\xi), \end{aligned}$$

from which we conclude that

$$\limsup_{t_0\to\infty} \sup_{\kappa\geq 1, t\geq t_0} \int_{t_0}^{t} \langle \mathbf{D} B_{s,\kappa}\rangle_s ds \leq 0. \tag{4.12}$$

As for the first term on the right hand side of (4.11), obviously for fixed t_0

$$\langle B_{t_0,\kappa}\rangle_{t_0} \to 0 \text{ for } \kappa \to \infty. \tag{4.13}$$

Combining (4.12) and (4.13) we conclude (by first fixing t_0) that

$$\limsup_{t\to\infty} \quad \langle B_{t,t^{\epsilon}}\rangle_t \leq 0,$$

whence we infer (4.9). $\square$

Next we “absorb” the ϵ of Lemma 4.4 into the δ and introduce the symbols

$$\begin{aligned} b_t(x,\xi) &= F_-\big(t^{2\delta} q^s(x,\xi)\big) F_-\big(t^{\nu} q^-(x,\xi)\big) \tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi), \\ b^1_t(x,\xi) &= -t^{-1} F'_-\big(t^{2\delta} q^s(x,\xi)\big) F_-\big(t^{\nu} q^-(x,\xi)\big) \tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi), \end{aligned} \tag{4.14}$$

where $0 < 2\delta < \min(\nu, 2\delta^s)$ with $\nu < 2/5$ and δ^s as in (2.14). Clearly

$$b_t(x,\xi) \in S_{unf}\Big(1, g_t^{1-\nu',\nu'}\Big) \subseteq S_{unif}\big(1, g_t^{1-\nu,\nu}\big); \quad \nu' = \nu - \delta.$$

We have the following integral estimate.

Lemma 4.5. *In the state $\psi_1(t) = L_1(t)\psi(t)$*

$$\int_1^{\infty} |\big\langle \big(b^1_t\big)^w(x,p)\big\rangle_t| dt < \infty. \tag{4.15}$$

Proof. We use the proofs of Lemmas 4.2 and 4.4. Notice that to leading order "the derivative" of the symbol

$$F_+\big(t^{2\delta}q^s(x,\xi)\big)F_-\big(t^\nu q^-(x,\xi)\big)\tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi)$$

is indeed non-positive, and that $F'_+ = -F'_-$. □

By combining Lemmas 4.1 and 4.4 we conclude the following localization result.

Proposition 4.6. *For any state $\psi = f(H)\psi$ obeying* (1.8) *and* (1.9) *with $f \in C_0^\infty(I_0)$ where I_0 is a sufficiently small neighborhood of E_0*

$$||\psi(t) - b_t^w(x,p)\psi(t)|| \to 0 \text{ for } t \to \infty. \tag{4.16}$$

Using the symbol $b_t(x,\xi)$ we can bound powers of γ, cf. (2.17). If we define $\gamma = \gamma(x,\xi)$ as in (2.3) upon substituting E by the symbols $h(r_0\hat x,\xi)$ we may consider the symbol

$$\gamma_t^\alpha(x,\xi) := \gamma^\alpha(x,\xi)b_t(x,\xi); \quad \alpha \in (\mathbf{N}\cup\{0\})^{2n-2}. \tag{4.17}$$

We have the bounds

$$||(\gamma_t^\alpha)^w(x,p)|| = O\big(t^{-\delta|\alpha|}\big). \tag{4.18}$$

5. Γ and its localization

With the assumption (2.8) we define operators G and Γ as follows: The right hand side of (2.18) is of the form

$$\gamma^{(m_0)} = \gamma_1 + \sum_{2\le|\alpha|\le m_0} c_\alpha\gamma^\alpha,$$

here c_α and γ^α depend smoothly of E. As done in (4.17) we substitute

$$E = h(r_0\hat x,\xi) \tag{5.1}$$

and multiply suitably by the factors $\tilde{\tilde g}_1(t^{-1}x)$ and $\tilde{\tilde g}_1(\xi)$ as introduced in Section 3 (with small supports). Precisely we pick $l \le n-1$ such that (2.8) holds and write

$$\gamma_1 = c_l(\xi - \xi(E_0))\cdot\omega_l(E_0) + r_E(x,\xi); \quad c_l = \partial_{\nu_l}\gamma_1(w,E_0)_{|w=0}.$$

Then we define the operator $G = G_t = \gamma_t^w(x,p)$ by the symbol

$$\begin{aligned}
&\gamma_t(x,\xi) = \gamma^1(x,\xi) + \gamma_t^2(x,\xi);\\
&\gamma^1(x,\xi) = (\xi - \xi(E_0))\cdot\omega_l(E_0),\\
&\gamma_t^2(x,\xi) = (c_l)^{-1}\Bigg(r_E(x,\xi) + \sum_{2\le|\alpha|\le m_0} c_\alpha\gamma^\alpha(x,\xi)\Bigg)\tilde{\tilde g}_1\big(t^{-1}x\big)\tilde{\tilde g}_2(\xi).
\end{aligned} \tag{5.2}$$

For the second term the substitution (5.1) is used. Let $\Gamma = \Gamma_t = \mathrm{Re}\,(G)$.

Clearly the quantization of this second term $B_1(t) = (\gamma_t^2)^w(x,p)$ is bounded.

We shall assume that

$$\delta(m_0+1)\geq 1, \tag{5.3}$$

where $\delta<2^{-1}\min(\nu,2\delta_s)$ is given as in Proposition 4.6.

We shall use the operator $L_1(t)$ given in (4.5). Let us introduce the notation $L_2(t)=b_t^w(x,p)$ for the quantization of the first symbol of (4.14). Let us also introduce the "bigger" localization operator

$$\begin{aligned}&L_3(t)=\Big(\tilde b_t\Big)^w(x,p);\\&\tilde b_t(x,\xi)=F_-\big(2^{-1}t^{2\delta}q^s(x,\xi)\big)F_-\big(2^{-1}t^{\nu}q^-(x,\xi)\big)\tilde g_1\big(t^{-1}x\big)\tilde g_2(\xi).\end{aligned}$$

Notice that also

$$\tilde b_t(x,\xi)\in S_{unif}\Big(1,g_t^{1-\nu',\nu'}\Big);\quad \nu'=\nu-\delta,$$

and that indeed for example

$$(I-L_3(t))L_2(t)L_1(t)=O\big(t^{-\infty}\big). \tag{5.4}$$

We obtain from (2.17), (5.3) and bounds like (4.18) that

$$L_3i[H,G]L_3=-L_3\tilde t^{-1}GL_3+O\big(t^{-2}\big), \tag{5.5}$$

where t is omitted in the notation and $\tilde t^{-1}$ is the Weyl quantization of the symbol

$$-\frac{\partial_\mu h(x,\xi)}{x\cdot\omega(h(x,\xi))}\beta_1^s(h(x,\xi))\tilde{\tilde g}_1\big(t^{-1}x\big)\tilde{\tilde g}_2(\xi).$$

We may assume that the supports of $\tilde{\tilde g}_1$ and $\tilde{\tilde g}_1$ are so small that

$$\operatorname{Re}\big(\tilde t^{-1}\big)\geq t^{-1}\operatorname{Re}\big(\tilde{\tilde g}_1\big(t^{-1}x\big)\tilde{\tilde g}_2(p)\big)+O\big(t^{-2}\big). \tag{5.6}$$

We have the following localization result.

