MARKOV TOWERS AND STOCHASTIC PROPERTIES OF BILLIARDS

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Dedicated to Anatole Katok on the occasion of his 60th birthday.

ABSTRACT. Markov partitions work most efficiently for Anosov systems or for Axiom A systems. However, for hyperbolic dynamical systems which are either singular or whose hyperbolicity is nonuniform, the construction of a Markov partition, which in these cases is necessarily countable, is a rather delicate issue even when such a construction exists. An additional problem is the use of a countable Markov partition for proving probabilistic statements. For a wide class of hyperbolic systems, L. S. Young, in 1998, constructed so called Markov towers, which she could apply successfully to establish nice, for instance, exponential correlation decay, and, moreover, as a consequence, a central limit theorem. The aim of this survey is twofold. First we show how the Markov tower construction is applicable for obtaining finer stochastic properties, like a local limit theorem of probability theory. Here the fundamental method is the study of the spectrum of the Fourier transform of the Perron–Frobenius operator. These ideas and results are applicable to all systems Young has been considering. Second, we survey the problem of recurrence of the planar Lorentz process. As an application of the results from the first part, we obtain a dynamical proof of recurrence for the finite horizon case. Here basically different proofs were given by K. Schmidt, in 1998, and J.-P. Conze, in 1999. As another application we can also treat the infinite horizon case, where already the global limit theorem is absolutely novel. It is not a central one, the scaling is $\sqrt{n \log n}$ in contrast to the classical $\sqrt{n}$ one. Beyond thus giving a rigorous proof for earlier heuristic ideas of P. Bleher, which used three delicate and hard hypotheses, we can also a) verify the local version of this limit theorem for the free flight function and b) prove the recurrence of the planar Lorentz process in the infinite horizon case.

1. INTRODUCTION

Since — following some ideas of Hadamard — M. Morse introduced the concept of symbolic dynamics, the method got more and more extensively used to study topological, and later also ergodic and stochastic, properties of dynamical systems possessing some hyperbolic behaviour.

On the one hand, "the idea of coding and semiconjugacies with topological Markov chains yields remarkably precise results concerning topological entropy, the growth of periodic orbits, the presence of orbits of various periods, and the structure of maps with zero topological entropy" [KH 95]. On the other hand, through the achievements of Bowen (cf. [B 75], Ruelle (cf. [R 78] and of Sinai (most notably his work [S 72] relating symbolic dynamics and Gibbs states of statistical physics), almost invertible semiconjugacies and conjugacies provided by Markov partitions made it possible to demonstrate exponential correlation decay and further nice and useful stochastic properties for Axiom A systems — and later for more general ones, too. Thus, for a long time it, quite naturally, seemed so that the construction of Markov partitions is 'the method' for obtaining effective statistical statements for more complicated systems as well.

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However, in course of the work of the Moscow school on billiards and of Benedicks and Young on the Hénon map it became clear that a Markov partition is a too delicate construction if one wants to relax the assumptions of smoothness or of the uniform hyperbolicity of the maps in question. In fact, a warning might have come earlier from Bowen's result showing that, even in the nicest systems, in the multidimensional ($d > 2$) case the boundaries of the elements of any Markov partition behave wildly, in particular, they are not smooth [B 78]. In addition, for instance, in discontinuous systems like billiards, the local invariant manifolds are, indeed, arbitrarily short, and as a consequence the Markov partition is necessarily countable and its elements are products of Cantor sets. Then to adapt the boundaries of a possible Markov partition in a Markov way is an extremely delicate issue even in the case when such a construction was successfully established (cf. [BS 80]).

It is not our aim to go into more details here since there exists a quite recent and excellent survey [ChY 00], which, on the one hand, gives a comprehensive historical overview, and, on the other hand, explains the way out: the 'weaker' construction of a Markov tower. This construction was designed by L. S. Young [Y 98] and it works for a wide class of systems with some hyperbolicity, among others for Anosov and Axiom A systems, two dimensional hyperbolic systems with singularities, two dimensional Sinai billiards with a finite horizon, hyperbolic unimodal maps and hyperbolic Hénon maps.

The construction of a Markov partition and the resulting symbolic dynamics opened the way in a straightforward manner to put the probabilistic arsenal of — appropriately mixing — stationary stochastic processes into action. In the case of a Markov tower this connection is not straightforward, and our actual aim is, indeed, to understand and to discuss the probabilistic approach in the case of a Markov tower. We note that, as we will see in subsection 2.2, the tower also leads to a (countable) Markov partition, but its properties, apart from its formal ones, are quite different from those of a traditional Markov partition. In particlular, it does not seem to provide a flexible Markov approximation. This is why in its applications new methods are needed and, in fact, their discussion is our main aim here.

The paper is organized as follows: in section 2 we recall the axioms of systems, which we are going to deal with, and briefly describe the tower construction. In section 3 we analyze how one can establish stochastic properties, in general, and further finer stochastic properties, like local limit theorems, in particular. The choice of local limit theorems may seem eventual but it is not. This will be clear from section 4, where we apply the local limit theorems to planar dispersing billiards. In doing so, beside attaining local theorems for them, we also obtain

- a dynamical proof of recurrence for the planar Lorentz process with a finite horizon (in fact, partly abstract ergodic-theoretic proofs were given by K. Schmidt [Sch 98] and Conze [Con 99], which were, however, also using the central limit theorem);
- the first rigorous proof for a noncentral limit theorem for the displacements of the planar Lorentz process with an infinite horizon; this result was conjectured by an earlier nonrigorous, heuristic argument of Bleher [B 92] based on three hard hypotheses (which, in fact, still do not follow from our approach);
- the first proof of recurrence for the planar Lorentz process with an infinite horizon.

2. Markov tower

2.1. **The Product Set.** The technique developed in [Y 98] allows to handle stochastic properties of systems

(1) whose every power is ergodic;
(2) which satisfy several technical assumptions well-known from hyperbolic theory;

(3) whose phase space X contains a subset Λ with a hyperbolic product structure;
(4) where the return time into Λ has an exponentially decaying tail.

