Minimal escape velocities for unitary evolution groups

Serge Richard*

October 11, 2003

Abstract

Starting from a strict Mourre inequality, the minimal escape velocity for a unitary evolution group in a Hilbert space is derived under some minimal conditions. If the self-adjoint generator H of this evolution is a Schrödinger operator and if the conjugate operator is the generator of dilations, then it follows that H has very good and easily understandable propagation properties. The striking fact is that no proof of the absence of singularly continuous spectrum of H is available yet under such weak conditions.

1 Introduction

This paper is a natural sequel of [12] on the minimal escape velocity for the evolution group generated by a self-adjoint operator in a Hilbert space. By improving part of the mentioned work (as suggested in [8]) and by applying these new results to some Schrödinger operators in $L^2(\mathbb{R}^n)$, we deduce some sharp propagation estimates. The minimal escape velocity is one variant of the generically called minimal velocity estimates, which are a key ingredient in the proof of asymptotic completeness for various models in quantum mechanics. We refer for example to [19], [21], [9] and [3] for their importance in the N-body problem, and to [11] and [17] for their use in some other anisotropic situations.

*Department of Theoretical Physics, University of Geneva, 1211 Geneva 4, Switzerland. Electronic mail: serge.richard@physics.unige.ch.
This work is partially supported by the Swiss National Science Foundation.

Let us first concentrate on Schrödinger operators and explain the interest of our estimates. We consider the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$, the usual Sobolev space $\mathcal{H}^2$ of order two on $\mathbb{R}^n$ and the generator A of the dilation group in $\mathcal{H}$. Let $V(Q)$ be a Δ-bounded operator with relative bound less than one, and let $H := -\Delta + V$ be the corresponding Schrödinger operator in $\mathcal{H}$ with domain $\mathcal{H}^2$. Assume that H is of class $C^1_u(A)$. The conditions of regularity of H with respect to A are explained in Section 2, but let us already mention that this requirement is very weak in the setting of the conjugate operator theory. Assume moreover that there exists an open interval J of $\mathbb{R}$ such that A is strictly conjugate to H on J. We show then that there exist a strictly positive constant $\upsilon_{\min}$ and a dense set of vectors φ in the spectral subspace $E_H(J)\mathcal{H}$ of $\mathcal{H}$ such that for each $\upsilon < \upsilon_{\min}$,

$$\big\|\chi(|Q| \le \upsilon t)e^{-iHt}\varphi\big\| \to 0 \quad \text{as} \quad t \to \infty, \tag{1}$$

where $\chi(|Q| \le \upsilon t)$ is the characteristic function of the ball in $\mathbb{R}^n$ centered at the origin and of radius equal to υt.

The physical interpretation of this result is that the probability of finding the state $e^{-iHt}\varphi$ in the growing ball goes to zero as the time t goes to infinity. In other words, the state $e^{-iHt}\varphi$ propagates to infinity or "flees the origin" [4] with a velocity at least equal to $\upsilon_{\min}$. Let us point out that the hypotheses of the previous estimate are easily fulfilled by Schrödinger operators with very general N-body potentials or cartesian potentials [17]. In the case where V is a two-body potential, the relation (1) is similar to some results obtained in [6].

The natural question which arises is about the nature of the spectrum of H on J. Do such propagation estimates imply the absence of singularly continuous spectrum on J ? We do not know the answer but two related works could corroborate a positive one. We mention first the paper [20] in which a connection is drawn between the time of sojourn of the state $e^{-iHt}\varphi$ in any finite region of the space and the absolutely continuous subspace of $\mathcal{H}$ with respect to H. Secondly, let us assume for a while that V is a bounded function on $\mathbb{R}^n$ satisfying $\lim_{|x|\to\infty} |x|V(x) = 0$. In that case, one shows that the relation (1) holds for the corresponding Schrödinger operator on any open interval J of $\mathbb{R}_+$ with 0 not in the closure of J (*cf.* Remark 4). But then, it has been proved in [16] that any Schrödinger operator $-\Delta + V$ in $L^2\big([0,\infty)\big)$, with a bounded function V satisfying $\lim_{x\to\infty} xV(x) = 0$, has purely absolutely continuous spectrum on $(0,\infty)$. Anyway, any proof (based on the method of the conjugate operator) of the absolute continuity of the spectrum of H on J requires a stronger condition than the $C^1_u(A)$-condition needed above. We refer to Chapter 7 of [1] for the most refined version of such results.

Let us now develop the abstract side of the minimal escape velocity. We consider two self-adjoint operators H and A in a Hilbert space $\mathcal{H}$ (with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$). The starting point is a strict Mourre inequality, *i.e.* the existence of an open interval J of $\mathbb{R}$ and of a strictly positive constant θ such that $\eta(H)[iH,A]\eta(H) \geq \theta\eta^2(H)$ for all smooth real functions η with support in J. In order to give an unambiguous meaning to that expression, a regularity condition on H with respect to A must be imposed : H has to be of class $C^1(A)$. But if H is only slightly more regular we are able to state our first main result. Let us denote by $C^\infty_c(J)$ the set of all smooth complex functions defined on J which have a compact support in J. We use the notations $\chi(A \leq c)$ and $\chi(A \geq c)$ for the spectral projections of the operator A on the intervals $(-\infty, c]$ and $[c,\infty)$ respectively.

Theorem 1 *Let H and A be self-adjoint operators in $\mathcal{H}$ with H of class $C^1_u(A)$. Assume that there exist an open interval J of $\mathbb{R}$ and $\theta > 0$ such that $\eta(H)[iH,A]\eta(H) \geq \theta\eta^2(H)$ for all real $\eta \in C^\infty_c(J)$. Let a and t be real numbers. Then for each real $\eta \in C^\infty_c(J)$ and for each $\upsilon < \theta$ one has*

$$\left\|\chi(A \leq a + \upsilon t)e^{-iHt}\eta(H)\chi(A \geq a)\right\| \to 0 \quad \text{as} \quad t \to \infty \tag{2}$$

uniformly in a.

The localization of the evolution in the spectrum of a conjugate operator has already a long history. We refer for example to [5], [14], [13] or more recently to [2] or [18] for different but related results. It is worth mentioning that in all those references, the operator H has to be more regular with respect to A than in Theorem 1. However, by requiring more regularity of H one may obtain a better control on the decrease of the norm in (2), *cf.* [12] for this kind of results. We point out that in this reference, H is at least of class $C^2(A)$.