Proposition 5.1. *Let ψ,ν and δ be given as in Proposition 4.6 and suppose (5.3). Then for all $\sigma\in(\nu',1-\nu')$, $\nu'=\nu-\delta$, and with $P=P_t=GG^*+G^*G$ where $G=G_t$ is given as above*

$$\big\|F_+\big(t^{2-2\sigma}P\big)\psi(t)\big\|\to 0 \text{ for } t\to\infty. \tag{5.7}$$

Proof. Using the calculus we compute (with some patience)

$$\begin{aligned}L_3i[H,P]L_3&=2\operatorname{Re}\left(L_3i[H,G]L_3G^*+G^*L_3i[H,G]L_3\right)\\&\quad+\operatorname{Re}\left(L_3i[H,G][G^*,L_3]-[G^*,L_3]i[H,G]L_3\right)\\&\quad+\operatorname{Re}\left(L_3i[H,G^*][G,L_3]-[G,L_3]i[H,G^*]L_3\right)\\&=2\operatorname{Re}\left(L_3i[H,G]L_3G^*+G^*L_3i[H,G]L_3\right)+c_t^w(x,p)+O\big(t^{2\nu'-3}\big),\end{aligned}$$

where

$$c_t(x,\xi)=c_t=2\operatorname{Re}\Big(\big\{\tilde b_t,\big\{\tilde b_t,\overline{\tilde\gamma_t}\big\}\big\}\{h,\tilde\gamma_t\}\Big)\in S_{unf}\big(t^{3\nu'-3},g_t^{1-\nu',\nu'}\big).$$

Applying (5.5) to the first two terms on the right hand side and symmetrizing yields

$$L_3 i[H,P]L_3 = -L_3\big\{P\,\mathrm{Re}\,\big(\tilde{t}^{-1}\big) + h.c.\big\}L_3 \\ + \mathrm{Re}\,\big(GO\big(t^{-2}\big) + G^*O\big(t^{-2}\big)\big) + O\Big(t^{2\nu'-3}\Big). \quad (5.8)$$

Notice that the contribution from $c_t^w(x,p)$ disappears and that we use

$$P\,\mathrm{Re}\,\big(\tilde{t}^{-1}\big) + h.c. = 2G\,\mathrm{Re}\,\big(\tilde{t}^{-1}\big)G^* + 2G^*\,\mathrm{Re}\,\big(\tilde{t}^{-1}\big)G + O\big(t^{-3}\big). \quad (5.9)$$

We shall use the scheme of the proof of Lemma 4.4. Consider with $\kappa = t^\epsilon$ for a small $\epsilon > 0$ the observable

$$A(t,\kappa) = L_1(t)^* F_+(B(t))L_2(t)^2 F_+(B(t))L_1(t);$$
$$B(t) = B(t;\kappa) = \bar{G}\bar{G}^* + \bar{G}^*\bar{G}, \quad \bar{G} = \bar{G}(t;\kappa) = \kappa^{-1}t^{1-\sigma}G_t.$$

As before we first compute the Heisenberg derivative treating κ as a parameter and split (with $L_j = L_j(t)$)

$$\mathbf{D}A(t,\kappa) = T_1(t,\kappa) + T_2(t,\kappa) + T_3(t,\kappa);$$
$$T_1 = L_1^*F_+(B(t))L_2^2(\mathbf{D}F_+(B(t)))L_1 + h.c.,$$
$$T_2 = L_1^*F_+(B(t))\big(\mathbf{D}L_2^2\big)L_2F_+(B(t))L_1,$$
$$T_3 = L_1^*F_+(B(t))L_2^2F_+(B(t))\mathbf{D}L_1 + h.c.$$

The analogue of (4.11) is

$$\langle A(t,\kappa)\rangle_t = \langle A(t_0,\kappa)\rangle_{t_0} + \int_{t_0}^t \langle T_1(s,\kappa) + T_2(s,\kappa) + T_3(s,\kappa)\rangle_s ds. \quad (5.10)$$

We shall prove that

$$\limsup_{t_0\to\infty}\sup_{t\geq t_0}\int_{t_0}^t \langle T_i(s,\kappa)\rangle_s ds \leq 0; \quad i = 1,2,3. \quad (5.11)$$

To do this we may replace $\mathbf{D}$ by the modified Heisenberg derivative

$$\mathbf{D}_3 = \frac{d}{dt} + i\big[\bar{H},\cdot\big]; \quad \bar{H} = L_3HL_3, \quad L_3 = L_3(t),$$

cf. (5.4) and arguments below for (5.17).

With this modification we first look at the most interesting bound (5.11) with $i=1$. We use (3.16) to write

$$\mathbf{D}_3F_+(B(t)) = F_+'^{\frac{1}{2}}(B(t))D(t)F_+'^{\frac{1}{2}}(B(t)) + R_1(t) + R_2(t) + R_3(t);$$
$$D(t) = \frac{2-2\sigma}{t}B(t) - L_3\big\{B(t)\,\mathrm{Re}\,\big(\tilde{t}^{-1}\big) + h.c.\big\}L_3. \quad (5.12)$$

Notice that here $R_3(t)$ is given by the integral representation (3.11) of Lemma 3.5 in terms of the bounded operator $D_r(t) = \mathbf{D}_3B(t) - D(t)$

which by (5.8) is of the form

$$\begin{aligned} D_r(t) &= \kappa^{-2}t^{2-2\sigma}\frac{d}{dt}P + \bigl\{\kappa^{-2}t^{2-2\sigma}L_3Hi[L_3,P] + h.c.\bigr\} \\ &+ \kappa^{-2}t^{2-2\sigma}\Bigl\{\operatorname{Re}\bigl(GO\bigl(t^{-2}\bigr)\bigr) + \operatorname{Re}\bigl(G^*O\bigl(t^{-2}\bigr)\bigr) + O\Bigl(t^{2\nu'-3}\Bigr)\Bigr\}. \end{aligned} \tag{5.13}$$

First we examine the contribution from the expectation of the term

$$\cdots L_2(s)^2\{R_1(s) + R_2(s)\}L_1(s) + h.c.$$

of the integrand of (5.11) (after substituting (5.12)). We may write, omitting here and henceforth the argument s,

$$\begin{aligned} i[D,B] &= -i\bigl[L_3\bigl\{B\operatorname{Re}\bigl(\tilde{t}^{-1}\bigr) + h.c.\bigr\}L_3, B\bigr] \\ &= -\bigl(L_3\bigl\{B\operatorname{Re}\bigl(\tilde{t}^{-1}\bigr) + h.c.\bigr\}i[L_3,B] + h.c.\bigr) \\ &- L_3\bigl\{B\operatorname{Re}\bigl(i\bigl[\tilde{t}^{-1},B\bigr]\bigr) + h.c.\bigr\}L_3. \end{aligned} \tag{5.14}$$