This class contains planar dispersing billiards with both bounded or unbounded free flight (i. e. with a finite resp. an infinite horizon), logistic interval maps, expanding maps with neutral fixed points, piecewise hyperbolic maps, Hénon attractors, their generalisations, and certain partially hyperbolic systems. The rest of this subsection is devoted to the precise definitions.

We start with describing precisely the models we are going to deal with. Let T be a $C^{1+\varepsilon}$ diffeomorphism with singularities of a compact Riemannian manifold X with boundary. More precisely, there exists a finite or countably infinite number of pairwise disjoint open regions $\{X_i\}$ whose boundaries are C^1 submanifolds of codimension 1, and finite volume such that $\cup X_i = X$, $T\big|_{\cup X_i}$ is $1-1$ and $T\big|_{X_i}$ can be extended to a $C^{1+\varepsilon}$-diffeomorphism of $\bar{X}_i$ onto its image. The Riemannian measure will be denoted by μ, and if $W \subset X$ is a submanifold, then μ_W will denote the induced measure. The invariant Borel probability measure will be denoted by ν.

Definition 1. *An embedded disk $\gamma \subset X$ is called an* unstable manifold *or an* unstable disk *if $\forall x,y \in \gamma,\ d(T^{-n}x, T^{-n}y) \to 0$ exponentially fast as $n \to \infty$; it is called a* stable manifold *or a* stable disk *if $\forall x,y \in \gamma,\ d(T^n x, T^n y) \to 0$ exponentially fast as $n \to \infty$. We say that $\Gamma^u = \{\gamma^u\}$ is a* continuous family of C^1 unstable disks *if the following hold:*

- *K^s is an arbitrary compact set; D^u is the unit disk of some $\mathbb{R}^n$;*
- *$\Phi^u : K^s \times D^u \to X$ is a map with the property that*
  - *Φ^u maps $K^s \times D^u$ homeomorphically onto its image,*
  - *$x \to \Phi^u \mid (\{x\} \times D^u)$ is a continuous map from K^s into the space of C^1 embeddings of D^u into X,*
  - *γ^u, the image of each $\{x\} \times D^u$, is an unstable disk.*

Continuous families of C^1 stable disks *are defined similarly.*

Definition 2. *We say that $\Lambda \subset X$ has a* hyperbolic product structure *if there exist a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ such that*

(i) $\dim \gamma^u + \dim \gamma^s = \dim X$
(ii) *the γ^u-disks are transversal to the γ^s-disks with the angles between them bounded away from 0;*
(iii) *each γ^u-disk meets each γ^s-disk in exactly one point;*
(iv) $\Lambda = (\cup \gamma^u) \cap (\cup \gamma^s)$.

Definition 3. *Suppose Λ has a hyperbolic product structure. Let Γ^u and Γ^s be the defining families for Λ. A subset $\Lambda_0 \subset \Lambda$ is called an s-*subset *if Λ_0 also has a hyperbolic product structure and its defining families can be chosen to be Γ^u and Γ^s_0 with $\Gamma^s_0 \subset \Gamma^s$; u-*subsets *are defined analogously. For $x \in \Lambda$, let $\gamma^u(x)$ denote the element of Γ^u containing x.*

Definition 4. *We call (X, T, ν) a* Young system, *if the following Properties* **(P1)–(P8)** *are true:*

(P1) There exists a $\Lambda \subset X$ with a hyperbolic product structure and with $\mu_\gamma\{\gamma \cap \Lambda\} > 0$ for every $\gamma \in \Gamma^u$.
(P2) There is a countable number of disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that
- on each γ^u-disk $\mu_{\gamma^u}\{(\Lambda \setminus \cup \Lambda_i) \cap \gamma^u\} = 0$;
- for each i, $\exists R_i \in \mathbb{Z}^+$ such that $T^{R_i}\Lambda_i$ is a u-subset of Λ;
- for each n there are at most finitely many i's with $R_i = n$;
- $\min R_i \geq$ some R_0 depending only on T

(P3) For every pair $x,y\in\Lambda$, we have a notion of *separation time* denoted by $s_0(x,y)$. If $s_0(x,y)=n$, then the orbits of x and y are thought of as being "indistinguishable" or "together" through their n^{th} iterates, while $T^{n+1}x$ and $T^{n+1}y$ are thought of as having been "separated." (This could mean that the points have moved a certain distance apart, or have landed on opposite sides of a discontinuity manifold, or that their derivatives have ceased to be comparable.) We assume:
  (i) $s_0\geq 0$ and depends only on the γ^s-disks containing the two points;
  (ii) the number of "distinguishable" n-orbits starting from Λ is finite for each n;
  (iii) for $x,y\in\Lambda_i$, $s_0(x,y)\geq R_i+s_0(T^{R_i}x,T^{R_i}y)$;

(P4) Contraction along γ^s disks. There exist $C>0$ and $\alpha<1$ such that for $y\in\gamma^s(x)$, $d(T^nx,T^ny)\leq C\alpha^n\ \ \forall n\geq 0$.

(P5) Backward contraction and distorsion along γ^u. For $y\in\gamma^u(x)$ and $0\leq k\leq n<s_0(x,y)$, we have
  (a) $d(T^nx,T^ny)\leq C\alpha^{s_0(x,y)-n}$;
  (b)

$$\log\prod_{i=k}^{n}\frac{\det DT^u(T^ix)}{\det DT^u(T^iy)}\leq C\alpha^{s_0(x,y)-n}.$$

(P6) Convergence of $D(T^i|\gamma^u)$ and absolute continuity of Γ^s.
  (a) for $y\in\gamma^s(x)$,

$$\log\prod_{i=n}^{\infty}\frac{\det T^u(T^ix)}{\det T^u(T^iy)}\leq C\alpha^n\quad\forall n\geq 0.$$

  (b) for $\gamma,\gamma'\in\Gamma^u$, if $\Theta:\ \gamma\cap\Lambda\to\gamma'\cap\Lambda$ is defined by $\Theta(x)=\gamma^s(x)\cap\gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x)=\prod_{i=0}^{\infty}\frac{\det DT^u(T^ix)}{\det DT^u(T^i\Theta x)}.$$

(P7) $\exists C_0>0$ and $\theta_0<1$ such that for some $\gamma\in\Gamma^u$,

$$\mu_\gamma\{x\in\gamma\cap\Lambda:R(x)>n\}\leq C_0\theta_0^n\quad\forall n\geq 0;$$

(P8) (T^n,ν) is ergodic $\forall n\geq 1$, where ν is the SRB-measure corresponding to the system (X,T).