Let us finally describe the content of this paper. In Section 2 we introduce some notations and definitions. The proof of Theorem 1 is given in Section 3. In certain situations, one has some interest in localizing the evolution in the spectrum of another self-adjoint operator B rather than in the spectrum of A. Section 4 is devoted to that question and Proposition 3 contains some sufficient conditions between A and B to that purpose. The last section is the application to Schrödinger operators. Our second main statement, Theorem 2, is exposed and proved. The relation (1) previously discussed is then a straightforward corollary of this theorem.

2 Some notations

Almost all the notations and definitions are borrowed from [1], to which we refer for details. For any positive integer k let $C^k(\mathbb{R})$ be the algebra of complex functions on $\mathbb{R}$ that are k times continuously differentiable. We also consider various subalgebras of $C^\infty(\mathbb{R}) := \bigcap C^k(\mathbb{R})$, namely : $C^\infty_{\rm pol}(\mathbb{R})$, the functions whose derivatives have at most polynomial growth at infinity, $\mathcal{S}^\mu(\mathbb{R})$ with $\mu \leq 0$, the symbols of degree μ, and $C^\infty_c(\mathbb{R})$, the functions with compact support. Let us recall that $f \in C^\infty(\mathbb{R})$ is a symbol of degree μ if for each k there exists a constant c_k such that $|f^{(k)}(x)| \leq c_k(1+x^2)^{\frac{\mu-k}{2}}$ for all $x \in \mathbb{R}$.

We collect some definitions related with the conjugate operator theory. $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators in $\mathcal{H}$ and $\{W_t\}_{t\in\mathbb{R}}$ is the unitary group in $\mathcal{H}$ generated by a self-adjoint operator A. For any $T \in \mathcal{B}(\mathcal{H})$, we write $T \in C_u(A)$, $T \in C^k(A)$ or $T \in C^k_u(A)$ if the mapping $\mathbb{R} \ni t \mapsto W_{-t}TW_t \in \mathcal{B}(\mathcal{H})$ is continuous in norm, strongly C^k or C^k in norm respectively. By assuming that $T \in C^1(A)$, the commutator $[iT, A]$, defined in form sense on the domain $D(A)$ of A, extends continuously to a bounded operator in $\mathcal{H}$. Let us mention that $T \in C^1_u(A)$ if and only if $T \in C^1(A)$ and $[iT, A]$ belongs to $C_u(A)$. A self-adjoint operator H in $\mathcal{H}$ is *of class* $C^k(A)$, resp. $C^k_u(A)$, if $(H-z)^{-1} \in C^k(A)$, resp. $(H-z)^{-1} \in C^k_u(A)$, for some, and then for all, $z \in \mathbb{C}\setminus\sigma(H)$. We have used the notation $\sigma(H)$ for the spectrum of H.

Let $\Phi : [1,\infty) \ni t \mapsto \Phi(t) \in \mathcal{B}(\mathcal{H})$ be an operator-valued mapping. We say that Φ $\big($or by a slight abuse of notation $\Phi(t)\big)$ belongs to $o(t^{-k})$ if $\|\Phi(t)\| \in \mathfrak{o}(t^{-k})$ or to $O(t^{-k})$ if $\|\Phi(t)\| \in \mathcal{O}(t^{-k})$, *i.e.* if $\lim_{t\to\infty} t^k\|\Phi(t)\| = 0$ or if $t^k\|\Phi(t)\| \leq c < \infty$ for all $t \geq 1$.

We shall use on $\mathbb{R}$ the *Fourier measure* $\underline{d}x := (2\pi)^{-1/2}dx$, where dx is the usual Lebesgue measure. Then a function $f : \mathbb{R} \to \mathbb{C}$ belongs to $L^1(\mathbb{R})$ if $\|f\|_{L^1} := \int_\mathbb{R} |f(x)|\underline{d}x < \infty$. For such a function, its Fourier transform $\mathscr{F}f \equiv \hat{f}$ is defined by $\hat{f}(x) := \int_\mathbb{R} e^{-ixy}f(y)\underline{d}y$. We recall that $\mathscr{F}$ extends canonically to an isomorphism of the space of tempered distributions $\mathscr{S}^*(\mathbb{R})$ onto itself.

For any complex Radon measure on $\mathbb{R}$ (simply called *measure*), we use the notation $\nu(x)\underline{d}x$ for $\nu(dx)$. With such a measure ν we associate its variation $|\nu|$, *i.e.* the smallest positive measure such that $|\nu(\Omega)| \leq |\nu|(\Omega)$ for each bounded and closed subset Ω of $\mathbb{R}$. The measure ν is *integrable on* $\mathbb{R}$ if $|\nu|(\mathbb{R}) < \infty$. The space of all integrable measures on $\mathbb{R}$ is identified with a subspace of $\mathscr{S}^*(\mathbb{R})$ by the formula $\langle f, \nu\rangle := \int_{\mathbb{R}} \overline{f(x)}\nu(x)\underline{d}x$, where f is any element of the space $\mathscr{S}(\mathbb{R})$ of tempered test functions on $\mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is the duality between $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}^*(\mathbb{R})$.

We are finally in position to recall a functional calculus. Let A be a self-adjoint operator in $\mathcal{H}$ and $f \in \mathscr{S}^*(\mathbb{R})$ such that $\hat{f}$ is an integrable measure on $\mathbb{R}$. Then for any $\varphi, \psi \in \mathcal{H}$, one has (*cf.* Definition 3.2.7 of [1]) :

$$\langle\varphi, f(A)\psi\rangle := \int_{\mathbb{R}} \langle\varphi, e^{iAx}\psi\rangle \hat{f}(x)\underline{d}x. \tag{3}$$

3 The abstract theory

We first consider a self-adjoint operator A in $\mathcal{H}$ and prove estimates for operators which have a certain regularity with respect to A. For some technicalities, we shall work on the domain $\mathcal{G} := \{\varphi \in \mathcal{H} \mid \mathrm{supp}_A\varphi \text{ is compact}\}$, which is dense in $\mathcal{H}$. In the sequel, it is assumed that a and s are real numbers with $s \geq 1$ and that $f, h, \eta, \ldots$ are real functions.

Lemma 1 *Consider a bounded operator $T \in C^1(A)$ and let h be a $C^\infty_{\mathrm{pol}}(\mathbb{R})$-function such that $\widehat{h'}$ is an integrable measure on $\mathbb{R}$. The commutator $\left[T, h\left(\frac{A-a}{s}\right)\right]$, defined in form sense on $\mathcal{G}$, extends then continuously to a bounded operator with norm less or equal to $\frac{1}{s}\|[T,A]\|\|\widehat{h'}\|_{L^1}$.*

In the following proofs, we write A_s for the operator $\frac{A-a}{s}$.