Substituted into the representation formulas (3.13) and (3.14) of Lemma 3.5 the first term to the right can be shown to contribute by terms of the form $\kappa^{-2}O(s^{-\infty})$ (using the factors of L_1 and L_2 and commutation), however the bound $\kappa^{-1}O\bigl(s^{\nu'-1-\sigma}\bigr)$ suffices. Here and henceforth $O\bigl(s^{-\tilde{\epsilon}}\bigr)$ refers to a term bounded by $Cs^{-\tilde{\epsilon}}$ uniformly in t (recall that B contains a factor $\kappa^{-2} = t^{-2\epsilon}$). To demonstrate this weaker bound we compute

$$\begin{aligned} &i[L_3,B] = \kappa^{-1}s^{1-\sigma}i[L_3,G]\bar{G}^* + \kappa^{-1}s^{1-\sigma}\bar{G}^*i[L_3,G] + h.c., \\ &i[L_3,G] = O\Bigl(s^{\nu'-1}\Bigr). \end{aligned}$$

Since the middle factor $\operatorname{Re}\bigl(\tilde{t}^{-1}\bigr) = O(s^{-1})$ we get the bound

$$\kappa^{-1}O\bigl(s^{1-\sigma}\bigr)O\bigl(s^{\nu'-2}\bigr) = \kappa^{-1}O\bigl(s^{\nu'-1-\sigma}\bigr).$$

We used that $\bar{G}$, $\bar{G}^*$ and B may be considered as bounded in combination with the resolvents of B; explicitly we exploited the uniform bounds (after commutation)

$$\begin{aligned} &||\bar{G}(B-z)^{-1}||, \quad ||\bar{G}^*(B-z)^{-1}|| \le C\frac{\langle z\rangle^{1/2}}{|\operatorname{Im} z|}, \\ &||(B-z)^{-1}|| \le C|\operatorname{Im} z|^{-1}, \quad ||B(B-z)^{-1}|| \le C. \end{aligned} \tag{5.15}$$

Similarly, since

$$\operatorname{Re}\bigl(i\bigl[\tilde{t}^{-1},B\bigr]\bigr) = \kappa^{-1}O\bigl(s^{-1-\sigma}\bigr)\bar{G}^* + \kappa^{-1}O\bigl(s^{-1-\sigma}\bigr)\bar{G} + h.c. \tag{5.16}$$

the second term to the right in (5.14) contributes by a term of the form $\kappa^{-1}O(s^{-1-\sigma})$.

Using the representation for $R_3 = R_3(s)$ and commutation we claim the bound

$$\cdots L_2^2 R_3 L_1 + h.c.$$
$$= \kappa^{-1} O\big(s^{-1}\big) + \kappa^{-1} O\big(s^{-1-\sigma}\big) + \kappa^{-2} O\Big(s^{2\nu'-1-2\sigma}\Big). \quad (5.17)$$

The contributions from the first two terms of (5.13) are $\kappa^{-2}O(s^{-\infty})$ and therefore in particular $\kappa^{-1}O(s^{-1})$. Let us elaborate on this weaker bound for the first term: Write

$$\kappa^{-2} s^{2-2\sigma} \frac{d}{ds} P = \kappa^{-1} s^{1-\sigma} \Big\{ \bar{G} \frac{d}{ds} G^* + \bar{G}^* \frac{d}{ds} G + h.c. \Big\},$$

and compute the time-derivative of the symbol $\tilde{\tilde{g}}_1(s^{-1}x)$ that defines the time-dependence of the symbol of G

$$\frac{d}{ds} \tilde{\tilde{g}}_1\big(s^{-1}x\big) = -s^{-2} x \cdot \big(\nabla \tilde{\tilde{g}}_1\big)\big(s^{-1}x\big).$$

The contribution from this expression is treated by using the factor $g_1(s^{-1}x)$ of L_1. First we may insert the j'th power of $F = \tilde{g}_1(s^{-1}x)$ next to a factor L_1. Then we place one factor of F next to any of the factors of the time-derivative of G by commuting through the resolvent of B, and repeat successively this procedure for the "errors" given in terms of intermediary commutators. At each step a factor of $\kappa^{-1}s^{\nu'-\sigma} = O\big(s^{\nu'-\sigma}\big)$ will be gained. (In fact for the first term of (5.13) treated here we have the stronger estimate $O\big(s^{-\sigma}\big)$.) This means that if we put $\sigma' = \sigma - \nu'$ then $h = s^{-\sigma'}$ will be an "effective Planck constant". Notice that

$$i\left[(B-z)^{-1}, F\right]$$
$$= \kappa^{-1} s^{1-\sigma} (B-z)^{-1} \Big\{ \bar{G} O\Big(s^{\nu'-1}\Big) + \bar{G}^* O\Big(s^{\nu'-1}\Big) + h.c. \Big\} (B-z)^{-1}.$$

Repeated commutation through such an expression by factors of F provides eventually the power $h^j = s^{-\sigma' j}$. Again the finite numbers of factors like $\bar{G}(B-z)^{-1}$ and $\bar{G}^*(B-z)^{-1}$ may be estimated by (5.15) before integrating with respect to the z–variable. We choose j so large that $\sigma'(j+1) \geq 1$.

The contribution to (5.17) from the second term of (5.13) may be treated very similarly.

Clearly the last term of (5.13) contributes by terms of the form of the last two terms to the right in (5.17).

Next we move the factors of L_2 next to those of L_1 (and other commutation) for the contribution to (5.11) from the first term to the right in (5.12) yielding, as a conclusion, that

$$\begin{aligned} &\langle T_1(s,\kappa)\rangle_s \leq \Big\langle \breve{\psi}, D(s)\breve{\psi} \Big\rangle + \kappa^{-1} O\big(s^{-1}\big) + O\Big(s^{\nu'-2}\Big); \\ &\breve{\psi} = \big(F_+^{2\prime}\big)^{\frac{1}{2}} (B(s)) L_2(s) L_1(s) \psi(s). \end{aligned} \quad (5.18)$$

Notice that commutation of $D(s)$ with the factors of $L_2(s)$, $F_+'^{\frac12}(B(s))$ and $\big(F_+^{2\prime}\big)^{\frac12}(B(s))$ (when symmetrizing) involves the calculus of Lemma 3.5 and the effective Planck constant $h = s^{-\sigma'}$ in a similar fashion as above.

For the first term on the right hand side of (5.18) we infer from (5.6) and (5.9) that

$$\Big\langle \breve\psi, D(s)\breve\psi \Big\rangle \leq C_1\kappa^{-2}s^{-1-2\sigma} + C_2 s^{-2}. \tag{5.19}$$

By combining (5.18) and (5.19) we finally conclude (5.11) for $i=1$.

As for (5.11) for $i=2$ we use Remark 4.3, the integral estimate of Lemma 4.5 and the factors of L_1. Notice that the leading (classical) term from differentiating the symbol b_t may be written as a sum of three terms: The contribution from "differentiating" the factor $F_-(t^\nu q^-(x,\xi))$ is non-positive, cf. Remark 4.3. The contribution from "differentiating" the first factor $F_-\big(t^{2\delta}q^s(x,\xi)\big)$ may after a symmetrization be treated by Lemma 4.5. The commutation through the factors of $F_+(B(s))$ (when symmetrizing) involves the calculus of Lemma 3.5 in a similar fashion as above. Finally the contribution from "differentiating" the last two factors are integrable due to the factors of L_1. We omit further details.

