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WEIL ASYMPTOTIC FORMULA FOR THE LAPLACIAN ON DOMAINS WITH ROUGH BOUNDARIES

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ABSTRACT. We study asymptotic distribution of eigenvalues of the Laplacian on a bounded domain in $\mathbb{R}^n$. Our main results include an explicit remainder estimate in the Weyl formula for the Dirichlet Laplacian on an arbitrary bounded domain, sufficient conditions for the validity of the Weyl formula for the Neumann Laplacian on a domain with continuous boundary in terms of smoothness of the boundary and a remainder estimate in this formula. In particular, we show that the Weyl formula holds true for the Neumann Laplacian on a Lip_α-domain whenever $(d-1)/\alpha < d$, prove that the remainder in this formula is $O(\lambda^{(d-1)/\alpha})$ and give an example where the remainder estimate $O(\lambda^{(d-1)/\alpha})$ is order sharp. We use a new version of variational technique which does not require the extension theorem.

INTRODUCTION

Let $-\Delta_{\mathrm{N}}$ be the Neumann Laplacian on a bounded domain $\Omega \subset \mathbb{R}^d$ and $N_{\mathrm{N}}(\Omega,\lambda)$ be the number of its eigenvalues which are strictly smaller than λ^2; if the number of these eigenvalues is infinite or $-\Delta_{\mathrm{N}}$ has essential spectrum below λ then we define $N_{\mathrm{N}}(\Omega,\lambda) := +\infty$. Similarly, let $-\Delta_{\mathrm{D}}$ be the Dirichlet Laplacian on Ω and $N_{\mathrm{D}}(\Omega,\lambda)$ be the number its eigenvalues lying below λ^2. We shall omit the lower index D or N and simply write Δ or $N(\Omega,\lambda)$ if the corresponding statement refers both to the Dirichlet and Neumann Laplacian.

According to the Weyl formula,

$$N(\Omega,\lambda) \;-\; C_{d,W}\,\mu_d(\Omega)\,\lambda^d \;=\; o(\lambda^d)\,, \qquad \lambda\to+\infty\,, \tag{0.1}$$

where $\mu_d(\Omega)$ is the d-dimensional Lebesgue measure of Ω and $C_{d,W}$ is the Weyl constant (see Subsection 1.1). If $N = N_{\mathrm{D}}$ then the Weyl formula holds for all bounded domains [BS]. If, in addition, the upper Minkowski dimension of the boundary is equal to $d_1 \in (d-1,d)$ then

$$N(\Omega,\lambda) \;-\; C_{d,W}\,\mu_d(\Omega)\,\lambda^d \;=\; O(\lambda^{d_1})\,, \qquad \lambda\to+\infty\,. \tag{0.2}$$

The asymptotic formula (0.2) with $N = N_{\mathrm{D}}$ is well known and is proved in many papers, for instance, in [BLi] and [Sa] where the authors obtained estimates with explicit constants. This formula remains valid for the Neumann

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Laplacian whenever Ω has the extension property (see below). Note that d_1 may well coincide with d, in which case (0.2) is useless.

If $N = N_{\mathrm{N}}$ then (0.1) is true only for domains with sufficiently regular boundaries. In the general case N_{N} does not satisfy (0.1); moreover, the Neumann Laplacian on a bounded domain may have a nonempty essential spectrum (see, for example, [HSS] or [Si]). The necessary and sufficient conditions for the absence of the essential spectrum in terms of capacities have been obtained in [M1]. In [BD] the authors proved that $N_{\mathrm{N}}(\Omega,\lambda)$ is polynomially bounded whenever the Sobolev space $W^{1,2}(\Omega)$ is embedded in $L^q(\Omega)$ for some $q>2$. If the log-Sobolev inequality holds on Ω then $N_{\mathrm{N}}(\Omega,\lambda)$ is exponentially bounded [Ma].

For domains Ω with sufficiently smooth boundaries, (0.1) is true for the both functions N_{D} and N_{N} and the remainder (i.e., the right hand side) is $O(\lambda^{d-1})$ [Iv1], [Se]. The proof is based on the study of propagation of singularities for the corresponding evolution equation (see [Iv3] or [SV]). If Ω has a rough boundary then the propagation of singularities near $\partial\Omega$ cannot be effectively described and one has to invoke the variational technique.

Let $\Omega_\delta^{\mathrm{b}}$ and $\Omega_\delta^{\mathrm{e}}$ be the internal and external δ-neighbourhoods of $\partial\Omega$ respectively. The classical variational proof of the Weyl formula involves covering the domain by a finite collection of disjoint cubes $\{Q_j\}_{j\in\mathcal{J}}$ and using the Dirichlet–Neumann bracketing. It is convenient to assume that $\{Q_j\}_{j\in\mathcal{J}}$ is the subset of the family of Whitney cubes covering $\Omega\bigcup\Omega_\delta^{\mathrm{e}}$ (see Theorem 3.3), which consists of the cubes Q_j such that $Q_j\bigcap\Omega\neq\emptyset$.

In view of the Rayleigh–Ritz variational formula, we have the estimates $\sum_{j\in\mathcal{J}_0} N_{\mathrm{D}}(Q_j,\lambda)\leqslant N_{\mathrm{D}}(\Omega,\lambda)\leqslant\sum_{j\in\mathcal{J}} N_{\mathrm{N}}(Q_j,\lambda)$, where $\{Q_j\}_{j\in\mathcal{J}_0}$ is the set of cubes Q_j lying inside Ω. If $\mu_d(\partial\Omega)=0$ then, estimating $N_{\mathrm{D}}(Q_j,\lambda)$ and $N_{\mathrm{N}}(Q_j,\lambda)$ for each j and taking $\delta=\lambda^{-1}$, we obtain (0.1) and (0.2) for the Dirichlet Laplacian. It is possible to get rid of the condition $\mu_d(\partial\Omega)=0$ but this requires additional arguments.

Similarly, the Rayleigh–Ritz formula implies that
$\sum_{j\in\mathcal{J}_0} N_{\mathrm{D}}(Q_j,\lambda)\leqslant N_{\mathrm{N}}(\Omega,\lambda)\leqslant\sum_{j\in\mathcal{J}_{m\delta}} N_{\mathrm{N}}(Q_j,\lambda)+N_{\mathrm{N}}(\bigcup_{j\in\mathcal{J}\setminus\mathcal{J}_{m\delta}} Q_j\bigcap\Omega,\lambda)$, where $\{Q_j\}_{j\in\mathcal{J}_{m\delta}}$ is the set of cubes lying inside $\Omega\setminus\Omega_{m\delta}^{\mathrm{b}}$. If for some $m\in\mathbb{N}$ and all sufficiently small positive δ there exist uniformly bounded extension operators from the Sobolev space $W^{1,2}(\Omega_{m\delta}^{\mathrm{b}})$ to $W^{1,2}(\Omega_{m\delta}^{\mathrm{b}}\bigcup\Omega_\delta^{\mathrm{e}})$ then $N_{\mathrm{N}}(\bigcup_{j\in\mathcal{J}\setminus\mathcal{J}_{m\delta}} Q_j\bigcap\Omega,\lambda)\leqslant N_{\mathrm{N}}(\bigcup_{j\in\mathcal{J}\setminus\mathcal{J}_{m\delta}} Q_j,C\lambda)=\sum_{j\in\mathcal{J}\setminus\mathcal{J}_{m\delta}} N_{\mathrm{N}}(Q_j,C\lambda)$, where C is a sufficiently large constant. If, in addition, $\mu_d(\partial\Omega)=0$ then, estimating the counting functions on the cubes and taking $\delta=\lambda^{-1}$, we obtain (0.1) and (0.2) for $N_{\mathrm{N}}(\Omega,\lambda)$.

However, the known extension theorems require certain regularity conditions on the boundary (for instance, it is sufficient to assume that $\partial\Omega$ belongs to the Lipschitz class or satisfies the cone condition). Domains with very irregular boundaries do not have the $W^{1,2}$-extension property, in which case the above scheme does not work the Neumann Laplacian. To the best of our knowledge, in all papers devoted to the Weyl formula for $N_{\mathrm{N}}(\Omega,\lambda)$ the authors either implicitly assumed that the domain has the $W^{1,2}$-extension property or directly applied a suitable extension theorem.

The main aim of this paper is to introduce a different technique which does not use an extension theorem. Instead of disjoint cubes, we cover the domain Ω by a family of relatively simple sets $S_m \subset \Omega$. For each of these sets the counting function $N(S_m,\lambda)$ can be effectively estimated from below and above. The sets S_m may overlap but, under certain conditions on Ω, the multiplicity of their intersection does not exceed a constant depending only on the dimension d.

This allows us to apply the Dirichlet–Neumann bracketing and obtain the Weyl asymptotic formula with a remainder estimate for the Neumann Laplacian on domains without the extension property (Theorem 1.3). The remainder term in this formula may well be of higher order than the first term. Then our asymptotic formula turns into an estimate for $N_{\mathrm{N}}(\Omega,\lambda)$. In particular, this may happen if $\Omega \in \mathrm{Lip}_\alpha$, that is, if $\partial\Omega$ coincides with the subgraph of a Lip_α-function in a neighbourhood of each boundary point. We prove that $N_{\mathrm{N}}(\Omega,\lambda) - C_{d,W}\,\mu_d(\Omega)\,\lambda^d = O(\lambda^{(d-1)/\alpha})$ whenever $\Omega \in \mathrm{Lip}_\alpha$ and $\alpha \in (0,1)$ (Corollary 1.6) and that this estimate is order sharp (Theorem 1.10). If $(d-1)/\alpha < d$ then the right hand side is $o(\lambda^d)$ and we have (0.1), otherwise $N_{\mathrm{N}}(\Omega,\lambda) = O(\lambda^{(d-1)/\alpha})$.

We also obtain a remainder estimate in (0.1) for the Dirichlet Laplacian (Theorem 1.8). This estimate holds true for all bounded domains and immediately implies (0.2).

For domains with smooth boundaries our variational method only gives the remainder estimate $O(\lambda^{d-1}\log\lambda)$; in order to obtain $O(\lambda^{d-1})$ one has to use more sophisticated results (see above). On the other hand, it can be applied to many other problems and combined with the technique developed in [BI], [Iv3], [Iv4], [Me], [Mi], [SV] or [Z] (see Section 5).

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1. Definitions and main results

1.1. **Basic definitions and notation.** Throughout the paper we assume that Ω is a bounded open connected subset (domain) of the d-dimensional Euclidean space $\mathbb{R}^d$ and that $d \geqslant 2$.

We shall be using the following notation.

- ω_d is the volume of the unit ball in $\mathbb{R}^d$ and $C_{d,W} := (2\pi)^{-d}\,\omega_d$ is the standard Weyl constant.
- If $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ then $x' := (x_1,\dots,x_{d-1})$ so that $x = (x',x_d)$.
- $\overline{\Omega}$ and $\partial\Omega$ are the closure and the boundary of Ω.
- $\mu_d(\Omega)$ denotes the d-dimensional volume of Ω and $D_\Omega := \operatorname{diam}\Omega$.

- $\operatorname{dist}(\Omega_1,\Omega_2) := \inf\limits_{x\in\Omega_1,\, y\in\Omega_2} |x-y|$ is the standard Euclidean distance.
- $\Omega^{\mathrm{b}}_\varepsilon := \{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)\leqslant\varepsilon\}$.
- C is the space of continuous functions.
- If Ω' is a $(d-1)$-dimensional domain, $f\in C(\overline{\Omega'})$, $b\in\mathbb{R}$ and $\alpha\in(0,1]$ then

$$\begin{aligned}\Gamma_f &:= \{x\in\mathbb{R}^d \mid x_d = f(x'),\ x'\in\Omega'\},\\ G_f &:= \{x\in\mathbb{R}^d \mid x_d < f(x'),\ x'\in\Omega'\},\\ G_{f,b} &:= \{x\in G_f \mid x_d > b\},\end{aligned}$$

$$\operatorname{Osc}(f,\Omega') := \tfrac12\big(\sup_{x\in\Omega'} f(x) - \inf_{x\in\Omega'} f(x)\big) \quad\text{and}\quad |f|_\alpha := \sup_{x,\,y\in\Omega'} \tfrac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

- $Q^{(n)}_a$ is the open n-dimensional cube with edges of length a parallel to the coordinate axes. If the size or the dimension of the cube $Q^{(n)}_a$ is not important for our purposes or evident from the context then we shall omit the corresponding index a or n. However, we shall always be assuming that the cube is open and that its edges are parallel to the coordinate axes.
- Lip_α is the space of functions f on a cube Q such that $|f|_\alpha<\infty$ and lip_α is the closure of Lip_1 in Lip_α with respect to the seminorm $|\cdot|_\alpha$.

Definition 1.1. Given a bounded function f on the cube $Q^{(n)}$ and $\delta>0$, we shall denote by $\mathcal{V}_\delta(f,Q^{(n)})$ the maximal number of disjoint cubes $Q^{(n)}(i)\subset Q^{(n)}$ such that $\operatorname{Osc}(f,Q^{(n)}(i))\geqslant\delta$ for each i. If $\operatorname{Osc}(f,Q^{(n)})<\delta$ then we define $\mathcal{V}_\delta(f,Q^{(n)}):=1$.

Definition 1.2. If τ is a positive nondecreasing function on $(0,+\infty)$, let $BV_{\tau,\infty}(Q)$ be the space spanned by all continuous functions f on $\overline{Q}$ such that $\mathcal{V}_{1/t}(f,Q)\leqslant\tau(t)$ for all $t>0$.

We shall briefly discuss the relation between $BV_{\tau,\infty}(Q)$ and known function spaces in Subsection 5.3.

Let X be a space of continuous real-valued functions defined on a cube $Q^{(d-1)}$. We shall say that Ω belongs to the class X and write $\Omega\in X$ if for each $z\in\partial\Omega$ there exists a neighbourhood $\mathcal{O}_z$ of the point z, a linear orthogonal map $U:\mathbb{R}^d\to\mathbb{R}^d$, a cube $Q^{(d-1)}_a\subset Q^{(d-1)}$, a function $f\in X$ and $b\in\mathbb{R}$ such that $U(\mathcal{O}_z\bigcap\Omega)=\{x\in G_{f,b}\mid x'\in Q^{(d-1)}_a\}$.

Since $\partial\Omega$ is compact, for every bounded set $\Omega\in BV_{\tau,\infty}$ there exists a finite collection of domains $\Omega_l\subset\Omega$, $l\in\mathcal{L}$, such that

(a) $\partial\Omega\subset\bigcup_{l\in\mathcal{L}}\overline{\Omega_l}$;
(b) for each l we have $U_l(\Omega_l)=G_{f_l,b_l}$, where $U_l:\mathbb{R}^d\to\mathbb{R}^d$ is a linear orthogonal map, $f_l\in BV_{\tau,\infty}(Q^{(d-1)}_{a_l})$ and $b_l<\inf f_l$;
(c) $a_l\leqslant D_\Omega$ and $\sup f_l - b_l\leqslant D_\Omega$ for all $l\in\mathcal{L}$.

Let us fix such a collection $\{\Omega_l\}_{l\in\mathcal{L}}$ and denote $n_\Omega := \#\mathcal{L}$ and

$$C_{\Omega,\tau} \;:=\; \sum_{l\in\mathcal{L}} \sup_{t>0} \left(\mathcal{V}_{1/t}(f_l, Q_{a_l}^{(d-1)})/\tau(t)\right) .$$

Let δ_Ω be the largest positive number such that $\Omega^{\mathrm{b}}_{\delta_\Omega} \subset \bigcup_{l\in\mathcal{L}} \Omega_l$, $\delta_\Omega \leqslant \sqrt{d}\, a_l$ and $2\delta_\Omega \leqslant \inf f_l - b_l$ for all $l \in \mathcal{L}$.

1.2. **Main results.** Throughout the paper we shall denote by C_d various constants depending only on the dimension d. Constants appearing in the most important estimates are numbered by an additional lower index; in our opinion, this makes our proofs more transparent. Their precise (but not necessarily best possible) values are given in Section 6.

Theorem 1.3. *If* $\Omega \in BV_{\tau,\infty}$ *and* $\lambda \geqslant \delta_\Omega^{-1}$ *then*

$$\begin{aligned}(1.1)\quad & |\, N_{\mathrm{N}}(\Omega,\lambda) - C_{d,W}\, \mu_d(\Omega)\, \lambda^d \,| \\ &\leqslant\; C_{d,9}\, C_{\Omega,\tau}\, n_\Omega^{1/2} \lambda \int_{(2D_\Omega)^{-1}}^{C_\Omega\, \lambda} t^{-2}\, \tau(t)\, \mathrm{d}t \;+\; C_{d,10}\, n_\Omega\, \lambda^{d-1} \int_0^{C_\Omega\, \lambda} \mu_d(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t\,,\end{aligned}$$

where $C_\Omega := 4\, C_{d,8}\, n_\Omega^{1/2}$ *. If, in addition,* $\Omega \subset \mathbb{R}^2$ *then there exists a positive constant* c *independent of* Ω *such that*

$$\begin{aligned}(1.2)\quad & |\, N_{\mathrm{N}}(\Omega,\lambda) - (4\pi)^{-1} \mu_2(\Omega)\, \lambda^2 \,| \;\leqslant\; c\, C_{\Omega,\tau}\, \tau(c\, n_\Omega^{1/2} \lambda) \\ &\qquad +\; c\, n_\Omega\, \lambda \left(D_\Omega + \int_0^{c\, n_\Omega^{1/2} \lambda} \mu_2(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t \right), \qquad \forall \lambda \geqslant \delta_\Omega^{-1} .\end{aligned}$$

Remark 1.4. For each continuous function f on a closed cube there exists a positive nondecreasing function τ such that $f \in BV_{\tau,\infty}$. Therefore Theorem 1.3 allows one to obtain an estimate of the form (1.1) for every domain $\Omega \in C$. In particular, this implies the following well known result: if $\Omega \in C$ then the essential spectrum of the Neumann Laplacian on Ω is empty.

