An interacting multielectron Anderson model

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Abstract

This article is a first tiny step towards a rigorous description of an interacting multielectron system in a random potential of Anderson type. Deterministic spectrum and the Wegner estimate for this model are proven.

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1 Introduction

This paper combines some techniques known from multiparticle Schrödinger operators with the random operator techniques for a single electron. We consider n interacting electrons moving under the influence of an additional random perturbation of Anderson type in the d-dimensional configuration space $\mathbb{R}^d$.

1.1 The interacting multielectron model

The Hamiltonian is of the form

$$H(\omega) := -\triangle + V_I - V_n(\omega), \tag{1.1}$$

where the Laplacian $-\triangle$ on $\mathbb{R}^{nd}$ describes the free kinetic energy of the n electrons.

$$V_I(x_1, ..., x_n) := \sum_{1 \leq k < l \leq n} v(x_k - x_l) \tag{1.2}$$

is a sum of repulsive two body interaction potentials $v(x) = v(-x) \geq 0$. We assume that V_I is infinitesimally $-\triangle$-bounded in $L^2(\mathbb{R}^{nd})$, like in the case $V_I = V_C$, where

$$V_C(x_1, ..., x_n) := \sum_{1 \leq k < l \leq n} \frac{1}{|x_k - x_l|} \tag{1.3}$$

is the repulsive Coulomb potential generated by the electrons. The random potential $V_n(\omega)$ should describe the interaction of n electrons with a potential created by a randomly chosen configuration (denoted by ω) of atoms or ions, hence it is written as

$$V_n(\omega) = \sum_{k=1}^{n} \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(k-1)} \otimes V_1(\omega) \otimes \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(n-k)}, \tag{1.4}$$

or $V_n(\omega, x_1, ..., x_n) = \sum_{k=1}^{n} V_1(\omega, x_k)$, where the one particle random potential $V_1(\omega)$ models the interaction of a single electron with the potential created by the configuration ω of atoms. Before we comment on the special form $V_1(\omega)$ takes in the Anderson model, we construct the probability space Ω, collecting all configurations ω of atoms under consideration:

1. Take a probability measure $\mathbb{P}_0$ on the Borel sets $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, such that $\mathbb{P}_0$ is supported in a finite interval.

2. Take a probability measure $\mathbb{P}_1$ on $\mathcal{B}(\mathbb{R}^d)$ and $R \in]0, \frac{1}{2}[$, such that

$$\mathbb{P}_1(\{x \in \mathbb{R}^d : |x| < R\}) = 1 \tag{1.5}$$

3. Define the probability space

$$\Omega := ((\mathbb{R} \times \mathbb{R}^d)^{\mathbb{Z}^d}, \mathcal{B}(\Omega), (\mathbb{P}_0 \otimes \mathbb{P}_1)^{\otimes \mathbb{Z}^d}) \tag{1.6}$$

as the product of the probability spaces. $\mathcal{B}(\Omega)$ is taken as the completion with respect to $\mathbb{P} = (\mathbb{P}_0 \otimes \mathbb{P}_1)^{\otimes \mathbb{Z}^d}$-zero-sets of the product σ-algebra.

Denote by ω' and ω'' the images of ω under the canonical projections $pr' : \Omega \longrightarrow \Omega' := (\mathbb{R}^{\mathbb{Z}^d}, \mathbb{P}_0^{\otimes \mathbb{Z}^d})$ and $pr'' : \Omega \longrightarrow \Omega'' := ((\mathbb{R}^d)^{\mathbb{Z}^d}, \mathbb{P}_1^{\otimes \mathbb{Z}^d})$ and by ω_j' and ω_j'', $j \in \mathbb{Z}^d$ the image of ω under the projections $pr_j' : \Omega \longrightarrow \mathbb{R}$ and $pr_j'' : \Omega \longrightarrow \mathbb{R}^d$, we write $\omega = (\omega', \omega'') = (\omega_j', \omega_j'')_{j \in \mathbb{Z}^d}$. Thus the random potential

$$V_1(\omega, x) := \sum_{j \in \mathbb{Z}^d} \omega_j' \wp(x - j - \omega_j'') \tag{1.7}$$

of Anderson type, acting in $L^2(\mathbb{R}^d)$ describes the interaction of a single electron with the atoms of a non crystalline solid. The random variables pr'_j and pr''_j determine the sort of atom and the displacement of the atom at lattice site $j \in \mathbb{Z}^d$. For a given realization $\omega \in \Omega$ the kind of atom at site $j \in \mathbb{Z}^d$ is given by ω'_j and the displacement at site j by ω''_j. Note, that due to assumption $|\omega''_j| \leq \frac{1}{2}$, we can loosely speak of an atom at site $j \in \mathbb{Z}^d$ even if $\omega''_j \neq 0$, because the is no doubt, to which lattice site $j \in Z^d$ an atom at $j + \omega''_j$ belongs to. We remark, that the product structure of Ω, we have assumed, is an equivalent way of getting independent, identical distributed families of random variables for sort and displacement of atoms - but this formulation is more convenient in the following proofs. As a consequence of the identical distribution of $(pr'_j)_{j\in\mathbb{Z}^d}$, the single site potential $\wp$ must satisfy a decay estimate at ∞, otherwise the series defining $V_1(\omega, x)$ does not converge. To ensure convergence and to exclude $V_1 = 0$, we will assume:

1. $\wp \geq 0$
2. There is a $c_0 > 0$, such that $c_0 1^0_{1+2R} \leq \wp$ holds, where 1^x_l denotes the characteristic function of a box $\Lambda(x, l)$ of sidelength l centered at x.
3. There are $c_1 < \infty$ and $\varsigma > d$, such that
$$\wp(x) \leq c_1 \langle x\rangle^{-\varsigma} := c_1(1 + |x|^2)^{-\frac{\varsigma}{2}} \tag{1.8}$$

Under these assumptions $V_1(\omega)$ is a bounded potential $\mathbb{P}$-almost everywhere (the displacements are bounded, so the series converges, see [Ze] Lemma 1.1), hence $|\mathcal{V}_1| := \mathbb{P} - \operatorname{ess\,sup}_{\omega\in\Omega} \|V_1(\omega)\|_{L^\infty}, |\mathcal{V}_n| := \mathbb{P} - \operatorname{ess\,sup}_{\omega\in\Omega} \|V_n(\omega)\|_{L^\infty} < \infty$.