Proof: By using the commutator expansions given in Theorem 5.5.3 of [1], one has the following equality in form sense on $\mathcal{G}$:

$$[T, h(A_s)] = \frac{1}{s}\int_0^1 d\tau \int_{\mathbb{R}} e^{iA_s\tau x}[T,A]e^{iA_s(1-\tau)x}\widehat{h'}(x)\underline{d}x.$$

Since $T \in C^1(A)$ the commutator $[T,A]$ extends continuously to a bounded operator in $\mathcal{H}$, and the estimate on the norm follows straightforwardly. ■

Corollary 1 *Assume that T and h satisfy the hypotheses of Lemma 1 and that h has support in $(-\infty, 0]$. Then the operator $\chi(A-a \geq 0)Th\left(\frac{A-a}{s}\right)$, defined on $\mathcal{G}$, extends continuously to a bounded operator with norm less or equal to $\frac{1}{s}\|[T,A]\|\|\widehat{h'}\|_{L^1}$.*

Proof: Since $\chi(x \geq 0)h\left(\frac{x}{s}\right) = 0$ for any $s \geq 1$ and all $x \in \mathbb{R}$, one has the equality (in form sense on $\mathcal{G}$) :

$$\chi(A - a \geq 0)Th(A_s) = \chi(A - a \geq 0)[T, h(A_s)].$$

The conclusion is then implied by Lemma 1. ∎

Lemma 2 *Consider a bounded operator $B \in C_u(A)$ and let h be a $L^\infty(\mathbb{R})$-function such that $\widehat{h}$ is an integrable measure on $\mathbb{R}$. Then the commutator $\left[B, h\left(\frac{A-a}{s}\right)\right]$ belongs to $o(s^0)$, uniformly in a.*

Proof: By using the functional calculus introduced in equation (3), one has :

$$\begin{aligned} \|[B, h(A_s)]\| &\leq \textstyle\int_{\mathbb{R}} \left\| e^{\frac{i}{s}Ax} B e^{-\frac{i}{s}Ax} - B \right\| |\widehat{h}(x)| \underline{d}x \\ \leq 2\|B\| \textstyle\int_{|x| \geq s^{1/2}} |\widehat{h}(x)| \underline{d}x &+ \textstyle\int_{|x| < s^{1/2}} \left\| e^{\frac{i}{s}Ax} B e^{-\frac{i}{s}Ax} - B \right\| |\widehat{h}(x)| \underline{d}x. \end{aligned} \tag{4}$$

The first term of (4) goes to 0 as s increases, and the second term is less or equal to $\sup_{|y|<s^{-1/2}} \left\| e^{iAy} B e^{-iAy} - B \right\| \|\widehat{h}\|_{L^1}$. By the regularity of B with respect to A, this belongs to $\mathfrak{o}(s^0)$. ∎

The next proposition imposes some apparently strong conditions on a certain function. But we shall show in a subsequent remark the existence of a class of functions satisfying all those requirements.

Proposition 1 *Let H be a self-adjoint operator in $\mathcal{H}$ of class $C_u^1(A)$. Assume that there exist an open interval J of $\mathbb{R}$ and $\theta > 0$ such that $\eta(H)[iH, A]\eta(H) \geq \theta\eta^2(H)$ for all $\eta \in C_c^\infty(J)$. Let f be a $C^\infty_{\text{pol}}(\mathbb{R})$-function such that $f' = -g^2$ for some bounded $C^\infty_{\text{pol}}(\mathbb{R})$-function g. Moreover assume that $\widehat{f'}, \widehat{g}$ and $\widehat{g'}$ are integrable measures on $\mathbb{R}$. Then for each $\eta \in C_c^\infty(J)$ the operator $\eta(H)\left[iH, f\left(\frac{A-a}{s}\right)\right]\eta(H)$ is well defined in form sense on $\mathcal{G}$ and satisfies the estimate*

$$\eta(H)\left[iH, f\left(\frac{A-a}{s}\right)\right]\eta(H) \leq \frac{\theta}{s}\eta(H) f'\left(\frac{A-a}{s}\right)\eta(H) + o(s^{-1}), \tag{5}$$

where $o(s^{-1})$ is uniform in a.

Proof: a) Let us first observe that $\eta(H)$ is a bounded operator in $\mathcal{H}$ which belongs to $C_u^1(A)$, *cf.* Corollary 6.2.6 (b) of [1]. It follows that the operator $\eta(H)[iH, f(A_s)]\eta(H)$ is well defined in form sense on $\mathcal{G}$. Indeed, $f(A_s)\eta(H)\varphi = \eta(H)f(A_s)\varphi + [f(A_s), \eta(H)]\varphi$, and the r.h.s. is well defined for all $\varphi \in \mathcal{G}$ (the commutator is bounded by Lemma 1).

b) Let $\tilde{\eta} \in C_c^\infty(J)$ be such that $\tilde{\eta}\eta = \eta$. We set $T := H\tilde{\eta}(H)$ $\big($which belongs to $C_u^1(A)\big)$ and denote by B the continuous extension of the operator formally given by $[iT, A]$ $\big(B$ belongs to $C_u(A)\big)$. One observes that $\eta(H)[iH, f(A_s)]\eta(H) = \eta(H)[iT, f(A_s)]\eta(H)$ and that the strict Mourre inequality can be rewritten as $\eta(H)B\eta(H) \geq \theta\eta^2(H)$ for all $\eta \in C_c^\infty(J)$.

Through the use of the commutator expansions of Theorem 5.5.3 of [1], one obtains the following equality in form sense on $\mathcal{G}$:

$$[iT, f(A_s)] = \frac{1}{s}R_s + \frac{1}{s}Bf'(A_s) \tag{6}$$

with $R_s = \int_0^1 d\tau \int_{\mathbb{R}} \left(e^{\frac{i}{s}A\tau x}Be^{-\frac{i}{s}A\tau x} - B\right) e^{iA_s x}\widehat{f'}(x)\underline{d}x$. Since the terms on the r.h.s. of (6) are bounded, the l.h.s. term of (5) extends continuously to $\eta(H)\left\{\frac{1}{s}R_s + \frac{1}{s}Bf'(A_s)\right\}\eta(H)$.