The next two corollaries are simple consequences of Theorem 1.3.

Corollary 1.5. *If* $\Omega \in BV_{\tau,\infty}$ *then there exists a constant* C_Ω *such that*

$$\begin{aligned}(1.3)\quad & |\, N_{\mathrm{N}}(\Omega,\lambda) - C_{d,W}\, \mu_d(\Omega)\, \lambda^d \,| \\ &\qquad\qquad \leqslant\; C_\Omega\, \lambda^{d-1} \int_{C_\Omega^{-1}}^{C_\Omega \lambda} \left(t^{-1} + t^{-d}\, \tau(t)\right) \mathrm{d}t\,, \qquad \forall \lambda \geqslant C_\Omega .\end{aligned}$$

Corollary 1.6. *If* $\alpha \in (0,1)$ *and* $\Omega \in \mathrm{Lip}_\alpha$ *then*

$$(1.4)\qquad N_{\mathrm{N}}(\Omega,\lambda) \;=\; C_{d,W}\, \mu_d(\Omega)\, \lambda^d \;+\; O\left(\lambda^{(d-1)/\alpha}\right), \qquad \lambda \to +\infty.$$

If $\alpha \in (0,1)$ *and* $\Omega \in \mathrm{lip}_\alpha$ *then*

$$(1.5)\qquad N_{\mathrm{N}}(\Omega,\lambda) \;=\; C_{d,W}\, \mu_d(\Omega)\, \lambda^d \;+\; o\left(\lambda^{(d-1)/\alpha}\right), \qquad \lambda \to +\infty.$$

Remark 1.7. If $\alpha \leqslant 1-d^{-1}$ then the asymptotic formula (1.4) turns into the estimate $N_{\mathrm{N}}(\Omega,\lambda) = O\left(\lambda^{(d-1)/\alpha}\right)$. Similarly, if $\alpha < 1-d^{-1}$ then (1.5) takes the form $N_{\mathrm{N}}(\Omega,\lambda) = o\left(\lambda^{(d-1)/\alpha}\right)$.

The following estimates for the Dirichlet Laplacian are much simpler. The inequality (1.6) seems to be new but results of this type are known to experts. Corollary 1.9 is an immediate consequence of Theorem 1.8; (1.7) also follows from (0.2).

Theorem 1.8. *For all* $\lambda > 0$ *we have*

$$|\, N_{\mathrm{D}}(\Omega,\lambda) - C_{d,W}\, \mu_d(\Omega)\, \lambda^d \,| \;\leqslant\; C_{d,11}\, \lambda^{d-1} \int_0^\lambda \mu_d(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t\,. \tag{1.6}$$

Corollary 1.9. *If* $\alpha \in (0,1)$ *and* $\Omega \in \mathrm{Lip}_\alpha$ *then*

$$N_{\mathrm{D}}(\Omega,\lambda) \;=\; C_{d,W}\, \mu_d(\Omega)\, \lambda^d \;+\; O\left(\lambda^{d-\alpha}\right), \qquad \lambda \to +\infty. \tag{1.7}$$

If $\alpha \in (0,1)$ *and* $\Omega \in \mathrm{lip}_\alpha$ *then*

$$N_{\mathrm{D}}(\Omega,\lambda) \;=\; C_{d,W}\, \mu_d(\Omega)\, \lambda^d \;+\; o\left(\lambda^{d-\alpha}\right), \qquad \lambda \to +\infty. \tag{1.8}$$

Note that $(d-1)/\alpha > d-\alpha$ whenever $\alpha \in (0,1)$. Therefore the remainder estimate in Corollary 1.9 is better than that in Corollary 1.6. The following theorem shows that the asymptotic formulae (1.4) and (1.5) are order sharp.

Theorem 1.10. *Let* $\alpha \in (0,1)$. *Then*

(1) *there exist a bounded domain* $\Omega \in \mathrm{Lip}_\alpha$ *and a positive constant* C_Ω *such that* $N_{\mathrm{N}}(\Omega,\lambda) \geqslant C_{d,W}\, \mu_d(\Omega)\, \lambda^d + C_\Omega^{-1}\, \lambda^{(d-1)/\alpha}$ *for all* $\lambda > C_\Omega$ *;*
(2) *for each nonnegative function* ϕ *on* $(0,+\infty)$ *vanishing at* $+\infty$ *there exist a bounded domain* $\Omega \in \mathrm{lip}_\alpha$ *and a positive constant* $C_{\phi,\Omega}$ *such that* $N_{\mathrm{N}}(\Omega,\lambda) \geqslant C_{d,W}\, \mu_d(\Omega)\, \lambda^d + C_{\phi,\Omega}^{-1}\, \phi(\lambda)\, \lambda^{(d-1)/\alpha}$ *for all* $\lambda > C_{\phi,\Omega}$.

Remark 1.11. In [BD] the authors proved that

$$0 < K_{\Omega,\mathrm{N}}(t,x,y) \leqslant C_\Omega\, t^{-(\alpha+d-1)/(2\alpha)}\,, \qquad \forall x,y \in \Omega, \quad \forall t \in (0,1], \tag{1.9}$$

whenever $\Omega \in \mathrm{Lip}_\alpha$ and $\alpha \in (0,1)$, where $K_{\Omega,\mathrm{N}}$ is the heat kernel of the Neumann Laplacian on Ω and C_Ω is a constant depending on Ω. The estimate (1.9) is order sharp as $t \to 0$ (see [BD], Example 6). Corollary 1.6 implies that there exists a constant C'_Ω such that

$$\int_\Omega K_{\Omega,\mathrm{N}}(t,x,x)\, \mathrm{d}x \leqslant C'_\Omega\, (t^{-d/2} + t^{-(d-1)/(2\alpha)})\,, \qquad \forall t \in (0,1].$$

In view of Theorem 1.10, this estimate is also order sharp. Since $d/2 < (\alpha+d-1)/(2\alpha)$ and $(d-1)/(2\alpha) < (\alpha+d-1)/(2\alpha)$, we see that integration of the heat kernel $K_{\Omega,\mathrm{N}}(t,x,x)$ improves its asymptotic properties.

1.3. **Further definitions and notation.** In the rest of the paper

- $\#T$ denotes the number of elements of the set T.
- If $\{T(i)\}_{i\in\mathcal{I}}$ is a finite family of sets $T(i)$ and $T := \bigcup_{i\in\mathcal{I}} T(i)$ then
$$\aleph\{T(i)\} := \sup_{x\in T} \left(\#\{i \in \mathcal{I} \,|\, x \in T(i)\}\right),$$
in other words, $\aleph\{T(i)\}$ is the multiplicity of the covering $\{T(i)\}_{i\in\mathcal{I}}$.
- If $s \in \mathbb{R}_+$ then $[s]$ is the entire part of s.
- $\operatorname{supp} f$ and ∇f denote the support and gradient of the function f.

The paper is organised as follows. In the next section we recall some well known results from spectral theory and estimate the counting function on 'model' domains. In Section 3 we discuss partitions of the domain Ω. In Section 4 we deduce the main theorems from the results of Sections 2 and 3. In the last section we extend our results to a wider class of domains and higher order operators and discuss other possible generalizations.

2. Variational formulae and related results

Recall that the Sobolev space $W^{1,2}(\Omega)$ is the space of functions $u \in L^2(\Omega)$ such that $\nabla u \in L^2(\Omega)$, endowed with the norm
$$\|u\|_{W^{1,2}(\Omega)} = (\|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{1/2}.$$
If Υ is a subset of $\partial\Omega$, let $W^{1,2}_{0,\Upsilon}(\Omega)$ be the closure in $W^{1,2}(\Omega)$ of the set
$$\{f \in W^{1,2}(\Omega) \mid \ \operatorname{supp} f \bigcap \Upsilon = \emptyset\}$$
and $W^{1,2}_0(\Omega) := W^{1,2}_{0,\partial\Omega}(\Omega)$. Obviously, $W^{1,2}_{0,\emptyset}(\Omega) = W^{1,2}(\Omega)$.

Let
$$N_{\mathrm{N,D}}(\Omega, \Upsilon, \lambda) \ := \ \sup(\dim E_\lambda) \tag{2.1}$$
where the supremum is taken over all subspaces $E_\lambda \subset W^{1,2}_{0,\Upsilon}(\Omega)$ such that
$$\|\nabla u\|^2_{L_2(\Omega)} \ < \ \lambda^2\, \|u\|^2_{L^2(\Omega)}\,, \qquad \forall u \in E_\lambda\,. \tag{2.2}$$
In view of the Rayleigh–Ritz variational formula, $N_{\mathrm{N,D}}(\Omega, \Upsilon, \lambda)$ can be thought of as the counting function of the Laplacian on the bounded domain Ω subject to Dirichlet boundary condition on Υ and Neumann boundary condition on the remaining part of the boundary. In particular, $N_{\mathrm{N,D}}(\Omega, \emptyset, \lambda) = N_{\mathrm{N}}(\Omega, \lambda)$ and $N_{\mathrm{N,D}}(\Omega, \partial\Omega, \lambda) = N_{\mathrm{D}}(\Omega, \lambda)$. Equivalently, (2.1) can be rewritten as
$$N_{\mathrm{N,D}}(\Omega, \Upsilon, \lambda) \ = \ \inf(\operatorname{codim} \tilde{E}_\lambda), \tag{2.3}$$
where the infimum is taken over all subspaces $\tilde{E}_\lambda \subset W^{1,2}_{0,\Upsilon}(\Omega)$ such that
$$\|\nabla u\|^2_{L_2(\Omega)} \ \geqslant \ \lambda^2\, \|u\|^2_{L^2(\Omega)}\,, \qquad \forall u \in \tilde{E}_\lambda\,. \tag{2.4}$$

Lemma 2.1. *Let $\{\Omega_i\}_{i\in\mathcal{I}}$ be a countable family of disjoint open sets $\Omega_j\subset\Omega$ such that $\mu_d(\Omega)=\mu_d(\bigcup_{i\in\mathcal{I}}\Omega_i)$. Then*

$$\sum_{i\in\mathcal{I}} N_{\mathrm{D}}(\Omega_i,\lambda) \;\leqslant\; N_{\mathrm{D}}(\Omega,\lambda) \;\leqslant\; N_{\mathrm{N}}(\Omega,\lambda) \;\leqslant\; \sum_{i\in\mathcal{I}} N_{\mathrm{N}}(\Omega_i,\lambda)$$

and $N_{\mathrm{N}}(\Omega,\lambda)\geqslant\sum_{j\in\mathcal{J}}N_{\mathrm{N,D}}(\Omega_j,\partial\Omega_j\setminus\partial\Omega,\lambda)$.

Lemma 2.1 is an elementary corollary of the Rayleigh–Ritz formula. The following lemma is less obvious.

Lemma 2.2. *Let $\{\Omega_i\}_{i\in\mathcal{I}}$ be a countable family of open sets $\Omega_j\subset\Omega$ such that $\mu_d(\Omega)=\mu_d(\bigcup_{i\in\mathcal{I}}\Omega_i)$, Υ be an arbitrary subset of $\partial\Omega$ and $\Upsilon_j:=\partial\Omega_j\bigcap\Upsilon$. If $\aleph\{\Omega_j\}\leqslant\varkappa<+\infty$ then $N_{\mathrm{N,D}}(\Omega,\Upsilon,\varkappa^{-1/2}\lambda)\leqslant\sum_{j\in\mathcal{J}}N_{\mathrm{N,D}}(\Omega_j,\Upsilon_j,\lambda)$.*

Proof. Denote by $\tilde E_{\lambda,j,\Omega}$ the subspace of functions $u\in W^{1,2}_{0,\Upsilon}(\Omega)$ such that $\|\nabla u\|^2_{L_2(\Omega_j)}\geqslant\lambda^2\,\|u\|^2_{L^2(\Omega_j)}$. We have $\varkappa\,\|u\|^2_{L^{1,2}(\Omega)}\geqslant\lambda^2\,\|u\|^2_{L^2(\Omega)}$ whenever $u\in\bigcap_{j\in\mathcal{J}}\tilde E_{\lambda,j,\Omega}$. Therefore, by (2.3),

$$N_{\mathrm{N}}(\Omega,\varkappa^{-1/2}\lambda)\;\leq\;\inf(\operatorname{codim}\bigcap_{j\in\mathcal{J}}\tilde E_{\lambda,j,\Omega})\;\leq\;\sum_{j\in\mathcal{J}}\inf(\operatorname{codim}\tilde E_{\lambda,j,\Omega}),$$

where the infimum are taken over all subspaces $\tilde E_{\lambda,j,\Omega}$ satisfying the above condition. If $\tilde E_{\lambda,j}$ is the intersection of the kernels of linear continuous functionals Λ_k on $W^{1,2}_{0,\Upsilon_j}(\Omega_k)$ and $E_{\lambda,j,\Omega}$ is the intersection of the kernels of linear continuous functionals $u\to\Lambda_k(u|_{\Omega_j})$ on $W^{1,2}_{0,\Upsilon}(\Omega)$ then $\operatorname{codim}\tilde E_{\lambda,j}\geqslant\operatorname{codim}E_{\lambda,j,\Omega}$ and $u|_{\Omega_j}\in\tilde E_{\lambda,j}$ whenever $u\in E_{\lambda,j,\Omega}$. This observation and (2.3) imply that $\inf(\operatorname{codim}\tilde E_{\lambda,j,\Omega})\leq N_{\mathrm{N,D}}(\Omega_j,\Upsilon_j,\lambda)$. □

Remark 2.3. Lemma 2.2 implies that $N_{\mathrm{N}}(\Omega,\varkappa^{-1/2}\lambda)\leqslant\sum_{j\in\mathcal{J}}N_{\mathrm{N}}(\Omega_j,\lambda)$ whenever $\bigcup_{j\in\mathcal{J}}\Omega_j\subset\Omega$, $\mu_d(\Omega)=\mu_d(\bigcup_{i\in\mathcal{I}}\Omega_i)$ and $\aleph\{\Omega_j\}\leqslant\varkappa$. It may well be the case that, under these conditions, $N_{\mathrm{N}}(\Omega,\lambda)\leqslant\sum_{j\in\mathcal{J}}N_{\mathrm{N}}(\Omega_j,\lambda)$. This conjecture looks plausible and is equivalent to the following statement: if $\Omega_1\subset\Omega$, $\Omega_2\subset\Omega$ and $\mu_d(\Omega)=\mu_d(\Omega_1)+\mu_d(\Omega_2)$ then $N_{\mathrm{N}}(\Omega_1,\lambda)+N_{\mathrm{N}}(\Omega_2,\lambda)\geqslant N_{\mathrm{N}}(\Omega,\lambda)$.

Remark 2.4. The first eigenvalue of the Neumann Laplacian $-\Delta_{\mathrm{N}}$ is always equal to 0 and the corresponding eigenfunction is identically equal to constant. Let $\lambda_{1,\mathrm{N}}(\Omega):=\inf\{\lambda\in\mathbb{R}_+\,|\,N_{\mathrm{N}}(\Omega,\lambda)>1\}$; if $-\Delta_{\mathrm{N}}$ has at least two eigenvalues lying below its essential spectrum (or the essential spectrum is empty) then $\lambda_{1,\mathrm{N}}(\Omega)$ coincides with the smallest nonzero eigenvalue of the operator $\sqrt{-\Delta_{\mathrm{N}}}$. By the spectral theorem, we have $\lambda_{1,\mathrm{N}}(\Omega)\geqslant\lambda$ if and only if $\int_\Omega|u(x)|^2\,\mathrm{d}x\leqslant\lambda^{-2}\int_\Omega|\nabla u(x)|^2\,\mathrm{d}x$ for all functions $u\in W^{1,2}(\Omega)$ such that $\int_\Omega u(x)\,\mathrm{d}x=0$. Note that $\int_\Omega|u(x)|^2\,\mathrm{d}x\leqslant\int_\Omega|u(x)-c|^2\,\mathrm{d}x$ for all $c\in\mathbb{C}$ whenever $\int_\Omega u(x)\,\mathrm{d}x=0$.

Definition 2.5. Denote by $\mathbf{P}(\delta)$ the set of all rectangles with edges parallel to the coordinate axes, such that the length of the maximal edge does not exceed δ. If f is a continuous function on $\overline{Q^{(d-1)}}$, let $\mathbf{V}(\delta, f)$ be the class of domains $V \subset G_f$ which can be represented in the form $V = G_{f,b}(Q_c^{(d-1)})$, where $Q_c^{(d-1)} \subset Q^{(d-1)}$, $c \leqslant \delta$, $b = \inf f - \delta$ and $\operatorname{Osc}(f, Q_c^{(d-1)}) \leqslant \delta/2$. We shall write $V \in \mathbf{V}(\delta)$ if $V \in \mathbf{V}(\delta, f)$ for some continuous function f. Finally, let $\mathbf{M}(\delta)$ be the class of open sets $M \subset \mathbb{R}^d$ such that $M \subset Q_\delta^{(d)}$ for some cube $Q_\delta^{(d)}$.

Lemma 2.6. *Let δ be an arbitrary positive number.*

(1) *If $P \in \mathbf{P}(\delta)$ then $N_{\mathrm{N}}(P, \lambda) = 1$ for all $\lambda \leqslant \pi\delta^{-1}$.*
(2) *If $V \in \mathbf{V}(\delta)$ then $N_{\mathrm{N}}(V, \lambda) = 1$ for all $\lambda \leqslant (1 + 2\pi^{-2})^{-1/2}\delta^{-1}$.*
(3) *If $M \in \mathbf{M}(\delta)$, $M \subset Q_\delta^{(d)}$ and $\Upsilon := \partial M \bigcap Q_\delta^{(d)}$ then we have $N_{\mathrm{N,D}}(M, \Upsilon, \lambda) \leqslant 1$ for all $\lambda \leqslant \pi\delta^{-1}$ and $N_{\mathrm{N,D}}(M, \Upsilon, \lambda) = 0$ for all $\lambda \leqslant (2^{-1} - 2^{-1}\delta^{-d}\mu_d(M))^{1/2}\,\pi\delta^{-1}$.*

Proof. If P is a rectangle then $\lambda_{1,\mathrm{N}} = \pi\, a^{-1}$, where a is the length of its maximal edge. This implies (1).

