Solitary wave dynamics in an external potential

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Abstract

We study the behavior of solitary-wave solutions of some generalized nonlinear Schrödinger equations with an external potential. The equations have the feature that in the absence of the external potential, they have solutions describing inertial motions of stable solitary waves.

We construct solutions of the equations with a non-vanishing external potential corresponding to initial conditions close to one of these solitary wave solutions and show that, over a large interval of time, they describe a solitary wave whose center of mass motion is a solution of Newton's equations of motion for a point particle in the given external potential, up to small corrections corresponding to radiation damping.

1 Introduction

In this paper we study the effective dynamics of solitary wave solutions of a class of generalized nonlinear Schrödinger equations with an external potential.

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These equations have the form

$$\mathrm{i}\partial_t\psi = (-\Delta + V)\psi - f(\psi) \ , \tag{1.1}$$

where $\psi : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{C}$, x denotes a point in space $\mathbb{R}^d$, $t \in \mathbb{R}$ is time, $\partial_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the spatial Laplacian, $V(x)$ is the external potential and is a real-valued, bounded slowly varying function on $\mathbb{R}^d$, and f is a map from the complex Sobolev space $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$ to $\mathrm{H}_{-1}(\mathbb{R}^d, \mathbb{C})$ such that $f(0) = 0$, and $f(\bar{\psi}) = \overline{f(\psi)}$, describing a nonlinear "self-interaction". Precise assumptions on V and f will be given in Section 2. Examples of nonlinearities of interest include local nonlinearities such as

$$f(\psi) = \lambda|\psi|^{2s}\psi, \quad 0 < s < \frac{2}{d}, \ \lambda > 0, \tag{1.2}$$

and Hartree-type nonlinearities

$$f(\psi) = \lambda(W * |\psi|^2)\psi, \quad \lambda > 0, \tag{1.3}$$

where W is of positive type, continuous, spherically symmetric potential function which tends to 0, as $|x| \to \infty$ and $W * g(x) := \int W(x-y)g(y)\,\mathrm{d}^d y$, denotes (spatial) convolution. Of course, λ can be scaled out by rescaling ψ.

We assume that the nonlinearity in (1.1) is such that the Cauchy problem has a unique, global solution, $\psi(x,t)$, in the space $\mathrm{C}(\mathbb{R}^+; \mathrm{H}_1(\mathbb{R}^d, \mathbb{C})) \cap \mathrm{C}^1(\mathbb{R}^+; \mathrm{H}_{-1}(\mathbb{R}^d, \mathbb{C}))$, given an initial condition $\psi(x, 0) = \psi_0(x) \in \mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$. Results on the Cauchy problem associated with (1.1) can be found in [18, 25, 11] (see Section 2 for a discussion).

For $V \equiv 0$, eq. (1.1) is the usual generalized nonlinear Schrödinger (or Hartree) equation. For self-focusing nonlinearities (as in example (1.2) or (1.3) with W of positive type and $\lambda > 0$ large enough), it can have stable solitary wave solutions of the form

$$\eta_\sigma(x,t) := \mathrm{e}^{\mathrm{i}(\frac{1}{2}v\cdot(x-a)+\gamma)}\eta_\mu(x-a), \tag{1.4}$$

where $\sigma := \{a, v, \gamma, \mu\}$, and $a = vt + a_0$, $\gamma = \mu t + \frac{v^2}{4}t + \gamma_0$, with $\gamma_0 \in [0, 2\pi)$, $a_0, v \in \mathbb{R}^d$ and $\mu \in \mathbb{R}^+$, constant, and where η_μ is a positive solution of the nonlinear eigenvalue problem

$$(-\Delta + \mu)\eta_\mu - f(\eta_\mu) = 0 \tag{1.5}$$

(see Section 2). Solutions of eq. (1.1) of the form (1.4) describe solitary waves traveling through space with a constant velocity v, and with an oscillatory phase given by $\mu t - \frac{1}{4}v^2 t$. Existence of such solutions, for a large class of nonlinearities has been established in [41, 6, 4, 5, 3, 24, 28, 1]. See Section 2 for an outline of results relevant for this paper.

In analyzing soliton-like solutions of eq. (1.1) we encounter two length scales: the size $\propto \mu^{-1/2}$ of the support of the function η_μ, which is determined by our choice of initial condition ψ_0, and the length scale $\propto (\sup |\nabla V|)^{-1}$ over which the external potential V varies appreciably. We will assume that the ratio

$$\epsilon_V = \frac{\sup |\nabla V(x)|}{\sqrt{\mu}}, \tag{1.6}$$

is *small*; *i.e.*, that the potential V varies little over the support of a solitary wave solution.

When V does not vanish, the wave $\eta_\sigma(x)$ in (1.4) does *not* solve eq. (1.1). However, we expect that if the initial condition

$$\psi_0(x) := \psi(x, 0) \tag{1.7}$$

for (1.1) is close to $\eta_{\sigma_0}(x)$ for some σ_0, in the sense that

$$\|\mathrm{e}^{-\mathrm{i}\frac{1}{2}v_0 \cdot x}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{H}_1} \leq \epsilon_0, \tag{1.8}$$

then for *all times* $0 \leq t \leq \frac{T}{\epsilon_V + \epsilon_0{}^2}$, where T is some positive constant, the solution $\psi(x,t)$ remains *close* to a solitary wave of the form (1.4) for some *time-dependent* parameters μ, v, a and γ determined by V and ψ_0. We will show, more specifically, that these parameters can be chosen to be solutions of the following system of ordinary differential equations

$$\dot{a} = v \ , \quad \frac{\dot{v}}{2} = -\nabla V(a) \ , \quad \dot{\mu} = 0 \ , \tag{1.9}$$

and

$$\dot{\gamma} = \mu + \frac{v^2}{4} - V(a) \ . \tag{1.10}$$

(up to error terms of size $\mathcal{O}({\epsilon_V}^2 + {\epsilon_0}^2)$), with initial conditions given by $a(0) = a_0$, $v(0) = v_0$, $\mu(0) = \mu_0$, and $\gamma(0) = \gamma_0$, with $a_0, v_0, \mu_0, \gamma_0$ as in (1.4). We observe that the first two equations in (1.9) are *Newton's equations of motion* for the trajectory $(a(t), v(t) = \dot{a}(t))$ of a point particle of mass $\frac{1}{2}$ moving in the external potential $V(a)$. The center of a solitary wave solution of (1.1) follows this trajectory, up to deviations $\mathcal{O}(\epsilon_V + \epsilon_0)$ due to "radiation damping".

We state here rigorously the result discussed above for a special class of nonlinearities. The general class of nonlinearities is introduced and discussed in the next section. We assume the external potential $V(x)$ satisfies the conditions

$$V \in \mathrm{C}^2 \text{ and } |\partial_x^\alpha V(x)| \leq C_\alpha {\epsilon_V}^{|\alpha|}, \quad \text{for } |\alpha| \leq 2. \tag{1.11}$$

In (1.11), $\epsilon_V > 0$ is the small parameter introduced in (1.6). In other words, we find it convenient to fix the size of the support (the 'width') of the solitary wave solution at $\mathcal{O}(1)$ and assume that the external potential $V(x)$ varies slowly. Let $\epsilon := \epsilon_V + \epsilon_0$.

Theorem 1. *Assume that the nonlinearity f is given by* (1.2)*, and assume that the potential V satisfies* (1.11) *with $\epsilon_V \ll 1$. Let I_0 be any closed, bounded interval in $(0, \infty)$. Let $\epsilon_0 \ll 1$ and the initial condition ψ_0 satisfy*

$$\|\mathrm{e}^{-\mathrm{i}\frac{1}{2}v_0 \cdot x}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{H}_1} < \epsilon_0, \tag{1.12}$$

for some $\sigma_0 \in \mathbb{R}^d \times \mathbb{R}^d \times [0, 2\pi) \times I_0$. Then there is a constant $T > 0$, independent of ϵ_V, ϵ_0 but possibly dependent on I_0, such that for times $0 \leq t \leq T(\epsilon_V + {\epsilon_0}^2)^{-1}$, the solution to (1.1) *with this initial condition is of the form*

$$\psi(x, t) = \mathrm{e}^{\mathrm{i}(\frac{1}{2}v \cdot (x-a) + \gamma)}(\eta_\mu(x - a) + w(x - a, t)), \tag{1.13}$$

where

$$\|w\|_{\mathrm{H}_1} = \mathcal{O}(\epsilon), \tag{1.14}$$

and where the parameters v, a, γ and μ satisfy the differential equations

$$\frac{1}{2}\dot{v} = -(\nabla V)(a) + \mathcal{O}(\epsilon^2), \tag{1.15}$$

$$\dot{a} = v + \mathcal{O}(\epsilon^2), \tag{1.16}$$

$$\dot{\gamma} = \mu - V(a) + \frac{1}{4}v^2 + \mathcal{O}(\epsilon^2), \tag{1.17}$$

$$\dot{\mu} = \mathcal{O}(\epsilon^2). \tag{1.18}$$

The same conclusions hold for nonlinearities of the form $f(\psi) = g(|\psi|^2)\psi + (W * |\psi|^2)\psi$ where W and g satisfy explicit conditions (see the discussion of the conditions in Section 2), provided an additional spectral condition is satisfied (see Condition (F) in Section 2).

The first result of this type was proved by Fröhlich, Tsai and Yau [16, 17] for the Hartree equation ((1.1) with (1.3)) under a spectral condition (see Condition (F) of Section 2). The choice of the Hartree type nonlinearity plays an important role in [16, 17]. For local, pure power nonlinearities and a small parameter, ϵ_V, Bronski and Jerrard [7] have shown that if an initial condition is of form (1.4), with $t = 0$, then the solution $\psi(x,t)$ of eq. (1.1) satisfies

$$\epsilon_V{}^{-d}|\psi(\frac{x}{\epsilon_V}, \frac{t}{\epsilon_V})|^2 \, \mathrm{d}^d x \to \|\eta_\mu\|^2_{\mathrm{L}^2} \delta_{a(t)} \tag{1.19}$$

in the C^{1*} (dual to C^1) topology, provided $a(t)$ satisfies the equation $\frac{1}{2}\ddot{a} = \nabla U(a)$, where $V(x) = U(\epsilon_V x)$ (see (1.9)).

Our approach is built on important developments in the nonlinear Schrödinger equation (NLS) in the last 20 years (see [11, 43] for reviews). We outline the landmark developments briefly here. Orbital stability of NLS solitary waves for $V = 0$ was proved by Cazenave and Lions [12] and M. Weinstein [47, 48], whose result was significantly extended by Grillakis, Shatah and Strauss [20, 21]. The next significant step was made by A. Soffer and M. Weinstein [38] who proved, under some restrictive conditions, asymptotic stability of nonlinear ground states for eq. (1.1) and by V. Buslaev and G. Perel'man [8] who, motivated by [38], proved asymptotic stability of NLS solitary waves (V=0) in one dimension, again under certain restrictive conditions. These results were significantly extended by Tsai and Yau [44, 45, 46], Soffer and Weinstein [39, 40], Cuccagna [13, 14], Buslaev and Perel'man [9], and Buslaev and C. Sulem [10]. Furthermore Perel'man [35], and Rodnianski, Schlag and Soffer [37], have obtained the first results on soliton scattering. Many of the issues touched upon in the present paper were studied also in work of Gustafson and Sigal [23] on dynamics of magnetic vortices.

We also mention interesting non-rigorous results by Pelinovsky, Afanasjev and Kivshar [33] and Pelinovsky and Grimshaw [34] on dynamics of NLS solitons near the borderline for stability. Nonrigorous results on dynamics of 'center of mass' of solitons were obtained by E. van Groesen and F. Mainardi and G. Derks and E. van Groesen [22, 15].

Our paper is organized as follows. In Section 2 we present the general hypotheses on the class of nonlinearities and formulate our main result under these hypothesis. In Section 3, we explain the Hamiltonian and variational aspects of the dynamics given by eq. (1.1) which play a role in our proof. In Section 4, we describe the symmetries of eq. (1.1) for $V \equiv 0$ and find the "zero modes" associated with these symmetries. Furthermore, we introduce and analyze a finite-dimensional manifold, $\mathrm{M_s}$, of stable solitary wave solutions to eq. (1.1). In Section 5, we introduce a convenient parameterization of functions in a small neighborhood of $\mathrm{M_s}$ in phase space. In Section 6, we transform the equation (1.1) to a moving frame, and then rewrite the resulting equation in terms of the parameters introduced in Section 5 In Sections 7 and 8, we control solutions of our equations of motion in a moving frame by constructing an approximately conserved Lyapunov functional. The proof of our main result is completed in Section 9. Some material of technical or review nature is collected in four appendices.

Remark about the notation: in this paper we consider equations, maps and functionals on complex spaces such as $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$, which sometimes are identified with real spaces; *e.g.*, $\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}) = \mathrm{H}_1(\mathbb{R}^d, \mathbb{R}) \oplus \mathrm{H}_1(\mathbb{R}^d, \mathbb{R})$, under the identification $\psi \leftrightarrow (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$. In this case the operator of multiplication by i^{-1} is identified with the operator

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.20}$$

(which defines a complex structure on the corresponding real spaces). Thus a real function η_μ is sometimes written as $(\eta_\mu, 0)$ and similarly an imaginary function as $\mathrm{i}\eta_\mu$ as $(0, \eta_\mu)$ (or even as $J\eta_\mu$). Fréchet derivatives are always understood to be defined on real spaces. They are denoted by primes. c will denote various constants which may depend only on the interval I_0 for the parameter μ_0.