c) Let us now observe that $\frac{1}{s}Bf'(A_s) = -\frac{1}{s}g(A_s)Bg(A_s) + o(s^{-1})$, where $o(s^{-1})$ is independent of a (we have used Lemma 2). Moreover, since $\eta(H) \in C^1(A)$, some commutator calculations based on Lemma 1 show that

$$\begin{aligned} &-\tfrac{1}{s}\eta(H)g(A_s)Bg(A_s)\eta(H) = -\tfrac{1}{s}g(A_s)\eta(H)B\eta(H)g(A_s) + O(s^{-2}) \\ &\leq -\tfrac{\theta}{s}g(A_s)\eta^2(H)g(A_s) + O(s^{-2}) = \tfrac{\theta}{s}\eta(H)f'(A_s)\eta(H) + O(s^{-2}), \end{aligned}$$

where $O(s^{-2})$ is independent of a.

d) It only remains to show that R_s belongs to $o(s^0)$ uniformly in a. One has that its norm is less or equal to

$$2\|B\| \int_{|x|\geq s^{1/2}} |\widehat{f'}(x)|\underline{d}x + \int_0^1 d\tau \int_{|x|<s^{1/2}} \left\|e^{\frac{i}{s}A\tau x}Be^{-\frac{i}{s}A\tau x} - B\right\| |\widehat{f'}(x)|\underline{d}x. \tag{7}$$

The first term of (7) goes to 0 as s increases. The second term of (7) is less or equal to $\sup_{|y|<s^{-1/2}} \left\|e^{iAy}Be^{-iAy} - B\right\| \|\widehat{f'}\|_{L^1}$, which belongs to $\mathfrak{o}(s^0)$ by the regularity of B with respect to A. One observes that both convergences are uniform in a. ■

Remark 1 Consider $g \in \mathcal{S}^\mu(\mathbb{R})$ for some $\mu < 0$. Since $g^2 \in \mathcal{S}^{2\mu}(\mathbb{R})$ and $g' \in \mathcal{S}^{\mu-1}(\mathbb{R})$, then $\widehat{g}, \widehat{g^2}$ and $\widehat{g'}$ are integrable measures on $\mathbb{R}$ (Proposition 5.4.5 of [1]). Moreover if $f(x) := -\int_0^x g^2(y)dy$, then f belongs to $C^\infty_{\mathrm{pol}}(\mathbb{R})$ and satisfies all the assumptions of Proposition 1.

Proposition 2 *Let H be a self-adjoint operator in $\mathcal{H}$ of class $C^1_u(A)$. Assume that there exist an open interval J of $\mathbb{R}$ and $\theta > 0$ such that $\eta(H)[iH, A]\eta(H) \geq \theta\eta^2(H)$ for all $\eta \in C_c^\infty(J)$. Let t be a real number with $t \geq 1$. Then for each $\eta \in C_c^\infty(J)$, for each $f \in L^\infty(\mathbb{R})$ with support in $(-\infty, 0]$ and for each $\upsilon < \theta$, one has*

$$f\left(\frac{A-a}{t} - \upsilon\right)\eta(H)e^{-iHt}\chi(A - a \geq 0) \in o(t^0)$$

uniformly in a.

The following proof is inspired from that of Theorem 1.1 of [12], but is considerably simpler in our situation.

Proof: a) Let g be a $C_c^\infty(\mathbb{R})$-function with support in $(\upsilon - \theta, 0)$ and such that $\int_{-\infty}^{\infty} g^2(y)dy = 1$. We set $h(x) = -\int_0^x g^2(y)dy$ and observe that h satisfies all conditions imposed on f in Proposition 1. Furthermore, since $h^{1/2}(x-\theta)f(x-\upsilon) = f(x-\upsilon)$ for all $x \in \mathbb{R}$, it is enough to prove that

$$h^{1/2}\left(\frac{A-a}{t} - \theta\right)\eta(H)e^{-iHt}\chi(A - a \geq 0) \in o(t^0) \tag{8}$$

uniformly in a.

b) Let us set $\Phi_s(t) := \eta(H)h(A_{t,s})\eta(H)$, with $A_{t,s}$ equal to $\frac{A-a-\theta t}{s}$. For each $\psi \in \mathcal{H}$, we define $\psi_t := e^{-iHt}\chi(A - a \geq 0)\psi$. Then (8) is equivalent to the statement that for all $\psi \in \mathcal{H}$,

$$\langle \psi_t, \Phi_t(t)\psi_t\rangle \leq \mathfrak{o}(t^0)\|\psi\|^2, \tag{9}$$

with $\mathfrak{o}(t^0)$ independent of ψ and a. One observes that

$$\begin{aligned}
\langle \psi_t, \Phi_s(t)\psi_t\rangle &= \langle \psi_0, \Phi_s(0)\psi_0\rangle + \int_0^t \frac{d}{d\tau}\langle \psi_\tau, \Phi_s(\tau)\psi_\tau\rangle d\tau \\
&= \langle \psi_0, \Phi_s(0)\psi_0\rangle - \frac{\theta}{s}\int_0^t \langle \psi_\tau, \eta(H)h'(A_{\tau,s})\eta(H)\psi_\tau\rangle d\tau \\
&\quad + \int_0^t \langle \psi_\tau, \eta(H)[iH, h(A_{\tau,s})]\eta(H)\psi_\tau\rangle d\tau.
\end{aligned} \tag{10}$$

By inserting (5) into (10) with a replaced by $a + \theta\tau$, we find that

$$\langle \psi_t, \Phi_s(t)\psi_t\rangle \leq \langle \psi_0, \Phi_s(0)\psi_0\rangle + \int_0^t \mathfrak{o}(s^{-1})\|\psi\|^2 d\tau$$

with $\mathfrak{o}(s^{-1})$ independent of a, τ and ψ. Moreover, with the help of Corollary 1, one gets that

$$\langle \psi_0, \Phi_s(0)\psi_0 \rangle \leq \frac{1}{s} \|\eta\|_{L^\infty} \|[\eta(H), A]\| \|\widehat{h'}\|_{L^1} \|\psi\|^2.$$

Hence, one has obtained that

$$\langle \psi_t, \Phi_s(t)\psi_t \rangle \leq \frac{c}{s} \|\psi\|^2 + t\, \mathfrak{o}(s^{-1}) \|\psi\|^2$$

with $\mathfrak{o}(s^{-1})$ and c independent of a, t and ψ. By setting $s = t$, this implies (9). ■

Proof of Theorem 1: Since $\chi\left(\frac{1}{t}x - v \leq 0\right) = \chi\left(x \leq vt\right)$ for any $t \geq 1$ and all $x \in \mathbb{R}$, the statement of the theorem is a special case of Proposition 2 with $f(\cdot) = \chi(\cdot \leq 0)$. ■

4 From one localization to another

The content of this section is inspired from Section 4.4.1 of [10]. The main difference is that the parameter a is not considered in that monograph.