Remark 5. *Some explanation: Properties* **(P1-7)** *do not involve the measure* ν*. Using them, in fact, Young constructs the SRB-measure* ν*, so the system satisfying* **(P1-7)** *is a Young sysstem if for the so constructed SRB-measure* **(P8)** *holds, too.*

2.2. **The Tower.** Now we will define the *Markov extension*, the actual *Markov tower*. Let $R:\ \Lambda\to\mathbb{Z}_+$ be the function which is R_i on Λ_i, and let

$$\Delta\stackrel{\text{def}}{=}\{(x,l):x\in\Lambda;\ \ l=0,1,\dots,R(x)-1\}$$

and define

$$F(x,l)=\begin{cases}(x,l+1) & \text{if}\quad l+1<R(x)\\ (T^Rx,0) & \text{if}\quad l+1=R(x)\end{cases}$$

We will refer to $\Delta_l=\{(x,j)\mid(x,j)\in\Delta,\ j=l\}$ as the l^{th} level of the tower Δ. Young also has a construction for $\tilde\nu$, the SRB measure of the extension, for which the pushforward is ν, and $J(F)\equiv 1$ except on $F^{-1}(\Delta_0)$. Thus, the Markov Tower is the dynamical system $(\Delta,F,\tilde\nu)$.

On the tower a *M*arkov partition $\mathcal{D}$ can be defined, with the following properties:

(a) $\mathcal{D}$ is a refinement of the partition Δ_l.

(b) By denoting by $\mathcal{D}_l$ the partition $\mathcal{D}|\Delta_l$, $\mathcal{D}_l = \{\Delta_{l,j} \mid j = 1, \ldots, j_l\}$ has only a finite number of elements and each one is the union of a collection of Λ_i's;
(c) $\mathcal{D}_l$ is a refinement of $F\mathcal{D}_{l-1}$;
(d) if x and y belong to the same element of $\mathcal{D}_l$, then $s_0(F^{-l}x, F^{-l}y) \geq l$;
(e) if $R_i = R_j$ for some $i \neq j$, then Λ_i and Λ_j belong to different elements of $\mathcal{D}_{R_i-1}$.

Let $\Delta^*_{l,j} = \Delta_{l,j} \cap F^{-1}(\Delta_0)$. We think of $\Delta_{l,j} \setminus \Delta^*_{l,j}$ as "moving upward" under F, while $\Delta^*_{l,j}$ returns to the base.

It is natural to *redefine the separation time* to be $s(x,y) \stackrel{\text{def}}{=}$ the largest n such that for all $i \leq n$, $F^i x$ and $F^i y$ lie in the same element of $\{\Delta_{l,j}\}$. We claim that **(P5)** is valid for $x, y \in \gamma^u \cap \Delta_{l,j}$ with s in the place of s_0. To verify this, first consider $x, y \in \Lambda$. We claim that $s(x,y) \leq s_0(x,y)$. If x, y do not belong to the same Λ_i, then this follows from rule (d) in the construction of $\mathcal{D}_l$; if $x, y \in \Lambda_i$, but $T^R x, T^R y$ are not contained in the same Λ_j, then $s(x,y) = R_i + s(T^R x, T^R y)$, which is $\leq s_0(x,y)$ by property **(P3)**,(iii) of s_0, and so on. In general, for $x, y \in \Delta_{l,j}$, let $x_0 = F^{-l}x$, $y_0 = F^{-l}y$ be the unique inverse images of x and y in Δ_0. Then by definition $s(x,y) = s(x_0, y_0) - l$, and what is said earlier on about x_0 and y_0 is equally valid for x and y.

*From here on s_0 is replaced by s and **(P5)** is modified accordingly.*

3. Stochastical Properties

As explained above, the tower construction actually provides a countable Markov partition. Below we first remind the reader how Young exploits this partition to obtain stochastic behaviour. The first goal is to establish exponential decay of correlations, where, of course, Property **(P7)** plays a decisive role.

3.1. The Perron–Frobenius Operator and the Doeblin–Fortet Property.

1. The starting point is to *factorise the dynamics* by a factorisation along stable manifolds of Δ. The advantage is that this dynamics will behave as an expanding map, an appropriate object to study via the Perron–Frobenius operator. Let $\bar{\Delta} := \Delta/\sim$ where $x \sim y$ iff $y \in \gamma^s(x)$. Since F takes γ^s-leaves to γ^s-leaves, the quotient dynamical system $\bar{F}: \bar{\Delta} \to \bar{\Delta}$ is clearly well defined.

The construction of $\bar{m}$ is not trivial but is quite standard. Young obtains it following [B 75]. A simple property of $\bar{m}$ is: let $\bar{m}|\bar{\Delta}_l$ be the measure induced from the natural identification of $\bar{\Delta}_l$ with a subset of $\bar{\Delta}_0$, so that $J(\bar{F}) \equiv 1$ except on $\bar{F}^{-1}(\bar{\Delta}_0)$, where $J(\bar{F}) = J(\overline{T^R} \circ \bar{F}^{-(R-1)})$. We note that for the factorised map Young also proves a distorsion property with a weaker constant $\beta:\ 1 > \beta > \sqrt{\alpha}$.

To investigate Birkhoff sums we have to associate a function $\bar{f}: \bar{\Delta} \to \mathbb{R}$ to each observable $f: M \to \mathbb{R}$. We can pull back f to the tower, and find an other function cohomologous to this one, which is constant along stable manifolds. This method is described for example in [PP 90].

2. The analytic tool of investigation is the *Perron–Frobenius operator*:

$$P(\bar{\varphi}(\bar{x})) = \sum_{\bar{y}: \bar{F}\bar{y} = \bar{x}} \frac{\bar{\varphi}(\bar{y})}{J\bar{F}(\bar{y})}.$$

3. The technique is based upon the *spectral properties of the Perron–Frobenius operator*. For this purpose we have to introduce *suitable Banach spaces*, where P has a nice spectrum. Actually for the method introduced by Doeblin and Fortet we need a pair of function spaces $\mathcal{C}$ and $\mathcal{L}$ (usually with some supremum-like and Lipschitz-like norms) such that $\mathcal{L} \leq \mathcal{C}$, $\|\cdot\|_{\mathcal{C}} \leq \|\cdot\|_{\mathcal{L}}$, and the inclusion of $\mathcal{L}$ into $\mathcal{C}$ be a compact operator.