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2 Main Result

In this section we formulate general assumptions on the nonlinearity f in (1.1) and state our main result under general assumptions. Our assumptions are rather abstract, each responsible for certain aspect of the problem and of our approach. Then we discuss specific nonlinearities for which these assumptions are satisfied

(A). (Energy) There exists a C^3-functional $F:\mathrm{H}_1 \mapsto \mathbb{R}$ such that $F'(\psi)=f(\psi)$, $\sup_{\|u\|_{\mathrm{H}_1}\le M}\|F''(u)\|_{\mathrm{B}(\mathrm{H}_1,\mathrm{H}_{-1})}<\infty$ and $\sup_{\|u\|_{\mathrm{H}_1}\le M}\|F'''(u)\|_{\mathrm{H}_1\mapsto \mathrm{B}(\mathrm{H}_1,\mathrm{H}_{-1})}\le \infty$ (here B is the space of bounded linear operators);

(B). (Symmetries) $F(\mathcal{T}\psi)=F(\psi)$, where $\mathcal{T}$ is a translation $\mathcal{T}_a^{\mathrm{tr}}: u(x)\mapsto u(x-a)\ \forall a\in\mathbb{R}^d$, a rotation $\mathcal{T}_R^{\mathrm{r}}: u(x)\mapsto u(R^{-1}x)\ \forall R\in \mathrm{SO}(d)$, a gauge transform $\mathcal{T}_\gamma^{\mathrm{g}}: u(x)\mapsto \mathrm{e}^{\mathrm{i}\gamma}u(x)\ \forall\gamma\in[0,2\pi)$, a boost transform $\mathcal{T}_v^{\mathrm{b}}: u(x)\mapsto \mathrm{e}^{\frac{\mathrm{i}}{2}v\cdot x}u(x)\ \forall v\in\mathbb{R}^d$, or a complex conjugation $\mathcal{T}^{\mathrm{c}}: u(x)\mapsto \bar{u}(x)$;

(C). (Existence of solitons) There is an interval $I\subset\mathbb{R}$ such that $\forall\mu\in I$ eq. (1.5) has a positive, spherically symmetric, $\mathrm{L}^2\cap\mathrm{C}^2$ solution η_μ, such that $\||x|^3\eta_\mu\|+\||x|^2|\nabla\eta_\mu|\|+\||x|^2\partial_\mu\eta_\mu\|<\infty\ \forall\mu\in I$ (here $\|\cdot\|$ denotes the L^2 norm);

(D). (Orbital stability) The solutions η_μ, $\mu\in I$, of (1.5) described in (C), satisfy

$$\partial_\mu\int\eta_\mu^2\,\mathrm{d}^dx>0; \tag{2.1}$$

(E). (GWP) Eq. (1.1) is globally well-posed in H_1 and in H_2;

(F). (Null space condition) $\forall\mu\in I$,

$$\mathrm{N}(\mathcal{L}_{\eta_\mu})=\mathrm{span}\{\big(0,\eta_\mu\big),\big(\partial_{x_j}\eta_\mu,0\big),j=1,...,d\} \tag{2.2}$$

where $\mathcal{L}_{\eta_\mu}:=-\Delta+\mu-f'(\eta_\mu)$, the Fréchet derivative of the map $\psi\mapsto(-\Delta+\mu)\psi-f(\psi)$ at η_μ.

We now discuss conditions (A)–(F), beginning with general remarks. Condition (A) allows us to define the conserved energy or Hamiltonian

$$\mathcal{H}_V(\psi)=\frac{1}{2}\int(|\nabla\psi|^2+V|\psi|^2)\,\mathrm{d}^dx-F(\psi), \tag{2.3}$$

and gives us the following estimates on the nonlinearities

$$|R_\eta^{(2)}(w)|\le c(M)\|w\|_{\mathrm{H}_1}^2,\quad |R_\eta^{(3)}(w)|\le c(M)\|w\|_{\mathrm{H}_1}^3, \tag{2.4}$$

and

$$\|N_\eta(w)\|_{\mathrm{H}_{-1}}\le C(M)\|w\|_{\mathrm{H}_1}^2 \tag{2.5}$$

for any $\eta \in \mathrm{H}_2(\mathbb{R}^d)$ and $w \in \mathrm{H}_1(\mathbb{R}^d)$. with $\|\eta\|_{\mathrm{H}_1} + \|w\|_{\mathrm{H}_1} \leq M$. Here

$$R^{(2)}_\eta(w) := F(\eta + w) - F(\eta) - \langle F'(\eta), w\rangle, \tag{2.6}$$

$$R^{(3)}_\eta(w) := F(\eta + w) - F(\eta) - \langle F'(\eta), w\rangle - \frac{1}{2}\langle F''(\eta)w, w\rangle \tag{2.7}$$

and

$$N_\eta(w) = F'(\eta + w) - F'(\eta) - F''(\eta)w. \tag{2.8}$$

Note that

$$N_\eta(w) = R^{(3)\prime}_\eta(w). \tag{2.9}$$

We can assume without loss of generality that $f'(0) = 0$. Then one can show that $I \subset \mathbb{R}_+$.

As was mentioned in the introduction, the nonlinearities of interest to us are local nonlinearities,

$$f(\psi)(x) = h(|\psi(x)|^2)\psi(x), \tag{2.10}$$

for some real function h on $\mathbb{R}_+$, and Hartree-type nonlinearities,

$$f(\psi) = (W * |\psi|^2)\psi, \tag{2.11}$$

where W is a fixed, real valued, spherically symmetric function, tending to 0 at ∞. More generally we consider nonlinearities the form

$$f(\psi)(x) = h(|\psi(x)|^2)\psi(x) + (W * |\psi|^2)(x)\psi(x). \tag{2.12}$$

We discuss now under which conditions on h and W in (2.12) Conditions (A)–(F) are satisfied.

Condition (A). For nonlinearities of type (2.12) the functional F is given by

$$F(\psi) = \frac{1}{2}\int H(|\psi|^2) + \frac{1}{2}(W * |\psi|^2)\,\mathrm{d}^d x, \tag{2.13}$$

where $H(s) = \int_0^s h(p)\,\mathrm{d}p$. The functional F is C^3 with the inequalities specified in Condition (A) satisfied, provided $h(s)$ is C^2 with $h^{(k)}(s) \leq c(1 + s^{q-k})$ $(k = 0, 1, 2)$, for some $q < 2/(d-2)$, and $d < 4$, and $W \in \mathrm{L}^p + \mathrm{L}^\infty$ for some $p > d/2$.

Condition (B). This condition is trivially satisfied for (2.12) with $W(x) = W(|x|)$.

Condition (C). Existence of a positive, spherically symmetric solitary wave solution to Eq (1.5), was proved in [4, 5, 6] for local nonlinearities (2.10) satisfying

$$-\infty < \lim_{r\to 0} h(r) < \mu,$$
$$-\infty \leq \lim_{r\to\infty} r^{-\alpha} h(r) \leq C, \tag{2.14}$$

for $0 < \alpha < 2/(d-2)$, if $d > 2$ and $\alpha \in (0,\infty)$, if $d = 1, 2$,

$$\exists \zeta > 0, \text{ such that } \int_0^\zeta h(r)\,\mathrm{d}r > \mu\zeta.$$

Additional results, for a large class of nonlinearities can be found in [41, 3, 24, 28, 1].

For nonlocal nonlinearities (2.11) (Hartree equation) with

$$W \in \mathrm{L}^p_{\mathrm{loc}},\ p \geq \frac{d}{2},\ W \to 0 \text{ as } |x| \to \infty, \tag{2.15}$$

existence of solutions was proved in [11, 19, 17, 27, 26, 1].

Moreover, for nonlinearities (2.10) or (2.11) satisfying conditions (2.14) or (2.15), solutions to (1.5) (solitary waves) are exponentially decaying at ∞ as $\mathcal{O}(\mathrm{e}^{-\sqrt{\mu}|x|})$; see [4, 32].

Condition (D). Condition (2.1) is a sufficient condition for orbital stability of solitary waves for (generalized) nonlinear Schrödinger equations (see [20, 21]). This condition is to be checked for each nonlinearity. In the special case of a pure power nonlinearity, $f(\psi) = \lambda|\psi|^{2s}\psi$, we have $\eta_\mu(x) = \mu^{\frac{1}{2s}}\eta_{\mu=1}(x\sqrt{\mu})$. Thus condition (D) reduces to the condition that $s < 2/d$.

Condition (E). Assume that f is of the form (2.12) with $h : [0,\infty) \mapsto \mathbb{R}$, smooth and satisfying $h(0) = 0$, and

$$|h'(r)| \leq C(1 + r^{\alpha-1})\ , \tag{2.16}$$

for some $\alpha \in [0, \frac{2}{d-2})$, for $d \geq 3$ and $\alpha \in [0,\infty)$ for $d = 1, 2$ and

$$h(r) \leq C(1 + r^{\beta})\ , \tag{2.17}$$

for some $\beta \in [0, \frac{2}{d})$. Furthermore assume that $W : \mathbb{R}^d \mapsto \mathbb{R}$ is an even potential such that

$$W \in \mathrm{L}^q(\mathbb{R}^d) + \mathrm{L}^\infty(\mathbb{R}^d), \tag{2.18}$$

for some $q \geq 1$, $q > d/4$ and let $W_+ = \max(0, W)$ s.t.

$$W_+ \in \mathrm{L}^r(\mathbb{R}^d) + \mathrm{L}^\infty(\mathbb{R}^d), \tag{2.19}$$

for some $r \geq 1$, $r \geq d/2$ for $d > 2$ and $r > 1$ for $d = 2$. Then (1.1) is globally well-posed in H_1 and H_2 (see [11, Chapter 6]).

Condition (F). This condition is more delicate. First we observe that

$$(\partial_{x_j}\eta_\mu, 0), (0, \eta_\mu) \in \mathrm{N}(\mathcal{L}_{\eta_\mu}) \ , \quad \forall j = 1, ...d \ , \tag{2.20}$$

due to the fact that η_μ breaks the translation and gauge symmetry of (1.5) (see Section 4).

Now we list some facts which are discussed in Appendix C. If η_μ is spherically symmetric and f is a local nonlinearity, $f(\psi)(x) = f(\psi(x))$, then $\mathcal{L}_\eta$ can have at most one zero eigenfunction in addition to (2.20). Hence

$$d + 1 \leq \dim \mathrm{N}(\mathcal{L}_{\eta_\mu}) \leq d + 2. \tag{2.21}$$

This extra zero eigenfunction is spherically symmetric and is also a zero eigenfunction of the ordinary differential operator

$$A_{\mu,0} = -\Delta_r + \mu - f'(\eta_\mu) \tag{2.22}$$

on $\mathrm{L}^2(\mathbb{R}_+, r^{d-1}\, \mathrm{d}r)$, where Δ_r is the radial Laplacian,

$$\Delta_r = \partial_r^2 + \frac{d-1}{r}\partial_r \ . \tag{2.23}$$

Thus

$$\mathrm{N}(\mathcal{L}_{\eta_\mu}) = \mathrm{span}\{(0, \eta_\mu), (\partial_{x_j}\eta_\mu, 0), (\mathrm{N}(A_{\mu,0}), 0)\}. \tag{2.24}$$

For local nonlinearities (2.10), $\mathrm{N}(A_{\mu,0}) = \{0\}$, if either $d = 1$ or

$$h'(r) + h''(r)r^2 > 0. \tag{2.25}$$

Alternative conditions on h for $d > 1$ are given in Appendix C. Thus, for local nonlinearities, if either $d = 1$ or (2.25) holds, then condition (F) is satisfied (see also [30, 47, 29, 42]).

In any case this extra degeneracy, if it happens, is non-generic. If (2.2) holds for some f, then it also holds for small perturbations of f. On the other hand we expect that if (2.2) fails for a given f then there are arbitrarily small perturbations of f for which (2.2) is satisfied.

It is easy to check that Condition (A)–(F) are satisfied for nonlinearity (1.2).

Let $\Sigma := \mathbb{R}^d \times \mathbb{R}^d \times [0, 2\pi) \times I$ and $\Sigma_0 := \mathbb{R}^d \times \mathbb{R}^d \times [0, 2\pi) \times I_0$, where I is a closed, bounded interval on the positive real axis, specified in Condition (C), and $\bar{I}_0$ is a closed interval contained in $I \backslash \partial I$. Recall $\epsilon := \epsilon_V + \epsilon_0$. Our main result is

Theorem 2. *Assume* (1.11) *and (A)–(F) are satisfied. Given $\epsilon_0 > 0$ and $\epsilon_V > 0$ defined in* (1.6) *such that $\epsilon \ll 1$ and an initial condition ψ_0 satisfying*

$$\|\mathrm{e}^{-\frac{\mathrm{i}}{2} v_0 \cdot x}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{H}_1} \leq \epsilon_0 \ , \tag{2.26}$$

for some $\sigma_0 := \{a_0, v_0, \gamma_0, \mu_0\} \in \Sigma_0$. Then there is a constant $T > 0$ (independent of ϵ_V, ϵ_0 but possibly dependent on I_0) such that for times $t(\epsilon_V + {\epsilon_0}^2) \leq T$ the solution to (1.1) *with this initial condition can be written in the form*

$$\psi(x,t) = \mathrm{e}^{\mathrm{i}(\frac{1}{2} v \cdot (x-a) + \gamma)}(\eta_\mu(x-a) + w(x-a,t)) \tag{2.27}$$

where

$$\|w\|_{\mathrm{H}_1} = \mathcal{O}(\epsilon) \tag{2.28}$$

and the parameters v, a, γ and μ satisfy the differential equations

$$\frac{1}{2}\dot{v} = -(\nabla V)(a) + \mathcal{O}(\epsilon^2), \tag{2.29}$$

$$\dot{a} = v + \mathcal{O}(\epsilon^2), \tag{2.30}$$

$$\dot{\gamma} = \mu - V(a) + \frac{1}{4}v^2 + \mathcal{O}(\epsilon^2), \tag{2.31}$$

$$\dot{\mu} = \mathcal{O}(\epsilon^2). \tag{2.32}$$

The proof of this theorem is given in Section 9.