As for (5.11) for $i=3$ we use the integral estimate (1.9) and commutation. We omit the details.

We conclude (5.11), and therefore by Proposition 4.6 the bound (5.7) first with σ replaced by $\sigma+\epsilon$ and then (since ϵ is arbitrary) by any σ as specified in the proposition. □

Corollary 5.2. *Under the conditions of Proposition 5.1 and with* $\Gamma = \Gamma_t = \mathrm{Re}\,(G)$

$$\big\|F_+\big(t^{1-\sigma}|\Gamma|\big)\psi(t)\big\| \to 0 \textit{ for } t\to\infty. \tag{5.20}$$

Proof. Let $\sigma\in(2\nu',1)$ be given. Fix $\sigma_1\in(2\nu',\sigma)$. By Proposition 5.1 it suffices to show that

$$\big\|F_+\big(t^{1-\sigma}|\Gamma|\big)F_-\big(t^{2-2\sigma_1}P\big)\big\| = O\big(t^{\sigma_1-\sigma}\big).$$

Clearly by the spectral theorem this estimate follows from

$$\big\|t^{1-\sigma_1}\Gamma F_-\big(t^{2-2\sigma_1}P\big)\big\| \leq 1,$$

which in turn follows from substituting $\Gamma = 2^{-1}(G+G^*)$ and then estimating

$$\begin{aligned}
&\big\|t^{1-\sigma_1}\Gamma F_-\big(t^{2-2\sigma_1}P\big)\big\|\\
&\leq 2^{-1}\big\|t^{1-\sigma_1}GF_-(\cdot)\big\| + 2^{-1}\big\|t^{1-\sigma_1}G^*F_-(\cdot)\big\|\\
&\leq 2^{-1}\big\|t^{2-2\sigma_1}F_-(\cdot)G^*GF_-(\cdot)\big\|^{1/2} + 2^{-1}\big\|t^{2-2\sigma_1}F_-(\cdot)GG^*F_-(\cdot)\big\|^{1/2}\\
&\leq \big\|F_-(\cdot)t^{2-2\sigma_1}PF_-(\cdot)\big\|^{1/2} \leq 1.
\end{aligned}$$

□

Remark 5.3. In the case of (2.9) we define Γ as follows: We pick $l \leq n-1$ such that (2.9) holds and write

$$\gamma_1 = c_l \frac{x}{\tilde{x}_n} \cdot \omega_l(E_0) + r_{t,E}(x,\xi);$$
$$c_l = \partial_{u_l}\gamma_1(w,E_0)_{|w=0}, \quad \tilde{x}_n = tk(E_0).$$

The operator $G = G_t = \gamma_t^w(x,p)$ is given by the symbol (using the substitution (5.1))

$$\begin{aligned} &\gamma_t(x,\xi) = \gamma_t^1(x,\xi) + \gamma_t^2(x,\xi); \qquad (5.21)\\ &\gamma_t^1(x,\xi) = t^{-1}x \cdot \omega_l(E_0),\\ &\gamma_t^2(x,\xi) = \frac{k(E_0)}{c_l}\Big(r_{t,E}(x,\xi) + \sum_{2\leq|\alpha|\leq m_0} c_\alpha \gamma^\alpha(x,\xi)\Big)\tilde{\tilde{g}}_1\big(t^{-1}x\big)\tilde{\tilde{g}}_2(\xi), \end{aligned}$$

cf. (5.2). One proves Proposition 5.1 with this G in the same way as before. Let $\Gamma = \mathrm{Re}\,(G)$. We have (5.20) for this Γ.

6. Mourre theory for Γ

Let Γ be as in Section 5 (assuming first (2.8)). The m_0 of (5.2) is here considered as arbitrary (but fixed); the condition (5.3) (needed before for dynamical statements) is not imposed.

We introduce for $0 < \bar{\delta} \leq 1$ the operators

$$\begin{aligned} &\bar{H} = t^{1-\bar{\delta}}\Gamma, \quad \bar{A} = \bar{a}_t^w(x,p); \qquad (6.1)\\ &\bar{a}_t(x,\xi) = t^{\bar{\delta}-1}\big(x\cdot\omega_l(E_0) + x\cdot\big(\omega_l(h(x,\xi)) - \omega_l(E_0)\big)\tilde{\tilde{g}}_1\big(t^{-1}x\big)\tilde{\tilde{g}}_2(\xi)\big). \end{aligned}$$

We shall need a specific construction of the functions $\tilde{\tilde{g}}_1$ and $\tilde{\tilde{g}}_2$ in the definition of $\bar{H}$ and $\bar{A}$ above in terms of a small parameter $\epsilon > 0$:

The factor $\tilde{\tilde{g}}_1(t^{-1}x)$ is the product of the n functions

$$\begin{aligned} &F_-\big(\epsilon^{-3}|t^{-1}x\cdot\omega_j(E_0)|\big); \quad j = 1,\dots,n-1,\\ &F_-\big(\epsilon^{-2}|t^{-1}x\cdot\omega_n(E_0) - k(E_0)|\big). \end{aligned} \qquad (6.2)$$

The factor $\tilde{\tilde{g}}_2(\xi)$ is the product of the n functions

$$\begin{aligned} &F_-\big(\epsilon^{-2}|(\xi-\xi(E_0))\cdot\omega_l(E_0)|\big),\\ &F_-\big(\epsilon^{-3}|(\xi-\xi(E_0))\cdot\omega_j(E_0)|\big); \quad j = 1,\dots,n-1, \quad j \neq l, \qquad (6.3)\\ &F_-\big(\epsilon^{-4}|(\xi-\xi(E_0))\cdot\omega_n(E_0)|\big). \end{aligned}$$

We have the following result (with $\langle\bar{A}\rangle = \big(1+\bar{A}^2\big)^{1/2}$).

Lemma 6.1. *There exists $\epsilon_0 > 0$ such that for all positive $\epsilon \leq \epsilon_0$ there exist constants $t_0, C > 1$ such that for all $t \geq t_0$ and $h \in C_0^\infty(\mathbf{R})$*

$$\big\|\langle\bar{A}\rangle^{-1}h\big(\bar{H}\big)\langle\bar{A}\rangle^{-1}\big\| \leq C\|h\|_{L^1}. \qquad (6.4)$$

Proof. We shall use the abstract theory of [M] with the conjugate operator $\bar{A}$ to obtain a globally uniform resolvent bound.