If we have such a pair of Banach spaces, and we can prove, that $\exists N, K$, and $\tau < 1$

$$\|P^N \bar{\varphi}\|_{\mathcal{L}} \leq \tau \|\bar{\varphi}\|_{\mathcal{L}} + K \|\bar{\varphi}\|_{\mathcal{C}}$$

then by knowing that $\forall i\ \ T^i$ is ergodic we have that the spectrum of P on $\mathcal{L}$ is contained in a disk with radius strictly smaller than one, except that 1 is an eigenvalue with multiplicity one, and the corresponding eigenfunction is the invariant density (cf. [I-TM 50]).

This kind of estimate captures the uniform expanding feature of the dynamics, or contraction of the P operator. In order to derive this so called *Doeblin–Fortet property of the transfer operator* (in the theory of dynamical systems often called the *Lasota–Yorke property*, cf. [LY 73]), Young uses an exponential factor in the function norms. $\bar{F}$ on the tower has Jacobian 1, when moving upwards, and the tower is usually unboundedly high, so we have to pretend expanding at least in the norms:

$$\|\bar{\varphi}\|_{\mathcal{C}} := \sup_{l,j} \left|\bar{\varphi}|_{\Delta_{l,j}}\right|_\infty e^{-l\varepsilon},$$

$$\|\bar{\varphi}\|_{\mathcal{L}} := \|\bar{\varphi}\|_{\mathcal{C}} + \left(\operatorname*{ess\ sup}_{\bar{x},\bar{y}\in\bar{\Delta}_{l,j}} \frac{|\bar{\varphi}(\bar{x})-\bar{\varphi}(\bar{y})|}{\beta^{s(\bar{x},\bar{y})}}\right) e^{-\varepsilon l}.$$

The denominator in the second definition is a natural distance for the points on the tower, so it is really a Lipschitz-like norm. The aforementioned trick is in the $e^{-l\varepsilon}$ term. This allows that a function on $\bar{\Delta}$ which is exponentially increasing with the height of the tower, to be in $\mathcal{C}$, if this growth is moderate. This also means, that P is contracting the norm in the middle of the tower, where the Jacobian is 1. The constant ε should be chosen carefully in order to hold back enough contraction when a Markov return occurs. Roughly speaking ε has to be smaller than the smallest positive Lyapunov exponent.

3.2. **The Central Limit Theorem.** The aforementioned spectral picture is essentially equivalent to the exponential decay of correlations for function pairs in $\mathcal{L}$. As a matter of fact, the boundedness of the functions in question is also needed. After the argument presented in [Y 98] this leads to the exponential correlation decay of bounded, piecewise Hölder functions on M. For the same class of functions Young immediately gets the *Central Limit Theorem* (CLT) by checking the conditions of a theorem by Keller [K 80]. Here we make a simple but important clarification. The traditional and by far the most widely used method for establishing the CLT is to apply Fourier transforms. Keller [K 80] (and thus also [Y 98]) can elude this by referring to a nice and useful theorem of Gordin [G 69]. (As a matter of fact, Fourier transform are, indirectly, still applied, since Gordin constructed a martingale approximation and used the martingale CLT. But in proving the CLT for martingales again Fourier transform is the method! Moreover, the error term in the martingale approximation is so large that it obviously excludes the applicability of this approach in proofs of finer statements, for instance, in those of local limit theorems.)

Let we recall two basic results from [Y 98]: the first one on the exponential decay of correlations and the second one on the CLT. Notation:

$$\mathcal{H}_\eta = \left\{\varphi : M \to \mathbb{R} \;\middle|\; \exists A > 0 \text{ such that for } \forall\, x,y \in M\ |\varphi(x)-\varphi(y)| \le Ad(x,y)^\eta\right\}.$$

Theorem 6. *([Y 98]) For any η, there exists $\tau < 1$ such that for all $\varphi,\psi \in \mathcal{H}_\eta$ there exists a $C = C(\varphi,\psi)$ such that*

$$\left|\int(\varphi\circ T^n)\psi d\nu - \int\varphi d\nu\int\psi d\nu\right| \le C\tau^n \qquad \forall n \ge 1$$

Theorem 7. *([Y 98]) Assume $\varphi \in \mathcal{H}_\eta$ and $\int\varphi d\nu = 0$. Then*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\varphi\circ T^i \Longrightarrow \mathcal{N}(0,\sigma^2)$$

for some $\sigma \geq 0$. *Moreover,* $\sigma = 0$ *iff* $\varphi = \psi \circ T - \psi$ *for some* $\psi \in L^2(\nu)$. *(Here* $\Longrightarrow$ *denotes weak convergence of probability distributions and* $\mathcal{N}(m, \sigma^2)$ *denotes the normal distribution with mean m and variance* σ^2*.)*

For simplicity, we have formulated these results for one-dimensional random variables, and their extension to vector valued functions is straightforward.

3.3. **Local limit theorem.** For illustrating a local CLT as compared to the widely used (global) CLT, take a simple symmetric random walk (SSRW) on $\mathbb{Z}^d$. So let $W_n = X_1 + \dots X_n$, where $X_1, \dots, X_n, \dots$ are independent, identically distributed random variables with the common distribution $P(X_i = \pm e_j) = \frac{1}{2d}$; $1 \leq j \leq d$ for all $i \in \mathbb{Z}_+$ (here the e_js are the standard unit vectors of $\mathbb{Z}^d$). Then, of course, the CLT says that $P(W_n \in \sqrt{n}A) \to \Phi(A)$ as $n \to \infty$, where $\Phi(A) = \int_A \phi(s)ds$ and $\phi(s) = (2\pi)^{-d/2} \exp(-\frac{s^2}{2})$ is the d-dimensional Gaussian density. In other words it describes the asymptotics of a sequence of sets increasing like $\sqrt{n}$. In contrast, for the SSRW the local CLT (LCLT) says that $n^{d/2} P(W_n = [s\sqrt{n}]) \to \phi(s)$ as $n \to \infty$, i. e. it describes the asymptotics of a sequence of sets of fixed size, consequently it is, indeed, local!