3 The Hamiltonian and variational structure of generalized nonlinear Schrödinger equations

It is well known that the generalized nonlinear Schrödinger equation is a Hamiltonian equation of motion on an infinite-dimensional phase space. In this paper we make extensive use of this fact and of the symplectic geometry of phase space, in particular of certain "submanifolds" in this space. We outline briefly some facts which are relevant for us.

In this section we consider the generalized nonlinear Schrödinger equation eq. (1.1),

$$\mathrm{i}\partial_t\psi = (-\Delta + V)\psi - f(\psi)\ ,$$

under Conditions (A) and (B) only.

We study eq. (1.1) on the space $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$. This space, considered as a real space ($\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}^2) = \mathrm{H}_1(\mathbb{R}^d, \mathbb{R}) \oplus \mathrm{H}_1(\mathbb{R}^d, \mathbb{R})$, $\psi \leftrightarrow (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)$), and equipped with the symplectic form

$$\omega(u, v) := \mathrm{Im} \int u\bar{v}\,\mathrm{d}^d x \tag{3.1}$$

(defined for u, v in the tangent space at a given point in $\mathrm{H}_1(\mathbb{R}^d; \mathbb{R}^2)$, which we can identify with $\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}^2)$ in our case), is a symplectic space.

Define the Hamiltonian functional on $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$ as

$$\mathcal{H}_V(\psi) := \frac{1}{2}\int |\nabla\psi|^2 + V|\psi|^2\,\mathrm{d}^d x - F(\psi), \tag{3.2}$$

where $F(\psi)$ is as in Condition (A); *i.e.*, $F'(\psi) = f(\psi)$. With these definitions eq. (1.1) can be written as

$$\partial_t\psi = J\mathcal{H}'_V(\psi), \tag{3.3}$$

where J is the operator on $\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}^2)$ (strictly speaking $J : (\mathrm{T}_\psi\mathrm{H}_1)^* \mapsto \mathrm{T}_\psi\mathrm{H}_1$) given by

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.4}$$

in block-diagonal notation. Thus J is a complex structure on $\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}^2)$ corresponding to the operator i^{-1} on $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$.

Observe that the space $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$ also has a real inner product (Riemannian metric)

$$\langle u, v\rangle := \mathrm{Re}\int u\bar{v}\,\mathrm{d}^d x, \tag{3.5}$$

so that $\omega(u, v) = \langle u, J^{-1}v\rangle$.

Since the Hamiltonian $\mathcal{H}_V(\psi)$ is autonomous (it does not explicitly depend on time t) and invariant under the gauge transformation $\mathcal{H}_V(\mathrm{e}^{\mathrm{i}\gamma}\psi) = \mathcal{H}_V(\psi)$ for all $\gamma \in [0, 2\pi)$, we have conservation of energy, $\mathcal{H}_V(\psi) = \text{const}$, and “mass” or “number of particles”, $\mathcal{N}(\psi) = \text{const}$, where

$$\mathcal{N}(\psi) := \frac{1}{2}\int |\psi|^2\,\mathrm{d}^d x \tag{3.6}$$

under the evolution of eq. (1.1).

Due to Condition (D) the solitary wave profiles η_μ described in Condition (C) are local minimizers of the Hamiltonian $\mathcal{H}_{V=0}(\psi)$ restricted to the spheres

$$\{\psi \in \mathrm{H}_1 : N(\psi) = m\} \tag{3.7}$$

for $m > 0$ (see [20], Thm 3). Hence they are critical points of the functional

$$\mathcal{E}_\mu(\psi) := \frac{1}{2}\int |\nabla\psi|^2 + \mu|\psi|^2 \,\mathrm{d}^d x - F(\psi), \tag{3.8}$$

where $\mu = \mu(m)$ is a Lagrange multiplier, and (1.5) is just the Euler-Lagrange equation for $\mathcal{E}_\mu(\psi)$.

Observe that the functional $\mathcal{E}_\mu(\psi)$ also arises as $(t_1 - t_0)\mathcal{E}_\mu(\phi) = \mathcal{S}_{V=0}(\phi \mathrm{e}^{\mathrm{i}\mu t})$ for any $\phi(x)$, where $\mathcal{S}_V$ is the action for eq. (1.1):

$$\mathcal{S}_V(\psi) := \int_{t_0}^{t_1} \left(\frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Im}\,\dot{\psi}\bar{\psi}\,\mathrm{d}^d x + \mathcal{H}_V(\psi) \right) \mathrm{d}t. \tag{3.9}$$

The functional $\mathcal{E}_\mu(\psi)$ will play an important role in our approach. We will use it as a Lyapunov functional in estimating the fluctuations w. Finally, we note that the operator $\mathcal{L}_\eta$ that appears in Condition (F) is the Hessian of $\mathcal{E}_\mu(\psi)$ at η_μ: $\mathcal{L}_\eta := \mathcal{E}_\mu''(\eta_\mu)$.

4 Symmetries, zero modes, and the manifold of solitary waves

In this section we introduce the manifold of solitary waves which is obtained by applying the generalized symmetry transforms (see below) to a fixed solitary wave. The tangent space to the manifold is introduced, and its inherited symplectic form is derived. Furthermore we prove the key fact that the inherited symplectic form is non-degenerate.

Starting from this section we will often use the abbreviation $\eta \equiv \eta_\mu$.

Eq. (1.1) with $V \equiv 0$ is invariant under spatial translations $\mathcal{T}_a^{\mathrm{tr}}$, gauge transformations $\mathcal{T}_\gamma^{\mathrm{g}}$, and Galilean transformations $\mathcal{T}_v^{\mathrm{gal}}$, where

$$\mathcal{T}_a^{\mathrm{tr}} : \psi(x,t) \mapsto \psi(x-a,t)\ , \qquad \mathcal{T}_\gamma^{\mathrm{g}} : \psi(x,t) \mapsto \mathrm{e}^{\mathrm{i}\gamma}\psi(x,t)\ , \tag{4.1}$$

$$\mathcal{T}_v^{\mathrm{gal}} : \psi(x,t) \mapsto \mathrm{e}^{\mathrm{i}(\frac{1}{2}v\cdot x - \frac{1}{4}|v|^2 t)}\psi(x-vt,t)\ . \tag{4.2}$$

(Transformations (4.1)–(4.2) map solutions of eq. (1.1) with $V=0$ into solutions of (1.1) with $V=0$.) The symmetries $\mathcal{T}_a^{\rm tr}$, $\mathcal{T}_\gamma^{\rm g}$, $\mathcal{T}_v^{\rm gal}$, have associated conserved quantities: field momentum, mass and 'center of mass motion', (*cf.* [17]),

$$\int \mathrm{i}\bar{\psi}\nabla\psi\,\mathrm{d}^d x, \quad \int |\psi|^2\,\mathrm{d}^d x, \quad \int \bar{\psi}(x+2\mathrm{i}t\nabla)\psi\,\mathrm{d}^d x. \tag{4.3}$$

For $f(\psi)=|\psi|^{2s}\psi$ and $V=0$, eq. (1.1) is also invariant under the scaling transformation

$$\mathcal{T}_\mu^{\rm s}:\psi(x,t)\mapsto \mu^{\frac{1}{2s}}\psi(\sqrt{\mu}x,\mu t)\ . \tag{4.4}$$

When the external potential is introduced into the problem, the translational and Galilean invariance are broken. In particular, conservation of the field momentum is replaced by the following 'Newton's law' (Ehrenfest's theorem)

$$\partial_t\langle\psi,-\mathrm{i}\nabla\psi\rangle = -\langle\psi,(\nabla V)\psi\rangle, \tag{4.5}$$

which plays an important role in our analysis, and which is proved in Appendix A.

Let $\mathcal{T}_v^{\rm b}:\psi(x)\mapsto \mathrm{e}^{\frac{\mathrm{i}}{2}v\cdot x}\psi(x)$, be the boost transform. We introduce the combined symmetry transformations $\mathcal{S}_{av\gamma}$:

$$p(x)\mapsto p_{av\gamma} := \mathcal{S}_{av\gamma}p = \mathcal{T}_a^{\rm tr}\mathcal{T}_v^{\rm b}\mathcal{T}_\gamma^{\rm g}p(x) = \mathrm{e}^{\mathrm{i}(\frac{1}{2}v\cdot(x-a)+\gamma)}p(x-a)\ . \tag{4.6}$$

Let $\eta_{av\mu\gamma} := \mathcal{S}_{av\gamma}\eta_\mu$. We define the manifold of solitary waves as

$$\mathrm{M_s} := \{\eta_{av\gamma\mu} : a,v,\gamma,\mu \in \mathbb{R}^d\times\mathbb{R}^d\times[0,2\pi]\times I\}\ . \tag{4.7}$$

The tangent space to this manifold at the solitary wave profile $\eta_\mu\in\mathrm{M_s}$ is given by

$$\mathrm{T}_{\eta_\mu}\mathrm{M_s} = \mathrm{span}(z_{\rm t},z_{\rm g},z_{\rm b},z_{\rm s})\ , \tag{4.8}$$

where

$$z_{\rm t} := \nabla_a\mathcal{T}_a^{\rm tr}\eta_\mu\big|_{a=0} = -\nabla\eta_\mu\ , \tag{4.9}$$

$$z_{\rm g} := \frac{\partial}{\partial\gamma}\mathcal{T}_\gamma^{\rm g}\eta_\mu\bigg|_{\gamma=0} = \mathrm{i}\eta_\mu\ , \tag{4.10}$$

$$z_{\rm b} := 2\nabla_v\mathcal{T}_v^{\rm gal}\eta_\mu\big|_{v=0,t=0} = \mathrm{i}x\eta_\mu\ , \tag{4.11}$$

and

$$z_{\mathrm{s}} := \partial_\mu \eta_\mu \ . \tag{4.12}$$

Symmetries of (1.1) which are broken by the solitary wave solution $\eta_\mu e^{i\mu t}$ lead to zero modes and "symplectically associated zero modes" of the Hessian $\mathcal{L}_\eta$. Namely we have for $\eta = \eta_\mu$

$$\mathcal{L}_\eta z_{\mathrm{t}} = 0 \ , \ \text{and } \mathcal{L}_\eta z_{\mathrm{g}} = 0 \ , \tag{4.13}$$

and

$$\mathcal{L}_\eta z_{\mathrm{b}} = 2\mathrm{i} z_{\mathrm{t}} \ , \quad \text{and } \mathcal{L}_\eta z_{\mathrm{s}} = \mathrm{i} z_{\mathrm{g}} \ . \tag{4.14}$$

The functions z_{b} and z_{s} are zero modes for the operator $(\mathrm{i}\mathcal{L}_\eta)^2$, *i.e.*,

$$\mathrm{i}\mathcal{L}_\eta(\mathrm{i}\mathcal{L}_\eta z_{\mathrm{b}}) = 0 \ , \quad \text{and } \mathrm{i}\mathcal{L}_\eta(\mathrm{i}\mathcal{L}_\eta z_{\mathrm{s}}) = 0 \ . \tag{4.15}$$

The relations (4.13) are proved by taking the derivatives of the equation $\mathcal{E}'_\mu(\mathcal{T}^{\mathrm{tr}}_a \mathcal{T}^{\mathrm{g}}_\gamma \eta_\mu) = 0$ with respect to the parameters a and γ, at $a = 0$ and $\gamma = 0$, and similarly for (4.14).

We have shown above that

$$\mathrm{T}_\eta \mathrm{M}_{\mathrm{s}} \subset \mathrm{N}_{\mathrm{gen}}(J\mathcal{L}_\eta) \ , \tag{4.16}$$

where

$$\mathrm{N}_{\mathrm{gen}}(J\mathcal{L}_\eta) := \mathrm{span}\{\bigcup_{n\geq 1} \mathrm{N}((J\mathcal{L}_\eta)^n)\} \ . \tag{4.17}$$

In what follows we denote $\sigma := \{a, v, \gamma, \mu\}$ and $\eta_\sigma := \eta_{av\gamma\mu}$. The manifold M_{s} inherits a symplectic form from (H_1, ω). This symplectic form is determined by the operator

$$\Omega^{-1}_{\eta_\sigma} := J^{-1} \restriction_{\mathrm{T}_{\eta_\sigma}\mathrm{M}_{\mathrm{s}}} \equiv \mathcal{P}_{\eta_\sigma} J^{-1} \mathcal{P}_{\eta_\sigma} \restriction_{\mathrm{T}_{\eta_\sigma}\mathrm{M}_{\mathrm{s}}}, \tag{4.18}$$

where $\mathcal{P}_{\eta_\sigma} : \mathrm{T}_{\eta_\sigma}\mathrm{H}_1 \mapsto \mathrm{T}_{\eta_\sigma}\mathrm{M}_{\mathrm{s}}$ is the L^2-orthogonal projection onto $\mathrm{T}_{\eta_\sigma}\mathrm{M}_{\mathrm{s}}$. Namely $\omega_{\eta_\sigma}(u, v) := \langle u, \Omega^{-1}_{\eta_\sigma} v\rangle$. The key fact here is that this symplectic form is non-degenerate, *i.e.*, the operator $J^{-1} \restriction_{\mathrm{T}_{\eta_\sigma}\mathrm{M}_{\mathrm{s}}}$, is invertible $\forall \eta_\sigma \in \mathrm{M}_{\mathrm{s}}$, as shown in Lemma 1 below. Define

$$m(\mu) := \frac{1}{2} \int |\eta_\mu|^2 \, \mathrm{d}^d x. \tag{4.19}$$

Lemma 1. *If $m'(\mu) > 0$, then $\Omega^{-1}_{\eta_\sigma}$ is invertible.*

Note that $m'(\mu) > 0$ is exactly the assumption for stability in Section 2, condition (D).