Assume now that $V \in \mathbf{V}(\delta, f)$, where f is a continuous function on $\overline{Q_c^{(d-1)}}$ and denote $b := \inf f - \delta$ and $P := Q_c^{(d-1)} \times (b, b+\delta)$. Clearly, $P \in \mathbf{P}(\delta)$. Let $u \in W^{1,2}(V)$ and c'_u the average of u over P. If $r \in [b, b+\delta]$ and $s \in [b+\delta, f(x')]$ then, by Jensen's inequality,

$$|u(x',s) - u(x',r)|^2 \;=\; \Big|\int_r^s \partial_t\, u(x',t)\,\mathrm{d}t\,\Big|^2 \;\leqslant\; (s-r)\int_b^{f(x')} |\partial_t\, u(x',t)|^2\,\mathrm{d}t\,.$$

Since $\int_b^{b+\delta}\int_{b+\delta}^{f}(s-r)\,\mathrm{d}s\,\mathrm{d}r = (\delta/2)\,(f-b-\delta)\,(f-b)$ and

$$0 \leqslant f - b - \delta = f - \inf f \leqslant 2\operatorname{Osc}(f, Q_c^{(d-1)}) \leqslant \delta\,,$$

we have

$$\int_b^{g(x')}\int_{g(x')}^{f(x')} |u(x',s) - u(x',r)|^2\,\mathrm{d}s\,\mathrm{d}r \leqslant\; \delta^3\int_b^{f(x')} |\partial_t\, u(x',t)|^2\,\mathrm{d}t\,.$$

In view of Remark 2.4 and (1), we also have

$$\int_P |u(x) - c'_u|^2\,\mathrm{d}x \;\leqslant\; \pi^{-2}\,\delta^2\int_P |\nabla u(x)|^2\,\mathrm{d}x. \tag{2.5}$$

Integrating the inequality

$$|u(x',s) - c'_u|^2 \;\leqslant\; (1+\gamma)\,|u(x',r) - c'_u|^2 + (1+\gamma^{-1})\,|u(x',s) - u(x',r)|^2$$

over $r \in [b, b+\delta]$, $s \in [b+\delta, f(x')]$ and $x' \in \Omega'$ and applying these two estimates, we obtain

$$\delta \int_{V\setminus P} |u(x) - c'_u|^2 \,\mathrm{d}x \ \leqslant (1+\gamma)\,\pi^{-2}\,\delta^3 \int_P |\nabla u(x)|^2 \,\mathrm{d}x \\ + (1+\gamma^{-1})\,\delta^3 \int_V |\partial_{x_d} u(x)|^2 \,\mathrm{d}x$$

for all $\gamma > 0$. Dividing both sides by δ and substituting $\gamma = \pi^2$, we see that $\int_{V\setminus P} |u(x) - c'_u|^2 \,\mathrm{d}x$ is estimated by $(1+\pi^{-2})\,\delta^2 \int_V |\nabla u(x)|^2 \,\mathrm{d}x$. Now (2) follows from (2.5) and Remark 2.4.

In order to prove (3), let us consider a function $u \in W^{1,2}(M)$ which vanishes near Υ and extend it by zero to the whole cube $Q_\delta^{(d)}$. Since $u \in W^{1,2}(Q_\delta^{(d)})$, (1) implies the first inequality (3). If c_u is the average of u over $Q_\delta^{(d)}$ then

$$\int_M |c_u|^2 \,\mathrm{d}x \ \leqslant\ \mu_d(M)\,\delta^{-d} \left(\int_M |c_u|^2 \,\mathrm{d}x + \int_{Q_\delta^{(d)}} |u(x) - c_u|^2 \,\mathrm{d}x \right). \tag{2.6}$$

Therefore Remark 2.4 and (1) imply that

$$\int_M |u(x)|^2 \,\mathrm{d}x \ \leqslant\ 2 \int_{Q_\delta^{(d)}} |u(x) - c_u|^2 \,\mathrm{d}x + 2 \int_M |c_u|^2 \,\mathrm{d}x \\ \leqslant\ 2 \left(1 + \mu_d(M)\,\delta^{-d} \left(1 - \mu_d(M)\,\delta^{-d}\right)^{-1} \right) \int_{Q_\delta^{(d)}} |u(x) - c_u|^2 \,\mathrm{d}x \\ \leqslant\ 2\,\pi^{-2}\,\delta^2 \left(1 - \mu_d(M)\,\delta^{-d}\right)^{-1} \int_M |\nabla u(x)|^2 \,\mathrm{d}x\,.$$

The second identity (3) follows from the above inequality and the Rayleigh–Ritz formula. □

Remark 2.7. The second estimate in Lemma 2.6(3) is sufficient for our purposes but is very rough. One can obtain a much more precise result in terms of capacities (see [M2], Chapter 10, Section 1).

Lemma 2.8. *Let* $\delta > 0$. *Then for all* $\lambda > 0$ *we have*

$$-\,C_{d,1}\left((\delta\lambda)^{d-1} + 1\right) \ \leqslant\ N(Q_\delta^{(d)}, \lambda) - C_{d,W}\,(\delta\lambda)^d \ \leqslant\ C_{d,1}\left((\delta\lambda)^{d-1} + 1\right).$$

Proof. Changing variables $\tilde{x} = \delta\,x$, we see that

$$N(\Omega, \delta\lambda) \ =\ N(\delta\Omega, \lambda)\,, \quad \text{where} \quad \delta\Omega := \{x \in \mathbb{R}^d \mid \delta^{-1}x \in \Omega\}\,. \tag{2.7}$$

Therefore it is sufficient to prove the required estimates only for $\delta = 1$. If $\Omega = \Omega' \times \Omega''$, $\Upsilon' \subset \partial\Omega'$ and $\Upsilon'' \subset \partial\Omega''$ then, separating variables, we obtain

$$N_{\mathrm{N,D}}(\Omega, \Upsilon, \lambda) \ =\ \int N_{\mathrm{N,D}}\left(\Omega', \Upsilon', \sqrt{\lambda^2 - \mu^2}\right) \mathrm{d}N_{\mathrm{N,D}}(\Omega'', \Upsilon'', \mu)\,, \tag{2.8}$$

where $\Upsilon = (\Upsilon' \times \partial\Omega'')\bigcup(\partial\Omega' \times \Upsilon'')$ and the right hand side is a Stieltjes integral. Using (2.8), explicit formulae for the counting functions on the unit interval and the identities

$$\int_0^\lambda (\lambda^2-\mu^2)^{n/2}\,\mathrm{d}\mu \;=\; \lambda^{n+1}\,\omega_{n+1}\,(2\,\omega_n)^{-1}\,, \qquad \forall n=1,2,\dots, \tag{2.9}$$

one can easily prove the required inequality by induction in d. □

Remark 2.9. Lemma 2.8 is an immediate consequence of well known results on spectral asymptotics in domains with piecewise smooth boundaries (see, for example, [Iv2] or [F]); a similar result holds true for higher order elliptic operators and operators with variable coefficients [V]. We have given an independent proof in order to find the explicit constant $C_{d,1}$.

3. Properties of domains and their partitions

3.1. **Besicovitch's and Whitney's theorems.** We shall use the following version of Besicovitch's theorem.

Theorem 3.1. *There are two constants $\mathcal{C}_n \geqslant 1$ and $\hat{\mathcal{C}}_n \geqslant 1$ depending only on the dimension n, such that for every compact set $K \subset \mathbb{R}^n$ and every positive function ρ on K one can find a finite subset $\mathcal{Y} \subset K$ and a family of cubes $\{Q^{(n)}_{\rho(y)}[y]\}_{y\in\mathcal{Y}}$ centred on y, which satisfy the following conditions:*

1. $K \subset \bigcup_{y\in\mathcal{Y}} Q^{(n)}_{\rho(y)}[y]$,
2. $\aleph\{K \bigcap Q^{(n)}_{\rho(y)}[y]\}_{y\in\mathcal{Y}} \leqslant \mathcal{C}_n$ *;*
3. *there exists a subset $\hat{\mathcal{Y}} \subset \mathcal{Y}$ such that $\#\mathcal{Y} \leqslant \hat{\mathcal{C}}_n(\#\hat{\mathcal{Y}})$ and the cubes $\{Q^{(n)}_{\rho(y)}[y]\}_{y\in\hat{\mathcal{Y}}}$ are mutually disjoint.*

Theorem 3.1 is proved in the same way as Besicovitch's theorem in [G], Chapter 1.

Corollary 3.2. *Let f be a continuous function on the closure $\overline{Q^{(d-1)}}$. Then for every $\varepsilon > 0$ there exists a finite family of cubes $\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}}$ such that*

1. $\bigcup_{x\in\mathcal{X}} \overline{Q^{(d-1)}}(x) = \overline{Q^{(d-1)}}$*;*
2. $\aleph\{Q^{(d-1)}(x)\} \leqslant C_{d,2}$*;*
3. $\#\mathcal{X} \leqslant C_{d,3}\,\mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}})$*;*
4. $\operatorname{Osc}(f, Q^{(d-1)}(x)) \leqslant \varepsilon$ *for each $x \in \mathcal{X}$.*

Proof. Without loss of generality we can assume that $Q^{(d-1)} = (-1,1)^{d-1}$ and $\operatorname{Osc}(f, Q^{(d-1)}) > \varepsilon$. Let us denote by $Q^{(d-1)}_t[y]$ the cube of the size t centred on y, define

$$\rho(y) := \inf\{t > 0 \mid \operatorname{Osc}(f, Q^{(d-1)} \bigcap Q^{(d-1)}_t[y]) = \varepsilon\}\,, \qquad y \in \overline{Q^{(d-1)}}\,,$$

apply Besicovitch's theorem to the set $K = \overline{Q^{(d-1)}}$ and find the sets $\mathcal{Y}$ and $\hat{\mathcal{Y}}$. If $y \in \mathcal{Y}$, denote $P^{(d-1)}[y] := Q^{(d-1)} \bigcap Q^{(d-1)}_{\rho(y)}[y]$ and assume that

$$P^{(d-1)}[y] \;=\; (a_1(y), b_1(y)) \times (a_2(y), b_2(y)) \times \cdots \times (a_{d-1}(y), b_{d-1}(y))\,,$$

where $-1 \leqslant a_j(y) < b_j(y) \leqslant 1$. Let $Q'(y)$ be the minimal cube such that $P^{(d-1)}(x) \subset Q'(y) \subset Q^{(d-1)}$ and $c(y) := \max_j\,(b_j(y) - a_j(y))$. We have

$$Q'(y) \;=\; (a'_1(y), b'_1(y)) \times (a'_2(y), b'_2(y)) \times \cdots \times (a'_{d-1}(y), b'_{d-1}(y))\,,$$

where

(-1) if $a_j(y) = -1$ then $a'_j(y) = -1$ and $b'_j(y) = a_j(y) + c(y)$;
(0) if $a_j(y) > -1$ and $b_j(y) < 1$ then $a'_j(y) = a_j(y)$ and $b'_j(y) = b_j(y)$;
(+1) if $b_j(y) = 1$ then $a'_j(y) = b_j(y) - c(y)$ and $b'_j(y) = 1$.

Let us consider the set $\Sigma = \{-1, 0, 1\}^{d-1}$ of all $(d-1)$-dimensional vectors $\sigma = (\sigma_1, \ldots, \sigma_{d-1})$ with entries σ_j equal to -1, 0 or 1. Denote by $\hat{\mathcal{Y}}_\sigma$ the set of points $y \in \hat{\mathcal{Y}}$ such that $a_j(y)$ and $b_j(y)$ satisfy the condition (σ_j) for all $j = 1, \ldots, d-1$. Since $\aleph\{P^{(d-1)}[y]\}_{y\in\hat{\mathcal{Y}}} = 1$, for each $\sigma \in \Sigma$ the cubes $\{Q'(y)\}_{y\in\hat{\mathcal{Y}}_\sigma} = 1$ are mutually disjoint. Therefore $\#\hat{\mathcal{Y}}_\sigma \leqslant \mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}})$ for all $\sigma \in \Sigma$ (see Definition 1.1) and, consequently, $\#\hat{\mathcal{Y}} \leqslant (\#\Sigma)\,\mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}}) \leqslant 3^{d-1}\,\mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}})$. This estimate and Theorem 3.1(3) imply that $\#\mathcal{Y} \leqslant 3^{d-1}\,\hat{\mathcal{C}}_{d-1}\,\mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}})$.

Since $\mathcal{Y} \subset \overline{Q^{(d-1)}}$, we have $1/2 \leqslant (b_j(y) - a_j(y))^{-1}(b_k(y) - a_k(y)) \leqslant 2$ for all $j, k = 1, \ldots, d-1$ and $y \in \mathcal{Y}$. Using this inequality, one can easily show by induction in d that every rectangle $P^{(d-1)}[y]$ coincides with the union of a finite collection of cubes $\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}_y}$ such that $\#\mathcal{X}_y \leqslant 2^{d-1}$ and $\aleph\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}_y} \leqslant 2^{d-1}$.

Let $\mathcal{X} := \bigcup_{y\in\mathcal{Y}} \mathcal{X}_y$. In view of the first two conditions of Theorem 3.1, the family $\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}}$ satisfies (1) and (2). The upper bound $\#\mathcal{Y} \leqslant 3^{d-1}\,\hat{\mathcal{C}}_{d-1}\,\mathcal{V}_\varepsilon(f, \overline{Q^{(d-1)}})$ implies (3). Finally, since $\operatorname{Osc}(f, P^{(d-1)}[y]) = \varepsilon$ and $Q^{(d-1)}(x) \subset P^{(d-1)}[y]$ whenever $x \in \mathcal{X}_y$, we have (4). $\square$

The following theorem is due to Whitney. It can be found, for example, in [St], Chapter VI, or [G], Chapter 1.

Theorem 3.3. *There exists a countable family of mutually disjoint cubes* $\{Q^{(d)}_{2^{-i}}(i,n)\}_{n\in\mathcal{N}(i),\,i\in\mathcal{I}}$ *such that* $\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \bigcup_{n\in\mathcal{N}_i} \overline{Q^{(d)}_{2^{-i}}(i,n)}$ *and*

$$Q^{(d)}_{2^{-i}}(i,n) \subset \{x \in \Omega \mid \sqrt{d}\,2^{-i} \leqslant \operatorname{dist}(x, \partial\Omega) \leqslant 4\sqrt{d}\,2^{-i}\}\,. \tag{3.1}$$

Here $\mathcal{I}$ is a subset of $\mathbb{Z}$ and $\mathcal{N}_i$ are some finite index sets.

3.2. **Auxiliary results.** In this subsection we shall prove several technical results concerning domains $G_{f,b}$.

Lemma 3.4. *Let f be a continuous function defined on the closure $\overline{Q_a^{(d-1)}}$. Then for every $\delta > 0$ and $m \in \mathbb{Z}_+$ there exists a finite family of cubes $\{Q^{(d-1)}(k)\}_{k\in\mathcal{K}_m}$ such that*

(1) $\bigcup_{k\in\mathcal{K}_m} \overline{Q^{(d-1)}(k)} = \overline{Q_a^{(d-1)}}$;
(2) $Q^{(d-1)}(k) \in \mathbf{P}(\delta)$ *for all* $k \in \mathcal{K}_m$;
(3) $\aleph\{Q^{(d-1)}(k)\}_{k\in\mathcal{K}_m} \leqslant C_{d,2}$;
(4) $\operatorname{Osc}(f, Q^{(d-1)}(k)) \leqslant 2^{m-1}\delta$ *for all* $k \in \mathcal{K}_m$;
(5) $\#\{k \in \mathcal{K}_m \mid \mu_{d-1}(Q^{(d-1)}(k)) \leqslant 2^{1-d}\,\delta^{d-1}\} \leqslant C_{d,3}\,\mathcal{V}_{2^{m-1}\delta}(f, Q_a^{(d-1)})$.