For stating our main theorem we have to fix some notations first. For a fixed $f : X \to \mathbb{R}^d$ denote the average $\int f d\nu = a$, and

$$S_n^f(x) = \sum_{k=0}^{n-1} f(T^k x)$$

the Birkhoff sum. Consider the smallest translated closed subgroup $V + r \subseteq \mathbb{R}^d$ which supports the values of f (V is the group and r is the translation). By ergodicity of all powers of T, the support of S_n is $V + nr$.

Theorem 8. *(*[SzV 03] *Suppose that*

1. (X, T, ν) *is a Young system (cf. subsection 2.1);*
2. f *is minimal: i. e. it is not cohomologous to a function for which the support in the above sense is strictly smaller.*
3. f *is nondegenerate: i. e. span* $\langle V \rangle = \mathbb{R}^d$*, and*
4. f *is bounded and Hölder continuous.*

Let $k_n \in V + nr$ *be such that* $\frac{k_n - na}{\sqrt{n}} \to k$*. Denote the distribution of* $S_n^f - k_n$ *by* υ_n*. Then*

$$n^{d/2} \upsilon_n \to \phi(k) l$$

where ϕ *is a nondegenerate normal density function with zero expectation, and* l *is the uniform measure on* V*: product of counting measures and Lebesgue measures. The convergence is meant in the weak topology.*

Remark 9. *For nonminimal functions we can obtain an analogous result. The limit measure on the right hand side in this case is not necessarily uniform.*

We know from classical analysis or from probability theory that the local behaviour of distributions (densities, measures,…) is connected to the tail behaviour of the corresponding Fourier transforms. Therefore, in the next subsection, we are going to study the Fourier transform of the Perron–Frobenius operator.

3.4. **The Fourier Transform of the Perron–Frobenius Operator.** When one wants to obtain finer results than the CLT, then Fourier transforms seem inevitable. Thus, in our setup, we have to define the Fourier transform of the Perron–Frobenius operator:

$$P_t(\bar{\varphi}) := P(e^{it\bar{f}}\bar{\varphi})$$

where f is the function for which the limit theorem is stated. Note that $P_0 = P$! Also, it is worth noting that $P_t^n \mathbf{1} = P^n(\exp(itS_n^{\bar{f}}))$ and, moreover, $\mathbb{E}P_t^n \mathbf{1} = \mathbb{E}P^n(\exp(itS_n^{\bar{f}})) = \exp(itS_n^{\bar{f}})$, the usual characteristic function of $S_n^{\bar{f}}$ where $S_n^{\bar{f}}$ denotes the Birkhoff sum for the function $\bar{f}$.

The heart of this method is to expand the leading eigenvalue of P_t — analogously to the Taylor expansions around 0 of characterisctic functions of probability theory. Before that, however, we have to ensure its existence. We proved in [SzV 03] that $t \mapsto P_t$ is continuous in the $\mathcal{L}$-norm, and by the stability of the spectrum, for small values of t there exists λ_t, the perturbed value of 1 as an eigenvalue with multiplicity one. By proving the Doeblin–Fortet inequality for P_t we get that the rest of the spectrum will lie in a disk, with radius smaller than 1, so it will not bother λ_t to be the leading eigenvalue.

If the function f is bounded and piecewise Hölder continuous on M, then we get the second order Taylor expansion for the Fourier transform:

$$P_t(\bar{\varphi}) = P(\bar{\varphi}) + itP\left(\bar{f}\bar{\varphi}\right) - \frac{t^2}{2}P\left(\bar{f}^2\bar{\varphi}\right) + o(t^2)\left\|\bar{f}^2\bar{\varphi}\right\|_{\mathcal{L}}.$$

From the assumptions it follows that $\left\|\bar{f}^2\bar{\varphi}\right\|_{\mathcal{L}}$ is finite. By an argument presented in various forms in [Nag 57], [KSz 83] and [GH 88] this leads to the expansion of λ_t.

The philosophy explained above can already be combined with the classical proof of the local CLT sketched in the Appendix. Indeed, term *II* is the same, term *I* corresponds to the CLT. Term *III* can be handled by using the ideas outlined above, while for bounding term *IIII* we have used some compactness arguments borrowed from [AD 01].

4. The Planar Lorentz Process

4.1. **Semidispersing billiards and Lorentz process.** In this subsection we summarize some basic properties of semidispersing billiards. Our aim is to introduce the most important concepts and fix the notation which is essentially borrowed from [KSSz 90]. Semidispersing billiards are more or less hyperbolic dynamical systems with singularities. Pesin's theory was extended to these systems in [KS 86].

A billiard is a dynamical system describing the motion of a point particle in a connected, compact domain $Q \subset \mathbb{T}^d$. The boundary of the domain in assumed to be piecewise C^3-smooth. Inside Q the motion is uniform while the reflection at the boundary ∂Q is elastic. As the absolute value of the velocity is a first integral of motion, the phase space of the billiard flow is fixed as $M = Q \times S^{d-1}$ – in other words, every phase point x is of the form $x = (q, v)$ with $q \in Q$ and $v \in \mathbb{R}^d$, $|v| = 1$. The Liouville probability measure μ on M is essentially the product of the Lebesgue measures, i. e. $d\mu = \text{const.}\,dqdv$. The resulting dynamical system $(M, S^{\mathbb{R}}, \mu)$ is the (toric) *billiard flow*.

Let $n(q)$ denote the unit normal vector of a smooth component of the boundary ∂Q at the point q, directed inwards Q. Throughout the paper we restrict our attention to *semidispersing billiards*: we require for every $q \in \partial Q$ the second fundamental form $K(q)$ of the boundary component to be nonnegative.