Proof. We prove that Ω_η^{-1} is invertible by showing that its matrix $\Omega_\eta^{-1}\big|_{\{z_k\}} := (\langle z_j, J^{-1}z_k\rangle)$ in the basis $\{z_1, \dots, z_{2d+2}\} \equiv \{z_{\mathrm{t}}, z_{\mathrm{b}}, z_{\mathrm{g}}, z_{\mathrm{s}}\}$ in $\mathrm{T}_\eta \mathrm{M}_{\mathrm{s}}$ is invertible. We compute

$$\Omega_\eta^{-1}\big|_{\{z_k\}} = \begin{pmatrix} 0 & -m\mathbb{1} & 0 & 0 \\ m\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & m' \\ 0 & 0 & -m' & 0 \end{pmatrix}, \tag{4.20}$$

This matrix is invertible provided $m' \neq 0$ ($m > 0$ always). Since $\Omega_{\eta_\sigma}^{-1}$ is related to Ω_η^{-1} by a similarity transform (see (5.6)), it is invertible as well. □ □

Corollary 2.1. *$\forall z \in \mathrm{T}_\eta \mathrm{M}_{\mathrm{s}}$, there exists at least one element, $z' \in \mathrm{T}_\eta \mathrm{M}_{\mathrm{s}}$, such that $\omega_\eta(z, z') \neq 0$.*

Proof. This follows directly from the non-degeneracy of Ω_η^{-1}. □ □

Remark 1. *The real form of the vectors* (4.9)–(4.12) *is*

$$(-\nabla\eta, 0), \quad (0, \eta), \quad (0, x\eta), \quad (\partial_\mu \eta, 0). \tag{4.21}$$

We abuse notation and denote both the real and complex representations of the vectors as $-\nabla\eta$, $-J\eta$, $-Jx\eta$ and $\partial_\mu\eta$, where for the real representation we interpret η as $(\eta, 0)$, and J is of the form (3.4)*, whereas in the complex notation $J = \mathrm{i}^{-1}$.*

Remark 2. *For the special case with $f(\psi) = \lambda|\psi|^{2s}\psi$,*

$$z_{\mathrm{s}} := \partial_\mu \mathcal{T}_\mu^s \eta\big|_{\mu=1,t=0} = \frac{1}{2}(\frac{1}{s} + x\cdot\nabla)\eta = \partial_\mu \eta_\mu\ . \tag{4.22}$$

Remark 3. *Equation* (1.1) *with $V = 0$ and with a nonlinearity of the form* (2.12)*, where W is spherically symmetric has also rotational symmetry: $\psi(x,t) \mapsto \psi(Rx, t)$, $R \in \mathrm{SO}(d)$. This symmetry does not play a role in our analysis since we consider only spherically symmetric solitary wave solutions of* (1.5)*.*

Remark 4. *Note that for $f(\psi) = \lambda|\psi|^{2s}\psi$ we can put the μ dependence on the same footing as the transformation induced for v, a and γ. Indeed, define*

$$\hat{\mathcal{S}}_{av\gamma\mu} = \mathcal{S}_{av\gamma} \circ \mathcal{T}_\mu^s. \tag{4.23}$$

Then $\eta_{av\gamma\mu} = \hat{\mathcal{S}}_{av\gamma\mu}\eta_1$.

5 Skew orthogonal decomposition

In this section we introduce the skew orthogonal decomposition of a solution ψ along the manifold $\mathrm{M_s}$ and in the skew orthogonal direction (see [2]), and show that this uniquely defines the solitary wave solution parameters appearing in the decomposition.

We recall the notation $\sigma = \{\gamma, a, v, \mu\}$ so that $\eta_\sigma := \eta_{av\gamma\mu}$. Define the δ-neighborhood

$$U_\delta = \{\psi \in \mathrm{H}_1 : \inf_{\sigma\in\Sigma_0} \|\psi - \eta_\sigma\|_{\mathrm{H}_1} \le \delta\} \tag{5.1}$$

of the manifold $\mathrm{M}_\mathrm{s}' := \{\eta_\sigma : \sigma \in \Sigma_0\}$, where

$$\Sigma_0 := \mathbb{R}^d \times \mathbb{R}^d \times [0, 2\pi) \times I_0. \tag{5.2}$$

Here I_0 is any bounded closed interval contained in $I\backslash\partial I$, (for the definition of I Condition (C)).

The main result of this section is

Proposition 1. *Let $\psi \in U_\delta$. For $\delta > 0$ sufficiently small, there exists a unique $\sigma = \sigma(\psi) \in \mathrm{C}^1(U_\delta, \Sigma)$ such that*

$$\omega(\psi - \eta_\sigma, z) = 0 \ \textit{i.e.,}\ \ \langle\psi - \eta_\sigma, J^{-1}z\rangle = 0\ , \forall z \in \mathrm{T}_{\eta_\sigma}\mathrm{M_s}\ . \tag{5.3}$$

Proof. We will use the following notation:

$$\sigma := \{\sigma_1, ..., \sigma_{2d+2}\} := \{a, v, \gamma, \mu\}, \tag{5.4}$$

$$\{z_{\mu,1}, ..., z_{\mu,2d+2}\} := \{z_\mathrm{t}, z_\mathrm{b}, z_\mathrm{g}, z_\mathrm{s}\}, \tag{5.5}$$

and

$$\partial_k = \partial_{\sigma_k},\ k = d+1, ..., 2d+2,\ \partial_k = \partial_{\sigma_k} + \frac{1}{2}\sigma_{k+d}\partial_{\sigma_{2d+1}},\ k = 1, ..., d. \tag{5.6}$$

Denote $z_{\sigma,j} := \mathcal{S}_{av\gamma} z_{\mu,j}$ so that $\partial_k\eta_\sigma = z_{\sigma,k} = \mathcal{S}_{av\gamma} z_{\mu,k}$. Clearly $\{z_{\sigma,j}\}$ is a basis in $\mathrm{T}_{\eta_\sigma}\mathrm{M_s}$.

We use the implicit function theorem for the map $G : \mathrm{H}_1 \times \Sigma \mapsto \mathbb{R}^{2d+2}$, defined by

$$G_j(\psi, \sigma) := \langle\psi - \eta_\sigma, J^{-1}z_{\sigma,j}\rangle \ \ \forall j = 1, ..., 2d+2. \tag{5.7}$$

We verify that (i) $G \in \mathrm{C}^1$, (ii) $G(\eta_{\sigma_0}, \sigma_0) = 0$ for any $\sigma_0 \in \Sigma$ and (iii) $\partial_\sigma G(\eta_{\sigma_0}, \sigma)|_{\sigma=\sigma_0}$ is invertible.

G is in C^1 in σ, since so are η_σ and $z_{\sigma,j}$, and G is C^1 in ψ since it is linear in ψ. So (i) follows.

(ii) follows directly from the definition of G.

To prove (iii), we use (5.6) to compute,

$$\partial_k G_j(\eta_{\sigma_0},\sigma)\big|_{\sigma=\sigma_0} = -\langle z_{\sigma_0,k}, J^{-1} z_{\sigma_0,j}\rangle = -\langle z_{\mu_0,k}, J^{-1} z_{\mu_0,k}\rangle, \tag{5.8}$$

where in the last equality we used that $\mathcal{S}_{av\gamma}$ is symplectic (see Remark 5 below). With the definition of Ω_η^{-1} in Section 4 we find

$$\partial_\sigma G(\eta_{\sigma_0},\sigma)\big|_{\sigma=\sigma_0} = -\Omega_\eta^{-1}\big|_{\{z_{\mu_0,k}\}_{k=1}^{2d+2}} . \tag{5.9}$$

Here we use the notation $\Omega_\eta^{-1}\big|_{\{z_{\mu_0,k}\}_{k=1}^{2d+2}}$ to denote the matrix of Ω_η^{-1} in the basis $\{z_{\mu_0,k}\}_{k=1}^{2d+2}$. Thus $\partial_\sigma G(\eta_{\sigma_0},\sigma)\big|_{\sigma=\sigma_0}$ is invertible for all σ_0 by Lemma 1. This shows (iii).

With properties (i)–(iii) the implicit function theorem implies that there exists a unique C^1 map $\sigma=\sigma(\psi)$, satisfying $G(\psi,\sigma(\psi))=0$ in a neighborhood, V_{σ_0}, of η_{σ_0}.

Now take $\sigma_0 = \{0,0,0,\mu_0\}$, with $\mu_0 \in I$. Denote $n := \{a,v,\gamma\}$ so that $\mathcal{S}_n := \mathcal{S}_{av\gamma}$. Then for all $\tilde{n}$ the map $\tilde{\sigma}$, defined on $\mathcal{S}_{\tilde{n}} V_{\sigma_0}$,

$$\tilde{\sigma}(\psi) := \tilde{n}\circ\sigma(\mathcal{S}_n^{-1}\psi) \tag{5.10}$$

where $\tilde{n}\circ\{n,\mu\} = \{\tilde{n}\circ n,\mu\}$ and $\tilde{n}\circ n$ is defined by $\mathcal{S}_{\tilde{n}}\circ\mathcal{S}_n =: \mathcal{S}_{\tilde{n}\circ n}$, solves equation (5.3) for any $\psi\in\mathcal{S}_{\tilde{n}}V_{\sigma_0}$. Since the neighborhood $\mathcal{S}_{\tilde{n}}V_{\sigma_0}$ with $\tilde{n}\in\mathbb{R}^d\times\mathbb{R}^d\times[0,2\pi)$ and $\mu_0\in I$ covers the neighborhood U_δ the statement of the proposition follows. □ □

So if we know that for a given initial condition (1.1) has a $C(\mathbb{R}, H_1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H_{-1}(\mathbb{R}^d))$ solution $\psi=\psi(t)$ which, for times $0\le t\le T$, stays in the neighborhood U_δ, then by Proposition 1 the solitary wave solution parameters σ trace out a unique C^1 trajectory $\sigma(\psi(t))$. We make the choice $\epsilon_0 \ll \delta$.

Remark 5. *The operator $\mathcal{S}_{av\gamma}$ is a symplectic, or canonical, operator; i.e., it leaves $\langle\cdot, J^{-1}\cdot\rangle$, unchanged: $\langle \mathcal{S}_{av\gamma}w, J^{-1}\mathcal{S}_{av\gamma}q\rangle = \langle w, J^{-1}q\rangle$. This follows from the fact that $[J,\mathcal{S}_{av\gamma}]=0$. Since $z\in T_{\mathcal{S}_{av\gamma}\eta_\mu}M_s$ implies that $\mathcal{S}_{av\gamma}^{-1}z \in T_{\eta_\mu}M_s$, we can write* (5.3) *in the form*

$$\langle \mathcal{S}_{av\gamma}^{-1}\psi - \eta_\mu, J^{-1}z\rangle = 0\ ,\ \forall z\in T_{\eta_\mu}M_s\ . \tag{5.11}$$

where η_μ and z depend on $\mu(t)$.

6 Equation of motion in the moving frame

Given a solution ψ to (1.1), we define the parameterization $\{\sigma, w\}$ for it by the equations

$$\psi - \eta_\sigma \perp J^{-1}\mathrm{T}_{\eta_\sigma}\mathrm{M}_\mathrm{s} \text{ and } w := \mathcal{S}_{av\gamma}^{-1}(\psi - \eta_\sigma). \tag{6.1}$$

By Proposition 1, this parameterization is well defined as long as $\psi \in U_\delta$. In this section we find the equation for the parameters $\{\sigma, w\}$. To this end we use the equation for the function u, defined by

$$u := \mathcal{S}_{av\gamma}^{-1}\psi. \tag{6.2}$$

We introduce the anti-self-adjoint generators

$$\mathcal{K}_j = \partial_{x_j}, \quad \mathcal{K}_{d+j} = -Jx_j, \quad \mathcal{K}_{2d+1} = -J, \quad \mathcal{K}_{2d+2} = \partial_\mu, \quad j = 1, ..., d \tag{6.3}$$

and the corresponding coefficients

$$\alpha_j = \dot{a}_j - v_j, \quad \alpha_{d+j} = -\frac{1}{2}\dot{v}_j - \partial_{x_j}V(a), \quad j = 1, ..., d, \tag{6.4}$$

$$\alpha_{2d+1} = \mu - \frac{1}{4}v^2 + \frac{1}{2}\dot{a}\cdot v - V(a) - \dot{\gamma}, \quad \alpha_{2d+2} = -\dot{\mu}. \tag{6.5}$$

Let

$$\alpha \cdot \mathcal{K} = \sum_{j=1}^{2d+2} \alpha_j \mathcal{K}_j \quad \text{and} \quad \underline{\alpha} \cdot \underline{\mathcal{K}} = \sum_{j=1}^{2d+1} \alpha_j \mathcal{K}_j. \tag{6.6}$$

The main result in this section is

Lemma 2. *If ψ satisfies* (1.1) *then u satisfies*

$$\dot{u} = J\big((-\Delta + \mu)u - f(u)\big) + \underline{\alpha} \cdot \underline{\mathcal{K}} u + J\mathcal{R}_V u, \tag{6.7}$$

where

$$\mathcal{R}_V := V(x + a) - V(a) - \nabla V(a) \cdot x = \mathcal{O}((\epsilon_V x)^2), \tag{6.8}$$