Proof. Let $\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}}$ be a family of cubes satisfying the conditions of Corollary 3.2 with $\varepsilon = 2^{m-1}\delta$. Assume that $Q^{(d-1)}(x) = Q_{a_x}^{(d-1)}$ with some $a_x > 0$ and denote by $\mathcal{X}_\delta$ the set of all indices $x \in \mathcal{X}$ such that $a_x \leqslant \delta$. For each $x \in \mathcal{X}\setminus\mathcal{X}_\delta$, we choose a positive integer m_x such that $a_x/m_x \in (\delta/2, \delta]$ and split the closed cube $\overline{Q^{(d-1)}(x)}$ into the union of m_x^{d-1} congruent closed cubes $\overline{Q_{a_x/m_x}^{(d-1)}(x,j)}$, $j = 1,\dots,m_x^{d-1}$. Let $Q_{a_x/m_x}^{(d-1)}(x,j)$ be the corresponding disjoint open cubes and

$$\{Q^{(d-1)}(k)\}_{k\in\mathcal{K}} := \{Q^{(d-1)}(k)\}_{x\in\mathcal{X}_\delta} \bigcup \{Q_{a_x/m_x}^{(d-1)}(x,j)\}_{x\in\mathcal{X}\setminus\mathcal{X}_\delta,\, j=1,\dots,m_x^{d-1}} .$$

Then (2) holds true and (1), (3), (4) and (5) follow from Corollary 3.2(1), Corollary 3.2(2), Corollary 3.2(4) and Corollary 3.2(3) respectively. □

Theorem 3.5. *Let f be a continuous function on $\overline{Q_a^{(d-1)}}$, $\delta \in (0, \sqrt{d}\,a]$ and $b \in [-\infty, \inf f - 2\delta]$. Then there exist countable families of sets $\{P_j\}_{j\in\mathcal{J}}$ and $\{V_k\}_{k\in\mathcal{K}}$ satisfying the following conditions:*

(1) $P_j \subset G_{f,b}$ *and* $P_j \in \mathbf{P}(\delta)$ *for all* $j \in \mathcal{J}$;
(2) $V_k \subset G_{f,b}$ *and* $V_k \in \mathbf{V}(\delta, f)$ *for all* $k \in \mathcal{K}$;
(3) $\aleph\{P_j\} \leqslant 3C_{d,2} + 1$ *and* $\aleph\{V_k\} \leqslant C_{d,2}$;
(4) $G_{f,b} \subset \bigcup_{j\in\mathcal{J},\,k\in\mathcal{K}} \left(\overline{P_j} \bigcup \overline{V_k}\right)$;
(5) $\#\{k \in \mathcal{K} \mid \mu_d(V_k) \leqslant 2^{1-d}\,\delta^d\} \leqslant C_{d,3}\,\mathcal{V}_{\delta/2}(f, Q_a^{(d-1)})$ *and*
$\#\{j \in \mathcal{J} \mid \mu_d(P_j) \leqslant (2\sqrt{d})^{-d}\,\delta^d\} \leqslant C_{d,3}\sum_{m=0}^{m_\delta} 2^m\,\mathcal{V}_{2^{m-1}\delta}(f, Q_a^{(d-1)})$,
where $m_\delta := \min\{m \in \mathbb{Z}_+ \mid 2^{m-1}\delta \geqslant \operatorname{Osc}(f, Q_a^{(d-1)})\}$.

Proof. Let $\{Q^{(d-1)}(k)\}_{k\in\mathcal{K}_m}$ be the same families of cubes as in Lemma 3.4, $c_k := \inf_{x\in Q^{(d-1)}(k)} f(x)$, $b_k = c_k - \delta$, $V_k := G_{f,b_k}(Q^{(d-1)}(k))$ and

$$P_{m,k,n} := Q^{(d-1)}(k) \times (c_k - n\delta, c_k - n\delta + \delta)\,,$$

where $k \in \bigcup_m \mathcal{K}_m$ and $n \in \mathbb{Z}_+$. Denote $\mathcal{N}_m := \{2^m + 1, \dots, 2^m + 2^{m+1}\}$. Lemma 3.4(4) implies that

$$\bigcup_{k\in\mathcal{K}_m,\, n\in\mathcal{N}_m} P_{m,k,n} \;\subset\; \{x \in G_f \mid 2^m\delta \leqslant f(x') - x_d \leqslant 2^{m+2}\delta\}\,, \tag{3.2}$$

for all $m=0,1,\dots,m_\delta$. Let $\mathcal{K}:=\mathcal{K}_0$, $\mathcal{J}_*:=\bigcup_{m=0}^{m_\delta}\mathcal{K}_m\times\mathcal{N}_m$ and $\{P_j\}_{j_*\in\mathcal{J}_*}:=\bigcup_{m=0}^{m_\delta}\{P_{m,k,n}\}_{k\in\mathcal{K}_m,n\in\mathcal{N}_m}$.

Assume that $x\in G_f$. If $f(x')-x_d\leqslant 2\delta$ then, by Lemma 3.4(1), we have $x\in\bigcup_{k\in\mathcal{K}}\left(\overline{V_k}\bigcup\overline{P_{0,k,2}}\right)$. If $f(x')-x_d>2^{m_\delta+1}\delta$ then

$$\operatorname{dist}(x,\Gamma_f)\geqslant f(x')-x_d-2\operatorname{Osc}(f,Q_a^{(d-1)})\geqslant f(x')-x_d-2^{m_\delta}\delta>2^{m_\delta}\delta\geqslant\delta.$$

Finally, if $2\delta\leqslant f(x')-x_d\leqslant 2^{m_\delta+1}\delta$ then $2^{m+1}\delta\leqslant f(x')-x_d\leqslant 2^{m+1}\delta+2^m\delta$ for some nonnegative integer $m\leqslant m_\delta$ and, in view of Lemma 3.4(1) and Lemma 3.4(4), we have $x\in\bigcup_{k\in\mathcal{K}_m,n\in\mathcal{N}_m}P_{m,k,n}$. Therefore

$$\{x\in G_f\mid \operatorname{dist}(x,\Gamma_f)\leqslant\delta\}\subset\bigcup_{j_*\in\mathcal{J}_*,\,k\in\mathcal{K}}\left(\overline{P_{j_*}}\bigcup\overline{V_k}\right). \tag{3.3}$$

Let us choose a constant $c\in(\delta/(2\sqrt{d}),\delta/\sqrt{d}]$ in such a way that $a/c\in\mathbb{N}$ and split the set $\overline{Q_a^{(d-1)}\times[b,+\infty)}$ into the union of congruent closed cubes $\overline{Q_c^{(d-1)}(i)}$ whose interiors $Q_c^{(d-1)}(i)$ are mutually disjoint. Let $\{P_j\}_{j\in\mathcal{J}}$ be the collection of all the rectangles P_{j_*} and all the cubes $Q_c^{(d-1)}(i)$ which are contained in $G_{f,b}$. Then (1) and (2) are obvious. The second inequality (3) and (5) follow from the corresponding statements of Lemma 3.4. The first inequality (3) is a consequence of (3.2), Lemma 3.4(3) and the identity $\aleph\{[2^m,2^{m+2}]\}_{i\in\mathbb{Z}_+}=3$. It remains to prove (4).

Let $x\in G_f$. If $\operatorname{dist}(x,\Gamma_f)\leqslant\delta$ then, by (3.3), either $x\in\overline{V_k}$ for some $k\in\mathcal{K}$ or $x\in\overline{P_{j^*}}$ for some $j^*\in\mathcal{J}^*$. Since $P_{j_*}\in\mathbf{P}(\delta)$ and $b\leqslant\inf f-2\delta$, in the latter case $P_{j_*}\subset G_{f,b}$. If $\operatorname{dist}(x,\Gamma_f)>\delta$ then the cube $Q_c^{(d-1)}(i)$, whose closure contains x, is a subset of $G_{f,b}$ because its diameter does not exceed δ. Therefore (4) holds true. □

In the two dimensional case we also have the following, more precise result.

Theorem 3.6. *Let the conditions of Theorem 3.5 be fulfilled and $d=2$. Then there exists countable families of sets $\{P_j\}_{j\in\mathcal{J}}$ and $\{V_k\}_{k\in\mathcal{K}}$ such that*

1. $P_j\subset G_{f,b}$ *and* $P_j\in\mathbf{P}(\delta)$ *for all* $j\in\mathcal{J}$;
2. $V_k\subset G_{f,b}$ *and* $V_k\in\mathbf{V}(\delta,f)$ *for all* $k\in\mathcal{K}$;
3. $\aleph(\{P_j\}_{j\in\mathcal{J}}\bigcup\{V_k\}_{k\in\mathcal{K}})\leqslant 2$;
4. $G_{f,b}\subset\bigcup_{j\in\mathcal{J},\,k\in\mathcal{K}}\left(\overline{P_j}\bigcup\overline{V_k}\right)$;
5. $\#\{k\in\mathcal{K}\mid\mu_2(V_k)\leqslant\delta^2/2\}\leqslant\mathcal{V}_{\delta/2}(f,Q_a^{(1)})$ *and* $\#\{j\in\mathcal{J}\mid\mu_2(P_j)\leqslant\delta^2/8\}\leqslant 6\,\mathcal{V}_{\delta/2}(f,Q_a^{(1)})+12a/\delta$.

Proof. In the two dimensional case we do not need Besicovitch's theorem because the 'cube' $Q_a^{(1)}$ coincides with an interval of the form $(b,b+a)$. Given $\varepsilon>0$, one can easily construct a finite family $\{Q^{(1)}(x)\}_{x\in\mathcal{X}}$ of disjoint subintervals $Q^{(1)}(x)\in(a,a+b)$ satisfying the conditions (1)–(4) of Corollary 3.2 with $C_{d,2}=C_{d,3}=1$. Therefore Lemma 3.4 remains valid if we substitute $C_{d,2}=C_{d,3}=1$.

Let $k \in \mathcal{K} := \mathcal{X}$ and b_k, $Q^{(1)}(k)$ and $V_k = G_{f,b_k}(Q^{(1)}(k))$ be the same as in the proof of Theorem 3.5. By the above, the first inequality in Theorem 3.5(5) holds true with $C_{d,3} = 1$. Therefore $\#\mathcal{K} \leqslant \mathcal{V}_{\delta/2}(f, Q^{(1)}_a) + 2a/\delta$ (the second term is the maximal number of intervals $Q^{(1)}(k)$ whose length exceeds $\delta/2$).

Let $V_f := \bigcup_{k\in\mathcal{K}} V_k$. The set $G_f \setminus V_f$ is a polygon with edges parallel to coordinate axes which has at most $2\,\mathcal{V}_{\delta/2}(f, Q^{(1)}_a)$ vertices lying on the horizontal lines $\{x \mid x_1 \in Q^{(1)}_a,\, x_2 = b_k\}$. Let us choose a constant $c \in (\delta/2, \delta]$ in such a way that $a/c \in \mathbb{N}$ and split the interval $Q^{(1)}_a$ into the union of a/c intervals (a_l, a_{l+1}) of length c; if $a < \delta$ then we take $(a_1, a_2) := Q^{(1)}_a$. Denote

$$\mathcal{K}'_l \;:=\; \{k \in \mathcal{K} \mid [a_{l-2}, a_{l+3}] \bigcap \overline{Q^{(1)}(k)} \neq \emptyset\}\,, \quad b_{k,l} := \min_{k\in\mathcal{K}'_l} b_k\,,$$

and $P_{k,l} := (a_l, a_{l+1}) \times (b_k, b'_k)$ where $b'_k := \min\{b_{k'} \mid b_{k'} > b_k,\, k' \in \mathcal{K}'_l\}$; we assume that $P_{k,l} := \emptyset$ whenever $b_k = \max\{b_{k'} \mid k' \in \mathcal{K}'_l\}$.

We have $\mathrm{dist}(x, \Gamma_f) > \delta$ whenever $x_1 \in [a_l, a_{l+1}]$ and $x_2 < b_{k,l}$. Therefore

$$\{x \in G_f \setminus V_f \mid \mathrm{dist}(x, \Gamma_f) \leqslant \delta,\, x_1 \in [a_l, a_{l+1}]\} \;\subset\; \bigcup_{k\in\mathcal{K}'_l} \overline{P_{k,l}}$$

and, consequently, (3.3) holds true with $\mathcal{J}_* := \bigcup_l \mathcal{K}'_l$ and $\{P_{j_*}\}_{j_*\in\mathcal{J}_*} := \bigcup_l \{P_{k,l}\}_{k\in\mathcal{K}'_l}$. For each fixed l the number of rectangles $P_{k,l}$ does not exceed $\#\mathcal{K}'_l - 1$. We also have $\sum_l (\#\mathcal{K}'_l - 1) \leqslant 6\,(\#\mathcal{K})$ because each point $x_1 \in Q^{(1)}_a$ belongs to at most six intervals $[a_{l-2}, a_{l+3}]$. Therefore

$$\#\mathcal{J}_* \;\leqslant\; 6\,(\#\mathcal{K}) \;\leqslant\; 6\,\mathcal{V}_{\delta/2}(f, Q^{(1)}_a) + 12a/\delta\,.$$

The rest of the proof repeats that of Theorem 3.5. □

3.3. **General domains.** We shall need the following elementary lemma.

Lemma 3.7. *Let h be a real-valued function on $\mathbb{R}_+$ and $0 < a \leqslant b$. If the function $th(t)$ is nondecreasing then*

$$\sum_{i\in\mathbb{Z} \mid a \leqslant 2^i \leqslant b} h(2^i) \;\leqslant\; 2\int_a^{2b} t^{-1}\, h(t)\, \mathrm{d}t\,.$$

Proof. We have $\sum_{a\leqslant 2^i\leqslant b} h(2^i) \;=\; 2\sum_{a\leqslant 2^i\leqslant b} (2^{-i} - 2^{-i-1})\,(2^i)\, h(2^i)$. Since the function $\tilde h(s) = s^{-1}\, h(s^{-1})$ is decreasing, the right hand side is estimated by $2\int_{(2b)^{-1}}^{a^{-1}} s^{-1}\, h(s^{-1})\, \mathrm{d}s = 2\int_a^{2b} t^{-1}\, h(t)\, \mathrm{d}t$. □

Corollary 3.8. *Let $\Omega \in BV_{\tau,\infty}$. Then for each $\delta \in (0, \delta_\Omega]$ there exist families of sets $\{P_j\}_{j\in\mathcal{J}}$ and $\{V_k\}_{k\in\mathcal{K}}$ satisfying the following conditions:*

(1) *for each j there exists $l \in \mathcal{L}$ such that $P_j \subset \Omega_l$ and $U_l(P_j) \in \mathbf{P}(\delta)$;*
(2) *for each k there exists $l \in \mathcal{L}$ such that $V_k \subset \Omega_l$ and $U_l(V_k) \in \mathbf{V}(\delta)$;*
(3) $\aleph\{P_j\} \leqslant n_\Omega\,(3C_{d,2} + 1)$ *and* $\aleph\{V_k\} \leqslant n_\Omega\, C_{d,2}$*;*

(4) $\Omega^{\rm b}_{\delta_0} \;\subset \bigcup\limits_{j\in\mathcal J,\, k\in\mathcal K} \left(\overline{P_j}\bigcup\overline{V_k}\right) \;\subset\; \Omega^{\rm b}_{\delta_1}\,,$

(5) $\#\mathcal K \leqslant C_{d,3}\, C_{\Omega,\tau}\, \tau(2/\delta) + n_\Omega\, C_{d,2}\, 2^{d-1}\, \delta^{-d}\, \mu_d(\Omega^{\rm b}_{\delta_1})$ *and*

$$\#\mathcal J \;\leqslant\; 4\, C_{d,3}\, C_{\Omega,\tau}\, \delta^{-1} \int_{(2D_\Omega)^{-1}}^{4/\delta} t^{-2}\, \tau(t)\, {\rm d}t + n_\Omega\, (3C_{d,2}{+}1)\, (2\sqrt d)^d\, \delta^{-d}\, \mu_d(\Omega^{\rm b}_{\delta_1})\,,$$

where $\delta_0 := \delta/\sqrt d$ *and* $\delta_1 := \sqrt d\, \delta + \delta/\sqrt d\,.$

Proof. Let $\Omega_l = U_l^{-1}(G_{f_l,\,b_l})$ be the sets introduced in Subsection 1.1. Given $\delta \in (0,\delta_\Omega]$, we apply Theorem 3.5 for each $l \in \mathcal L$ and denote by $\{P_j\}_{j\in\mathcal J(l)}$ and $\{V_k\}_{k\in\mathcal K(l)}$ the families of subsets of Ω_l, which satisfy the conditions of Theorem 3.5 in an appropriate orthogonal coordinate system.

Let $\mathcal J'(l) := \{j \in \mathcal J(l) \mid {\rm dist}(P_j, \partial\Omega) \leqslant \delta_0\}$,

$$\{P_j\}_{j\in\mathcal J} := \bigcup_{l\in\mathcal L} \{P_j\}_{j\in\mathcal J'(l)} \quad \text{and} \quad \{V_k\}_{k\in\mathcal K} := \bigcup_{l\in\mathcal L} \{V_k\}_{k\in\mathcal K(l)}\,.$$

Then each of the conditions (1)–(3) is a consequence of the corresponding condition in Theorem 3.5.

If $x \notin \bigcup_{l\in\mathcal L} \Omega_l$ then ${\rm dist}(x,\partial\Omega) \geqslant \delta_\Omega > \delta_0$. If $x \in \Omega_l \bigcap \Omega^{\rm b}_{\delta_0}$ then, by Theorem 3.5(4), we have $x \in \bigcup_{j\in\mathcal J(l),\,k\in\mathcal K(l)} \left(\overline{P_j}\bigcup\overline{V_k}\right)$. In this case $x \in \bigcup_{j\in\mathcal J'(l),\,k\in\mathcal K(l)} \left(\overline{P_j}\bigcup\overline{V_k}\right)$ because ${\rm diam}\, P_j \leqslant \sqrt d\,\delta$. Therefore $\Omega^{\rm b}_{\delta_0}$ is a subset of $\bigcup_{j\in\mathcal J,\,k\in\mathcal K} \left(\overline{P_j}\bigcup\overline{V_k}\right)$. The estimates $\sup_{x\in V_k} {\rm dist}(x,\partial\Omega) \leqslant \sqrt d\,\delta$ and ${\rm diam}\, P_j \leqslant \sqrt d\,\delta$ imply the second inclusion (4).