The boundary ∂Q defines a natural cross section for the billiard flow. Namely consider

$$\partial M = \{(q, v) \mid q \in \partial Q, \langle v, n(q)\rangle \geq 0\}.$$

This set actually has a natural bundle structure (cf. [BChSzT]). The Poincaré section map T, also called the *billiard map* is defined as the first return map on ∂M. The invariant measure for the map is denoted by μ_1, and we have $d\mu_1 = \text{const.}\,|\langle v, n(q)\rangle|\,dq dv$. Throughout the paper we work with this discrete time dynamical system $(\partial M, T, \mu_1)$. Recall the usual notation: for $(q,v) \in M$ one denotes $\pi(q,v) = q$ the natural projection.

The *Lorentz process* is the natural $\mathbb{Z}^d$ cover of a toric billiard. More precisely: consider $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ the factorisation by $\mathbb{Z}^d$. Its fundamental domain D is a d-dimensional cube (semiopen, semiclosed) in $\mathbb{R}^d$, so $\mathbb{R}^d = \cup_{z\in\mathbb{Z}^d}(D+z)$, where $D+z$ is the translated fundamental domain.

By denoting $\tilde{Q} = \Pi^{-1}Q$, $\tilde{M} = \tilde{Q} \times S^{d-1}$, etc., the Lorentz dynamics is $(\tilde{M}, \{\tilde{S}^t \mid t \in \mathbb{R}\}, \tilde{\mu})$ and its Poincaré section map is $(\partial\tilde{M}, \tilde{T}, \tilde{\mu}_1)$. The *free flight function* $\tilde{\psi} : \partial\tilde{M} \to \mathbb{R}^d$ is defined as follows: $\tilde{\psi}(\tilde{x}) = \tilde{q}(T\tilde{x}) - \tilde{q}(\tilde{x})$. The *discrete free flight function* $\tilde{\kappa} : \partial\tilde{M} \to \mathbb{Z}^d$ is defined as follows: $\tilde{\kappa}(\tilde{x}) = \iota(\tilde{T}\tilde{x}) - \iota(\tilde{x})$, where $\iota(\tilde{x}) = z$ if $\tilde{x} \in Dz$. Observe finally, that $\tilde{\psi}$ and $\tilde{\kappa}$ are invariant under the $\mathbb{Z}^d$ action, so there are ψ and κ functions defined on ∂M, such that $\tilde{\psi} = \Pi^*\psi$ and $\tilde{\kappa} = \Pi^*\kappa$. Actually for our purposes it will be more convenient to choose the fundamental domain in such a way that $\partial\tilde{Q} \cap \partial D = \emptyset$. In this way κ will be continuous.

4.2. **LCLT and recurrence for the $d = 2$ finite horizon case.** Consider the Lorentz process starting from the fundamental cell, a fixed isomorphic version of the fundamental domain. In the domain the starting phase point is random, it is distributed according to the invariant measure of the corresponding torus-billiard. The relative position of the Lorentz particle after the n^{th} collision is S_n^{ψ}, the discrete position is S_n^{κ}. So the event $A_n = (S_n^{\kappa} = 0)$ means that after the n^{th} collision the particle is again in the fundamental domain. Recurrence means that it happens almost surely.

Theorem 10. [Sch 98], [Con 99], [SzV 03] *The planar Lorentz process with a finite horizon is recurrent.*

For the proof we will use a stronger version of the well-known Borel–Cantelli lemma. This version is due to Lamperti:

Lemma 11. *(*[Spi 64]*) If for the sequence of events* $\{A_n\}$

$$\sum_{k=1}^{\infty} \nu(A_k) = \infty$$

and some asymptotic independence holds:

$$\liminf_{n\to\infty} \frac{\sum\limits_{j,k=1}^{n} \nu(A_j A_k)}{\left(\sum\limits_{k=1}^{n} \nu(A_k)\right)^2} < c$$

then A_n happens infinite often almost surely:

$$\nu\left((q,v) \in \partial M \;\middle|\; \exists n_k \to \infty \;\; (q,v) \in \bigcap_{k=0}^{\infty} A_{n_k} \right) = 1$$

To check the first condition we have to deal with the asymptotic probabilities of the events $S_n^{\kappa} = 0$. This is exactly the region of the local limit theorem. Restrict ourselves to the $d = 2$ finite horizon case! This case is known to be a Young system i. e. it satisfies **(P1)–(P8)**, so the first condition of our LCLT is satisfied. For the second condition we need to check the minimality of κ:

Theorem 12. *([SzV 03]) κ is minimal in the class of ψ.*

The proof is quite involved. Surprisingly it is related to arguments in [BChS 91], [BSp 96] and [B 00] for establishing the nondegeneracy of the covariance matrix in the CLT. To prove minimality, for each sublattice $\mathbb{L} \subset \mathbb{Z}^2$ of finite index $n = \mathbb{L} : \mathbb{Z}^2$, we needed a periodic point $n|\mathrm{per}(x)$ such that the Birkhoff sum $S^{\kappa}_{\mathrm{per}(x)}(x) \notin \mathbb{L}$. To find this point we used again Markov properties of Λ. Details can be found in [SzV 03].

Since κ is minimal and the values span the plane it is nondegenerate. Since it is piecewise constant it is also Hölder continuous, so the LCLT applies. This gives $\nu(S^{\kappa}_n = 0) \sim \frac{\mathrm{const}}{n}$, and the sum clearly diverges.

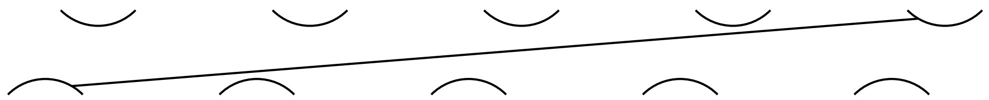
The second condition of Lamperti's Borel–Cantelli lemma contains intersection of recurrence events, so we also had to prove a LCLT for joint distributions.

5. Infinite Horizon

Chernov [Ch 99] observed that the planar infinite horizon Sinai billiard also satisfies **(P1)–(P8)**, so according to theorem 8 the LCLT also holds exactly the same manner as stated for the finite horizon case. Nota bene: for bounded functions $f : M \to \mathbb{R}^d$ satisfying the conditions in Theorem 8 the asymptotics of the Birkhoff sum $\nu(S^f_n = 0) \sim \frac{c}{n^{d/2}}$.