Proof. For this proof, we will use complex notation, *i.e.*, $J = \mathrm{i}^{-1}$, and in particular we note that $[J, \mathcal{S}_{av\gamma}] = 0$. Let $\psi_a(x) = \psi(x + a)$, and let $\phi = \frac{1}{2}v \cdot x + \gamma$. With this notation $u = \mathrm{e}^{-\mathrm{i}\phi}\psi_a$. Differentiating (6.2), we find

$$\mathrm{i}\dot{u} = \mathrm{e}^{-\mathrm{i}\phi}(-\Delta + \mu + V_a)\psi_a - f(\psi_a) + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\dot{a}\cdot\nabla\psi_a + (\frac{1}{2}\dot{v}\cdot x + \dot{\gamma} - \mu)u\ , \tag{6.9}$$

where $V_a(x) = V(x+a)$ and $\mathrm{e}^{-\mathrm{i}\phi} f(\psi_a) = f(u)$, by Condition (B). We rewrite the potential, using (6.8), in the form

$$V(x+a) = V(a) + \nabla V(a) \cdot x + \mathcal{R}_V(x). \tag{6.10}$$

Now use

$$\mathrm{e}^{-\mathrm{i}\phi} \nabla \psi_a = \nabla(\mathrm{e}^{-\mathrm{i}\phi}\psi_a) + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\psi_a \nabla\phi \tag{6.11}$$

and

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\phi}\Delta\psi_a &= \Delta(\mathrm{e}^{-\mathrm{i}\phi}\psi_a) + \mathrm{i}\nabla\phi\cdot\nabla(\mathrm{e}^{-\mathrm{i}\phi}\psi_a) + \mathrm{i}\mathrm{e}^{-\mathrm{i}\phi}\nabla\phi\cdot\nabla\psi_a && (6.12)\\ &= \Delta(\mathrm{e}^{-\mathrm{i}\phi}\psi_a) + 2\mathrm{i}\nabla\phi\cdot\nabla(\mathrm{e}^{-\mathrm{i}\phi}\psi_a) - |\nabla\phi|^2\mathrm{e}^{-\mathrm{i}\phi}\psi_a\ , && (6.13)\end{aligned}$$

where $\phi = \frac{1}{2}v\cdot x + \gamma$, to conclude (6.7). Note that the parameters and their time derivatives, as well as $V(a)$ and $\nabla V(a)$ are collected into $\underline{\alpha}$. $\square$ $\square$

We now re-parameterize the non-linear Schrödinger equation into separate equations for σ and w. Recalling that $u = \eta + w$, $\partial_t \eta = \dot{\mu}\partial_\mu \eta$, $\mathcal{E}'_\mu(\eta) = 0$, and $\mathcal{E}'_\mu(\eta+w) = \mathcal{L}_\eta w + N_\eta(w)$, where $\mathcal{L}_\eta = \mathcal{E}''_\mu(\eta)$, we can rewrite (6.7) as

$$\dot{\mu}\partial_\mu\eta + \dot{w} = J\mathcal{L}_\eta w + JN_\eta(w) + \underline{\alpha}\cdot\underline{\mathcal{K}}(\eta+w) + J\mathcal{R}_V(\eta+w) \tag{6.14}$$

We collect the linear terms acting on w as $\mathcal{L}_{\eta,\sigma} w$, where

$$\mathcal{L}_{\eta,\sigma} := \mathcal{L}_\eta + \mathcal{R}_V + J^{-1}\underline{\alpha}\cdot\underline{\mathcal{K}}, \tag{6.15}$$

and the remaining terms into a source term

$$q(\sigma) := \alpha\cdot\mathcal{K}\eta + J\mathcal{R}_V\eta, \tag{6.16}$$

to obtain

$$\dot{w} = J\mathcal{L}_{\eta,\sigma} w + N_\eta(w) + q(\sigma). \tag{6.17}$$

The equations for the parameters are obtained using the skew orthogonality condition and (6.14). Let $z \in \mathrm{T}_\eta\mathrm{M}_\mathrm{s}$. Upon recalling that $\langle Jz, J\mathcal{L}_\eta w\rangle = 0$, and so

$$0 = \partial_t\langle Jz, w\rangle = \langle Jz, \dot{w}\rangle + \dot{\mu}\langle J\partial_\mu z, w\rangle, \tag{6.18}$$

we find

$$\dot{\mu}\langle Jz, \partial_\mu\eta\rangle - \underline{\alpha}\cdot\langle Jz, \underline{\mathcal{K}}\eta\rangle = \dot{\mu}\langle J\partial_\mu z, w\rangle + \langle z, N_\eta(w)\rangle + \underline{\alpha}\cdot\langle Jz, \underline{\mathcal{K}}w\rangle + \langle z, \mathcal{R}_V(\eta+w)\rangle.$$

Recall that $\partial_\mu \eta = \mathcal{K}_{2d+2}\eta$, so the left-hand side is $\sum_j \langle Jz, \mathcal{K}_j \eta\rangle \alpha_j = \sum_j \langle Jz, z_j\rangle \alpha_j$, where we used $\mathcal{K}_j\eta = z_j$. Now let z be one of the basis vectors, $z = z_k$. Then the inner product on the left-hand side coincides with the definition of $(\Omega_\eta^{-1})_{kj}$. Furthermore, using $\underline{\mathcal{K}}^* = -\underline{\mathcal{K}}$, and $[\underline{\mathcal{K}}, J] = 0$, we combine $\underline{\alpha} \cdot \langle Jz, \underline{\mathcal{K}}w\rangle$ and and $\dot{\mu}\langle J\partial_\mu z, w\rangle$, into $\alpha \cdot \langle \mathcal{K}z, Jw\rangle$. The result is

$$\sum_{j=1}^{2d+2} (\Omega^{-1})_{kj}\alpha_j = \langle z_k, N_\eta(w) + \mathcal{R}_V(w+\eta)\rangle + \alpha \cdot \langle \mathcal{K}z_k, Jw\rangle. \tag{6.19}$$

Replacing α by the explicit expression, (6.19) reads, for $k = 2d{+}1$ and $k = 1, .., d$,

$$\dot{\mu} = (m'(\mu))^{-1}\left(\langle \eta, JN_\eta(w) + J\mathcal{R}_V w\rangle - \alpha \cdot \langle \mathcal{K}\eta, w\rangle\right), \tag{6.20}$$

$$\frac{1}{2}\dot{v}_k = -\partial_{x_k} V(a) + (m(\mu))^{-1}\left(\langle \partial_k \eta, N_\eta(w) + \mathcal{R}_V w\rangle - \alpha \cdot \langle \mathcal{K}\partial_k\eta, Jw\rangle \right. \\ \left. + \langle \partial_k\eta, \mathcal{R}_V\eta\rangle\right), \tag{6.21}$$

where $m(\mu) := 2^{-1}\|\eta\|^2$, and we have used $\langle J\eta, \mathcal{R}_V\eta\rangle = 0$. For $k = d+1, ..., 2d$, $k = 2d+2$, we use the expressions for $\dot{\mu}$ and $\dot{v}_k$ obtained above to find

$$\dot{a}_k = v_k + (m(\mu))^{-1}\left[\langle x_k\eta, JN_\eta(w) + J\mathcal{R}_V w\rangle + \alpha \cdot \langle \mathcal{K}x_k\eta, w\rangle\right], \tag{6.22}$$

$$\dot{\gamma} = \mu - \frac{1}{4}v^2 + \frac{1}{2}\dot{a}\cdot v - V(a) - (m'(\mu))^{-1}\left[\langle \partial_\mu\eta, N_\eta(w) + \mathcal{R}_V w\rangle \right. \\ \left. -\alpha \cdot \langle \mathcal{K}\partial_\mu\eta, Jw\rangle + \langle \partial_\mu\eta, \mathcal{R}_V\eta\rangle\right],$$

where we used $\langle x_k\eta, J\mathcal{R}_V\eta\rangle = 0$, and $\eta(x) = \eta(|x|)$ so that $\langle x\eta, \eta\rangle = 0$. Furthermore, observe that all terms containing w and $\mathcal{R}_V$ are of higher order. We abbreviate this as follows

$$\dot{\sigma} = X(\sigma) - \delta X(\sigma, w). \tag{6.23}$$

For the estimates used later, we note that $\dot{\sigma_j} - X_j(\sigma) = \alpha_j$. We formalize the above calculation in the following Proposition:

Proposition 2. *(1) The parameters σ and the fluctuation w (defined in* (6.1)*) satisfy the equations*

$$\dot{\sigma} = X(\sigma) - \delta X(\sigma, w), \tag{6.24}$$

and

$$\dot{w} = J\mathcal{L}_{\eta,\sigma}w + JN_\eta(w) + q(\sigma). \tag{6.25}$$

Here $\mathcal{L}_{\eta,\sigma}$, $q(\sigma)$ and $N_\eta(w)$ are given by (6.15), (6.16) *and* (2.5) *respectively, and (with Ω_η defined in Lemma 1)*

$$\delta X_j(\sigma, w) = \sum_{k=1}^{2d+2} (\Omega_\eta)_{jk} [\langle z_k, N_\eta(w) + \mathcal{R}_V(w+\eta)\rangle + \alpha \cdot \langle \mathcal{K} z_k, Jw\rangle], \qquad (6.26)$$

$\forall j = 1, ..., 2d+2$, *with* $\{z_k\}_{k=1}^{2d+2} := \{z_{\rm t}, z_{\rm b}, z_{\rm g}, z_{\rm s}\}$, *and* $\mathcal{R}_V$ *given by* (6.8).

(2) The vector field δX satisfies the following estimate for $\|w\|_{\rm H_1} \le 1$:

$$\delta X = \mathcal{O}(|\alpha|\|w\| + {\epsilon_V}^2 + \|w\|^2_{\rm H_1}) \ , \qquad (6.27)$$

where $|\alpha| := \max_{j=1,...,2d+2} |\alpha_j|$.

Proof. (1) is the result of the calculation done in (6.14)–(6.23). In particular (6.26) follows from (6.19).

(2) Estimate (6.27) follows directly from (6.26), together with the facts that $\|\mathcal{R}_V z_k\| = \mathcal{O}({\epsilon_V}^2)$ and $\|N_\eta(w)\|_{\rm H_{-1}} \le c\|w\|^2_{\rm H_1}$ for $\|w\|_{\rm H_1} \le 1$ (see (2.5)). □ □

The goal is to show that $\sup_{t\in(0,T)} |\delta X| = \mathcal{O}({\epsilon_V}^2 + {\epsilon_0}^2)$ and $\sup_{t\in(0,T)} \|w\|_{\rm H_1} = \mathcal{O}(\epsilon_V + \epsilon_0)$, for some $T = \mathcal{O}(1/(\epsilon_V + {\epsilon_0}^2))$.

7 Approximate Conservation of a Lyapunov functional

In this section, we show that the Lyapunov functional $\mathcal{E}_\mu(u) - \mathcal{E}_\mu(\eta_\mu)$, is approximately conserved. Recall $\eta = \eta_\mu$ is the solitary wave profile (see Section 1) and u is the solution ψ of eq. (1.1), transformed to the moving frame: $u := \mathcal{S}^{-1}_{av\gamma}\psi$ ($\mathcal{S}_{av\gamma}$ is defined in Section 4). We use the skew-orthogonal decomposition of u (see Section 5): $u := \mathcal{S}^{-1}_{av\gamma}\psi = \eta + w$, with

$$\omega_\eta(w, z) = 0 \ \text{ for all } z \in {\rm T}_\eta {\rm M}_{\rm s}, \qquad (7.1)$$

provided $\psi \in U_\delta$. The main result of this section is the following

Proposition 3. *Let $\psi \in U_\delta$ solve eq.* (1.1) *and let u, w and η be defined as above. Then*

$$\partial_t(\mathcal{E}_\mu(u) - \mathcal{E}_\mu(\eta_\mu)) = \mathcal{O}(|\alpha|\|w\|^2_{\rm H_1} + {\epsilon_V}^2\|w\|_{\rm H_1} + \epsilon_V\|w\|^2_{\rm H_1}). \qquad (7.2)$$

To prove this proposition we use the following

Lemma 3. *Let u be defined as above. Then*

$$\partial_t \mathcal{E}_\mu(u) = \frac{1}{2}\dot\mu\|u\|^2 - \langle(\frac{1}{2}\dot v + \nabla V_a)\mathrm{i}u, \nabla u\rangle. \tag{7.3}$$

of Proposition 3. We first recall that η is a critical point to $\mathcal{E}_\mu(u)$, thus

$$\partial_t \mathcal{E}_\mu(\eta) = \frac{1}{2}\dot\mu\|\eta\|^2. \tag{7.4}$$

Using Lemma 3, we find

$$\partial_t(\mathcal{E}_\mu(u) - \mathcal{E}_\mu(\eta)) = A - B. \tag{7.5}$$

where $A := 2^{-1}\dot\mu(\|u\|^2 - \|\eta\|^2)$ and $B := \langle(2^{-1}\dot v + \nabla V_a)\mathrm{i}u, \nabla u\rangle$. First we use the decomposition $u = \eta + w$, the condition $0 = \langle \mathrm{i}z_{\mathrm{g}}, w\rangle = \langle \eta, w\rangle$ (from (7.1)), and the estimate $|\dot\mu| = |\alpha_{2d+2}| \leq |\alpha|$ to obtain

$$A = \frac{1}{2}\dot\mu\|w\|^2 = \mathcal{O}(|\alpha|\|w\|^2). \tag{7.6}$$

For the term B, recall that $0 = \langle \mathrm{i}z_{\mathrm{t}}, w\rangle = \langle \mathrm{i}\nabla\eta, w\rangle$, and furthermore that $\langle \mathrm{i}q\eta, \nabla\eta\rangle = 0$ for any real-valued function $q \in \mathrm{L}^\infty$. Then

$$B = \langle(\frac{1}{2}\dot v + \nabla V_a)\mathrm{i}w, \nabla w\rangle + \langle(\nabla V_a)\mathrm{i}\eta, \nabla w\rangle + \langle(\nabla V_a)\mathrm{i}w, \nabla\eta\rangle. \tag{7.7}$$

Now, we use that $\nabla V(a)\cdot\langle \mathrm{i}w, \nabla\eta\rangle = 0 = \nabla V(a)\cdot\langle \mathrm{i}\eta, \nabla w\rangle$, to obtain

$$\begin{aligned} B = (\frac{1}{2}\dot v + \nabla V(a))\cdot\langle \mathrm{i}w, \nabla w\rangle + \langle(\nabla V_a - \nabla V(a))\mathrm{i}w, \nabla w\rangle + \\ \langle(\nabla V_a - \nabla V(a))\mathrm{i}\eta, \nabla w\rangle + \langle(\nabla V_a - \nabla V(a))\mathrm{i}w, \nabla\eta\rangle. \end{aligned} \tag{7.8}$$