In order to prove (5), let us denote by M_δ the smallest positive integer such that $2^{M_\delta-1}\delta \geqslant D_\Omega$. By Theorem 3.5(5), we have

$$\#\{j \in \bigcup_{l\in\mathcal L} \mathcal J(l) \mid \mu_d(P_j) \leqslant 2^{1-d}\,\delta^d\} \leqslant C_{d,3}\, C_{\Omega,\tau} \sum_{m=0}^{M_\delta} 2^m\, \tau((2^{m-1}\delta)^{-1})\,.$$

Since $2^{M_\delta-1}\delta \leqslant 2D_\Omega$, applying Lemma 3.7 with $a = (2D_\Omega)^{-1}\delta$, $b = 2$ and $h(t) = t^{-1}\,\tau(\delta^{-1}t)$, we obtain

$$\#\{j \in \bigcup_{l\in\mathcal L} \mathcal J(l) \mid \mu_d(P_j) \leqslant 2^{1-d}\,\delta^d\} \;\leqslant\; 4\, C_{d,3}\, C_{\Omega,\tau}\, \delta^{-1} \int_{(2D_\Omega)^{-1}}^{4/\delta} t^{-2}\, \tau(t)\, {\rm d}t\,.$$

Now the second estimate (5) follows from the first inequality (3) and the second inclusion (4). Similarly, the first estimate (5) is a consequence of the second inequality (3), the second inclusion (4) and the first inequality in Theorem 3.5(5). $\square$

Corollary 3.9. *Let* $\Omega \in BV_{\tau,\infty}$ *and* $\Omega \in \mathbb R^2$. *Then for each* $\delta \in (0,\delta_\Omega]$ *there exist families of sets* $\{P_j\}_{j\in\mathcal J}$ *and* $\{V_k\}_{k\in\mathcal K}$ *satisfying the conditions* (1), (2) *and* (4) *of Corollary* 3.8 *such that*

(3′) $\aleph\,(\{P_j\}_{j\in\mathcal J} \bigcup \{V_k\}_{k\in\mathcal K}) \leqslant 2\, n_\Omega\,;$

(5′) $\#\mathcal{K} \leqslant C_{\Omega,\tau}\,\tau(2/\delta) + 2\,n_\Omega\,\delta^{-2}\,\mu_2(\Omega^{\rm b}_{\delta_1})$ *and*
$\#\mathcal{J} \ \leqslant\ 6\,C_{\Omega,\tau}\,\tau(2/\delta) \ +\ 12\,D_\Omega/\delta \ +\ 16\,n_\Omega\,\delta^{-2}\,\mu_2(\Omega^{\rm b}_{\delta_1})\,.$

Proof. The corollary is proved in the same way as Corollary 3.8, with the use of Theorem 3.6 instead of Theorem 3.5. □

Our proof of Theorem 1.8 is based on the following simple lemma.

Lemma 3.10. *Let Ω be an arbitrary domain. Then for every $\delta > 0$ there exists a family of sets $\{M_k\}_{k\in\mathcal{K}}$ satisfying the following conditions:*

(1) $M_k \subset \Omega$ *and* $M_k \in \mathbf{M}(\delta)$ *for each* $k \in \mathcal{K}$ *;*
(2) $\aleph\{M_j\} = 1$ *;*
(3) $\Omega^{\rm b}_{\delta_0} \ \subset \bigcup\limits_{k\in\mathcal{K}} \overline{M_k} \ \subset\ \Omega^{\rm b}_{\delta_1}$ *, where* $\delta_0 := \delta/\sqrt{d}$ *and* $\delta_1 := \sqrt{d}\,\delta + \delta/\sqrt{d}\,.$

Proof. Consider an arbitrary cover of $\mathbb{R}^d$ by closed cubes $\overline{Q^{(d)}_\delta(k)}$ with disjoint interiors $Q^{(d)}_\delta(k)$ and define $\{M_k\}_{k\in\mathcal{K}} := \{\Omega \bigcap Q^{(d)}_\delta(k)\}_{k\in\mathcal{K}}$, where $\mathcal{K}$ the set of indices k such that $\Omega^{\rm b}_{\delta_0} \bigcap Q^{(d)}_\delta(k) \neq \emptyset$. □

4. Spectral asymptotics

4.1. **Estimates of the counting function.** In this section we shall always assume that $\delta_0 := \delta/\sqrt{d}$, $\delta_1 := \sqrt{d}\,\delta + \delta/\sqrt{d}$ and denote

$$R_\Omega(\lambda,\delta_1) \ :=\ 3\,(4\sqrt{d})\,C_{d,1}\int_{\delta_1}^\infty \left(s^{-1}\lambda^{d-1} + s^{-d}\right)\,\mathrm{d}(\mu_d(\Omega^{\rm b}_s))\,, \tag{4.1}$$

where $\int \left(s^{-1}\lambda^{d-1} + s^{-d}\right)\,\mathrm{d}(\mu_d(\Omega^{\rm b}_s))$ is understood as a Stieltjes integral.

Theorem 4.1. *If $\Omega \in \mathbb{R}^d$ is an arbitrary domain and $\delta > 0$ then*

$$N(\Omega,\lambda) - C_{d,W}\,\mu_d(\Omega)\,\lambda^d \ \geqslant\ -\,R_\Omega(\lambda,\delta_1) - C_{d,W}\,\mu_d(\Omega^{\rm b}_{4\delta_1})\,\lambda^d\,, \quad \forall\lambda > 0\,, \tag{4.2}$$

and

$$N_{\rm D}(\Omega,\lambda) - C_{d,W}\,\mu_d(\Omega)\,\lambda^d \ \leqslant\ R_\Omega(\lambda,\delta_1) \ +\ ((4d)^d + 2)\,\delta^{-d}\,\mu_d(\Omega^{\rm b}_{4\delta_1}) \tag{4.3}$$

for all $\lambda \leqslant \delta^{-1}$. If $\Omega \in BV_{\tau,\infty}$ and $\delta \in (0,\delta_\Omega]$ then

$$\begin{aligned} N_{\rm N}(\Omega,\lambda) - C_{d,W}\,\mu_d(\Omega)\,\lambda^d \ &\leqslant\ R_\Omega(\lambda,\delta_1) \ +\ (4d)^d\,\delta^{-d}\,\mu_d(\Omega^{\rm b}_{4\delta_1}) \\ &\quad +\ C_{d,6}\,n_\Omega\,\delta^{-d}\,\mu_d(\Omega^{\rm b}_{\delta_1}) \ +\ 8\,C_{d,3}\,C_{\Omega,\tau}\,\delta^{-1}\int_{(2D_\Omega)^{-1}}^{4/\delta} t^{-2}\,\tau(t)\,\mathrm{d}t \end{aligned} \tag{4.4}$$

for all $\lambda \leqslant \min\{1, C_{d,9}^{1/2}\,n_\Omega^{-1/2}\}\,\delta^{-1}$.

Proof. Let $Q^{(d)}_{2^{-i}}(i,n)$ be the Whitney cubes introduced in Theorem 3.3,

$$\mathcal{I}^-_\delta := \{i \in \mathcal{I} \mid \sqrt{d}\,2^{-i} \leqslant \delta_0/4\}\,, \quad \mathcal{I}^+_\delta := \{i \in \mathcal{I} \mid \sqrt{d}\,2^{-i} > \delta_1\}\,,$$

$\mathcal{I}_\delta^0 := \mathcal{I}\setminus(\mathcal{I}_\delta^+\bigcup\mathcal{I}_\delta^-)$ and $\Omega_\delta^\sigma := \bigcup_{i\in\mathcal{I}_\delta^\sigma}\bigcup_{n\in\mathcal{N}_i} Q^{(d)}_{2^{-i}}(i,n)$, where $\sigma=+\,,\ \sigma=0$ or $\sigma=-\,$. The set Ω_δ^σ are mutually disjoint and $\overline{\Omega}=\overline{\Omega_\delta^+}\bigcup\overline{\Omega_\delta^0}\bigcup\overline{\Omega_\delta^-}\,$. By virtue of (3.1),

$$\Omega_\delta^-\subset\Omega^{\mathrm{b}}_{\delta_0}\,,\quad \Omega_\delta^0\subset\Omega^{\mathrm{b}}_{4\delta_1}\setminus\Omega^{\mathrm{b}}_{\delta_0/4}\,,\quad \Omega\setminus\Omega^{\mathrm{b}}_{4\delta_1}\subset\Omega_\delta^+\subset\Omega\setminus\Omega^{\mathrm{b}}_{\delta_1}\,. \tag{4.5}$$

and

$$\#\mathcal{N}_i\ \leqslant\ 2^{i\,d}\left(\mu_d(\Omega^{\mathrm{b}}_{4\sqrt{d}\,2^{-i}})-\mu_d(\Omega^{\mathrm{b}}_{\sqrt{d}\,2^{-i}})\right)\,,\quad \forall i\in\mathcal{I}\,. \tag{4.6}$$

In view of the second inclusion (4.5), we have

$$\sum_{i\in\mathcal{I}_\delta^0}\#\mathcal{N}_i\ \leqslant\ (4\sqrt{d}\,\delta_0^{-1})^d\,\mu_d(\Omega^{\mathrm{b}}_{4\delta_1})\ =\ (4d)^d\,\delta^{-d}\,\mu_d(\Omega^{\mathrm{b}}_{4\delta_1})\,. \tag{4.7}$$

Since $\aleph\{\,[\sqrt{d}\,2^{-i},4\sqrt{d}\,2^{-i}]\,\}_{i\in\mathbb{Z}}\ =\ 3$ and $\Omega^{\mathrm{b}}_s\ =\ \Omega^{\mathrm{b}}_{D_\Omega}$ for all $s\ \geqslant\ D_\Omega\,$, the inequalities (4.6) imply that

$$\sum_{i\in\mathcal{I}_\delta^+}((2^i)^{1-d}\lambda^{d-1}+1)\,\#\mathcal{N}_i\ \leqslant\ 3\,(4\sqrt{d})\int_{\delta_1}^\infty(s^{-1}\lambda^{d-1}+s^{-d})\,\mathrm{d}(\mu_d(\Omega^{\mathrm{b}}_s)) \tag{4.8}$$

for all $\lambda>0\,$.

By Lemma 2.1,

$$\begin{aligned} &N(\Omega,\lambda)-C_{d,W}\,\mu_d(\Omega)\,\lambda^d\\ &\qquad\geqslant\ -C_{d,W}\,\mu_d(\Omega\setminus\Omega_\delta^+)\,\lambda^d\ +\ \left(N_{\mathrm{D}}(\Omega_\delta^+,\lambda)-C_{d,W}\,\mu_d(\Omega_\delta^+)\,\lambda^d\right)\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &N_{\mathrm{D}}(\Omega,\lambda)-C_{d,W}\,\mu_d(\Omega)\,\lambda^d\\ &\qquad\leqslant\ N_{\mathrm{N,D}}(\Omega\setminus\Omega_\delta^+,\partial\Omega,\lambda)\ +\ \left(N_{\mathrm{N}}(\Omega_\delta^+,\lambda)-C_{d,W}\,\mu_d(\Omega_\delta^+)\,\lambda^d\right). \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &N_{\mathrm{N}}(\Omega,\lambda)-C_{d,W}\,\mu_d(\Omega)\,\lambda^d\\ &\qquad\leqslant\ N_{\mathrm{N}}(\Omega\setminus\Omega_\delta^+,\lambda)\ +\ \left(N_{\mathrm{N}}(\Omega_\delta^+,\lambda)-C_{d,W}\,\mu_d(\Omega_\delta^+)\,\lambda^d\right) \end{aligned} \tag{4.11}$$

Lemma 2.1 implies that

$$\begin{aligned} \sum_{n\in\mathcal{N}_i,\,i\in\mathcal{I}_\delta^+}\left(N_{\mathrm{D}}(Q^{(d)}_{2^{-i}}(i,n),\lambda)-C_{d,W}\,(2^{-i}\lambda)^d\right)\ &\leqslant\ N(\Omega_\delta^+,\lambda)-C_{d,W}\,\mu_d(\Omega_\delta^+)\,\lambda^d\\ &\leqslant\ \sum_{n\in\mathcal{N}_i,\,i\in\mathcal{I}_\delta^+}\left(N_{\mathrm{N}}(Q^{(d)}_{2^{-i}}(i,n),\lambda)-C_{d,W}\,(2^{-i}\lambda)^d\right). \end{aligned}$$

In view of Lemma 2.8, the right and left hand sides are estimated from below and above by $\pm\,C_{d,1}\sum_{i\in\mathcal{I}_\delta^+}\left((2^i)^{1-d}\lambda^{d-1}+1\right)\#\mathcal{N}_i$. Therefore, by (4.8),

$$|\,N(\Omega_\delta^+,\lambda)-C_{d,W}\,\mu_d(\Omega_\delta^+)\,\lambda^d\,|\ \leqslant\ R_\Omega(\lambda,\delta_1)\,,\qquad \forall\lambda>0\,. \tag{4.12}$$

Since $\Omega\setminus\Omega^{\mathrm b}_{4\delta_1}\subset\Omega^+_\delta$, the lower bound (4.2) is an immediate consequence of (4.9) and (4.12).

Assume that $\lambda\leqslant\delta^{-1}$. Let $\{M_k\}_{k\in\mathcal K}$ be the family of sets introduced in Lemma 3.10 and

$$\{S_m\}_{m\in\mathcal M_{\mathrm D}} \;:=\; \{Q^{(d)}_{2^{-i}}(i,n)\}_{n\in\mathcal N_j,\,i\in\mathcal I^0_\delta}\bigcup\{M_k\}_{k\in\mathcal K}\,.$$

Lemma 3.10(3) and (4.5) imply that $\bigcup_{m\in\mathcal M_{\mathrm D}}S_m\;=\;\Omega\setminus\Omega^+_\delta$. In view of Lemma 3.10(2), we have $\aleph\{S_m\}_{m\in\mathcal M_{\mathrm D}}\leqslant 2$. Consequently, by Lemma 2.2,

$$N_{\mathrm{N,D}}(\Omega\setminus\Omega^+_\delta,\partial\Omega,\lambda)\;\leqslant\;\sum_{m\in\mathcal M_{\mathrm D}}N_{\mathrm{N,D}}(S_m,\Upsilon_m,\sqrt2\,\lambda)\,,$$

where $\Upsilon_m\;=\;\partial S_m\bigcap\partial\Omega$. Since each set S_m belongs either to $\mathbf P(d^{-1/2}\delta_1)$ or to $\mathbf M(\delta)$, Lemma 2.6 implies that $N_{\mathrm N}(S_m,\Upsilon_m,\sqrt2\,\lambda)\leqslant 1$. Moreover, if $S_m\in\mathbf M(\delta)$ then, in view of Lemma 2.6(3), $N_{\mathrm N}(S_m,\Upsilon_m,\sqrt2\lambda)>0$ only if $\mu_d(S_m)\;\geqslant\;\delta^d-4\pi^{-2}\,\delta^{d+2}\lambda^2$. By Lemma 3.10(3), the number of set $M\in\{M_k\}_{k\in\mathcal K}$ satisfying this estimate does not exceed

$$\bigl(1-4\pi^{-2}\,\delta^2\lambda^2\bigr)^{-1}\,\delta^{-d}\,\mu_d(\Omega^{\mathrm b}_{\delta_1})\;\leqslant\;2\,\delta^{-d}\,\mu_d(\Omega^{\mathrm b}_{\delta_1})$$

Taking into account (4.7), we obtain

$$N_{\mathrm{N,D}}(\Omega\setminus\Omega^+_\delta,\partial\Omega,\lambda)\;\leqslant\;(4d)^d\,\delta^{-d}\,\mu_d(\Omega^{\mathrm b}_{4\delta_1})+2\,\delta^{-d}\,\mu_d(\Omega^{\mathrm b}_{\delta_1})\,.$$

This estimate, (4.10) and (4.12) imply (4.3).

In order to prove (4.4), let us consider the family of sets $\{P_j\}_{j\in\mathcal J}$ and $\{V_k\}_{k\in\mathcal K}$ constructed in Corollary 3.8 and define

$$\{S_m\}_{m\in\mathcal M_{\mathrm N}}\;:=\;\{Q^{(d)}_{2^{-i}\delta}(i,n)\}_{n\in\mathcal N_j,\,i\in\mathcal I^0_\delta}\bigcup\{P_j\}_{j\in\mathcal J}\bigcup\{V_k\}_{k\in\mathcal K}\,.$$

Corollary 3.8(4) and (4.5) imply that $\bigcup_{m\in\mathcal M}S_m=\Omega\setminus\Omega^+_\delta$. In view of Corollary 3.8(3), we have $\aleph\{S_m\}_{m\in\mathcal M}\leqslant n_\Omega\,C^2_{d,4}$. Consequently, by Lemma 2.2,

$$N_{\mathrm N}(\Omega\setminus\Omega^+_\delta,\lambda)\;\leqslant\;\sum_{m\in\mathcal M_{\mathrm N}}N_{\mathrm N}(S_m,n_\Omega^{1/2}\,C_{d,4}\,\lambda)\,.$$

Since each set S_m belongs either to $\mathbf V(\delta)$ or to $\mathbf P(d^{-1/2}\delta_1)$, Lemma 2.6 implies that $N_{\mathrm N}(S_m,n_\Omega^{1/2}\,C_{d,4}\,\lambda)\;=\;1$ whenever $n_\Omega^{1/2}\,C_{d,4}\,\lambda\;\leqslant\;C_{d,5}\,\delta^{-1}$. Estimating $\#\mathcal M$ with the use of (4.7) and Corollary 3.8(5) and applying the inequalities

$$(\delta/4)\,\tau(\delta/2)\;=\;\tau(\delta/2)\int_{2/\delta}^{4/\delta}t^{-2}\,\mathrm dt\;\leqslant\;\int_{2/\delta}^{4/\delta}t^{-2}\,\tau(t)\,\mathrm dt\;\leqslant\;\int_{(2D_\Omega)^{-1}}^{4/\delta}t^{-2}\,\tau(t)\,\mathrm dt\,,$$

we see that

$$\begin{aligned}(4.13)\quad N_{\mathrm N}(\Omega\setminus\Omega^+_\delta,\lambda)\;\leqslant\;&8\,C_{d,3}\,C_{\Omega,\tau}\,\delta^{-1}\int_{(2D_\Omega)^{-1}}^{4/\delta}t^{-2}\,\tau(t)\,\mathrm dt\\&+\;(4d/\delta)^d\,\mu_d(\Omega^{\mathrm b}_{4\delta_1})\;+\;C_{d,6}\,n_\Omega\,\delta^{-d}\,\mu_d(\Omega^{\mathrm b}_{\delta_1})\end{aligned}$$

for all $\lambda \leqslant C_{d,7}\, n_\Omega^{-1/2}\, \delta^{-1}$. Now (4.4) follows from (4.11) and (4.12). $\square$