But the ψ or the κ free flight functions are not covered by this theorem since here κ is not bounded, ψ is even not Hölder. This is not a surprise, since in this case a new phenomenon shows up. Former heuristic arguments by Bleher [B 92] already suggested to expect that the moving particle is superdiffusive namely $\frac{S^{\psi}_n}{\sqrt{n \log n}}$ will have a limit distribution. The reason is the following:

The infinite horizon condition is equivalent to the existence of collision free orbits in the phase space M. These orbits form corridors in $\mathbb{R}^2$. Large free flights can occur by "crossing" one of these corridors. The smaller the angle with the direction of the corridor is, the longer is the free flight.

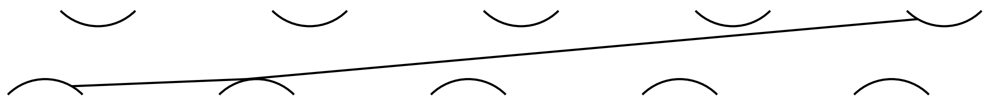


Computing the invariant measure for the small angle sets gives the asymptotics $\nu(|\kappa| = u) \sim \frac{c}{u^3}$. This means that $\int |\kappa|^2 d\nu = \infty$, but any power with smaller exponent is integrable. As a matter of fact, the distribution of κ is in the nonnormal domain of attraction of the normal law, and its Fourier transform is:

$$\hat{\kappa} \stackrel{\mathrm{def}}{=} \int e^{it\kappa} d\nu = 1 + c|t|^2 \log|t| + O(|t|^2)$$

(where c is a constant matrix) which means that if we would add independent copies of the same distribution then $\frac{S^*_n}{\sqrt{n \log n}}$ would tend to a gaussian law. The log factor comes from the fact that the truncated $\kappa^x = \kappa \mathbf{1}_{|\kappa| \le x}$ has a variance of order $\log x$.

Direct geometrical calculations show that, when κ is large, then the order of $\kappa \circ T$ is between $\sqrt{\kappa}$ and κ^2. (There is a case when the trajectory hits the neighbouring scatterer on the same side of the corridor before "crossing".



In this case we change the Poincaré section and consider the sum of the small and the necessarily large free flight vector as $\kappa \circ T$.) An important observation is that typically the next free flight will be in the regime of $\sqrt{\kappa}$. More precisely, for any $\delta > \frac{1}{2}$ the probability $\nu(\kappa \circ T > u^{\delta} \mid \kappa = u) \to 0$ as $u \to \infty$. Even more the conditional expectation $\mathbb{E}_{\nu}(\kappa \circ T \mid \kappa)$ is of order $\sqrt{\kappa}$. This means that though $|\kappa|^2$ is not integrable, nevertheless, $\int \kappa(\kappa \circ T) d\nu < \infty$. The hope that the autocorrelation may have a fast decay has lead to the conjecture that S_n^{κ} asymptotically behaves the same way as the sum of independent copies of κ.

5.1. **Limit theorems: global and local, and recurrence for the $d = 2$ infinite horizon case.** To reach the aforementioned limit theorem, and moreover the local limit theorem for κ we used the symbolic space $\bar{\Delta}$ as constructed in [Y 98], and investigated carefully how and where large values of $\bar{\kappa}$ appear on the tower. Clearly κ is bounded on Λ. Since in one step it can grow at most to it is square, or shrink to it is square root the time needed to reach the $\kappa = u$ set from Λ is about $c \log\log u$, and before returning to Λ also the same amount of time is needed. By **(P7)** it is immediate that both the measure of phase points, which spend k iterates in the corridor before returning to the base of the tower, and both the measure of phase points visiting the corridor k times before returning to the base is exponentially small in k.

We had to replace the function norms in the definition of $\mathcal{C}$ and $\mathcal{L}$ to refer to $\bar{\kappa}$ on the tower. Instead of Young's $e^{-\varepsilon l}$ factor we used

$$\prod_{k=1}^{l} \min\left(e^{-\varepsilon}, \bar{\kappa}^{-\delta} \circ \bar{F}^{-k}\right).$$

Observe, that if κ is bounded and δ is small enough this gives back $e^{-\varepsilon l}$. In that way we managed to achieve that $t \mapsto P_t$ be a countinuous mapping in both function norms. This was not the case with the original norms, if the horizon is infinite. So P_t can be considered as a perturbation of P, and this also extends to the leading eigenvalue λ_t.

To obtain that the asymptotic behaviour is the same as in the case of the sum of independent copies, we needed to show that

$$\lambda_t = 1 + c|t|^2 \log|t| + O(|t|^2)$$

has the same kind of behaviour as the dynamically untouched $\hat{\kappa}$. This is the key of the proof. While sums of independent copies give the product Fourier transform $\hat{\kappa}^n$, that of the Birkhoff sums give $\int P_t^n(\mathbf{1}) d\nu$. The operator can be approximated by $\lambda_t^n \bar{\nu}$ (here $\bar{\nu}$ is the leading projection operator) up to an exponentially small error term coming from the rest of the spectrum. Summarising: if $\hat{\kappa}(t)$ and λ_t behave the same way as $t \to 0$, then the sum of independent copies and the Birkhoff sum S_n^{κ} behave the same way as $n \to \infty$.

There is a quite involved proof of the λ_t expansion which is based on correlation estimates of powers of the truncated $\bar{\kappa}$ and the eigenfunction related to λ_t. Details will appear in a technical paper.

Theorem 13. *Suppose that the direction vectors of infinite collision free flights span the plane. Let $A \subset \mathbb{R}^2$. Then*

$$\nu(S_n^\kappa \in \sqrt{n \log n} A) \to \int_A \phi(k) dk$$

where ϕ is a nondegenerate gaussian density with zero expectation.

Remark 14. *The problem of the limiting behaviour of displacements in the case of an infinite horizon has raised the interest of several people using different methods (very interesting works are* [B 92] *and* [ZE 97]*). It is worth mentioning that the computational method of* [ZE 97] *forecasts a non-Gaussian limit under the same scaling. .*

Theorem 15. *Suppose that the direction vectors of infinite collision free flights span the plane. Let $k_n \in \mathbb{Z}^2$ such that $\frac{k_n}{\sqrt{n \log n}} \to k$. Then*

$$n \log n \ \ \nu(S_n^\kappa = k_n) \to \phi(k)$$

where ϕ is a nondegenerate gaussian density with zero expectation.

