SYMPLECTIC STRUCTURES AND CHIRAL FORMULATIONS OF EINSTEIN-MATTER EQUATIONS

DR LEE MCCULLOCH-JAMES

Abstract. Equivalent chiral Lagrangians for (complex) Einstein equations are collected and related by their identical (infinite dimensional) symplectic structures. Viewing Plebanski's Lagrangian, which employs an $sl(2, C)$-valued field variable alternative to the tetrad as a constrained BF theory, extensions to other (super) gauge groups follow naturally. Accordingly, scalar and spinor matter field couplings to an Einstein-Maxwell Lagrangian employing $gl(2, C)$-valued field variables are investigated. Employing instead, spin fields as dynamical variables and symplectic techniques a succinct derivation of the Witten-Nester two-form as a Noether charge is realised.

1. Introduction

The renewed interest in re-formulating Einstein's field equations has been stimulated by influences such as the developments in the study of gauge theories, the construction of half-flat solutions in the 1980's by, for example, Penrose, Newman and Plebanski [?] and the recasting of the Hamiltonian formulation of general relativity in terms of new variables by Ashtekar [?]. The latter, itself a response to the first two influences, reintroduced the idea of regarding the connection and bases of two forms as primary dynamical variables with the metric a secondary derived variable. The novelty of such schemes is the focus on so-called complex chiral actions and complex versions of Einstein's equations.

Of these investigations Plebanski's, [?] formulation in the mid 1970's of a chiral first order variational principle for general relativity in which the basic field variables are $sl(2, C)$ valued two-form, connection form and a spinor-valued zero-form is of particular interest. The chiral nature of the formulation is in the sense that local Lorentz representations involve only $SL(2, C)$ and not its conjugate, $\overline{SL(2, C)}$. Viewed as a 'constrained' BF theory this lends itself to natural generalisations as in the Einstein-Maxwell theory of Robinson, [?] that employs $sl(2, C) \oplus C$ field variables. Constrained here is in the sense that a multiplier term is needed (spoiling the precise BF form of the Lagrangian) to ensure that the basic two form field variable is derived from a co-frame. Chiral N=1 and N=2 Supergravity theories, [?], [?], [?] amongst others have been formulated. Such chiral formulations lead to complex vacuum field equations for a complex metric with the real theory recovered only upon imposing (by hand) reality conditions. Included in subsequent developments have been chiral Lagrangian formulations, in which the primary variables are spin $\frac{3}{2}$ fields and $sl(2, C)$ valued connections, [?, ?, ?]. Goldberg, [?] using a Witten-Nester two form associated to the anti-self-dual part of the Levi-Civita connection

Key words and phrases. two spinors, Noether charge, symplectic, Witten-Nester form, BF theory, Einstein-Maxwell Lagrangian, Grassman, Plebanski,Weyl spinor, chiral.

derived also semi-chiral Lagrangian.

Einstein's field equations as Euler-Lagrange equations for some Lagrangian field theory have associated Noether identities so nothing is gained by looking at their integrability conditions that are automatically satisfied due to their associated Bianchi identities. Accordingly in a complementary development it has proven useful to view Einsteins equations in another way asking for equations that have Einstein's field equations themselves as integrability conditions. Penrose, [?] and his collaborators' twistorial approach to Einstein's vacuum field equations has focused on the spin $\frac{3}{2}$ zero rest-mass field equations which can have the Einstein vacuum equations, with or without cosmological constant, as integrability conditions.

The structure of this article explores certain aspects of these investigations by deploying a phase space construction of Crnkovic and Witten [?]. Using their invariant symplectic 2-form associated to the phase space of solutions to the equations of motion one may readily relate the various chiral Lagrangian formulations. In this scheme, the (pre) symplectic potential is transcribed as the boundary term that results from the 'integration by parts' when implementing the variation of the Lagrangian, while the (pre)symplectic 2-form structure is its functional exterior derivative. In general this two-form is degenerate and motions along these degenerate directions correspond to gauge transformations of the theory, [?]. It is possible however to quotient this phase space by the integral manifolds of the degenerate directions and obtain a natural (non-degenerate) symplectic structure on the resulting phase space, [?].

In Chapter 2 a presentation of the Einstein-Cartan equations in terms of two spinor valued three form equations introduces the notation. Further a clarification of the semi-chiral Lagrangian formulation of the Einstein-Weyl equations, [?] is made with a view to applying the results to the work of Robinson, [?].

The recent proliferation of formulations of vaccuum Einstein equations has been reviewed by Peldan, [?] emphasising the Hamiltonian aspects of the theories. In Chapter 3 the Lagrangian formulations are collected and related by their identical symplectic structures, the general theory of which is also reviewed.

Work on chiral variational principles has been extended to include various matter fields by Capovilla et al [?] and Pillin, [?]. Following their prescriptions, the effects of coupling (charged) Higgs and fermionic matter fields to the $GL(2,C)$ formulation of electromagnetism and gravity by Robinson, [?] are presented in Chapter 4. In doing so the necessity to restrict the gauge group to $SL(2,C) \otimes U(1)$ in order to recover the real theory is clarified, as is how the formulation is distinct from those theories which use a linear connection to incorporate Maxwell, [?]. The notion of a spinor density is employed to describe the charged complex (Higgs) scalar field and the theory as developed previously by Plebanski, [?] is summarised in the appendix.

In Chapter 5, by employing symplectic techniques and the chiral Lagrangian of Jacobson et al, [?] its associated quasi-local charges are derived and their relation to the Witten-Nester forms, [?] is clarified thus providing an alternative derivation