Recall that $-\alpha_{d+j} := \frac{1}{2}\dot v_j + \partial_j V(a)$ (see (6.5)). Since $|\alpha_j| \leq |\alpha|$, the first term on the right hand side is $\mathcal{O}(|\alpha|\|w\|^2_{\mathrm{H}_1})$. Since $\nabla V_a = \mathcal{O}(\epsilon_V)$ the second term on the right-hand side is $\mathcal{O}(\epsilon_V\|w\|^2_{\mathrm{H}_1})$. Finally due to $\nabla V_a - \nabla V(a) = \mathcal{O}({\epsilon_V}^2|x|)$ and $|x|\eta, |x|\nabla\eta \in \mathrm{L}^2$, (Condition (C)) the third and fourth terms are $\mathcal{O}({\epsilon_V}^2\|w\|_{\mathrm{H}_1})$. Collecting these estimates we arrive at

$$B = \mathcal{O}(|\alpha|\|w\|^2_{\mathrm{H}_1} + {\epsilon_V}^2\|w\|_{\mathrm{H}_1} + \epsilon_V\|w\|^2_{\mathrm{H}_1}). \tag{7.9}$$

Relations (7.5), (7.6) and (7.9) imply (7.2). □ □

It remains to prove Lemma 3. To this end we note the following

Lemma 4. *Let* ψ *be a solution to eq.* (1.1). *Then*

$$\partial_t \frac{1}{2} \int V|\psi|^2 = \langle (\nabla V) \mathrm{i}\psi, \nabla\psi \rangle. \tag{7.10}$$

Proof. (7.10) is obtained by integrating the relation

$$\partial_t (V|\psi|^2) = \mathrm{i}\nabla \cdot (V\bar{\psi}\nabla\psi - V\psi\nabla\bar{\psi}) - \mathrm{i}(\nabla V) \cdot (\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}), \tag{7.11}$$

which follows from the nonlinear Schrödinger equation (1.1). □ □

of Lemma 3. Note that the identity

$$\mathcal{H}_V(\mathcal{S}_{av\gamma}^{-1}\psi) + \frac{1}{2}\mu\|\mathcal{S}_{av\gamma}^{-1}\psi\|^2 - \frac{1}{2}\int V|\mathcal{S}_{av\gamma}^{-1}\psi|^2 = \mathcal{E}_\mu(u). \tag{7.12}$$

holds for all $u = \mathcal{S}_{av\gamma}^{-1}\psi$. We observe the following relations

$$\|\mathcal{S}_{av\gamma}^{-1}\psi\|^2 = \|\psi\|^2, \quad \int V|\mathcal{S}_{av\gamma}^{-1}\psi|^2 = \int V_{-a}|\psi|^2 \tag{7.13}$$

and

$$2\mathcal{H}_V(\mathcal{S}_{av\gamma}^{-1}\psi) = 2\mathcal{H}_V(\psi) + \frac{1}{4}v^2\|\psi\|^2 - v \cdot \langle \mathrm{i}\psi, \nabla\psi\rangle + \int (V_{-a} - V)|\psi|^2 \tag{7.14}$$

Inserting the above relations into (7.12) we find

$$2\mathcal{H}_V(\psi) + (\frac{1}{4}v^2 + \mu)\|\psi\|^2 - v \cdot \langle \mathrm{i}\psi, \nabla\psi\rangle - \int V|\psi|^2 = 2\mathcal{E}_\mu(u). \tag{7.15}$$

We now take the time derivative of the above relation. Using the fact that $\mathcal{H}_V(\psi)$ and $\|\psi\|^2$ are conserved quantities, and using Lemma 4 and Ehrenfest's theorem (4.5) we find

$$(\frac{1}{2}\dot{v}\cdot v + \dot{\mu})\|\psi\|^2 - \dot{v}\cdot\langle \mathrm{i}\psi, \nabla\psi\rangle + v\cdot\langle \nabla V\psi, \psi\rangle - 2\langle (\nabla V)\mathrm{i}\psi, \nabla\psi\rangle = 2\partial_t\mathcal{E}_\mu(u). \tag{7.16}$$

Collecting terms of the form $(\frac{1}{2}\dot{v} + \nabla V)$ and replacing ψ with $\mathcal{S}_{av\gamma}u$ gives

$$\dot{\mu}\|u\|^2 + \langle \mathrm{i}(\dot{v} + 2\nabla V_a) \cdot \nabla u, u\rangle = 2\partial_t\mathcal{E}_\mu(u). \tag{7.17}$$

By observing that $\langle q\mathrm{i}\nabla u, u\rangle = -\langle q\mathrm{i}u, \nabla u\rangle$ for real $q \in \mathrm{L}^\infty$, we arrive at the result of the lemma. □ □

8 Lower bound on the Lyapunov functional

Proposition 4. *Let η and w be as in Proposition 3. Then there exist constants $\rho > 0$ and $c > 0$ independent of ϵ_V and ϵ_0 such that for $\|w\|_{\mathrm{H}_1} \le 1$,*

$$|\mathcal{E}_\mu(\eta + w) - \mathcal{E}_\mu(\eta)| \ge \frac{\rho}{2}\|w\|^2_{\mathrm{H}_1} - c\|w\|^3_{\mathrm{H}_1} \ . \tag{8.1}$$

Proof. We expand $\mathcal{E}_\mu(u)$ around η. Using that η is a critical point to $\mathcal{E}_\mu$ ($\mathcal{E}'_\mu(\eta) = 0$), we write

$$\mathcal{E}_\mu(\eta + w) - \mathcal{E}_\mu(\eta) = \frac{1}{2}\langle w, \mathcal{L}_\eta w\rangle + R^{(3)}_\eta(w) \ , \tag{8.2}$$

where, recall, $\mathcal{L}_\eta := \mathcal{E}''_\mu(\eta)$ and where $R^{(3)}_\eta(w)$ is defined in (2.7). From condition (A) and (2.4) we have for $\|w\|_{\mathrm{H}_1} \le 1$

$$|R^{(3)}_\eta(w)| \le c\|w\|^3_{\mathrm{H}_1} \ . \tag{8.3}$$

Let $\mathrm{Y}_\eta := \{w \in \mathrm{H}_1(\mathbb{R}^d) : \omega_\eta(w, z) = 0, \forall z \in \mathrm{T}_\eta\mathrm{M}_{\mathrm{s}}, \|w\|_{\mathrm{H}_1} = 1\}$. It is shown in Appendix D (see [48]) that

$$\rho := \inf_{w \in \mathrm{Y}_\eta} \langle w, \mathcal{L}_\eta w\rangle > 0. \tag{8.4}$$

Hence, for w that satisfy (7.1) we have the following coercivity estimate

$$\langle w, \mathcal{L}_\eta w\rangle \ge \rho\|w\|^2_{\mathrm{H}_1}. \tag{8.5}$$

Using this estimate and the bound (8.3) on R_η in (8.2) we arrive at (8.1). $\square$ $\square$

Remark 6. *Since $\sigma_{\mathrm{ess}}(\mathcal{L}_\eta) = [\mu, \infty)$, we have that $\rho \le \mu$. We expect that for a wide class of nonlinearities $\rho \ge c\mu$ for some constant $c > 0$.*

9 Upper bound on $\|w\|_{\mathrm{H}_1}$ and proof of the main result

In this section we prove the main result of the paper, by providing an upper bound on $\|w\|_{\mathrm{H}_1}$. To achieve this, we use both the approximate conservation of $\mathcal{E}_\mu$ and the lower bound on the Lyapunov functional. For vector functions $s \mapsto w(s) \in \mathrm{H}_1$, and $s \mapsto \alpha(s) \in \mathbb{R}^{2d+2}$ we introduce the norms $|||w|||_{\mathrm{H}_1} := \sup_{s\in[0,t]} \|w(s)\|_{\mathrm{H}_1}$ and $|\alpha|_\infty := \sup_{s\le t} |\alpha(s)|$. We state the main result of this section

Proposition 5. *Assume ϵ_V, ϵ_0 are sufficiently small. There are constants $c, c' < \infty$, independent of ϵ_V and ϵ_0 such that for $t \le c(\epsilon_V + {\epsilon_0}^2)^{-1}$,*

$$\|w\|_{\mathrm{H}_1} \le c'(\epsilon_V + \epsilon_0), \tag{9.1}$$

$$|\alpha|_\infty \le c'({\epsilon_V}^2 + {\epsilon_0}^2), \tag{9.2}$$

where $w = \mathcal{S}^{-1}_{av\gamma}\psi - \eta_\mu$ and $\alpha_j = \dot{\sigma}_j - X_j(\sigma)$ (see (6.24)).

Denote

$$\triangle\mathcal{E} := \mathcal{E}_{\mu_0}(u_0) - \mathcal{E}_{\mu_0}(\eta_{\mu_0}) \ , \tag{9.3}$$

where $u_0 := \mathcal{S}^{-1}_{av\gamma}\psi\big|_{t=0}$. We begin with two simple auxiliary lemmas.

Lemma 5. *There exists a constant $c > 0$ independent of ϵ_V and ϵ_0 such that $|\triangle\mathcal{E}|$ satisfies the inequality*

$$|\triangle\mathcal{E}| \le c{\epsilon_0}^2. \tag{9.4}$$

Proof. To estimate $\triangle\mathcal{E}$, we use the fact that $\mathcal{E}'_{\mu_0}(\eta_{\mu_0}) = 0$ to obtain

$$\begin{aligned} 2\triangle\mathcal{E} &= 2(\mathcal{E}_{\mu_0}(\eta_{\mu_0} + w_0) - \mathcal{E}_{\mu_0}(\eta_{\mu_0}) - \langle \mathcal{E}'_{\mu_0}(\eta_{\mu_0}), w_0\rangle) \\ &= \|\nabla w_0\|^2 + \mu_0\|w_0\|^2 - R^{(2)}_\eta(w_0). \end{aligned} \tag{9.5}$$

Since $\|w_0\|_{\mathrm{H}_1} \le 1$, the estimate (2.4) gives

$$\|R^{(2)}_\eta(w_0)\| \le c\|w_0\|^2_{\mathrm{H}_1} \ . \tag{9.6}$$

This together with the identity (9.5) and the fact that $\|w_0\|_{\mathrm{H}_1} \le \epsilon_0 < 1$ gives the estimate (9.4). □ □

Lemma 6. *Let $\rho > 0$ be the coercivity constant given in (8.4). There exists a constant c independent of ϵ_V and ϵ_0, such that for $\|w\|_{\mathrm{H}_1} \le 1$*

$$\rho|\|w\||^2_{\mathrm{H}_1} \le c{\epsilon_0}^2 + ct({\epsilon_V}^2|\|w\||_{\mathrm{H}_1} + (\epsilon_V + |\alpha|_\infty)|\|w\||^2_{\mathrm{H}_1}) + c|\|w\||^3_{\mathrm{H}_1} \tag{9.7}$$

Proof. Using the approximate conservation of the Lyapunov functional (the time integral of (7.2)), and pulling sup of norms out of the time integral we obtain

$$|\mathcal{E}_\mu(\eta_\mu + w) - \mathcal{E}_\mu(\eta_\mu)| \le |\triangle\mathcal{E}| + ct({\epsilon_V}^2|\|w\||_{\mathrm{H}_1} + (\epsilon_V + |\alpha|_\infty)|\|w\||^2_{\mathrm{H}_1}) \tag{9.8}$$

where $|\triangle\mathcal{E}|$ is defined in (9.3). Now we substitute (9.8) into the lower bound (8.1), use the initial condition estimate (9.4) and take the $\sup_{s\in[0,t]}$ of the resulting expression to obtain (9.7). □ □

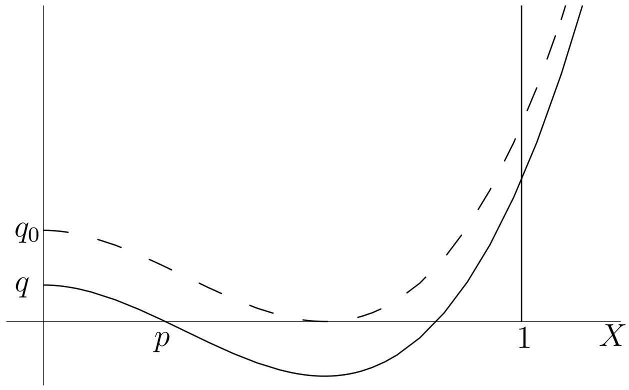

Figure 1: Schematic graph of the left-hand side of (9.9).

of Proposition 5. Using the triangle inequality we derive from (9.7) that the function $X(t) := |||w|||_{\mathrm{H}_1}$ satisfies the equation

$$0 \leq \rho {\epsilon_V}^2 + c{\epsilon_0}^2 - \rho X^2 + cX^3 \tag{9.9}$$

for times $ct(\epsilon_V + |\alpha|) \leq \rho/4$, provided $X \leq 1$. The graph of the right-hand side of (9.9) is shown in Figure 1. Thus for $\rho{\epsilon_V}^2 + c{\epsilon_0}^2 < q_0$, see Figure 1, $X = |||w|||_{\mathrm{H}_1} \leq p$ if $X|_{t=0} \leq p$, where p is the smallest positive zero of the left-hand side of (9.9) (see Figure 1), provided $p \leq 1$. For $\epsilon_0 + \epsilon_V$ sufficiently small, $c'(\epsilon_V + \rho^{-1/2}\epsilon_0) \leq p \leq c(\epsilon_V + \rho^{-1/2}\epsilon_0)$ and therefore

$$\|w\|_{\mathrm{H}_1} \leq |||w|||_{\mathrm{H}_1} \leq c(\epsilon_V + \frac{1}{\sqrt{\rho}}\epsilon_0) \tag{9.10}$$

provided $\|w_0\|_{\mathrm{H}_1} \leq 1$. Substituting (9.10) into (6.27) we obtain from (6.24) and the notation $\alpha := \dot{\sigma} - X(\sigma)$ that

$$|\alpha|_\infty \leq c({\epsilon_V}^2 + \rho^{-1}{\epsilon_0}^2), \tag{9.11}$$

for times $t \leq c(\epsilon_V + \rho^{-1}{\epsilon_0}^2)^{-1}$. This completes the proof. $\square$ $\square$

A Ehrenfest's theorem

In this appendix we prove Ehrenfest's theorem, eq. (4.5). Denote $I_\psi(t) := \langle \psi, -\mathrm{i}\partial_{x_j}\psi\rangle$. Let ψ solve eq. (1.1) and assume that $\psi \in \mathrm{C}(\mathbb{R}, \mathrm{H}_2) \cap \mathrm{C}^1(\mathbb{R}, \mathrm{L}^2)$.