Let us recall from Lemma 7.2.15 of [1] that if T is a bounded operator belonging to $C^1(A)$, the closure of the symmetric, densely defined operator T^*AT $\big(D(T^*AT) \supset D(A)\big)$ is a self-adjoint operator which we still denote by T^*AT. Moreover, $D(A)$ is a core for this operator. Therefore, if H is of class $C^1(A)$ and $\tilde{\eta} \in C_c^\infty(\mathbb{R})$, the operator $\tilde{\eta}(H)A\tilde{\eta}(H)$, defined on $D(A)$, admits a unique self-adjoint extension $\big($*cf.* Theorem 6.2.5 of [1] for the proof that $\tilde{\eta}(H)$ belongs to $C^1(A)\big)$. We also mention (Proposition 7.2.16 of the same reference) that if $\eta \in C_c^\infty(\mathbb{R})$, then $\eta(H)$ belongs to $C^1\big(\tilde{\eta}(H)A\tilde{\eta}(H)\big)$.

Lemma 3 *Let H and A be self-adjoint operators in $\mathcal{H}$ with H of class $C^1(A)$. Let $\eta, \tilde{\eta}$ be $C_c^\infty(\mathbb{R})$-functions such that $\tilde{\eta}\eta = \eta$, and let f be a $C^\infty(\mathbb{R})$-function such that $f = 0$ in a neighbourhood of $-\infty$ and $f = 1$ in a neighbourhood of $+\infty$. Then one has*

$$\left\{ f\left(\frac{A-a}{s}\right) - f\left(\frac{\tilde{\eta}(H)A\tilde{\eta}(H) - a}{s}\right) \right\} \eta(H) \in O(s^{-1})$$

uniformly in a.

Proof: Let us set $A_s := \frac{A-a}{s}$, $\tilde{A}_s := \frac{\tilde{\eta}(H)A\tilde{\eta}(H)-a}{s}$, $R(z) := (A_s - z)^{-1}$ and $\tilde{R}(z) := (\tilde{A}_s - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. For space reasons, we write η for $\eta(H)$

and $\tilde{\eta}$ for $\tilde{\eta}(H)$. Let φ, ψ be elements of $\mathcal{H}$. By using Theorem 6.1.4 (b) of [1] for any integer $r \geq 1$, one has that $\langle \varphi, \{f(A_s) - f(\tilde{A}_s)\}\eta\psi\rangle$ is equal to

$$\begin{aligned}&\textstyle\sum_{k=0}^{r-1} \frac{1}{\pi k!} \int_{\mathbb{R}} f^{(k)}(\lambda) \Im \langle \varphi, i^k \{R(\lambda + i) - \tilde{R}(\lambda + i)\}\eta\psi\rangle d\lambda \\ &+ \textstyle\frac{1}{\pi(r-1)!} \int_0^1 d\mu \int_{\mathbb{R}} \mu^{r-1} f^{(r)}(\lambda) \Im \langle \varphi, i^r \{R(\lambda + i\mu) - \tilde{R}(\lambda + i\mu)\}\eta\psi\rangle d\lambda.\end{aligned}$$

Moreover, one observes that there exist two constants c_1 and c_2, independent of a and s, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\begin{aligned}
&|\langle \varphi, \{R(z) - \tilde{R}(z)\}\eta\psi\rangle| = |\langle \{\tilde{A}_s - A_s\}R(\overline{z})\varphi, \tilde{R}(z)\eta\psi\rangle| \\
&= |\langle \eta\{\tilde{A}_s - A_s\}R(\overline{z})\varphi, \tilde{R}(z)\psi\rangle + \frac{1}{s}\langle \{R(\overline{z}) - \tilde{R}(\overline{z})\}\varphi, \tilde{\eta}[\eta, A]\tilde{\eta}\tilde{R}(z)\psi\rangle| \\
&\leq \frac{1}{s}|\langle \eta[A, \tilde{\eta}]R(\overline{z})\varphi, \tilde{R}(z)\psi\rangle| + \frac{c_1}{s}\{\|R(\overline{z})\varphi\| + \|\tilde{R}(\overline{z})\varphi\|\}\|\tilde{R}(z)\psi\| \\
&\leq \frac{c_2}{s}\|R(\overline{z})\varphi\|\|\tilde{R}(z)\psi\| + \frac{c_1}{s}\|\tilde{R}(\overline{z})\varphi\| \ \|\tilde{R}(z)\psi\|,
\end{aligned}$$

where we have used that $[\tilde{R}(z), \eta] = \frac{1}{s}\tilde{R}(z)\tilde{\eta}[\eta, A]\tilde{\eta}\tilde{R}(z)$. By using then the Hölder inequality and the identity (*cf.* Chap. XIII.7, Example 2 of [15]) valid for any self-ajoint operator K :

$$\int_{\mathbb{R}} \|(K - \lambda - i\mu)^{-1}\varphi\|^2 d\lambda = \frac{\pi}{|\mu|}\|\varphi\|^2,$$

we find that for $\mu \neq 0$,

$$\left| \int_{\mathbb{R}} f^{(k)}(\lambda) \Im \langle \varphi, i^k \{R(\lambda + i\mu) - \tilde{R}(\lambda + i\mu)\}\eta\psi\rangle d\lambda \right| \leq \frac{d}{s|\mu|}\|\varphi\|\|\psi\|.$$

with $d = \pi(c_1 + c_2)\|f^{(k)}\|_{L^\infty}$. By choosing $r \geq 2$, one has $\int_0^1 \mu^{r-1}\frac{1}{|\mu|}d\mu < \infty$, and we have therefore obtained that

$$|\langle \varphi, \{f(A_s) - f(\tilde{A}_s)\}\eta(H)\psi\rangle| \leq \frac{c}{s}\|\varphi\|\|\psi\|$$

for some c independent of a and s. ■

Let us recall from Theorem 6.2.10 of [1] that if A and B are self-adjoint operators in $\mathcal{H}$ with B of class $C^1(A)$, then $D(A) \cap D(B)$ is a core for B.

Lemma 4 *Let A and B be self-adjoint operators in $\mathcal{H}$. Assume that*

i) B is of class $C^1(A)$ and $A \leq B$ on $D(A) \cap D(B)$, *ii)* $\left[h\left(\frac{B}{s}\right), A\right] \in O(s^0)$ *for each* $h \in C_c^\infty(\mathbb{R})$.