4.2. **Two dimensional domains.** If $d = 2$, $\tau(t) = t$ and $\delta \asymp \lambda^{-1}$ then the first term on the right hand side of (4.13) coincides with $c\,\lambda\,\log\lambda$, where c is some constant. On the other hand, for two dimensional domains with smooth boundaries we have $N_{\mathrm{N}}(\Omega^{\mathrm{b}}_{\lambda^{-1}}, \lambda) \sim \lambda$ as $\lambda \to \infty$ (see, for example, [SV]). The following lemma gives a refined estimate for $N_{\mathrm{N}}(\Omega\setminus\Omega^+_\delta, \lambda)$, which does not contain the logarithmic factor.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^2$, $\Omega \in BV_{\tau,\infty}$, $\delta \in (0, \delta_\Omega]$ and Ω^+_δ be defined as in Subsection 4.1. Then for all $\lambda \leqslant \frac{\sqrt{2}}{3}\, n_\Omega^{-1/2}\delta^{-1}$ we have*

$$(4.14)\quad N_{\mathrm{N}}(\Omega\setminus\Omega^+_\delta, \lambda) \;\leqslant\; 7\, C_{\Omega,\tau}\, \tau(2/\delta) + (64 + 18\, n_\Omega)\, \delta^{-2}\, \mu_2(\Omega^{\mathrm{b}}_{4\delta_1}) + 12\, D_\Omega/\delta\,.$$

Proof. Applying the same arguments as in the proof of Theorem 4.1 but using Corollary 3.9 instead of Corollary 3.8, one obtains (4.14) instead of (4.13). $\square$

4.3. **Proof of Theorems 1.3, 1.8 and Corollary 1.5.** Integrating by parts in the Stieltjes integral and changing variables $s = t^{-1}$, we obtain

$$(4.15)\quad \int_\varepsilon^\infty (s^{-1}\lambda^{d-1} + s^{-d})\, \mathrm{d}(\mu_d(\Omega^{\mathrm{b}}_s)) \;+\; (\varepsilon^{-1}\lambda^{d-1} + \varepsilon^{-d})\, \mu_d(\Omega^{\mathrm{b}}_\varepsilon)$$
$$= \int_0^{\varepsilon^{-1}} (\lambda^{d-1} + d\, t^{d-1})\, \mu_d(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t\,, \qquad \forall \varepsilon > 0\,.$$

Therefore $\big((4\delta_1)^{-1}\lambda^{d-1} + (4\delta_1)^{-d}\big)\, \mu_d(\Omega^{\mathrm{b}}_{4\delta_1}) \leqslant (\lambda^{d-1} + d\, \delta_1^{1-d}) \int_0^{\delta_1^{-1}} \mu_d(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t$ and $\int_{\delta_1}^\infty (s^{-1}\lambda^{d-1} + s^{-d})\, \mathrm{d}(\mu_d(\Omega^{\mathrm{b}}_s)) \leqslant (\lambda^{d-1} + d\, \delta_1^{1-d}) \int_0^{\delta_1^{-1}} \mu_d(\Omega^{\mathrm{b}}_{t^{-1}})\, \mathrm{d}t$. Applying these inequalities and the estimates (4.2)–(4.4) with $\delta_1^{-1} = \lambda$ or $\delta^{-1} = C_{d,8}\, n_\Omega^{1/2}\, \lambda$, we obtain (1.1) and (1.6). The estimate (1.2) is proved in the same manner, using (4.14) instead of (4.13). Finally, since $\int_a^b t^{-2}\, \tau(t)\, \mathrm{d}t \leqslant b^{d-2} \int_a^b t^{-d}\, \tau(t)\, \mathrm{d}t$, (1.3) is a consequence of (1.1) and the following lemma.

Lemma 4.3. *If $\Omega \in BV_{\tau,\infty}$ then*

$$\mu_d(\Omega^{\mathrm{b}}_\varepsilon) \;\leqslant C_{d,2}\, 3^d\, n_\Omega\, D_\Omega^{d-1}\varepsilon + C_{d,3}\, 3^d\, C_{\Omega,\tau}\, \varepsilon^d\, \tau(\varepsilon^{-1})\,, \qquad \forall \varepsilon > 0\,.$$

Proof. Assume first that f is a continuous function on the closed cube $\overline{Q^{(d-1)}_a}$. Let $\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}}$ be the same family of cubes as in Corollary 3.2, $\Gamma_f(x) := \{z \in \Gamma_f \mid z' \in Q^{(d-1)}(x)\}$ and $\mathcal{X}_\varepsilon := \{x \in \mathcal{X} \mid Q^{(d-1)}(x) \in \mathbf{P}(\varepsilon)\}$.

If $\mathrm{dist}(y, \Gamma_f) \leqslant \varepsilon$ then $\mathrm{dist}(y, \Gamma_f(x)) \leqslant \varepsilon$ for some $x \in \mathcal{X}$. Therefore

$$\mu_d\left(\{y \in Q^{(d-1)}_a \mid \mathrm{dist}(y, \Gamma_f) \leqslant \varepsilon\}\right)$$
$$\leqslant \sum_{x\in\mathcal{X}} \mu_d\left(\{y \in Q^{(d-1)}_a \mid \mathrm{dist}(y, \Gamma_f(x)) \leqslant \varepsilon\}\right).$$

The set $\{y \in Q^{(d-1)}_a \mid \mathrm{dist}(y, \Gamma_f(x)) \leqslant \varepsilon\}$ lies in the ε-neighbourhood of the rectangle $Q^{(d-1)}(x) \times \big(\inf_{z\in Q^{(d-1)}(x)} f(z)\,, \sup_{z\in Q^{(d-1)}(x)} f(z)\big)$. In view of Corollary 3.2(4), the measure of this ε-neighbourhood does not exceed $3\varepsilon\,(a_x+2\varepsilon)^{d-1}$, where a_x is the length of the edge of $Q^{(d-1)}(x)$. Therefore

$$\mu_d\left(\{y\in Q_a^{(d-1)}\mid \operatorname{dist}(y,\Gamma_f)\leqslant\varepsilon\}\right)\leqslant 3^d\,\varepsilon^d\,(\#\mathcal{X}_\varepsilon)+\sum_{x\in\mathcal{X}\setminus\mathcal{X}_\varepsilon}3^d\,\varepsilon\,a_x^{d-1}\,.$$

Now the obvious inequality $\sum_{x\in\mathcal{X}}a_x^{d-1}\leqslant a^{d-1}\,\aleph\{Q^{(d-1)}(x)\}_{x\in\mathcal{X}}$ and Corollary 3.2(3) imply that

$$\mu_d(\{y\in\mathbb{R}^d\mid \operatorname{dist}(y,\Gamma_f)\leqslant\varepsilon\})\ \leqslant\ C_{d,2}\,3^d\varepsilon\,a^{d-1}+C_{d,3}\,3^d\varepsilon^d\,\mathcal{V}_\varepsilon(f,Q_a^{(d-1)})\,.$$

Since $\Omega_\varepsilon^{\mathrm{b}}=\bigcup_{l\in\mathcal{L}}\{x\in\Omega\mid \operatorname{dist}(x,\Gamma_{f_l})\leqslant\varepsilon\}$, where f_l are the functions introduced in Subsection 1.1, the lemma follows from this inequality. $\square$

4.4. **Proof of Corollaries 1.6 and 1.9.** Let $\Omega\in\operatorname{Lip}_\alpha$, f_l be the functions introduced in Subsection 1.1 and $|\Omega|_\alpha:=\max_l|f_l|_\alpha$, where $|\cdot|_\alpha$ is the seminorm defined in Subsection 1.1. If $x\in G_{f_l}$ and $\operatorname{dist}(x,(y',f_l(y')))\leqslant\delta$ then

$$f_l(x')-x_d\ \leqslant\ |x_d-f_l(y')|+|f_l(y')-f_l(x')|\ \leqslant\ \delta+\delta^\alpha\,|f_l|_\alpha\,. \tag{4.16}$$

Therefore $\{x\in G_{f_l}\mid \operatorname{dist}(x,\Gamma_{f_l})\leqslant\delta\}\subset\{x\in G_{g_l}\mid f_l(x')-x_d\leqslant\delta+\delta^\alpha\,|f_l|_\alpha\}$ and, consequently $\mu_d(\{x\in G_{f_l}\mid \operatorname{dist}(x,\Gamma_{f_l})\leqslant\delta\})\leqslant a^{d-1}\,(\delta+\delta^\alpha\,|f_l|_\alpha)$. This immediately implies the following lemma.

Lemma 4.4. *If $\Omega\in\operatorname{Lip}_\alpha$ and $\delta\leqslant\delta_\Omega$ then $\mu_d(\Omega_\delta^{\mathrm{b}})\leqslant n_\Omega\,D_\Omega^{d-1}\,(\delta+\delta^\alpha\,|\Omega|_\alpha)$.*

If $Q_c^{(d-1)}\subset Q_{a_l}^{(d-1)}$ then $\operatorname{diam}Q_c^{(d-1)}=d^{1/2}\,c$ and

$$2\operatorname{Osc}(f,Q_c^{(d-1)})\ \leqslant\ \sup_{x',y'\in Q_c^{(d-1)}}|f_l(x')-f_l(y')|\ \leqslant\ d^{\alpha/2}\,c^\alpha\,|f|_\alpha\,. \tag{4.17}$$

Therefore $c^{d-1}\geqslant d^{(1-d)/2}\,|f|_\alpha^{(1-d)/\alpha}\,\delta^{(d-1)/\alpha}$ whenever $\operatorname{Osc}(f,Q_c^{(d-1)})\geqslant\delta/2$ and, consequently,

$$\mathcal{V}_{\delta/2}(f,Q_a^{(d-1)})\ \leqslant\ d^{(d-1)/2}\,a^{d-1}\,|f|_\alpha^{(d-1)/\alpha}\,\delta^{(1-d)/\alpha}+1\,. \tag{4.18}$$

The inequality (4.18) implies the following result.

Lemma 4.5. *If $\Omega\in\operatorname{Lip}_\alpha$ and*

$$\tau(t)\ =\ 2^{(1-d)/\alpha}\,d^{(d-1)/2}\,D_\Omega^{d-1}\,|\Omega|_\alpha^{(d-1)/\alpha}\,t^{(d-1)/\alpha}+1 \tag{4.19}$$

then $\Omega\in BV_{\infty,\tau}$ and $C_{\Omega,\tau}\leqslant n_\Omega$.

Clearly, (1.4) follows from (1.1) and Lemma 4.5. Similarly, (1.6) and Lemma 4.4 imply (1.7). It remains to prove (1.5) and (1.8).

Assume that $\Omega\in\operatorname{lip}_\alpha$. Then for each $\varepsilon>0$ we can find functions $f_{l,1}^{(\varepsilon)}\in\operatorname{Lip}_1$ and $f_{l,2}^{(\varepsilon)}\in\operatorname{Lip}_\alpha$ such that $f_l=f_{l,1}^{(\varepsilon)}+f_{l,2}^{(\varepsilon)}$ and $|f_{l,2}^{(\varepsilon)}|_\alpha\leqslant\varepsilon$. Obviously,

$\mathcal{V}_\delta(f^{(\varepsilon)}_{l,1}+f^{(\varepsilon)}_{l,2},Q)\ \leqslant\ \mathcal{V}_{\delta/2}(f^{(\varepsilon)}_{l,1},Q)\ +\ \mathcal{V}_{\delta/2}(f^{(\varepsilon)}_{l,2},Q)$. Therefore (4.18) implies that

$$\begin{aligned}(4.20)\quad \mathcal{V}_\delta(f_l,Q^{(d-1)}_{a_l})\ &\leqslant\ d^{(d-1)/2}\,D_\Omega^{d-1}\left(\varepsilon^{(d-1)/\alpha}\,\delta^{(1-d)/\alpha}+C_\varepsilon^{d-1}\,\delta^{1-d}\right)+2\\ &\leqslant\ \varepsilon^{(d-1)/\alpha}\,\tau_\varepsilon(\delta^{-1})\,,\end{aligned}$$

where $C_\varepsilon:=\max_l\,|f^{(\varepsilon)}_{l,2}|_1$, and

$$\tau_\varepsilon(t)\ :=\ d^{(d-1)/2}\,D_\Omega^{d-1}\left(t^{(d-1)/\alpha}+C_{\varepsilon,\Omega}\,\varepsilon^{(1-d)/\alpha}\,t^{d-1}\right)+2\,\varepsilon^{(1-d)/\alpha}\,.$$

We also have

$$|f_l(x')-f_l(y')|\leqslant\varepsilon\,|x'-y'|^\alpha+|f^{(\varepsilon)}_{l,2}|_1\,|x'-y'|\,,\qquad\forall x',y'\in Q^{(d-1)}_{a_l}\,.$$

Therefore, instead of (4.16), we obtain $f_l(x')-x_d\leqslant\delta+\delta\,|f^{(\varepsilon)}_{l,2}|_1+\delta^\alpha\,\varepsilon$. This implies that $\mu_d(\Omega^{\mathrm{b}}_\delta)\leqslant n_\Omega\,D_\Omega^{d-1}\,(\delta+C_\varepsilon\,\delta+\delta^\alpha\,\varepsilon)$ whenever $\delta\leqslant\delta_\Omega$.

In view of (4.20), we have $\Omega\in BV_{\infty,\tau_\varepsilon}$ and $C_{\Omega,\tau_\varepsilon}\leqslant\varepsilon^{(d-1)/\alpha}\,n_\Omega$. Choosing a sufficiently large constant C and applying (4.2)–(4.4) with $\delta=C\,\lambda^{-1}$ and $\tau=\tau_\varepsilon$, we see that

$$\begin{aligned}|\,N_{\mathrm{N}}(\Omega,\lambda)-C_{d,W}\,\mu_d(\Omega)\,\lambda^d\,|\ &\leqslant\ \varepsilon^{(d-1)/\alpha}\,C'_\Omega\,\lambda^{(d-1)/\alpha}+C'_{\Omega,\varepsilon}\,\lambda^{d-1}\,,\\ |\,N_{\mathrm{D}}(\Omega,\lambda)-C_{d,W}\,\mu_d(\Omega)\,\lambda^d\,|\ &\leqslant\ \varepsilon\,C'_\Omega\,\lambda^{d-\alpha}+C'_{\Omega,\varepsilon}\,\lambda^{d-1}\end{aligned}$$

for all $\lambda>1$, where C'_Ω is a constant depending only on the domain Ω and $C'_{\Omega,\varepsilon}$ is a constant depending on Ω and ε . Since ε can be made arbitrarily small, these inequalities imply (1.5) and (1.8). □

4.5. **Proof of Theorem 1.10.** Let $Q^{(d-1)}_1=(0,1)^{d-1}$, $\alpha\in(0,1)$ and p be a sufficiently large positive integer. In particular, we shall be assuming that $p\geqslant\max\{\alpha^{-1},(1-\alpha)^{-1}\}$ and, consequently,

$$(4.21)\qquad 2^{1-\alpha p}\leqslant1\,,\quad\left(1-2^{-\alpha p}\right)^{-1}\ \leqslant\ 2\,,\quad\left(1-2^{(1-\alpha)\,p}\right)^{-1}\ \leqslant\ 2$$

and

$$(4.22)\quad\left(2^{(1-\alpha)\,(n+1)p}-1\right)\left(2^{(1-\alpha)\,p}-1\right)^{-1}\ \leqslant\ 2^{1+(1-\alpha)\,np}\,,\qquad\forall n=1,2,\dots$$

Given $j\in\mathbb{Z}_+$, let us denote by $\mathcal{K}_j$ the set of nonnegative integer vectors $\mathbf{k}=(k_1,\dots,k_{d-1})\in\mathbb{Z}_+^{d-1}$ such that $\max_i\,k_i\leqslant2^{jp}-1$ and consider the $(d-1)$-dimensional cubes

$$Q(j,\mathbf{k})\ :=\ \{x'\in\mathbb{R}^{d-1}\mid 2^{jp}x'-\mathbf{k}\in Q^{(d-1)}_1\}\,,\qquad\mathbf{k}\in\mathcal{K}_j\,.$$

with edges of length 2^{-jp}. For each fixed $j\in\mathbb{Z}_+$ and $\mathbf{k}\in\mathcal{K}_j$ the cubes $Q(j,\mathbf{k})$ are disjoint and $\overline{Q^{(d-1)}_1}=\bigcup_{\mathbf{k}\in\mathcal{K}_j}\overline{Q(j,\mathbf{k})}$.

Let $\psi\in\mathrm{Lip}_1$ be a nonnegative Lipschitz function on $Q^{(d-1)}_1$ vanishing on the boundary $\partial Q^{(d-1)}_1$, $a_\psi:=\sup\psi$ and $b_{\psi,p}:=\sqrt{d}\,2^{3-(1-\alpha)p}\,(|\psi|_1+a_\psi)$. We shall be assuming that p is large enough so that $a_\psi > b_{\psi,p}$. Let us extend ψ by 0 to the whole space $\mathbb{R}^{d-1}$ and define

$$g_j(x') \;:=\; \sum_{\mathbf{k}\in\mathcal{K}_j} \psi(2^{jp}x'-\mathbf{k})\,, \qquad f_n(x') \;:=\; \sum_{j=0}^{n} \varepsilon_j\, 2^{-\alpha\, jp}\, g_j(x')$$

and $f(x') := \lim_{n\to\infty} f_n(x') = \sum_{j=0}^\infty \varepsilon_j\, 2^{-\alpha jp}\, g_j(x')$, where $\{\varepsilon_j\}$ is a nonincreasing sequence such that $\varepsilon_j \in [0,1]$ and

$$2^{(1-\alpha)([j/2]-j)p} \;\leqslant\; \varepsilon_{[j/2]} \;\leqslant\; 2\,\varepsilon_j\,, \qquad \forall j = 1,2,\dots \tag{4.23}$$

Note that the condition (4.23) is fulfilled whenever $\{\varepsilon_j\}$ is a sufficiently slowly decreasing sequence.