ϕ depends only on the corridor geometry. When computing the covariance one considers only the directions and widths of corridors, and the bounding points of the corridors. This is a finite set of points on the scatterers, the geometry of this finite set, and the curvature of the scatterers at these points are involved, but nothing else. The sum of $\frac{1}{n \log n}$ diverges so the recurrence follows using an analogous argument as for the finite horizon case. In the case when all corridors are parallel one has to apply a nonisotropic scaling.

Theorem 16. *Suppose, that all collision-free flights in the plane are parallel to the unit vector w. Consider the linear transformation B_n which has the matrix $\begin{pmatrix} \sqrt{n \log n} & 0 \\ 0 & \sqrt{n} \end{pmatrix}$ in the basis $w, w^\perp$. Let $k_n \in \mathbb{Z}^2$ such that $B_n^{-1} k_n \to k$. Then*

$$\det B_n \ \ \nu(S_n^\kappa = k_n) = n\sqrt{\log n} \ \ \nu(S_n^\kappa = k_n) \to \phi(k)$$

where ϕ is a nondegenerate gaussian density with zero expectation.

The sum of $\frac{1}{n\sqrt{\log n}}$ is also divergent, so recurrence is also obtained in this case.

Remark 17. *It is worth noting that in a recent manuscript of Gouëzel* [G 02] *a related problem was investigated for $1-D$ piecewise expanding maps with a neutral fixed point. The behaviour of billiard orbits in corridors is analogous to that of orbits of these $1-D$ maps near the neutral fixed point. Essential difficulties in our case arise from a) the larger dimension of the space; b) the not quite explicit form of the billiard dynamics; and c) emphatically from the fact that in our case the function of interest is unbounded with a quite long tail whereas in* [G 02] *it is bounded. His setup is more general since he is also considering stable limit laws in general, like* [AD 01]*. The restriction of our interest to the nonnormal domain of attraction of the Gaussian law came from the fact that our main concern was the free flight function.*

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APPENDIX: A CLASSICAL LOCAL CLT

Here we recall, in the simplest case, the classical proof of Gnedenko [G 48] of a local CLT.

For simplicity, consider the case $d = 1$. Following the notations of subsection 3.3, our goal here is to prove that, as $n \to \infty$,

$$\sqrt{n}P(W_n = k_n) \to \phi(k)$$

if $\frac{k_n}{\sqrt{n}} \to k$. Heuristically one expects that

$$\sqrt{n}P(W_n = k_n) = \sqrt{n}P(\frac{W_n}{\sqrt{n}} = \frac{k_n}{\sqrt{n}}) \approx \sqrt{n}\phi(\frac{k_n}{\sqrt{n}})\frac{2}{\sqrt{n}} = \frac{1}{\pi}\int \exp(-is\frac{k_n}{\sqrt{n}})\gamma(s)ds$$

where $\gamma(s) = \exp(-\frac{s^2}{2})$ is the standard gaussian characteristic function. Here we used the gaussian approximation suggested by the CLT and the Fourier inversion formula.

For a proof, let us turn to characteristic functions of W_n. Denote by $\xi(t)$ the common characteristic function of the variables X_i. Then

$$\sqrt{n}P(W_n = k_n) = \sqrt{n}\frac{1}{2\pi}\int_{|t|\le\frac{\pi}{2}} \exp(-itk_n)\xi^n(t)dt$$

and by substituting $s = t\sqrt{n}$, this is equal to

$$= \frac{1}{2\pi}\int_{|s|\le\sqrt{n}\frac{\pi}{2}} \exp(is\frac{k_n}{\sqrt{n}})\xi^n(\frac{s}{\sqrt{n}})ds$$

We emphasize that it is fundamental to precisely know the support of the values of the variables X_i (i. e. the minimal lattice containing these values) since the form of the inversion formula depends on this. Moreover, the fact that the X_is take their values on a lattice of span 2 was used in our first heuristic formula, too. (This information is encapsulated in the minimality condition of Theorem 7.) To prove our desired statement we write

$$\left|2\pi\left[\sqrt{n}P(W_n = k_n) - 2\phi(\frac{k_n}{\sqrt{n}})\right]\right|$$

$$\le \int_{|s|\le A}\left|\xi^n(\frac{s}{\sqrt{n}}) - \gamma(s)\right|ds + \int_{|s|\ge A}\gamma(s)ds + \int_{A\le|s|\le\varepsilon\sqrt{n}}\left|\xi^n(\frac{s}{\sqrt{n}})\right|ds + \int_{\varepsilon\sqrt{n}\le|s|\le\sqrt{n}\frac{\pi}{2}}\left|\xi^n(\frac{s}{\sqrt{n}})\right|ds$$

$$= I + II + III + IIII$$

For making the right hand side sufficiently small, we will first select A to be sufficiently large and then ε sufficiently small. Thus II can be made arbitrarily small, and, for fixed A, I will also be small by the CLT (as a matter of fact, the smallness of I would also follow from our forthcoming argument for handling III. In fact, III is a quite interesting term. By expanding $\xi(t)$ in a power series in the neighbourhood of 0, one can easily see that, if ε is sufficiently small, then for $|s| \le \varepsilon$ one has

$$|\xi^n(\frac{s}{\sqrt{n}})| \le 1 - \frac{s^2}{4} \le \exp(-\frac{s^2}{4})$$

As a consequence one obtains

$$III \le \int_{A\le|s|\le\varepsilon\sqrt{n}} \exp(-\frac{s^2}{4})ds \le \int_{A\le|s|} \exp(-\frac{s^2}{4})ds$$

and this term is also small if A is large. Finally, the smallness of $IIII$ follows from the fact that, in the interval $\varepsilon\sqrt{n} \leq |s| \leq \sqrt{n}\frac{\pi}{2}$, the term $\left|\xi(\frac{s}{\sqrt{n}})\right|$ is uniformly bounded away from 1 from above, and consequently for the integrand in $IIII$ we have an exponentially collapsing upper bound.

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