Then $I_\psi(t)$ is in C^1, and after a straightforward calculation using eq. (1.1), we find

$$\dot{I}_\psi = \int -|\psi|^2 \partial_{x_j} V \,\mathrm{d}^d x =: L_\psi(t). \tag{A.1}$$

By the fundamental theorem of calculus, we obtain

$$I_\psi(t) = I_\psi(t_0) + \int_{t_0}^{t} L_\psi(s)\,\mathrm{d}s. \tag{A.2}$$

Now take an initial condition $\psi_0 \in \mathrm{H}_1$. Pick $\psi_{0,n} \in \mathrm{H}_2$ such that $\psi_{0,n} \to \psi_0$ in H_1. Then by Theorems 5.2 and 4.2 of [11], the solutions ψ_n corresponding to the initial conditions $\psi_{0,n}$ satisfy $\psi_n \in \mathrm{C}(\mathbb{R}, \mathrm{H}_2) \cap \mathrm{C}^1(\mathbb{R}, \mathrm{L}^2)$ and $\psi_n \to \psi$ in $\mathrm{C}(\mathbb{R}, \mathrm{H}_1)$. Thus since

$$I_{\psi_n}(t) = I_{\psi_n}(t_0) + \int_{t_0}^{t} L_{\psi_n}\,\mathrm{d}s, \tag{A.3}$$

we have

$$I_\psi(t) = I_\psi(t_0) + \int_{t_0}^{t} L_\psi\,\mathrm{d}s. \tag{A.4}$$

We furthermore observe that both $I_\psi(t)$ and $L_\psi(t)$ are continuous in t for $\psi \in \mathrm{C}(\mathbb{R}, \mathrm{H}_1) \cap \mathrm{C}^1(\mathbb{R}, \mathrm{H}_{-1})$. Hence (A.4) implies that $I_\psi(t)$ is C^1 and satisfies $\dot{I}_\psi(t) = L_\psi(t)$. This proves (4.5). $\square$

B Minimization under constraint and spectrum of Hessian

In this appendix we show that the operator $\mathcal{L}_\eta := \mathcal{E}_\mu''(\eta)$ has exactly one negative eigenvalue. The argument below is probably well-known, but we did not find it in the literature. Let X be a Banach space and $K \in \mathrm{C}^3(\mathrm{X}, \mathbb{R})$ be a given functional. Define the set

$$\mathrm{M} = \{u \in \mathrm{X} : K(u) = 0\}\,. \tag{B.1}$$

We have the following

Proposition 6. *Let $\mathcal{E}$ be a C^2 functional on* X. *Assume there is a Hilbert space,* H, *such that* $\mathrm{H} \supset \mathrm{X}$, *densely, and that the Hessian quadratic form*

Hess $\mathcal{E}(u)(\alpha,\beta)$, $\alpha,\beta \in \mathrm{X}$, *defines a self-adjoint operator* $\mathcal{E}''(u)$ *on* H *such that* $\langle \alpha, \mathcal{E}''(u)\beta\rangle = \mathrm{Hess}\,\mathcal{E}(u)(\alpha,\beta)$, $\forall \alpha \in \mathrm{X}$, $\beta \in \mathrm{D}(\mathcal{E}''(u)) \subset \mathrm{X}$. *Let* η *be a minimizer of* $\mathcal{E}$ *on the set* $\mathrm{M} \subset \mathrm{X}$ *defined in* (B.1). *Assume* $K'(\eta) \neq 0$. *Then the Hessian operator* $\mathcal{E}''(\eta)$ *has at most one negative eigenvalue.*

Proof. Let η be a minimizer of $\mathcal{E}$ on M. Then η satisfies

$$\mathcal{E}'(\eta) = 0 \text{ and } \mathcal{E}''(\eta) \geq 0\ . \tag{B.2}$$

Here $\mathcal{E}'(\eta) : \mathrm{T}_\eta\mathrm{M} \mapsto \mathbb{R}$ and $\mathcal{E}''(\eta) : \mathrm{T}_\eta\mathrm{M} \mapsto (\mathrm{T}_\eta\mathrm{M})^*$, where

$$\mathrm{T}_\eta\mathrm{M} = \{\,\partial_s\eta_s|_{s=0} : \eta_s \in \mathrm{C}^1([0,\epsilon],\mathrm{M}),\ \eta_{s=0} = \eta\}\ . \tag{B.3}$$

We claim that $\mathrm{T}_\eta\mathrm{M}$ can be written as

$$\mathrm{T}_\eta\mathrm{M} = \{\xi \in \mathrm{X} : \langle K'(\eta), \xi\rangle = 0\} =: K'(\eta)^\perp\ . \tag{B.4}$$

Indeed, if $\xi \in \mathrm{T}_\eta\mathrm{M}$ then there exists η_s as in (B.3) with $\xi = \partial_s|_{s=0}\eta_s$ and therefore

$$0 = \partial_s K(\eta_s)|_{s=0} = \langle K'(\eta), \xi\rangle\ . \tag{B.5}$$

On the other hand if $\xi \in K'(\eta)^\perp$ then we can find η_s such that $\eta_{s=0} = \eta$, $\partial_s|_{s=0}\eta_s = \xi$ and $K(\eta_s) = 0$ by solving the equation $f(a,s) = 0$ where

$$f(a,s) := \frac{1}{s^2}K(\eta + s\xi + s^2 aK'(\eta)) \tag{B.6}$$

for a and setting $\eta_s = \eta + s\xi + s^2aK'(\eta)$. The latter equation has a unique solution for s sufficiently small since $f(b,0) = 0$, where $b = -\langle \xi, K''(\eta)\xi\rangle/\|K'(\eta)\|^2$ and $\partial_a f(b,0) = \|K'(\eta)\|^2$, and therefore the implicit function theorem is applicable.

Now, the second equation in (B.2) can be rewritten as

$$\inf_{\xi \in K'(\eta)^\perp} \langle \xi, \mathcal{E}''(\eta)\xi\rangle \geq 0. \tag{B.7}$$

According to the max-min principle, the number of non-positive eigenvalues of $\mathcal{E}''(\eta)$ is less or equal to the co-dimension of $K'(\eta)^\perp$, which is 1. $\square$ $\square$

In our case $\mathrm{X} = \mathrm{H}_1(\mathbb{R})$, $\mathrm{H} = \mathrm{L}^2(\mathbb{R}^d)$, $\mathcal{E} := \mathcal{E}_\mu$, where

$$\mathcal{E}_\mu(u) = \frac{1}{2}\int |\nabla u|^2 + \mu|u|^2\,\mathrm{d}^d x - F(\psi)\ , \tag{B.8}$$

and η is a minimizer to $\mathcal{E}_\mu$ with the constraint

$$K(u) := \frac{1}{2}\int |u|^2 \, \mathrm{d}^d x - m = 0 \; . \tag{B.9}$$

This implies that $\mathcal{L}_\eta := \mathcal{E}''_\mu(\eta)$ has at most one negative eigenvalue.

Proposition 7. *Under condition (D) $\mathcal{L}_\eta$ has exactly one negative eigenvalue.*

Proof. By Proposition 6 has at most one negative eigenvalue. One the other hand, since $\mathcal{L}_\eta \partial_\mu \eta = -\eta$, we have

$$\langle \partial_\mu \eta, \mathcal{L}_\eta \partial_\mu \eta \rangle = -\frac{1}{2}\partial_\mu \int \eta^2 \, \mathrm{d}^d x < 0, \tag{B.10}$$

by Condition (D). Therefore by a variational principle $\mathcal{L}_\eta$ has at least one negative eigenvalue. Thus $\mathcal{L}_\eta$ has exactly one negative eigenvalue. □ □

C The null space of $\mathcal{L}_\eta$

In this appendix we discuss condition (F) (see (2.2)). We represent complex functions $u(x) = u_1(x) + \mathrm{i}u_2(x)$ as real vectors $(u_1(x), u_2(x))$. In this representation the operator $\mathcal{L}_\eta$ takes the form

$$\mathcal{L}_\eta = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \; , \tag{C.1}$$

(on $\mathrm{L}^2(\mathbb{R}^d, \mathbb{R}) \oplus \mathrm{L}^2(\mathbb{R}^d, \mathbb{R})$), where

$$L_1 = -\Delta + \mu - f^{(1)}(\eta) \; , \tag{C.2}$$

and

$$L_2 = -\Delta + \mu - f^{(2)}(\eta), \tag{C.3}$$

with $f^{(1)}(\eta) := \partial_{\mathrm{Re}\,\psi}(\mathrm{Re}\, f)(\eta)$ and $f^{(2)}(\eta) := \partial_{\mathrm{Im}\,\psi}(\mathrm{Im}\, f)(\eta)$. The diagonal form of $\mathcal{L}_\eta$ follows from the diagonal form of $f'(\eta)$:

$$f'(\eta) = \mathrm{diag}(f^{(1)}(\eta), f^{(2)}(\eta)). \tag{C.4}$$

The latter follows from the relation $f(\mathcal{T}^{\mathrm{c}}\psi) = \mathcal{T}^{\mathrm{c}} f(\psi)$, where $\mathcal{T}^{\mathrm{c}}$ is complex conjugation, and the fact that η is real. This relation, in turn, follows from $F(\mathcal{T}^{\mathrm{c}}\psi) = F(\psi)$ (see Condition (B)).

The matrix operator (C.1) is then extended to $L^2(\mathbb{R}^d, \mathbb{C}) \oplus L^2(\mathbb{R}^d, \mathbb{C})$. The operators L_1 and L_2 are self-adjoint, with essential spectra given by

$$\sigma_{\text{ess}}(L_j) = [\mu, \infty), \quad j = 1, 2. \tag{C.5}$$

Relation (2.20) implies that

$$\partial_{x_j} \eta \in \mathrm{N}(L_1), \ \forall j = 1, ..., d, \tag{C.6}$$

and

$$\eta \in \mathrm{N}(L_2). \tag{C.7}$$

From now on, we assume that f is a local nonlinearity. The fact that $\eta > 0$, implies by a Perron-Frobenius argument (see [36]) that

$$L_2 \geq 0 \text{ and } \mathrm{N}(L_2) = \mathbb{C}\eta. \tag{C.8}$$

Thus it remains to analyze the operator L_1. To begin with, we observe that since $\mathcal{L}_\eta$ has exactly one negative eigenvalue, relation (C.8) implies that L_1 has exactly one negative eigenvalue (the same as $\mathcal{L}_\eta$).

First we consider the case $d = 1$. Then the zero mode, η', of L_1 has exactly one zero (at $x = 0$) and consequently (by Sturm-Liouville theory) L_1 has exactly one negative eigenvalue, as we concluded above from general considerations. (Remember that the lowest eigenvalue in our case is simple, and that the nonlinearity is local)

Theorem 3 ([47]). *Let $d = 1$. Then* $\mathrm{N}(L_1) = \mathbb{C}\eta'$.

Proof. The proof follows Weinstein [47]. We know one solution, η'. The fact that $\eta''(0) \neq 0$ allow us to choose the first linearly independent solution to be $w_1 = \eta'/\eta''(0)$. Then $w_1(0) = 0$, $w_1'(0) = 1$. Consider a second linearly independent solution, w_2, with $w_2(0) = 1$. Since the Wronskian

$$W[w_1, w_2] = w_1 w_2' - w_1' w_2 \tag{C.9}$$

is constant with respect to x, and since

$$W[w_1, w_2](x = 0) = w_1(0)w_2'(0) - w_1'(0)w_2(0) = -w_2(0) = -1 \ , \tag{C.10}$$

we have $W[w_1, w_2] = -1$ for all x, and therefore

$$\eta'(x)w_2'(x) - \eta''(x)w_2(x) = -\eta''(0) > 0 \ . \tag{C.11}$$

The last equation can be rewritten as

$$(\frac{w_2}{\eta'})' = \frac{-\eta''(0)}{(\eta')^2}. \tag{C.12}$$

Now, for $\epsilon > 0$ and $x > 0$,

$$w_2(x) = w_2(\epsilon)\frac{\eta'(x)}{\eta'(\epsilon)} - \eta'(x)\eta''(0)\int_\epsilon^x \frac{1}{(\eta'(\tilde{r}))^2}\,\mathrm{d}\tilde{r}. \tag{C.13}$$

Since $\eta'(x) < 0$ for $x > 0$, $\eta''(0) < 0$, and $|\eta'(x)| \leq C\mathrm{e}^{-c|x|}$, we have

$$w_2(x) \leq w_2(\epsilon)\frac{\eta'(x)}{\eta'(\epsilon)} - \eta'(x)\eta''(0)\int_\epsilon^x \frac{\mathrm{d}s}{C^2\mathrm{e}^{-2cs}} \to -\infty \quad \text{as} \quad x \to \infty\,. \tag{C.14}$$

Hence $w_2 \notin \mathrm{N}(L_1)$ and we are done. $\square$ $\square$

Next, we consider the case $d \geq 2$. We use the assumption that η is spherically symmetric. In this case the operator L_1 (and also L_2 and $\mathcal{L}_\eta$) is spherically symmetric; *i.e.*, it commutes with the action of the rotation group $\mathrm{SO}(d)$ on $\mathrm{L}^2(\mathbb{R}^d)$. As a result it can be decomposed into a direct sum of ordinary differential operators corresponding to the eigenfunction expansion of the Laplacian $\Delta_{\mathrm{S}^{d-1}}$ on the sphere S^{d-1}. Denote the orthonormal eigenfunctions of $-\Delta_{\mathrm{S}^{d-1}}$ corresponding to the eigenvalues $\lambda_k = k(d-2+k)$ by $Y_k(\theta)$, $\theta \in \mathrm{S}^{d-1}$. Then the operator L_1 can we written as the direct sum

$$L_1 = \oplus A_{\mu,k} \tag{C.15}$$

acting on the direct sum $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}^d x) = \bigoplus_{k=0}^\infty \mathrm{L}^2(\mathbb{R}_+, r^{d-1}\mathrm{d}r) \otimes Y_k(\theta)$, where the operators $A_{\mu,k}$ are defined on $\mathrm{L}^2(\mathbb{R}_+, r^{d-1}\mathrm{d}r)$ by

$$A_{\mu,k} = -\Delta_r + \mu + V_k(r), \tag{C.16}$$

and where $\Delta_r = \partial_r^2 + (d-1)r^{-1}\partial_r$ is the radial Laplacian in $\mathbb{R}^d$ and $V_k(r) := -f'(\eta)(r) + \lambda_k r^{-2}$. Clearly $A_{\mu,k}^* = A_{\mu,k}$ and $\sigma_{\mathrm{ess}}(A_{\mu,k}) = [\mu, \infty)$.