Let f *and* g *be* $C^\infty(\mathbb{R})$*-functions such that* $\max(\operatorname{supp} g) < \min(\operatorname{supp} f)$. *Moreover assume that* $f = 1$ *in a neighbourhood of* $+\infty$ *and that* g *has compact support. Then there exists* $c < \infty$ *independent of* a *and* s *such that*

$$\left\| g\left(\frac{B}{s}\right) f\left(\frac{A-a}{s}\right) \right\| \leq \frac{c}{s}(1+|a|). \tag{11}$$

Proof: Let $\tilde{g}$ be in $C_c^\infty(\mathbb{R})$ such that $\max(\operatorname{supp} \tilde{g}) < \min(\operatorname{supp} f)$, $\tilde{g}g = g$ and $0 \leq \tilde{g} \leq 1$. Then the operator $\tilde{g}\left(\frac{B}{s}\right)$ belongs to $C^1(A)$ and $\tilde{g}\left(\frac{B}{s}\right) \frac{A}{s} \tilde{g}\left(\frac{B}{s}\right)$, defined on $D(A)$, admits a unique self-adjoint extension which we denote by $\tilde{A}_s$ (*cf.* the observations made before Lemma 3).

It follows from hypothesis *i)* that $\tilde{A}_s \leq \tilde{g}\left(\frac{B}{s}\right) \frac{B}{s} \tilde{g}\left(\frac{B}{s}\right) < \min(\operatorname{supp} f)$ on $D(A)$, and therefore that $f(\tilde{A}_s) = 0$ for any $s \geq 1$. In order to obtain (11), it is hence enough to show that there exists $c < \infty$ independent of a and s such that

$$\left\| g\left(\frac{B}{s}\right) \{f(A_s) - f(\tilde{A}_s)\} \right\| \leq \frac{c}{s}(1+|a|).$$

The rest of the proof is now analogous to that given in Lemma 3 and we shall only point out the minor difference. Let us set $R(z) := (A_s - z)^{-1}$ and $\tilde{R}(z) := (\tilde{A}_s - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. One has, in form sense on $\mathcal{H}$, that

$$\begin{aligned} g\left(\frac{B}{s}\right) \{R(z) - \tilde{R}(z)\} &= \tilde{R}(z) g\left(\frac{B}{s}\right) (\tilde{A}_s - A_s) R(z) \\ &+ \frac{1}{s} \tilde{R}(z) \tilde{g}\left(\frac{B}{s}\right) \left[A, g\left(\frac{B}{s}\right)\right] \tilde{g}\left(\frac{B}{s}\right) \{R(z) - \tilde{R}(z)\}, \end{aligned}$$

and that

$$\tilde{R}(z) g\left(\frac{B}{s}\right) (\tilde{A}_s - A_s) R(z) = \frac{1}{s} \tilde{R}(z) g\left(\frac{B}{s}\right) \left\{\left[A, \tilde{g}\left(\frac{B}{s}\right)\right] + a\right\} R(z).$$

Hypothesis *ii)* is now used in order to obtain a uniform bound for the commutators. ■

We now refine Lemma 4 to the case where B dominates only a localized version of A.

Lemma 5 *Let* H, A *and* B *be self-adjoint operators in* $\mathcal{H}$ *and let* $\eta, \tilde{\eta}$ *be* $C_c^\infty(\mathbb{R})$*-functions such that* $\tilde{\eta}\eta = \eta$. *Assume that* *i)* H *is of class* $C^1(A)$ *and of class* $C^1(B)$*,* B *is of class* $C^1(A)$*,*

ii) *the operators* $(B \pm i)^{-1} A\tilde{\eta}(H)$ *defined on* $D(A)$ *extend continuously to bounded operators in* $\mathcal{H}$*,*

iii) $\tilde{\eta}(H)A\tilde{\eta}(H) \leq B$ *on* $D\big(\tilde{\eta}(H)A\tilde{\eta}(H)\big) \cap D(B)$*,*

iv) $\left[h\left(\frac{B}{s}\right), A\right] \in O(s^0)$ *for each* $h \in C_c^\infty(\mathbb{R})$*.*

Let f *and* g *be* $C^\infty(\mathbb{R})$*-functions such that* $\max(\operatorname{supp} g) < \min(\operatorname{supp} f)$*. Moreover assume that* $f = 1$ *in a neighbourhood of* $+\infty$ *and that* g *has a compact support. Then there exists* $c < \infty$ *independent of* a *and* s *such that*

$$\left\| g\left(\frac{B}{s}\right) f\left(\frac{A-a}{s}\right) \eta(H) \right\| \leq \frac{c}{s}(1+|a|).$$

Proof: For space reasons, we write $\tilde{\eta}$ for $\tilde{\eta}(H)$. One has

$$\begin{aligned} &\left\| g\left(\frac{B}{s}\right) \left\{ f\left(\frac{A-a}{s}\right) - f\left(\frac{\tilde{\eta}A\tilde{\eta}-a}{s}\right) + f\left(\frac{\tilde{\eta}A\tilde{\eta}-a}{s}\right) \right\} \eta(H) \right\| \\ &\leq \|\eta\|_{L^\infty} \left\| g\left(\frac{B}{s}\right) f\left(\frac{\tilde{\eta}A\tilde{\eta}-a}{s}\right) \right\| + O(s^{-1}), \end{aligned} \tag{12}$$

where we have used Lemma 3 and thus obtained that $O(s^{-1})$ is independent of a.

In order to deal with the first term of (12) we shall use Lemma 4 with $\tilde{\eta}A\tilde{\eta}$ instead of A. It follows from hypotheses *i)* and *ii)* that B is of class $C^1(\tilde{\eta}A\tilde{\eta})$ (the proof is similar to that of Lemma 4.4.7 of [10]). Thus we only have to prove that $\left[h\left(\frac{B}{s}\right), \tilde{\eta}A\tilde{\eta}\right]$ belongs to $O(s^0)$ for each $h \in C_c^\infty(\mathbb{R})$. This commutator is equal to $\big($in form sense on $D(A)\big)$:

$$\left[h\left(\frac{B}{s}\right), \tilde{\eta}\right] A\tilde{\eta} + \tilde{\eta}\left[h\left(\frac{B}{s}\right), A\right]\tilde{\eta} + \tilde{\eta}A\left[h\left(\frac{B}{s}\right), \tilde{\eta}\right]. \tag{13}$$