We claim that for all small enough ϵ

$$i\big[\bar{H},\bar{A}\big]\geq 2^{-1};\quad t\geq t_0=t_0(\epsilon). \tag{6.5}$$

To see this we notice that clearly the first term in (5.2) and the first term of the symbol $\bar{a}$ contribute by

$$i\Big[t^{1-\bar{\delta}}\big(\gamma^1\big)^w(x,p),t^{\bar{\delta}-1}x\cdot\omega_l(E_0)\Big]=1,$$

so it remains to estimate

$$\Big\|i\Big[t^{1-\bar{\delta}}\big(\mathrm{Re}\,\big(\gamma_t^2\big)\big)^w(x,p),\bar{A}\Big]\Big\|\leq 4^{-1};\quad t\geq t_0, \tag{6.6}$$

and

$$\Big\|i\Big[t^{1-\bar{\delta}}\big(\gamma^1\big)^w(x,p),\bar{A}-t^{\bar{\delta}-1}x\cdot\omega_l(E_0)\Big]\Big\|\leq 4^{-1};\quad t\geq t_0. \tag{6.7}$$

Let us denote by $a_t(x,\xi)$ the Weyl symbol of the operator in (6.6) or the one in (6.7). We have in both cases that $a_t\in S_{unif}\big(1,g_t^{1,0}\big)$, so it suffices to show (cf. [Hö, Theorem 18.6.3] and the proof of [DG, Proposition D.5.1]) that

$$\sup_{x,\xi\in\mathbf{R}^n,t\geq t_0}|a_t(x,\xi)|\leq\nu_0, \tag{6.8}$$

where ν_0 is a (universal) small positive constant associated for example to the L^2 –boundedness result [Hö, Theorem 18.6.3].

For (6.8) we note the uniform bounds

$$\begin{aligned}
&h(x,\xi)-E_0=O\big(\epsilon^4\big),\\
&t\partial_{x_j}h(x,\xi)=O\big(\epsilon^2\big),\\
&\partial_{\xi_j}h(x,\xi)=O\big(\epsilon^2\big)\text{ for }j\leq n-1,\quad \partial_{\xi_n}h(x,\xi)=O\big(\epsilon^0\big),\\
&\gamma_j(x,\xi)=O\big(\epsilon^2\big),\quad t\partial_x\gamma_j(x,\xi)=O\big(\epsilon^0\big),\quad \partial_\xi\gamma_j(x,\xi)=O\big(\epsilon^0\big),
\end{aligned}$$

on the support of the function $\tilde{\tilde{g}}_1(t^{-1}x)\tilde{\tilde{g}}_2(\xi)$ given by (6.2) and (6.3). Here we used (1.3) and (1.4), and the notation

$$x_j=x\cdot\omega_j(E_0),\quad \xi_j=\xi\cdot\omega_j(E_0).$$

By estimating the leading term of the symbol using these bounds we may show (with some patience) that

$$\sup_{x,\xi\in\mathbf{R}^n,t\geq t_0}|a_t(x,\xi)|\leq C\epsilon, \tag{6.9}$$

from which (6.8) and (therefore) (6.5) follow.

As for the boundedness of second commutators required by the Mourre theory we have the bound

$$\big\|i\big[i\big[\bar{H},\bar{A}\big],\bar{A}\big]\big\|=O\Big(t^{\bar{\delta}-1}\Big)=O(1). \tag{6.10}$$

Using (6.5) and (6.10) we readily obtain by keeping track of constants in the method of [M] that for some positive constant C

$$\left\| \langle \bar{A} \rangle^{-1} \big(\bar{H} - z\big)^{-1} \langle \bar{A} \rangle^{-1} \right\| \leq C; \quad \operatorname{Im} z \neq 0, \quad t \geq t_0. \tag{6.11}$$

Representing $h\big(\bar{H}\big) = \pi^{-1} \lim_{\epsilon \downarrow 0} \int h(\lambda) \operatorname{Im} \Big(\big(\bar{H} - \lambda - i\epsilon\big)^{-1}\Big) d\lambda$ and then using (6.11) we conclude (6.4). □

Remark. Although stated for concrete operators $\bar{H}$ and $\bar{A}$ clearly there is an abstract version of Lemma 6.1; the important properties are (6.5) and (6.10).

Corollary 6.2. *Suppose $h_1, h_2 \in C_0^\infty(\mathbf{R})$ and $0 \leq \sigma < \bar{\delta} \leq 1$. Then there exists $\epsilon_0 > 0$ such that for all positive $\epsilon \leq \epsilon_0$ there exists $C > 0$ such that for all $t \geq 1$*

$$\left\| h_1\big(\bar{A}\big) h_2\Big(t^{\bar{\delta}-\sigma} \bar{H}\Big) \right\| \leq C t^{(\sigma - \bar{\delta})/2}. \tag{6.12}$$

Remark 6.3. In the case of (2.9) we introduce (with Γ as in Remark 5.3)

$$\begin{aligned} &\bar{H} = t^{1-\bar{\delta}}\Gamma, \quad \bar{A} = \bar{a}_t^w(x,p); \\ &\bar{a}_t(x,\xi) = t^{\bar{\delta}}\big((p - \xi(E_0)) \cdot \omega_l(E_0) + b(x,\xi)\tilde{\tilde{g}}_1\big(t^{-1}x\big)\tilde{\tilde{g}}_2(\xi)\big), \\ &b(x,\xi) = (p - \xi(h(x,\xi))) \cdot \omega_l(h(x,\xi)) - (p - \xi(E_0)) \cdot \omega_l(E_0). \end{aligned} \tag{6.13}$$

Here the factor $\tilde{\tilde{g}}_1(t^{-1}x)$ is the product of the n functions

$$\begin{aligned} &F_-\big(\epsilon^{-2}|t^{-1}x \cdot \omega_l(E_0)|\big), \\ &F_-\big(\epsilon^{-3}|t^{-1}x \cdot \omega_j(E_0)|\big); \quad j = 1, \cdots, n-1, \quad j \neq l, \\ &F_-\big(\epsilon^{-2}|t^{-1}x \cdot \omega_n(E_0) - k(E_0)|\big), \end{aligned}$$

while the factor $\tilde{\tilde{g}}_2(\xi)$ is the product of

$$\begin{aligned} &F_-\big(\epsilon^{-3}|(\xi - \xi(E_0)) \cdot \omega_j(E_0)|\big); \quad j = 1, \cdots, n-1, \\ &F_-\big(\epsilon^{-4}|(\xi - \xi(E_0)) \cdot \omega_n(E_0)|\big). \end{aligned}$$

One verifies (6.12) under the same conditions as in Corollary 6.2 along the same line as before.

7. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on Proposition 4.6, and Corollaries 5.2 and 6.2 (with the assumption (2.8)).

We recall the assumptions of Proposition 4.6: $0 < 2\delta < \min(\nu, 2\delta^s)$ with $\nu < 2/5$ and δ^s as in (2.14).

Lemma 7.1. *With $\bar{A} = \bar{A}_t$ given in terms of any (small) $\epsilon > 0$ and of $\bar{\delta} = \delta$ (with δ as above) by either (6.1) (in the case of (2.8)) or (6.13) (in the case of (2.9))*

$$\lim_{t \to \infty} ||F_+\big(|\bar{A}|\big)\psi(t)|| = 0, \tag{7.1}$$

where $\psi = f(H)\psi$ is given as in Proposition 4.6 (with f strongly supported depending on ϵ).

Proof. We fix δ_1 such that $2\delta < 2\delta_1 < \min(\nu, 2\delta^s)$. Let $b_t(x,\xi)$ be given by (4.14) in terms of δ_1 and ν.