Now observe that

$$(\partial_{x_j}\eta)(x) = \hat{x}_j\eta'(r) \in \mathrm{L}^2(\mathbb{R}_+, r^{d-1}\mathrm{d}r) \otimes Y_1(\theta), \tag{C.17}$$

where $\hat{x} = x|x|^{-1} \in \mathrm{S}^{d-1}$. Hence

$$\eta' \in \mathrm{N}(A_{\mu,1}). \tag{C.18}$$

Since $\eta'(r) < 0$ for $r > 0$, we have by an extension of the Perron-Frobenius (or Sturm-Liouville) theory (see [31]) that zero is the lowest eigenvalue of $A_{\mu,1}$ and is simple. Thus

$$A_{\mu,1} \geq 0 \text{ and } \mathrm{N}(A_{\mu,1}) = \mathbb{C}\eta'. \tag{C.19}$$

Next, since for $k \geq 2$

$$A_{\mu,k} - A_{\mu,1} = \frac{\lambda_k - \lambda_1}{r^2} > 0, \tag{C.20}$$

we conclude that $A_{\mu,k} > 0$ for $k \geq 2$.

Theorem 4. $\mathrm{N}(A_{\mu,0})$ *is trivial provided* $f(\psi) = h(|\psi|^2)\psi$ *with* h *satisfying*

$$h'(r) + h''(r)r > 0, \;\; r > 0. \tag{C.21}$$

Proof. Assume there exists $\xi \in \mathrm{L}^2$ such that $A_{\mu,0}\xi = 0$. Then 0 is the second eigenvalue of $A_{\mu,0}$, and so the corresponding eigenfunction ξ has exactly one zero in $(0, \infty)$, say at r_0. Observe the following properties

i. $\xi \perp \mathrm{Ran}(A_{\mu,0})$;

ii. $A_{\mu,0}\eta = 2h'(\eta^2)\eta^3$;

iii. $A_{\mu,0}\partial_\mu \eta = -\eta$.

Properties ii and iii follow from the equation

$$(-\Delta_r + \mu)\eta - f(\eta) = 0 \tag{C.22}$$

for η.

Properties i–iii imply the relation

$$\langle \xi, (h'(\eta^2)\eta^2 - \alpha)\eta \rangle = 0, \;\; \forall \alpha \in \mathbb{R}. \tag{C.23}$$

Since η is monotonically decreasing from some $\eta(0) > 0$ at 0 to 0 as $r \to \infty$ (see *e.g.*, [4]), and since $h'(s)s$ is monotonically increasing function of s by condition (C.21), we can choose α such that the monotonically decreasing function $h'((\eta(r))^2)\eta^2(r) - \alpha$ has a zero exactly at r_0. In that case the left hand side of (C.23) is non-zero, which leads to a contradiction. Thus the equation $A_{\mu,0}\xi = 0$ has no nontrivial solutions in L^2. □ □

Finally we present the conditions under which McLeod's [29] uniqueness proof of positive solitons implies that $A_{\mu,0}$ has trivial null space in the case $d > 1$. There exists $\alpha > 0$ such that

$$\begin{aligned} \mu s - sh(s^2) > 0, \text{ for } 0 < s < \alpha, \\ \mu s - sh(s^2) < 0, \text{ for } \alpha < s < \infty, \\ (sh(s^2) - \mu s)' > 0, \text{ when } s = \alpha, \end{aligned} \tag{C.24}$$

and for each $S > \alpha$, $\exists \lambda = \lambda(S) \in \mathrm{C}((\alpha, \infty), \mathbb{R}_+)$ such that

$$K(s, \lambda) \geq 0 \text{ for } s \in (0, S), \quad K(s, \lambda) \leq 0 \text{ for } s \in (S, \infty), \tag{C.25}$$

where

$$K(s, \lambda) = \mu s + \lambda s^3 h'(s^2) - sh(s^2).$$

D Coercivity of $\mathcal{L}_\eta$

The goal of this appendix is to prove the following result, essentially due to [48].

Proposition 8. *There is $\rho' > 0$ such that if w satisfies $\omega(w, z) = 0$ $\forall z \in \mathrm{T}_\eta \mathrm{M}_\mathrm{s}$, then*

$$\langle w, \mathcal{L}_\eta w \rangle \geq \rho' \|w\|^2_{\mathrm{H}_1}. \tag{D.1}$$

Proof. We break the proof into three steps. The proof utilizes the fact that $\mathcal{L}_\eta$ has exactly one non-degenerate negative eigenvalue and the assumption (F); *i.e.*, that $\mathrm{N}(\mathcal{L}_\eta) = \mathrm{span}\{(0, \eta), (\partial_{x_j} \eta, 0), j = 1, ..., d\}$.

Step 1. *Let* $\mathrm{X}_1 = \{w \in \mathrm{H}_1 : \|w\| = 1 \, , \langle (\eta, 0), w \rangle = 0\}$. *Then*

$$\inf_{w \in \mathrm{X}_1} \langle w, \mathcal{L}_\eta w \rangle = 0 \, . \tag{D.2}$$

Proof. Let $\alpha := \inf_{w \in \mathrm{X}_1} \langle w, \mathcal{L}_\eta w \rangle$, (see (C.1)). Clearly $\nu \leq \alpha \leq 0$, where $\nu < 0$ is the negative eigenvalue of $\mathcal{L}_\eta$ (see Appendix B). That $\alpha \leq 0$ is clear, as $w = (0, \eta)/\|\eta\| \in \mathrm{X}_1$ yields $\langle w, \mathcal{L}_\eta w \rangle = 0$. Moreover $\alpha \neq \nu$. Indeed if $\alpha = \nu$ then the minimizer, v, of (D.2) would be an eigenfunction of $\mathcal{L}_\eta$ corresponding to the smallest eigenvalue, ν. Since $(\eta, 0) \perp v$, $(\eta, 0) \perp \mathrm{N}(\mathcal{L}_\eta)$ and since ν is the only negative eigenvalue of $\mathcal{L}_\eta$ (see Proposition 6), we conclude that $(\eta, 0)$ is in the spectral subspace of $\mathcal{L}_\eta$ corresponding to the interval $[\delta, \infty)$ for some $\delta > 0$.

Therefore $\mathcal{L}_\eta^{-1}(\eta,0)$ is well defined and $\langle(\eta,0),\mathcal{L}_\eta^{-1}(\eta,0)\rangle > 0$. On the other hand the equation $\mathcal{L}_\eta(\partial_\mu\eta,0) = -(\eta,0)$ implies that

$$\langle(\eta,0),\mathcal{L}_\eta^{-1}(\eta,0)\rangle = -m'(\mu) < 0 \tag{D.3}$$

by Condition (D) which contradicts $\langle(\eta,0),\mathcal{L}_\eta(\eta,0)\rangle > 0$. Hence $\alpha = \nu$ is impossible.

To show that $\alpha = 0$ we use the Euler-Lagrange equations corresponding to (D.2)

$$\mathcal{L}_\eta w = \alpha w + \beta(\eta,0) \tag{D.4}$$

where α and β are the Lagrange multipliers corresponding to $\|w\| = 1$ and $\langle(\eta,0),w\rangle = 0$ respectively. Assume $\nu < \alpha < 0$. If $\beta = 0$, then α would be a negative eigenvalue in $(\nu,0)$ which contradicts that ν is the only negative eigenvalue. Thus $\beta \neq 0$. Given $\nu < \alpha < 0$, $\beta \neq 0$, we can solve the Euler-Lagrange equation as

$$w = \beta(\mathcal{L}_\eta - \alpha)^{-1}(\eta,0). \tag{D.5}$$

The inner product of the equation above with $(\eta,0)$, the orthogonality relation $\langle w,(\eta,0)\rangle = 0$, and $\beta \neq 0$, give

$$0 = \langle(\eta,0),(\mathcal{L}_\eta + |\alpha|)^{-1}(\eta,0)\rangle =: q(|\alpha|). \tag{D.6}$$

$q(\lambda)$ is analytic in $\lambda \in (0,|\nu|)$, and hence differentiable. Moreover, it is monotonically decreasing, since

$$q'(\lambda) = -\langle(\eta,0),(\mathcal{L}_\eta + \lambda)^{-2}(\eta,0)\rangle = -\|(\mathcal{L}_\eta + \lambda)^{-1}(\eta,0)\|^2 < 0. \tag{D.7}$$

Furthermore by (D.3) $q(0) = \langle(\eta,0),\mathcal{L}_\eta^{-1}(\eta,0)\rangle < 0$. Thus $q(|\alpha|) \neq 0$, for $\alpha \in (\nu,0)$, which contradicts (D.6). Hence $\alpha = 0$. □ □

Step 2. *Let* $\mathrm{X} := \{w \in \mathrm{H}_1(\mathbb{R}^d,\mathbb{C}) : \|w\| = 1,\ \omega(w,z) = 0,\ \forall z \in \mathrm{T}_\eta\mathrm{M}_\mathrm{s}\}$. *Then*

$$\inf_{w\in\mathrm{X}}\langle w,\mathcal{L}_\eta w\rangle > 0\ . \tag{D.8}$$

Proof. The Euler-Lagrange equation corresponding to (D.8) is

$$\mathcal{L}_\eta w = \alpha w + \sum_k \gamma_k J z_k \tag{D.9}$$

where $\{z_k\}$ is a basis for $\mathrm{T}_\eta\mathrm{M}_\mathrm{s}$. Here α and $\{\gamma_k\}$ are the Lagrange multipliers corresponding to the constraints $\|w\| = 1$ and $\omega(w,z_k) = 0$ $\forall k$ respectively.

Note that $\alpha = \langle w, \mathcal{L}_\eta w\rangle$, and that $\mathrm{X} \subset \mathrm{X}_1$, hence $\alpha \geq 0$. Assume that $\alpha = 0$, and that one $\gamma_j \neq 0$. Then for at least one $z_k \in \mathrm{T}_\eta \mathrm{M}_\mathrm{s}$, we have

$$\langle z_k, \mathcal{L}_\eta w\rangle = \gamma_j (\Omega_\eta)_{jk} \neq 0, \tag{D.10}$$

for some k, which contradicts $\langle z_k, \mathcal{L}_\eta w\rangle = \langle \mathcal{L}_\eta z_k, w\rangle = 0 \ \forall k$. Here we have used that $\det \Omega_\eta \neq 0$, and that z_k is either a zero-eigenfunction or an associated zero-mode for $\mathcal{L}_\eta$. Thus either $\alpha > 0$ or $\gamma_j = 0$. Consider the latter case. In this case

$$\mathcal{L}_\eta w = 0. \tag{D.11}$$

which implies that $w \in \mathrm{N}(\mathcal{L}_\eta)$. Since $\mathrm{N}(\mathcal{L}_\eta) \subset \mathrm{T}_\eta \mathrm{M}_\mathrm{s}$, the relation $\omega(w, z_k) = 0$ for all $z_k \in \mathrm{T}_\eta \mathrm{M}_\mathrm{s}$, contradicts the non-degeneracy of Ω_η^{-1} (see Corollary 2.1). Thus $\alpha > 0$. □ □

Step 3. End of Proof. Eq. (D.8) implies that there exists a $\rho'' > 0$ such that

$$\langle w, \mathcal{L}_\eta w\rangle \geq \rho'' \|w\|^2 \ , \tag{D.12}$$

for some $\rho'' = \rho''(\mu)$. To improve the coercivity from L^2 to H_1, we let $0 < \delta < 1$, and estimate $\langle w, \mathcal{L}_\eta w\rangle$ using (D.12) as

$$(1-\delta)\rho'' \|w\|^2 + \delta \langle w, \mathcal{L}_\eta w\rangle \leq \langle w, \mathcal{L}_\eta w\rangle \ . \tag{D.13}$$

Upon using the explicit form of $\mathcal{L}_\eta$ we find that

$$\langle w, \mathcal{L}_\eta w\rangle \geq \|\nabla w\|^2 - C_\mu \|w\|^2, \tag{D.14}$$

where

$$C_\mu = \sup_x (\mu + |f'(\eta)|). \tag{D.15}$$

The last two estimates with $\delta := \rho''(1 + \rho'' + C_\mu)^{-1}$ imply

$$\langle w, \mathcal{L}_\eta w\rangle \geq \rho' \|w\|^2_{\mathrm{H}_1} \ , \tag{D.16}$$

where $\rho' = \rho''(1+\rho''+C_\mu)^{-1}$. This concludes the proof of Proposition 8. □ □

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