By hypothesis *iv)* the second term of (13) is bounded uniformly in s. So let us concentrate on the first term (the third one being similar). Let φ, ψ belong to $D(A)$ and let r be a strictly positive integer. By using Theorem 6.1.4 (b) of [1], the term $\langle \varphi, \left[h\left(\frac{B}{s}\right), \tilde{\eta}\right] A\tilde{\eta}\psi\rangle$ is equal to

$$\begin{aligned} &\textstyle\sum_{k=0}^{r-1} \frac{1}{\pi k!} \int_{\mathbb{R}} h^{(k)}(\lambda) \Im \langle \varphi, i^k \left[\left(\frac{B}{s} - \lambda - i\right)^{-1}, \tilde{\eta}\right] A\tilde{\eta}\psi\rangle d\lambda \\ &+ \textstyle\frac{1}{\pi (r-1)!} \int_0^1 d\mu \int_{\mathbb{R}} \mu^{r-1} h^{(r)}(\lambda) \Im \langle \varphi, i^r \left[\left(\frac{B}{s} - \lambda - i\mu\right)^{-1}, \tilde{\eta}\right] A\tilde{\eta}\psi\rangle d\lambda. \end{aligned} \tag{14}$$

Let us observe that for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\left[\left(\frac{B}{s} - z\right)^{-1}, \tilde{\eta}\right] = \left(\frac{B}{s} - z\right)^{-1} [\tilde{\eta}, B](B - sz)^{-1},$$

where $[\tilde{\eta}, B]$ extends continuously to a bounded operator. By inserting the first resolvent equation

$$(B - s\lambda - is\mu)^{-1} = \left\{I + (s\lambda + is\mu + i)(B - s\lambda - is\mu)^{-1}\right\}(B + i)^{-1}$$

and by taking into account hypothesis *ii)*, one obtains that for any $s \geq 1$,

$$\left|\left\langle \varphi, \left[\left(\frac{B}{s} - \lambda - i\mu\right)^{-1}, \tilde{\eta}\right] A\tilde{\eta}\psi \right\rangle\right| \leq \frac{c}{|\mu|^2}\{|\lambda| + |\mu| + 1)\}\|\varphi\|\|\psi\|$$

where c is independent of s. Finally, by using this estimate in (14) with $r \geq 3$ one finds that

$$\left|\left\langle \varphi, \left[h\left(\frac{B}{s}\right), \tilde{\eta}\right] A\tilde{\eta}\psi \right\rangle\right| \leq c'\|\varphi\|\|\psi\|$$

for some constant c' independent of s. ■

Proposition 3 *Let H, A and B be self-adjoint operators in $\mathcal{H}$ such that H is of class $C^1_u(A)$ and of class $C^1(B)$, B of class $C^1(A)$ and $B \geq 0$. Assume that there exist an open interval J of $\mathbb{R}$ and $\theta > 0$ such that $\eta(H)[iH, A]\eta(H) \geq \theta\eta^2(H)$ for all $\eta \in C_c^\infty(J)$. Let t be a real number with $t \geq 1$ and let $\eta, \tilde{\eta}$ be $C_c^\infty(J)$-functions such that $\tilde{\eta}\eta = \eta$. Assume moreover that*

- *i) the operators $(B \pm i)^{-1}A\tilde{\eta}(H)$ defined on $D(A)$ extend continuously to bounded operators in $\mathcal{H}$,*
- *ii) $\tilde{\eta}(H)A\tilde{\eta}(H) \leq B$ on $D\big(\tilde{\eta}(H)A\tilde{\eta}(H)\big) \cap D(B)$,*
- *iii) $\left[h\left(\frac{B}{t}\right), A\right] \in O(t^0)$ for each $h \in C_c^\infty(\mathbb{R})$.*

Then for each positive $v < \theta$, there exists $c < \infty$ independent of a and t such that

$$\left\|\chi(B \leq vt)e^{-iHt}\eta(H)\chi(A \geq a)\right\| \leq \mathfrak{o}(t^0) + \frac{c}{t}|a|,$$

where $\mathfrak{o}(t^0)$ is uniform in a.

Proof: Let $v' \in (v, \theta)$ and let g be a $C_c^\infty(\mathbb{R}, [0,1])$-function such that $\operatorname{supp} g \subset (-\infty, v')$ and $g = 1$ on $[0, v]$. Let f be a $C^\infty(\mathbb{R}, [0,1])$-function such that $\max(\operatorname{supp} g) < \min(\operatorname{supp} f)$ and $f(x) = 1$ for all $x \geq v'$. Since $\chi(B \leq vt) = \chi\left(\frac{B}{t} \leq v\right) = \chi\left(\frac{B}{t} \leq v\right) g\left(\frac{B}{t}\right)$, one has

$$\left\|\chi(B \leq vt) e^{-iHt} \eta(H) \chi(A \geq a)\right\| \leq \left\| g\left(\frac{B}{t}\right) f\left(\frac{A-a}{t}\right) \eta(H)\right\| + \left\|\left\{1 - f\left(\frac{A-a}{t}\right)\right\} e^{-iHt} \eta(H) \chi(A \geq a)\right\|.$$

By Lemma 5, there exists a constant $c < \infty$ independent of a and t such that the first term on the r.h.s. is less or equal to $\frac{c}{t}(1 + |a|)$. Since $\{1 - f(\cdot + v')\}$ has support in $(-\infty, 0]$, one obtains from Proposition 2 that the second term on the r.h.s belongs to $\mathfrak{o}(t^0)$ uniformly in a. ■

Remark 2 Since $\chi(A \geq a)\chi(A \geq 0) = \chi(A \geq a)$ for any $a \geq 0$, the statement of Proposition 3 can be rewritten in such a situation : for each $a \geq 0$ and each $v < \theta$, one has

$$\chi(B \leq vt) e^{-iHt} \eta(H) \chi(A \geq a) \in o(t^0),$$

where $o(t^0)$ is uniform in a.

5 Application to Schrödinger operators

We consider the Hilbert space $L^2(\mathbb{R}^n)$ and the Sobolev space of order two on $\mathbb{R}^n$ denoted by $\mathcal{H}^2$. We recall that for $j \in \{1, \ldots, n\}$, Q_j is the operator of multiplication by the variable x_j, $P_j := -i\nabla_j$ is a component of the momentum operator and $-\Delta$ is equal to P^2. For any real number a, let us define a_- which is equal to $\max\{-a, 0\}$.