By Proposition 4.6 it suffices to show that

$$||F_+\big(|\bar{A}|\big)b_t^w(x,p)|| \to 0 \text{ for } t \to \infty,$$

and therefore in turn

$$||\bar{A}b_t^w(x,p)|| = O\big(t^{\delta-\delta_1}\big).$$

For the latter bound one easily check that the symbol of $\bar{A}b_t^w(x,p)$ belongs to

$$S_{unf}\Big(t^{\delta-\delta_1}, g_t^{1-\nu',\nu'}\Big); \quad \nu' = \nu - \delta_1.$$

□

Now, we first fix δ as above and conclude from Lemma 7.1 that

$$||\psi(t) - F_-\big(|\bar{A}|\big)\psi(t)|| \to 0 \text{ for } t \to \infty, \tag{7.2}$$

where $\psi = f(H)\psi$ is given as in Proposition 4.6. This holds for $f \in C_0^\infty(I_0)$; $I_0 = I_0(\epsilon)$.

Next we fix any $\sigma \in (0,\delta)$ in agreement with Corollary 5.2 which means that

$$||F_+\big(|t^{1-\sigma}\Gamma|\big)\psi(t)|| \to 0 \text{ for } t \to \infty. \tag{7.3}$$

Here the input of δ in Proposition 5.1 say δ_1 (needed to fix the m_0 in the definition of the Γ of Corollary 5.2) is different; we need to have $\sigma > \nu'$, $\nu' = \nu_1 - \delta_1$, for which $\delta_1 < \delta$ is needed. The construction of this Γ depends on the same ϵ as above, cf. Section 6.

Combining (7.2) and (7.3) leads to

$$||\psi(t) - F_-\big(|\bar{A}|\big)F_-\big(|t^{1-\sigma}\Gamma|\big)\psi(t)|| \to 0 \text{ for } t \to \infty. \tag{7.4}$$

By combining Corollary 6.2 and (7.4) we conclude (by finally fixing $\epsilon > 0$ sufficiently small) that

$$||\psi(t)|| = 0 \text{ for } t \to \infty, \tag{7.5}$$

and therefore that $\psi = 0$ proving Theorem 1.1.

Remark 7.2. With the assumption (2.9) we proceed similarly using Remarks 5.3 and 6.3, and Lemma 7.1.

8. Proof of Theorem 1.2

We shall here elaborate on the derivation of Theorem 1.2 from our general result Theorem 1.1.

First we remove the singularity at $x = 0$ by defining

$$h(x,\xi) = 2^{-1}\xi^2 + \tilde{V}(x); \quad \tilde{V}(x) = F_+(|x|)V(\hat{x}),$$

where (as before) V is a Morse function on S^{n-1}. (See Remarks 8.3 for extensions.) In this case clearly the hypotheses (H1)–(H3) of Section 2 are satisfied, and (H4) holds for any critical point $\omega_l \in C_r$ and energy $E > V(\omega_j)$ upon putting $\omega(E) = \omega_l$, $\xi(E) = k(E)\omega_l$ and $k(E) = \sqrt{2(E - V(\omega_l))}$.

For (1.6) we put

$$g(u, \eta, E) = \sqrt{2(E - V(\omega_l))} - \sqrt{2E - \eta^2 - 2V(\omega_l + u)},$$

yielding (1.7) with

$$A(E) = k(E)^{-1} \begin{pmatrix} V^{(2)}(\omega_l) & 0 \\ 0 & I \end{pmatrix}.$$

We may choose an orthonormal basis in $\{\omega_l\}^\perp \subseteq \mathbf{R}^n$ for which $V^{(2)}(\omega_l)$ is diagonal, say $V^{(2)}(\omega_l) = \operatorname{diag}(q_1, \dots, q_{n-1})$. The eigenvalues of $B(E)$ take the form

$$\begin{aligned} \beta_j^+(E) &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 2q_j/(E - V(\omega_l))} \text{ or} \\ \beta_j^-(E) &= -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 2q_j/(E - V(\omega_l))} \end{aligned} \tag{8.1}$$

say with $\sqrt{\zeta} := i\sqrt{-\zeta}$ if $\zeta < 0$.

Clearly the hypothesis (H5) is the non-degeneracy condition, $q_j \neq 0$ for all j, while hypothesis (H6) amounts to $q_j < 0$ for some j, i.e. ω_l is a local maximum or a saddle point of V.

As for (H7) one easily checks that there exists a smooth basis of eigenvectors of $B(E)^{tr}$ for $E - V(\omega_l) \in (0, \infty) \setminus \{2q_1, \dots, 2q_{n-1}\}$.

Elementary analyticity arguments show that given any $m \in \{2, 3, \dots\}$ the set of resonances of order m for any of the eigenvalues of $B(E)$ is discrete in $(V(\omega_l), \infty)$.

In conclusion, the hypotheses (H1)–(H8) are satisfied for any local maximum or saddle point ω_l of a Morse function V for $E_0 \in (V(\omega_l), \infty) \setminus \mathcal{D}$ where $\mathcal{D}$ is discrete in $(V(\omega_l), \infty)$.

Due to the possible existence of bound states we change the definition of P_l to be

$$P_l = s - \lim_{t \to \infty} e^{itH} \chi_l(\hat{x}) e^{-itH} E_{ac}(H),$$

where $E_{ac}(H)$ is the orthogonal projection onto the absolutely continuous subspace of H, see [H] and [ACH, Theorem C.1]. This gives (1.12) with the left hand side replaced by $E_{ac}(H)$.

Now, to get (1.14) it suffices by Theorem 1.1 to verify (1.13) for any $E_0 \in (V(\omega_l), \infty)$. Invoking the discreteness of the set of eigenvalues of H on the complement of the set of critical values of V, cf. [ACH, Theorem C.1], one may easily conclude (1.13) from the following statement:

Consider any open set $I_0 \subseteq (V(\omega_l), \infty)$ such that

$$I_0 \cap (\sigma_{pp}(H) \cup V(C_r)) = \emptyset.$$

Let $\mathcal{H}_0$ be the closure of the subspace of states $\psi = f(H)\psi$, $f \in C_0^\infty(I_0)$, obeying (1.8) and (1.9). Then for all $\psi = P_l f(H)\psi$ where $f \in C_0^\infty(I_0)$

$$\psi \in \mathcal{H}_0. \tag{8.2}$$

We shall verify (8.2) by showing that indeed $\psi = P_l f(H)\psi$ obeys (1.8) and (1.9). We shall proceed a little more generally than needed in that we here assume that the $\mathcal{U}_0$ of (1.9) is given by

$$\mathcal{U}_0 = \mathcal{U}_\epsilon = \tilde{\mathcal{C}}_\epsilon \times \mathbf{R}^n;$$

$$\tilde{\mathcal{C}}_\epsilon = \{x \in \mathbf{R}^n \setminus \{0\} | \hat{x} \in \mathcal{C}_\epsilon\}, \quad \mathcal{C}_\epsilon = \big\{\omega \in S^{n-1} \mid |\omega - \omega_l| < \epsilon\big\},$$

where $\epsilon > 0$ is taken so small that $\mathcal{C}_\epsilon \cap C_r = \{\omega_l\}$.