Lemma 4.6. *We have*

(1) $g_j = 0$ *on* $\partial Q(j,\mathbf{k})$ *for all* $\mathbf{k}\in\mathcal{K}_n$ *and* $j \geqslant n$*;*
(2) $0 \leqslant f(x') - f_n(x') \leqslant 2\,\varepsilon_{n+1}\, 2^{-\alpha\,(n+1)p}\, a_\psi \leqslant \varepsilon_{n+1}\, 2^{-\alpha\, np}\, a_\psi$*;*
(3) $|f_n|_\beta \leqslant 2^{1+(\beta-\alpha)np}\,(|\psi|_1 + a_\psi)$ *for all* $\beta\in[\alpha,1]$*;*
(4) $f \in \mathrm{Lip}_\alpha$ *and* $|f|_\alpha \leqslant 2\,(|\psi|_1 + a_\psi)$*;*
(5) $f \in \mathrm{lip}_\alpha$ *whenever* $\varepsilon_j \to 0$ *as* $j\to\infty$*;*
(6) $2\,\mathrm{Osc}\,(f_{n-1}, Q(n,\mathbf{k})) \leqslant \varepsilon_n\, 2^{-\alpha\, np}\, b_{\psi,p}$ *for all* $\mathbf{k}\in\mathcal{K}_n$*.*

Proof. (1) is obvious and (2) immediately follows from (4.21). In order to prove (3), let us fix $\beta\in[\alpha,1]$, denote $n' := \max\{j \mid 2^{-jp} \geqslant |x'-y'|\}$, $n'' := \min\{n, n'\}$ and estimate

$$\begin{aligned}\sum_{j=0}^{n} \frac{|g_j(x')-g_j(y')|}{2^{\alpha jp}\,|x'-y'|^\beta} \;&=\; \sum_{j=0}^{n''} \frac{|g_j(x')-g_j(y')|}{2^{\alpha jp}\,|x'-y'|^\beta} \;+\; \sum_{j=n''+1}^{n} \frac{|g_j(x')-g_j(y')|}{2^{\alpha jp}\,|x'-y'|^\beta}\\ &\leqslant\; |\psi|_1 \sum_{j=0}^{n''} 2^{(1-\alpha)jp}\,|x'-y'|^{1-\beta} \;+\; a_\psi \sum_{j=n''+1}^{n} 2^{-\alpha jp}\,|x'-y'|^{-\beta}\,.\end{aligned}$$

In view of (4.22), the first term on the right hand side is estimated by $|\psi|_1 \sum_{j=0}^{n''} 2^{(1-\alpha)jp+(1-\beta)np} \leqslant 2^{1+(\beta-\alpha)np}|\psi|_1$. If $n\leqslant n'$ then the second term on the right hand side vanishes; if $n > n'$ then, by (4.21), it does not exceed $2\,a_\psi\, 2^{-\alpha(n''+1)p}|x'-y'|^{-\beta} \leqslant 2\,a_\psi\, 2^{(\beta-\alpha)(n''+1)p} \leqslant 2^{1+(\beta-\alpha)np} a_\psi$. Thus,

$$\sum_{j=0}^{n} \frac{|g_j(x')-g_j(y')|}{2^{\alpha jp}\,|x'-y'|^\beta} \;\leqslant\; 2^{1+(\beta-\alpha)np}\,(|\psi|_1 + a_\psi)\,. \tag{4.24}$$

This estimate immediately implies (3) and (4). The inclusion (5) is also a consequence of (4.24) because $|f-f_n|_\alpha \leqslant \varepsilon_{n+1}\,\sup_{x',y'} \sum_{j=0}^\infty \frac{|g_j(x')-g_j(y')|}{2^{\alpha jp}\,|x'-y'|^\alpha}$.

Finally, in view of (4.23) and (4.24), we have

$$(|\psi|_1 + a_\psi)^{-1}|f_j|_1 \;\leqslant\; 2^{1+(1-\alpha)[j/2]p} \;+\; \varepsilon_{[j/2]}\, 2^{1+(1-\alpha)jp} \;\leqslant\; \varepsilon_j\, 2^{3+(1-\alpha)jp}\,. \tag{4.25}$$

Since $\mathrm{diam}\, Q(n,\mathbf{k}) = \sqrt{d}\, 2^{-np}$, (4.25) with $j = n-1$ implies (6). □

Let $\Omega := G_{f,0}$, $\Omega_{n,\mathbf{k}} := \{x \in \Omega \mid x' \in Q(n,\mathbf{k})\,,\ x_d \in (f_{n-1}(x'), f(x'))\}$, $\Upsilon_{n,\mathbf{k}} := \partial\Omega_{n,\mathbf{k}} \setminus \partial\Omega$ and Ω_{n-1} be the interior of $\Omega \setminus \left(\bigcup_{\mathbf{k}\in\mathcal{K}_n} \Omega_{n,\mathbf{k}}\right)$.

Denote $a_{n,\mathbf{k}} := \sup\limits_{x'\in Q(n,\mathbf{k})} f_{n-1}(x')$ and consider the function

$$u_{n,\mathbf{k}}(x) \;:=\; \begin{cases} \sin\left(2^{\alpha\, np}(x_d - a_{n,\mathbf{k}})/\varepsilon_n\right), & x_d \geqslant a_{n-1,\mathbf{k}}\,, \\ 0\,, & x_d < a_{n-1,\mathbf{k}}\,, \end{cases}$$

on $\Omega_{n,\mathbf{k}}$. We have $u_{n,\mathbf{k}}(x) \in W^{1,2}(\Omega_{n,\mathbf{k}})$ and, in view of Lemma 4.6(1), $u_{n,\mathbf{k}} = 0$ on $\Upsilon_{n,\mathbf{k}}$. Applying Lemma 4.6(2) and Lemma 4.6(6), we see that

$$\begin{aligned} \int_{\Omega_{n,\mathbf{k}}} |\nabla u_{n,\mathbf{k}}(x)|^2\,\mathrm{d}x &= \varepsilon_n^{-2}\, 2^{2\alpha\, np} \int_{Q(n,\mathbf{k})} \int_0^{f(x')-a_{n,\mathbf{k}}} \cos^2\left(2^{\alpha\, np}\, x_d/\varepsilon_n\right) \mathrm{d}x_d\, \mathrm{d}x' \\ &\leqslant\; \varepsilon_n^{-2}\, 2^{2\alpha\, np} \int_{Q(n,\mathbf{k})} \int_0^{\varepsilon_n\, 2^{-\alpha\, np}(g_n(x')+a_\psi)} \cos^2\left(2^{\alpha\, np} x_d/\varepsilon_n\right) \mathrm{d}x_d\, \mathrm{d}x' \\ &=\; \varepsilon_n^{-1}\, 2^{\alpha\, np}\, 2^{-(d-1)\, np} \int_{Q_1^{(d-1)}} \int_0^{\psi(x')+a_\psi} \cos^2 x_d\, \mathrm{d}x_d\, \mathrm{d}x' \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_{n,\mathbf{k}}} |u_{n,\mathbf{k}}(x)|^2\,\mathrm{d}x \;&=\; \int_{Q(n,\mathbf{k})} \int_0^{f(x')-a_{n,\mathbf{k}}} \sin^2\left(2^{\alpha\, np} x_d/\varepsilon_n\right) \mathrm{d}x_d\, \mathrm{d}x' \\ &\geqslant\; \int_{Q(n,\mathbf{k})} \int_0^{\varepsilon_n 2^{-\alpha\, np}(g_n(x')-b_{\psi,p})} \sin^2\left(2^{\alpha\, np} x_d/\varepsilon_n\right) \mathrm{d}x_d\, \mathrm{d}x' \\ &=\; \varepsilon_n\, 2^{-\alpha\, np}\, 2^{-(d-1)\, np} \int_{Q_1^{(d-1)}} \int_0^{\psi(x')-b_{\psi,p}} \sin^2 x_d\, \mathrm{d}x_d\, \mathrm{d}x'\,. \end{aligned}$$

Therefore $\int_{\Omega_{n,\mathbf{k}}} |\nabla u_{n,\mathbf{k}}(x)|^2\,\mathrm{d}x \leqslant c_{\psi,p}^2\, \varepsilon_n^{-2}\, 2^{2\alpha np} \int_{\Omega_{n,\mathbf{k}}} |u_{n,\mathbf{k}}(x)|^2\,\mathrm{d}x$, where

$$c_{\psi,p} \;:=\; \left(\frac{\int_{Q_1^{(d-1)}} \int_0^{\psi(x')+a_\psi} \cos^2 x_d\; \mathrm{d}x_d\, \mathrm{d}x'}{\int_{Q_1^{(d-1)}} \int_0^{\psi(x')-b_{\psi,p}} \sin^2 x_d\; \mathrm{d}x_d\, \mathrm{d}x'} \right)^{1/2}.$$

This implies that $N_{\mathrm{N,D}}(\Omega_{n,\mathbf{k}}, \Upsilon_{n,\mathbf{k}}, \lambda) \geqslant 1$ whenever $\lambda \geqslant c_{\psi,p}\, \varepsilon_n^{-1}\, 2^{\alpha np}$.

Assume that $\lambda \in \left[c_{\psi,p}\, \varepsilon_n^{-1}\, 2^{\alpha np},\ c_{\psi,p}\, \varepsilon_{n+1}^{-1}\, 2^{\alpha(n+1)p}\right)$ and, using Lemma 2.1, estimate

$$N_{\mathrm{N}}(\Omega, \lambda) \;\geqslant\; N_{\mathrm{D}}(\Omega_{n-1}, \lambda) \;+\; \sum_{\mathbf{k}\in\mathcal{K}_n} N_{\mathrm{N,D}}(\Omega_{n,\mathbf{k}}, \Upsilon_{n,\mathbf{k}}, \lambda)\,.$$

By the above, the second term on the right hand side is not smaller than $\#\mathcal{K}_n = 2^{(d-1)\, np} \geqslant \left(c_{\psi,p}\, 2^{\alpha p}\right)^{(1-d)/\alpha}\, \varepsilon_{n+1}^{(d-1)/\alpha}\, \lambda^{(d-1)/\alpha}$. On the other hand, in view of Theorem 1.8, Lemma 4.4 and Lemma 4.6(3) with $\beta = \alpha$, we have

$$N_{\mathrm{D}}(\Omega_{n-1}, \lambda) \;\geqslant\; C_{d,W}\, \mu_d(\Omega_{n-1})\, \lambda^d \;-\; C_d\left(|\psi|_1 + a_\psi + 1\right) \lambda^{d-\alpha}$$

for all sufficiently large λ. Finally, by Lemma 4.6(2),

$\mu_d(\Omega)\,\lambda^d \;-\; \mu_d(\Omega_{n-1})\,\lambda^d \;\leqslant\; \varepsilon_n\, 2^{-\alpha\,(n-1)p} a_\psi\, \lambda^d \;\leqslant\; a_\psi\, c_{\psi,p}\,(\varepsilon_n/\varepsilon_{n+1})\, 2^{2\alpha p}\,\lambda^{d-1}\,.$

Since $\varepsilon_n \leqslant \varepsilon_{[(n+1)/2]} \leqslant 2\,\varepsilon_{n+1}$, the above estimates imply that

$$\begin{aligned}(4.26)\quad N_{\mathrm{N}}(\Omega,\lambda) \;\geqslant\; & C_{d,W}\,\mu_d(\Omega_n)\,\lambda^d \;+\; \left(c_{\psi,p}\,2^{\alpha p}\right)^{(1-d)/\alpha}\varepsilon_{n+1}^{(d-1)/\alpha}\,\lambda^{(d-1)/\alpha}\\ &-\; C_d\,(|\psi|_1 + a_\psi + 1)\,\lambda^{d-\alpha} \;-\; C_{d,W}\,a_\psi\,c_{\psi,p}\,2^{2\alpha p+1}\,\lambda^{d-1}\end{aligned}$$

for all $\lambda \in \left[c_{\psi,p}\,\varepsilon_n^{-1}\,2^{\alpha n p},\; c_{\psi,p}\,\varepsilon_{n+1}^{-1}\,2^{\alpha(n+1)p}\right)$.

By Lemma 4.6(4), $\Omega \in \mathrm{Lip}_\alpha$ and we have $(d-1)/\alpha > d-\alpha > d-1$. Therefore taking $\varepsilon_0 = \varepsilon_1 = \cdots = 1$, we obtain a domain satisfying the conditions of Theorem 1.10(1). If ϕ is a nonnegative function on $(0,+\infty)$ and $\phi(\lambda)\to 0$ as $\lambda\to\infty$ then we can choose a sequence ε_n converging to zero and satisfying (4.23) in such a way that the function $\phi(\lambda)\,\lambda^{(d-1)/\alpha}$ and the last two terms in (4.26) are estimated by $\left(c_{\psi,p}\,2^{\alpha p}\right)^{(1-d)/\alpha}\varepsilon_{n+1}^{(d-1)/\alpha}\,\lambda^{(d-1)/\alpha}$ for all $\lambda \in \left[c_{\psi,p}\,\varepsilon_n^{-1}\,2^{\alpha n p},\; c_{\psi,p}\,\varepsilon_{n+1}^{-1}\,2^{\alpha(n+1)p}\right)$ and all sufficiently large n. In view of Lemma 4.6(5), this proves Theorem 1.10(2). □

5. Remarks and generalisations

5.1. **Poincaré inequality.** According to the Poincaré inequality,

$$(5.1)\quad \int_\Omega |u|^2\,\mathrm{d}x \leqslant c_\Omega \int_\Omega |\nabla u|^2\,\mathrm{d}x \quad \text{whenever } u\in W^{1,2}(\Omega) \text{ and } \int_\Omega u\,\mathrm{d}x = 0,$$

where c_Ω is a positive constant. By Remark 2.4, the Poincaré inequality (5.1) on a domain Ω holds true if and only if the zero eigenvalue of the Neumann Laplacian is isolated and $c_\Omega \geqslant \lambda_{1,\mathrm{N}}^{-2}(\Omega)$.

Lemma 5.1. *Let Ω satisfies* (5.1) *and $\tilde\Omega\subset\mathbb{R}^d$. If there exist an invertible map $F:\Omega\to\tilde\Omega$ and a constant C_F such that $|F(x)-F(y)|\leqslant C_F\,|x-y|$ for all $x,y\in\Omega$ and $|F^{-1}(x)-F^{-1}(y)|\leqslant C_F\,|x-y|$ for all $x,y\in\tilde\Omega$ then $\tilde\Omega$ also satisfies* (5.1) *with a positive constant $c_{\tilde\Omega} = C_d\,C_F^{2d+2}c_\Omega$.*

Proof. Let $v\in W^{1,2}(\tilde\Omega)$, $u(x) := v(F^{-1}(x))$ and $c_u := \int_\Omega u(x)\,\mathrm{d}x$. Under the conditions of the lemma the maps F and F^{-1} are differentiable almost everywhere. Changing variables and estimating the Jacobians, we obtain

$$\int_{\tilde\Omega} |v(y)-c_u|^2\,\mathrm{d}y \;\leqslant\; C_d\,C_F^d \int_\Omega |u(x)-c_u|^2\,\mathrm{d}x$$

and

$$\int_{\tilde\Omega} |\nabla v(y)|^2\,\mathrm{d}y \;\geqslant\; C_d\,C_F^{-d-2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x\,.$$

These two estimates and the Poincaré inequality (5.1) imply that

$$\int_{\tilde\Omega} |v(y)|^2\,\mathrm{d}y \;\leqslant\; \int_{\tilde\Omega} |v(y)-c_u|^2\,\mathrm{d}y \;\leqslant\; C_d\,C_F^{2d+2}c_\Omega \int_{\tilde\Omega} |\nabla v(y)|^2\,\mathrm{d}y$$

whenever $\int_{\tilde\Omega} v\,\mathrm{d}y = 0$. □

Lemma 5.1 allows one to extend Theorem 1.3 to more general domains.

Theorem 5.2. *Assume that there exists a finite collection of domains* $\Omega_l \subset \Omega$ *such that*

(a) $\partial\Omega \subset \bigcup_l \overline{\Omega_l}$ *;*
(b′) *for each* l *there exist an invertible map* $F_l : \mathbb{R}^d \to \mathbb{R}^d$ *satisfying the conditions of Lemma* 5.1 *such that* $F_l(\Omega_l) = G_{f_l,b_l}$ *, where* $f_l \in BV_{\tau,\infty}(Q_{a_l}^{(d-1)})$ *and* $b_l < \inf f_l$ *;*
(c) $a_l \leqslant D_\Omega$ *and* $\sup f_l - b_l \leqslant D_\Omega$ *for all* $l \in \mathcal{L}$*.*

Then (1.1) *holds true.*

Proof. Let C_{F_l} be the constant introduced in Lemma 5.1 and $C := \max_l C_{F_l}$. Under conditions of the theorem, Corollary 3.8 remains valid if we replace U_l with F_l and take $\delta_n := C^{-1}\delta_n$. Since (5.1) is equivalent to the identity $N_{\mathrm{N}}(\Omega, c_\Omega^{-2}) = 1$, Lemma 2.6 and Lemma 5.1 imply that $N_{\mathrm{N}}(S_m, \lambda) = 1$ for all $\lambda \leqslant c'_\Omega\,\delta^{-1}$, where S_m are the same sets as in the proof of Theorem 4.1 and c'_Ω is a constant depending on the domain Ω. Therefore, using the same arguments as in Subsection 4.1, we obtain the estimates (4.2) and (4.4) with some other constants (which may depend on Ω). In the same way as in Subsection 4.3, these estimates imply (1.1). □

The following example shows that Theorem 5.2 is not just a formal generalization of Theorem 1.3.