Theorem 2 *Let $V(Q)$ be a Δ-bounded operator with relative bound less than one, and let $H := -\Delta + V$ be the corresponding Schrödinger operator in $L^2(\mathbb{R}^n)$ with domain $\mathcal{H}^2$. Assume that H is of class $C_u^1(A)$, with $A := \frac{1}{2}(P \cdot Q + Q \cdot P)$ the generator of dilation, and that there exist an open interval J of $\mathbb{R}$ and $\theta > 0$ such that $\eta(H)[iH, A]\eta(H) \geq \theta \eta^2(H)$ for all $\eta \in C_c^\infty(J)$. Let a and t be real numbers with $t \geq 1$. Then there exists $v_{min} > 0$ such that for each $\eta \in C_c^\infty(J)$ and each $v < v_{min}$ one has*

$$\left\|\chi(|Q| \leq vt) e^{-iHt} \eta(H) \chi(A \geq a)\right\| \leq \mathfrak{o}(t^0) + \frac{c}{t} a_-,$$

where $\mathfrak{o}(t^0)$ is uniform in a, and where c is a positive constant independent of a and t.

Proof: This theorem is an application of Proposition 3. Let b be any strictly positive number and let $\tilde{\eta}$ be a $C_c^\infty(J)$-function such that $\tilde{\eta}\eta = \eta$. The first step consists in verifying that the positive operator $B := b\langle Q\rangle \equiv b(1+Q^2)^{1/2}$ is of class $C^1(A)$, and that H is of class $C^1(B)$. This can be easily obtained with the help of Theorem 6.3.4 (a) of [1]. Secondly, let us observe that hypothesis *ii)* of Proposition 3 is fulfilled if the operator $\langle Q\rangle^{-1/2}\tilde{\eta}(H)A\tilde{\eta}(H)\langle Q\rangle^{-1/2}$ is bounded and if the value of b is chosen equal to its norm. But this new condition is quite standard and can be easily proved with some commutators calculations (statement *i)* of Lemma 6.2 of [17] may help). The other requirements of Proposition 3 are then also easily checked. One finishes the proof by setting $v_{\min} := \frac{\theta}{b}$ and by taking into account Remark 2 and the fact that if $v < \frac{v'}{b}$ then $\chi\big(b\langle Q\rangle \leq v't\big)\chi\big(|Q| \leq vt\big) = \chi\big(|Q| \leq vt\big)$ for t large enough . ■

Remark 3 One obtains the estimate (1) by observing that the set of vectors of the form $\eta(H)\chi(A \geq a)\psi$ with $\eta \in C_c^\infty(J), a \in \mathbb{R}$ and $\psi \in \mathcal{H}$ is dense in the subspace $E_H(J)\mathcal{H}$ of $\mathcal{H}$.

Remark 4 Theorem 9.4.10 of [1] contains some sufficient conditions on the potential V such that $-\Delta + V$ is a N-body Hamiltonians of class $C_u^1(A)$. In particular, if V is a bounded real function on $\mathbb{R}^n$ satisfying $\lim_{|x|\to\infty} |x|V(x) = 0$, then the corresponding two-body Hamiltonian H is of class $C_u^1(A)$. Its essential spectrum is equal to $[0, \infty)$ and all its eigenvalues are negative and can accumulate only on 0; moreover, the operator A is strictly conjugate to H on any open interval J of $\mathbb{R}_+$ with 0 not in the closure of J (*cf.* Corollary 1.4 of [7] and Theorem 7.2.9 and Corollary 7.2.11 of [1]). Hence Theorem 2 applies and H has very good propagation properties on J.

Acknowledgments : The author thanks W.O. Amrein for critical reading of the text and for his constant support.

References

[1] W.O. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser Verlag (1996).

[2] A. Boutet de Monvel, V. Georgescu, J. Sahbani, *Boundary values of resolvent families and propagation properties*, C. R. Acad. Sci. Paris **322** (Série 1), 289-294 (1996).

[3] J. Dereziński, C. Gérard, *Scattering Theory of Classical and Quantum N-Particle Systems*, Springer-Verlag Berlin Heidelberg (1997).

[4] J.D. Dollard, *On the definition of scattering subspaces in nonrelativistic quantum mechanics*, J. Math. Phys. **18** (2), 229-232 (1977).

[5] V. Enss, *Asymptotic Completeness for Quantum Mechanical Potential Scattering*, Commun. Math. Phys. **61**, 285-291 (1978).

[6] V. Enss, *Asymptotic Observables on Scattering States*, Commun. Math. Phys. **89**, 245-268 (1983).

[7] R. Froese, I. Herbst, *Exponential Bounds and Absence of Positive Eigenvalues for N-Body Schrödinger Operators*, Commun. Math. Phys. **87**, 429-447 (1982).

[8] V. Georgescu, *Review on the paper of Hunziker, Sigal and Soffer* [12], Results from MathSciNet: MR1720738 (2001g:47129).

[9] C. Gérard, *Sharp Propagation Estimates for N-Particle Systems*, Duke Math. Journal **67** (3), 483-515 (1992).

[10] C. Gérard, I. Laba, *Multiparticle Quantum Scattering in Constant Magnetic Fields*, Mathematical Surveys and Monographs, Volume 90, American Mathematical Society (2002).

[11] C. Gérard, F. Nier, *Scattering theory for the perturbations of periodic Schrödinger operators*, J. Math. Kyoto Univ. **38** (4), 595-634 (1998).

[12] W. Hunziker, I.M. Sigal, A. Soffer, *Minimal Escape Velocities*, Comm. Partial Differential Equations **24** (11&12), 2279-2295 (1999).

[13] A. Jensen, *Propagation Estimates for Schrödinger-type Operators*, Transactions of the American Mathematical Society **291** (1), 129-144 (1985).

[14] E. Mourre, *Opérateurs conjugués et propriétés de propagation*, Commun. Math. Phys. **91**, 279-300 (1983).

[15] M. Reed, B. Simon, *Methods of Modern Mathematical Physics IV : Analysis of Operators*, Academic Press (1978).

[16] C. Remling, *The Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators with Decaying Potentials*, Commun. Math. Phys. **193**, 151-170 (1998).

[17] S. Richard, *Spectral and Scattering Theory for Schrödinger Operators with Cartesian Anisotropy*, preprint 03-69 at www.ma.utexas.edu/mp_arc.

[18] J. Sahbani, *Théorèmes de Propagation, Hamiltoniens Localement Réguliers et Applications*, Thèse de Doctorat de l'Université Paris VII (1996).

[19] I.M. Sigal, A. Soffer, *Local decay and velocity bounds*, Preprint, Princeton University (1988).

[20] K.B. Sinha, *On the absolutely and singularly continuous subspaces in scattering theory*, Ann. Inst. Henri Poincaré **XXVI** (3), 263-277 (1977).

[21] E. Skibsted, *Propagation Estimates for N-body Schroedinger Operators*, Commun. Math. Phys. **142**, 67-98 (1991).