Pick $\tilde{f} \in C_0^\infty(I_0)$ such that $0 \le \tilde{f} \le 1$ and $\tilde{f} = 1$ in a neighborhood of supp (f). Let $r \in C^\infty(\mathbf{R}^n)$ be given in terms of any $F_+ \in \mathcal{F}_+$ by

$$r(x) = \int_0^{|x|} F_+(s)ds + \int_0^1 F_-(s)ds. \tag{8.3}$$

(Notice that $r(x) = |x|$ for $|x| \ge 1$.) Let

$$p_{||} = \frac{1}{2}(\nabla r \cdot p + h.c.), \quad \tilde{p}_{||} = \tilde{f}(H) p_{||} \tilde{f}(H).$$

Lemma 8.1. *Let $\chi_l \in C_0^\infty(\mathcal{C}_\epsilon)$ be given with $0 \le \chi_l \le 1$ and $\chi_l = 1$ in a neighborhood of ω_l, and $\tilde{g}_2 \in C_0^\infty(\mathbf{R})$ by*

$$\tilde{g}_2(s) = \tilde{f}\big(2^{-1}s^2 + V(\omega_l)\big) 1_{(0,\infty)}(s).$$

Let real-valued $g_1^-, g_1^+ \in C_0^\infty(\mathbf{R})$ be given with

$$c_+^- < \tilde{c}_-; \quad c_+^- = \sup\big(\mathrm{supp}\,(g_1^-)\big), \quad \tilde{c}_- = \inf(\mathrm{supp}\,(\tilde{g}_2)),$$

$$c_-^+ > \tilde{c}_+; \quad c_-^+ = \inf\big(\mathrm{supp}\,(g_1^+)\big), \quad \tilde{c}_+ = \sup(\mathrm{supp}\,(\tilde{g}_2)).$$

Let $F_+ \in \mathcal{F}_+$, $F_- \in \mathcal{F}_-$ and

$$C > 2\sqrt{2(\sup(\mathrm{supp}\,(f)) - \min(V))}.$$

Then, in the state $\psi(t) = e^{-itH} P_l f(H)\psi$

$$\int \big\langle r^{-1-\delta} \big\rangle_t dt < \infty; \quad \delta > 0, \tag{8.4}$$

$$\int |\langle p \cdot r^{(2)} p \rangle_t| dt < \infty, \tag{8.5}$$

$$\int \Big\langle r|\nabla \tilde{V}|^2 \Big\rangle_t dt < \infty, \tag{8.6}$$

$$\int \Big\langle \tilde{\chi}_l r^{-\frac{1}{2}} \big(\eta^2 + u^2\big) r^{-\frac{1}{2}} \tilde{\chi}_l \Big\rangle_t dt < \infty; \quad \tilde{\chi}_l = \chi_l(\hat{x}) F_+(r), \tag{8.7}$$

$$\int_1^\infty -t^{-1} \big\langle F_-'\big(C^{-1}t^{-1}r\big) \big\rangle_t dt < \infty, \tag{8.8}$$

$$\int_1^\infty t^{-1} ||g\big(\tilde{p}_{||}\big) F_-\big(C^{-1}t^{-1}r\big)\psi(t)||^2 dt < \infty; \tag{8.9}$$

$$g \in C_0^\infty((-\infty,0)), \quad \bar{g} = g,$$

$$\int_1^\infty t^{-1}||\big(1-\tilde{g}_2\big(\tilde{p}_{||}\big)\big)F_-\big(C^{-1}t^{-1}r\big)\tilde{\chi}_l\psi(t)||^2 dt < \infty, \tag{8.10}$$

$$\int_1^\infty t^{-1}||B^-(t)\psi(t)||^2 dt < \infty; \quad B^-(t) = g_1^-\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big), \tag{8.11}$$

$$\int_1^\infty t^{-1}||B^+(t)\psi(t)||^2 dt < \infty; \quad B^+(t) = g_1^+\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big). \tag{8.12}$$

Proof. For (8.4), (8.5) and (8.6) we refer to [H] and [ACH, Theorem C.1]. The bound (8.7) follows from those estimates by Taylor expansion.

As for (8.8) we consider the "propagation observable"

$$\Phi(t) = f(H)F_-\big(C^{-1}t^{-1}r\big)f(H).$$

We may bound its Heisenberg derivative as

$$\mathbf{D}\Phi(t) \geq -\epsilon t^{-1} f(H)F_-'\big(C^{-1}t^{-1}r\big)f(H) + O\big(t^{-2}\big); \quad \epsilon > 0.$$

As for (8.9) we consider the observable

$$\Phi(t) = \tilde{f}(H)g\big(\tilde{p}_{||}\big)t^{-1}rF_-\big(C^{-1}t^{-1}r\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H).$$

We write its Heisenberg derivative as

$$\mathbf{D}\Phi(t) = T_1 + T_2 + T_3;$$

$$T_1 = \tilde{f}(H)\big(\mathbf{D}g\big(\tilde{p}_{||}\big)\big)t^{-1}rF_-\big(C^{-1}t^{-1}r\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H) + h.c.,$$

$$T_2 = 2^{-1}\tilde{f}(H)g\big(\tilde{p}_{||}\big)t^{-1}r\big(\mathbf{D}F_-\big(C^{-1}t^{-1}r\big)\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H) + h.c.,$$

$$T_3 = 2^{-1}\tilde{f}(H)g\big(\tilde{p}_{||}\big)\big(\mathbf{D}\big(t^{-1}r\big)\big)F_-\big(C^{-1}t^{-1}r\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H) + h.c.,$$

and notice the identities

$$\mathbf{D}r = p_{||}, \quad \mathbf{D}p_{||} = p \cdot r^{(2)}p + O\big(r^{-3}\big). \tag{8.13}$$

Using (8.4), (8.5), the second identity of (8.13) and (3.11) we readily obtain after symmetrization that

$$\int_1^\infty |\langle T_1\rangle_t| dt < \infty. \tag{8.14}$$

As for the the term T_2 we use the first identity of (8.13) and (8.8) to derive

$$\int_1^\infty |\langle T_2\rangle_t| dt < \infty. \tag{8.15}$$

For the term T_3 we compute using the first identity of (8.13) and (3.11)

$$T_3 = \mathrm{Re}\Big(t^{-1}\tilde{f}(H)g\big(\tilde{p}_{||}\big)\big(p_{||} - t^{-1}r\big)F_-\big(C^{-1}t^{-1}r\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H)\Big) + O\big(t^{-2}\big)$$

$$\leq -\epsilon t^{-1}\tilde{f}(H)g\big(\tilde{p}_{||}\big)F_-\big(C^{-1}t^{-1}r\big)g\big(\tilde{p}_{||}\big)\tilde{f}(H) + O\big(t^{-2}\big); \epsilon > 0. \tag{8.16}$$

We conclude (8.9) from (8.14), (8.15) and (8.16).

The bound (8.10) follows from elementary energy bounds, Taylor expansion and the previous estimates. (For this we need (8.9) to deal with the "region" where $p_{||}^2$ energetically has the right size, but $p_{||} < 0$.)