Example 5.3. Let f be a nowhere differentiable Lip_α-function on the interval $[0,1]$. Assume that $f > 1$ and consider the domain

$$\Omega \;:=\; \{(\varphi, r) \in \mathbb{R}^2 \mid \varphi \in (0,1)\,,\; 1 < r < f(\varphi)\}\,,$$

where (φ, r) are the polar coordinates on $\mathbb{R}^2$. If $y_1 = r\sin\varphi$ and $y_2 = r\cos\varphi$ are the standard Cartesian coordinates on $\mathbb{R}^2$ then the map which takes the point with polar coordinates (φ, r) into the point with Cartesian coordinates $(y_1, y_2) = (\varphi, r)$ satisfies the conditions of Lemma 5.1. Therefore, by Theorem 5.2, we have (1.1).

On the other hand, if (x_1, x_2) are arbitrary Cartesian coordinates on $\mathbb{R}^2$ then $x_1(\varphi, r) = r\sin(\varphi + \varphi_0)$ and $x_2(\varphi, r) = r\cos(\varphi + \varphi_0)$ for some $\varphi_0 \in [0, 2\pi)$. For every subinterval $(a,b) \subset (0,1)$ there exist at least two different points $\varphi_1, \varphi_2 \in (a,b)$ such that $x_1(\varphi_1, f(\varphi_1)) = x_1(\varphi_2, f(\varphi_2))$ (otherwise the function $x_1(\varphi, f(\varphi))$ would be monotone on (a,b) and, by Lebesgue's theorem, almost everywhere differentiable). Since $x_2(\varphi_1, f(\varphi_1)) \ne x_2(\varphi_2, f(\varphi_2))$, we see that the set $\{r = f(\varphi)\}$ cannot be represented as the graph of a continuous function in Cartesian coordinates.

Nowhere differentiable functions $f \in \mathrm{Lip}_\alpha$ do exist. For instance, the function $f(t) := \sum_{n=0}^\infty 10^{-n}\,\mathrm{dist}(10^n t, \mathbb{Z})$ is not differentiable at each $t \in \mathbb{R}$ (see [W] or [RS-N], Chapter 1, Section 1) but $f \in \mathrm{Lip}_\alpha(\mathbb{R})$ for all $\alpha \in (0,1)$.

5.2. **Higher order operators.** Let us consider, instead of the Laplacian, a homogeneous elliptic nonnegative operator $A(D_x)$ of degree $2m$ with real constant coefficients and denote by Q_A its quadratic form (we use the standard notation $D_x := -i\,\partial_x$). Let $W^{m,2}(\Omega)$ be the Sobolev space, $W_0^{m,2}(\Omega)$ be the closure of C_0^∞ in $W^{m,2}(\Omega)$ and $A_{\rm N}$ and $A_{\rm D}$ be the self-adjoint operators in the space $L^2(\Omega)$ generated by the quadratic form Q_A with domains $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$ respectively. Then the results of Section 2 remain valid with the following modifications.

(i) In the definitions of $N_{\rm N,D}$, $N_{\rm N}$, $N_{\rm D}$ and in Lemma 2.2 we replace the Dirichlet form $\int_\Omega |\nabla u|^2\,{\rm d}x$ with Q_A, $W^{1,2}(\Omega)$ with $W^{m,2}(\Omega)$, λ^2 with λ^{2m}, and $\varkappa^{-1/2}$ with $\varkappa^{-1/(2m)}$.

(ii) The kernel of the operator $A_{\rm N}$ is the space $\mathcal{P}_m(\Omega)$ of all polynomials on Ω whose degree is strictly smaller than m. Therefore we have $\int_\Omega |u(x)|^2\,{\rm d}x \leqslant \lambda^{-2m}\,Q_A[u]$ for all $u\in W^{1,2}(\Omega)\ominus\mathcal{P}_m(\Omega)$ if and only if $\lambda_{1,{\rm N}}(\Omega)\geqslant\lambda$, where $\lambda_{1,{\rm N}}(\Omega)$ is the first nonzero eigenvalue of $A_{\rm N}$. If p_u is the projection of $u\in L_2(\Omega)$ onto the subspace $\mathcal{P}_m(\Omega)$ then $\|u-p_u\|_{L^2(\Omega)}\leqslant\|u-p\|_{L^2(\Omega)}$ for all $p\in\mathcal{P}_m(\Omega)$ (cf. Remark 2.4).

(iii) Let $C_{A,W} := (2\pi)^{-d}\,\mu_d\{\xi\in\mathbb{R}^d : A(\xi)<1\}$. Then there exists a constant $C_{A,Q}$ such that

$$-\,C_{A,Q}\,(\delta\lambda)^{d-1}\leqslant N(Q_\delta^{(d)},\lambda)-C_{A,W}\,(\delta\lambda)^d\leqslant C_{A,Q}\,(\delta\lambda)^{d-1},\quad \forall\lambda>\delta^{-1},$$

for all $\delta>0$ (see Remark 2.9).

(iv) Instead of Lemma 2.6 we have the following result.

Lemma 5.4. *There exists a constant c_A depending only on the operator A and the dimension d such that the following statements hold true.*

(1) *If $P\in\mathbf{P}(\delta)$ then $N_{\rm N}(P,\lambda)=\dim\mathcal{P}_m$ for all $\lambda\leqslant c_A\,\delta^{-1}$.*

(2) *If $V\in\mathbf{V}(\delta)$ then $N_{\rm N}(V,\lambda)=\dim\mathcal{P}_m$ for all $\lambda\leqslant c_A\,\delta^{-1}$.*

(3) *If $M\in\mathbf{M}(\delta)$, $M\subset Q_\delta^{(d)}$ and $\Upsilon:=\partial M\bigcap Q_\delta^{(d)}$ then we have $N_{\rm N,D}(M,\Upsilon,\lambda)\leqslant\dim\mathcal{P}_m$ for all $\lambda\leqslant c_A\,\delta^{-1}$ and $N_{\rm N,D}(M,\Upsilon,\lambda)=0$ for all $\lambda\leqslant(1-c_A^{-1}\,\delta^{-d}\mu_d(M))_+^{1/(2m)}\,c_A\,\delta^{-1}$.*

Proof. We shall denote by C various constants depending only on A and d.

Since $A(\xi)\leqslant C\sum_{j=1}^d\xi_j^{2m}$, it is sufficient to prove the lemma assuming that $A(D_x)=A_m(D_x):=\sum_{j=1}^d D_{x_j}^{2m}$. Then (1) is easily obtained by separation of variables. If $u\in W^{m,2}(Q_\delta^{(d)})$, $u\equiv0$ outside M and p_u is the projection of u onto the subspace $\mathcal{P}_m(M)$ then

$$\begin{aligned}\int_M |p_u|^2\,{\rm d}x \;&\leqslant\; \mu_d(M)\sup_{x\in Q_\delta^{(d)}}|p_u(x)|^2 \;\leqslant\; C\,\mu_d(M)\,\delta^{-d}\int_{Q_\delta^{(d)}}|p_u|^2\,{\rm d}x\\ &=\; C\,\mu_d(M)\,\delta^{-d}\left(\int_M|p_u|^2\,{\rm d}x+\int_{Q_\delta^{(d)}}|u-p_u|^2\,{\rm d}x\right).\end{aligned}$$

Applying (ii) and this estimate instead of Remark 2.4 and (2.6), we obtain (3) in the same way as Lemma 2.6(3).

In order to prove (2), let us assume that $V = G_{f,b}(Q_c^{(d-1)})$ with $c \leqslant \delta$, $b = \inf f - \delta$ and $\operatorname{Osc} f \leqslant \delta/2$ and consider a function $u \in W^{m,2}(V)$. Let $p_{u;r,k}(x')$ be the projection of the function $\partial_{x_d}^k u(x',r) \in L^2(Q_c^{(d-1)})$ onto the subspace $\mathcal{P}_{m-k}(Q_c^{(d-1)})$, $p_{u;r}(x) := \sum_{k=0}^{m-1} \frac{1}{k!}(x_d-r)^k\, p_{u;r,k}(x')$ and $v_r(x) := \sum_{k=0}^{m-1} \frac{1}{k!}(x_d-r)^k\, \partial_{x_d}^k u(x',r)$, where $r \in [b, b+\delta]$ and $x_d \in [b, f(x')]$. We have

$$|u(x) - p_{u;r}(x)|^2 \;\leqslant\; 2\,|u(x) - v_r(x)|^2 + 2\,|v_r(x) - p_{u;r}(x)|^2 . \tag{5.2}$$

Since $|x_d - b| \leqslant 2\delta$, Jensen's inequality implies that

$$\begin{aligned} |u(x) - v_r(x)|^2 \;&=\; ((m-1)!)^{-2}\,\Big|\int_r^{x_d} (x_d - t)^{m-1}\,\partial_{x_d}^m u(x',t)\,\mathrm{d}t\,\Big|^2 \\ &\leqslant\; ((m-1)!)^{-2}\,|x_d - r| \int_r^{x_d} (x_d-t)^{2m-2}\,|\partial_{x_d}^m u(x',t)|^2\,\mathrm{d}t \\ &\qquad\qquad \leqslant\; ((m-1)!)^{-2}\,(2\delta)^{2m-1} \int_b^{f(x')} |\partial_{x_d}^m u(x)|^2\,\mathrm{d}x_d . \end{aligned}$$

In view of (ii) and (1), we also have

$$\int_{Q_c^{(d-1)}} |\partial_{x_d}^k u(x) - p_{u;r,k}(x')|^2\,\mathrm{d}x' \;\leqslant\; C\,\delta^{2m-2k}\, Q_{A'_{m-k}}[\partial_{x_d}^k u(x)]$$

for all $k = 0,\ldots,m-1$, where $A'_{m-k}(D_{x'}) := \sum_{j=1}^{d-1} D_{x_j}^{2m-2k}$ and $Q_{A'_{m-k}}$ is the quadratic form of A'_{m-k} with domain $W^{m-k,2}(Q_c^{(d-1)})$. Therefore, integrating (5.2) over $r \in [b, b+\delta]$, $x_d \in [b, f(x')]$, $x' \in Q_c^{(d-1)}$ and estimating $|x_d - r| \leqslant 2\delta$, we obtain

$$\begin{aligned} &\delta^{-1} \int_b^{b+\delta}\!\! \int |u(x) - p_{u;r}(x)|^2\,\mathrm{d}x\,\mathrm{d}r \\ &\qquad\qquad \leqslant\; C\,\delta^{2m} \int_V |\partial_{x_d}^m u(x)|^2\,\mathrm{d}x \;+ C\,\delta^{2m} \sum_{k=0}^{m-1} \sum_{|\alpha|=m} \int_P |\partial_x^\alpha u(x)|^2\,\mathrm{d}x , \end{aligned}$$

where $P = Q_c^{(d-1)} \times (b, b+\delta)$. Since the L_2-norms of the mixed derivatives $\partial_x^\alpha u(x)$ on a rectangle are estimated by the L_2-norms of the derivatives $\partial_{x_j}^m$, this estimate and (ii) imply (2). $\square$

Applying the same arguments as in Section 4 and using (iii) and Lemma 5.4, we obtain the following result.

Theorem 5.5. *Let A be a homogeneous nonnegative elliptic differential operator of order $2m$ with real constant coefficients. If $N_{\mathrm{N}}(\lambda,\Omega)$ and $N_{\mathrm{D}}(\lambda,\Omega)$*

denote the number of eigenvalues of the corresponding self-adjoint operator lying below λ^{2m} *then Theorems* 1.3, 1.8 *and Corollaries* 1.5, 1.6, 1.9 *holds true with* $C_{d,W} := C_{A,W}$.

5.3. **Other function spaces.** Let $B^{\alpha}_{p,q}$ be the Besov space and $BV_{\beta,\infty} := BV_{\tau_\beta,\infty}$ where $\tau_\beta(t) = (t^\beta + 1)$ and $\beta \in (0,+\infty)$. Lemma 4.5 implies that $B^{\alpha}_{\infty,\infty} = \mathrm{Lip}_\alpha \subset BV_{(d-1)/\alpha,\infty}$. Estimating the norm of the embedding $B^{\alpha}_{p,\infty}(Q^{(d-1)}_a) \hookrightarrow C(Q^{(d-1)}_a)$ for $\alpha p > d-1$ and $a > 0$, one can also show that $B^{\alpha}_{p,\infty} \subset BV_{(d-1)/\alpha,\infty}$ whenever $\alpha p > d-1$.

5.4. **Open problems.**

5.4.1. *The spaces* $BV_{\tau,\infty}$. The space $BV_{\beta,\infty}$ or $BV_{\tau,\infty}$ (under certain conditions on the function τ) is a Banach space with respect to an appropriate norm. Similar spaces have been considered in the dimension one, but we could not find references in the multidimensional case. It would be interesting to find a more constructive description of these spaces and to investigate their properties.

5.4.2. *More general domains.* The crucial point in our proof of Theorem 1.3 is the construction of the families $\{S_m\}_{\mathcal{M}}$ such that

(i) $\Omega^{\mathrm{b}}_\delta \subset \bigcup_m S_m \subset \Omega$,
(ii) $\aleph\{S_m\}_{\mathcal{M}} \leqslant C$,
(iii) $N_{\mathrm{N}}(S_m,\lambda) \leqslant C'$ whenever $\lambda \leqslant C''\delta^{-1}$,

where C , C' and C'' are some constants independent of $\delta \in \mathbb{R}_+$.

The remainder estimate in the Weyl formula for the Neumann Laplacian depends on the behaviour of $\#\mathcal{M}$ as $\delta \to 0$. In this paper we were assuming that Ω is the union of subgraphs of continuous functions, used Lemma 2.6 in order to prove (iii) and applied Corollary 3.2 in order to estimate $\aleph\{S_m\}$ and $\#\mathcal{M}$. Theorem 3.1 allows one to construct families of open sets S_m satisfying (i)–(iii) for many other domains Ω . It should be possible to find less restrictive sufficient conditions which guarantee the existence of such families and imply an asymptotic formula for $N_{\mathrm{N}}(\Omega,\lambda)$.

5.4.3. *Operators with variable coefficients.* Our main goal was to estimate the contribution of $\partial\Omega$ to the Weyl formula. In the interior part of Ω we used the old fashioned variational technique based on the Whitney decomposition and Dirichlet–Neumann bracketing. There are much more advanced methods of studying the asymptotic behaviour of the spectral function at the interior points (see the monographs [Iv3], [SV] or the recent papers [BI], [Iv4]), which are applicable to operators with variable coefficients.

Freezing the coefficients at an arbitrary point $x \in S_m$, we see that (iii) remains valid for a uniformly elliptic operator A with variable coefficients, provided that the corresponding quadratic form is homogeneous, the coefficients are uniformly continuous, δ is sufficiently small and $\operatorname{diam} S_m \leqslant c\,\delta$

with some constant c independent of δ. Using this observation and applying a more powerful technique in the interior of Ω, one can try to extend our results to operators with variable coefficients.

5.4.4. *Reminder estimate for the Dirichlet Laplacian.* It is not difficult to construct a bounded domain Ω such that $\lim_{\delta\to 0} |\delta^{-\alpha}\,\mu_d(\Omega_\delta^{\mathrm{b}})| = C'$ and

$$N_{\mathrm{D}}(\Omega,\lambda) - C_{d,W}\,\mu_d(\Omega)\,\lambda^d \;\geqslant\; -C^{-1}\,\lambda^{d-\alpha}\,, \qquad \forall\lambda > C\,, \tag{5.3}$$

where C and C' are some positive constants. For example, it can be done by considering a cube with a sequence of 'cracks' converging to the outer boundary, which get denser as the outer boundary is approached (similar domains were studied in [LV] and [MV]). For such a domain the estimate (1.7) is order sharp. It would be interesting find a domain $\Omega \in \mathrm{Lip}_\alpha$ satisfying (5.3) (cf. Theorem 1.10). Note that in the known examples disproving the so-called Berry conjecture (see, for instance, [BLe] or [LV]) the domain does not belong to the class Lip_α.

6. Constants

Throughout the paper $C_{d,W}$ is the Weyl constant (see Subsection 1.1),

$$C_{d,1} := \sum_{n=0}^{d-1} \frac{n!\,(d-n)!}{d!}\,C_{n,W}\,, \qquad C_{0,W} := 1\,,$$

$C_{d,2} = 2^{d-1}\,\mathcal{C}_{d-1}$ and $C_{d,3} = 6^{d-1}\,\hat{\mathcal{C}}_{d-1}$ where $\mathcal{C}_{d-1}$ and $\hat{\mathcal{C}}_{d-1}$ are the constants introduced in Theorem 3.1,

$$C_{d,4} := (4\,C_{d,2}+2)^{1/2}\,, \quad C_{d,5} := \min\left\{(1+2\pi^{-2})^{-1/2},\ \pi(1+d^{-1})^{-1}\right\}\,,$$

$$C_{d,6} := 2^{d-1}\,C_{d,2} + (3\,C_{d,2}+1)\,(2\sqrt{d})^d\,, \quad C_{d,7} := C_{d,4}^{-1}\,C_{d,5}\,,$$

$$C_{d,8} := \max\{1, C_{d,7}^{-1/2}\}\,, \quad C_{d,9} := 8\,C_{d,3}\,C_{d,8}\,,$$

$$C_{d,10} := (d+1)\left(12\sqrt{d}\,C_{d,1} + 4\,C_{d,W} + (4^d d^d + C_{d,6})\,(4d^{1/2}+4d^{-1/2})^d\right),$$

$$C_{d,11} := (d+1)\left(12\sqrt{d}\,C_{d,1} + 4\,C_{d,W} + (4^d d^d + 2)\,(4d^{1/2}+4d^{-1/2})^d\right).$$

Remark 6.1. If ρ is continuous then Theorem 3.1 holds true with $\mathcal{C}_n = 2^n$ and $\hat{\mathcal{C}}_n = 4^n$ (see [G]). Since the function ρ in the proof of Corollary 3.2 is continuous, all our results remain valid for $C_{d,2} = 4^{d-1}$ and $C_{d,3} = 24^{d-1}$.

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