As for (8.11) we consider

$$\Phi(t) = \tilde{f}(H)\tilde{g}_2\big(\tilde{p}_{||}\big)F\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H); \quad F(s') = \int_{-\infty}^{s'} g_1^-(s)^2 ds.$$

We write its Heisenberg derivative as

$$\begin{aligned}
&\mathbf{D}\Phi(t) = T_1 + T_2;\\
&T_1 = \tilde{f}(H)\big(\mathbf{D}\tilde{g}_2\big(\tilde{p}_{||}\big)\big)F\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H) + h.c.,\\
&T_2 = \tilde{f}(H)\tilde{g}_2\big(\tilde{p}_{||}\big)\big(\mathbf{D}F\big(t^{-1}r\big)\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H).
\end{aligned}$$

Using (8.4), (8.5), the second identity of (8.13) and (3.11) as for (8.9) we obtain that

$$\int_1^\infty |\langle T_1\rangle_t| dt < \infty. \tag{8.17}$$

As for the the term T_2 we compute using the first identity of (8.13) and (3.11)

$$\begin{aligned}
T_2 &= t^{-1}\tilde{f}(H)B^-(t)^*\big(p_{||} - t^{-1}r\big)B^-(t)\tilde{f}(H) + O\big(t^{-2}\big)\\
&\geq t^{-1}B^-(t)^*\Big(\tilde{p}_{||}1_{[\tilde{c}_-,\infty)}\big(\tilde{p}_{||}\big) - c_+^-\tilde{f}(H)^2\Big)B^-(t) + O\big(t^{-2}\big)\\
&\geq \epsilon t^{-1}B^-(t)^*B^-(t) + O\big(t^{-2}\big); \quad \epsilon = \tilde{c}_- - c_+^-.
\end{aligned} \tag{8.18}$$

Clearly (8.11) follows by combining (8.17) and (8.18).

As for (8.12) we may proceed similarly using

$$\Phi(t) = \tilde{f}(H)\tilde{g}_2\big(\tilde{p}_{||}\big)F\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H); \quad F(s') = \int_{-\infty}^{s'} g_1^+(s)^2 ds.$$

□

Corollary 8.2. *Let ψ, $\chi_l \in C_0^\infty(\mathcal{C}_\epsilon)$ and $\tilde{g}_2$ be given as in Lemma 8.1. Let $g_1 \in C_0^\infty(\mathbf{R})$ be given such that $0 \leq g_1 \leq 1$ and $g_1 = 1$ in an open interval containing* supp $(\tilde{g}_2)$. *Then*

$$||\psi(t) - g_1\big(t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\chi_l(\hat{x})\tilde{f}(H)\psi(t)|| \to 0 \textit{ for } t \to \infty. \tag{8.19}$$

Proof. From the very definition of ψ we have

$$||\psi(t) - \chi_l(\hat{x})\tilde{f}(H)\psi(t)|| \to 0 \text{ for } t \to \infty.$$

Next, from [H, Theorems 4.10 and 4.12] we learn that

$$||\psi(t) - \tilde{g}_2\big(\tilde{p}_{||}\big)\chi_l(\hat{x})\tilde{f}(H)\psi(t)|| \to 0 \text{ for } t \to \infty. \tag{8.20}$$

Whence to show (8.19) it suffices to verify that

$$||\big\{g_1\big(t^{-1}r\big) - g_1\big(\tilde{p}_{||}\big)\big\}\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H)\psi(t)|| \to 0 \text{ for } t \to \infty,$$

which in turn is reduced (by a standard density argument using that the energy bounds the momentum) to verifying that for all constants C large enough

$$||F_-\big(C^{-1}t^{-1}r\big)\big\{g_1\big(t^{-1}r\big) - g_1\big(\tilde{p}_{||}\big)\big\}\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H)\psi(t)|| \to 0 \text{ for } t \to \infty. \quad (8.21)$$

For (8.21) we consider the observable

$$\Phi_C(t) = \tilde{f}(H)\tilde{g}_2\big(\tilde{p}_{||}\big)F_-\big(C^{-1}t^{-1}r\big)\big(\tilde{p}_{||} - t^{-1}r\big)^2 F_-\big(C^{-1}t^{-1}r\big)\tilde{g}_2\big(\tilde{p}_{||}\big)\tilde{f}(H).$$

Using Lemma 8.1 as well as the proof of this lemma we easily show that

$$\left|\int_1^\infty |\frac{d}{dt}\langle\Phi_C(t)\rangle_t\right| dt, \quad \int_1^\infty t^{-1}\langle\Phi_C(t)\rangle_t dt < \infty,$$

from which we conclude that along some sequence $t_k \to \infty$ indeed $\langle\Phi_C(t_k)\rangle_{t_k} \to 0$, and then in turn that

$$\langle\Phi_C(t)\rangle_t \to 0. \quad (8.22)$$

We easily obtain (8.21) using (8.22), (3.10) and commutation. □

Now, one may easily verify (8.2) for $\psi = P_l f(H)\psi$ as follows: We introduce a partition $f = \sum f_i$ of sharply localized f_i's and for each of these a "slightly larger" $\tilde{f}_i$. Using these functions and the states $\psi_i = P_l f_i(H)\psi$ as input in Corollary 8.2 the bounds (1.8) follow from the conclusion of the corollary and [H, Theorems 4.10 and 4.12]. As for (1.9) we may use the same partition and then conclude the result from Lemma 8.1 (applied with $\tilde{f}$ replaced by $\tilde{f}_i$).

Remarks 8.3.

(1) Using the Mourre estimate [ACH, Theorem C.1] one may easily include a short-range perturbation $V_1 = O\big(|x|^{-1-\delta}\big)$, $\delta > 0$, $\partial_x^\alpha V_1 = O(|x|^{-2})$, $|\alpha| = 2$, to the Hamiltonian H. In particular Theorem 1.2 holds for the strictly homogeneous case as discussed in Section 1.

(2) The non-degeneracy condition at ω_l is important for the method of proof presented in this paper. However it is not important that the set of critical points C_r is finite; it suffices that ω_l is an isolated non-degenerate critical point and that $V(C_r)$ is countable.

(3) At a local maximum we proved a somewhat better result in [HS1] (by a different method): A larger class of perturbations was included and we imposed a somewhat weaker condition than the non-degeneracy condition. The method of [HS1] yielded only a limited result at saddle points. Although there are indications that this method of proof might be extended to included

Theorem 1.2 (by using a certain complicated iteration scheme) the proof presented in this paper is probably much simpler.

(4) The components of the γ of (2.3) may be taken of the form

$$\gamma_j = \eta_j + \sqrt{2(E - V(\omega_l))}\beta_j^{\#}(E)u_j,$$

where $\beta_j^{\#}(E)$ is given by one of the expressions of (8.1). In particular both of the conditions (2.8) and (2.9) are satisfied in the potential case.

(5) We applied the Sternberg linearization procedure in [HS2] to the equations (1.6) in the case of a local minimum. In this case the union of all resonances (of all orders and for all eigenvalues) is discrete on $(V(\omega_l), \infty)$. One needs to exclude this set of resonances to construct a smooth Sternberg diffeomorphism, see for example [N, Theorem 9]. The construction of the symbol $\gamma^{(m)}$ in (2.18) may be viewed as a rudiment of this procedure. However, the union of all resonances at a local maximum or a saddle point ω_l is dense in $(V(\omega_l), \infty)$, and for that reason the smooth Sternberg diffeomorphism (defined at non-resonance energies) would not be suited for quantization. Although not elaborated, one may essentially view $\gamma^{(m)}$ as being constructed by a C^m Sternberg diffeomorphism.

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