1.2 Which phenomena in random media are due to interacting electrons?

A motivation for the study of this interacting n-electron model in an Anderson background comes from the discovery of "universality classes" in the measurement of the conductivity σ of doted semiconductors at low temperatures. Most of the experimental results can be fitted very well to a function of the type
$$\sigma(n) = \begin{cases} 0 & \text{if } n \leq n_c \\ \sigma_c(\frac{n}{n_c} - 1)^\nu & \text{if } n \geq n_c \end{cases}$$
where n is the donator concentration and σ_c, n_c are fit parameters. For experimental data see e.g. [RATB] on P doted Si or [KKSK] on Si doted $Al_{0,3}Ga_{0,7}As$. The astonishing fact about the exponent ν is, that in all these

experiments it could be chosen to be roughly 1 or $\frac{1}{2}$. A heuristic explanation of the first class $\nu \sim 1$ argues, that one can neglect electron-electron interactions and approximate this situation in the well known Anderson model for one electron. In contrast the $\nu \sim \frac{1}{2}$ case should be an effect of the electron-electron interaction.
The experimental results mentioned above are interpreted as a metal-insulator transition for electrons in three dimensional random media. But for electrons confined to a two dimensional space it is still an open question if there is such a transition and if this is an effect of interaction.

1.3 Rigorous results

A rigorous proof of this behavior of the conductivity in disordered systems in the framework of the Anderson model is far away from the present techniques. The detection of absolutely continuous spectrum in the Anderson model as a sign for nonzero conductivity remains an open problem, which has been mentioned in Barry Simon's list of problems in mathematical physics [Si] in the twenty-first century. In the one electron Anderson model deterministic spectrum and localization for low energies have been proven, see [St] for a summary of recent results. The crucial ingredient of current localization proofs in the continuous Anderson model is an estimate of the norm of localized resolvents arising from a multiscale analysis procedure. The starting point for multiscale analysis is Fröhlich and Spencer's paper [FrSp] for discrete Anderson models in $l^2(\mathbb{Z}^d)$, which is made applicable for continuous Anderson models in $L^2(\mathbb{R}^d)$ by Combes, Hislop and Kirsch, Stollmann, Stolz [CoHi], [KSS1] and [KSS2]. An important ingredient in the multiscale analysis is the Wegner estimate, serving as an a priori bound.
In contrast to the one electron models, there are at least to my knowledge no rigorous results for the multielectron Anderson model. In the next section we check some basic properties of the multielectron Anderson model, which enable us to apply the general theory of ergodic operator families, implying deterministic spectrum. Chapter 3 lists the differences between one and many electron model in a proof of Wegner's estimate.

2 Selfadjointness, ergodicity and deterministic spectrum of H

In this section we will prove the existence of nonrandom sets Σ, Σ_c, Σ_{ac},... such that $\Sigma = \sigma(H(\omega))$, $\Sigma_c = \sigma_c(H(\omega))$,... for $\mathbb{P}$-almost every $\omega \in \Omega$. This goal is achieved by use of the theory of ergodic operator families. A first step is the proof of selfadjointness for the operators $H(\omega)$ $\mathbb{P}$-almost everywhere.

Theorem 2.1. *For $\mathbb{P}$-almost every realization $\omega \in \Omega$ the Hamiltonian*

$$H(\omega) = -\triangle + V_C - V_n(\omega) \tag{2.1}$$

is bounded from below, selfadjoint on $\mathcal{D}(-\triangle)$ and $C_0^\infty(\mathbb{R}^{nd})$ is a core for $H(\omega)$.

Proof. In multiparticle Schrödinger theory, see e. g. [HuSi], it is well known, that the Coulomb potential $V_C = \sum_{1 \le k < l \le n} \frac{1}{|x_k - x_l|}$ is a Kato potential, i.e. for each $\alpha > 0$ there is a $\beta(\alpha) < \infty$, such that $\|V_C \Psi\| \le \alpha \|-\triangle \Psi\| + \beta(\alpha)\|\Psi\|$ for each $\Psi \in C_0^\infty(\mathbb{R}^{nd})$, but the closure $H_0^2(\mathbb{R}^{nd})$ of $C_0^\infty(\mathbb{R}^{nd})$ with respect to the H^2-norm equals $H^2(\mathbb{R}^{nd})$, see [Wl] chap. 3.2, so this extends to

$$\|V_C \Psi\| \le \alpha \| - \triangle \Psi\| + \beta(\alpha)\|\Psi\| \tag{2.2}$$

for all $\Psi \in \mathcal{D}(-\triangle) = H^2(\mathbb{R}^{nd})$. From (2.2) as an application of the Kato-Rellich theorem, we see, that $H_0 = -\triangle + V_C \ge -\min_{\alpha > 0} \max\{\frac{\beta(\alpha)}{1-\alpha}, \beta(\alpha)\}$ is selfadjoint on $\mathcal{D}(-\triangle)$ and $C_0^\infty(\mathbb{R}^{nd})$ is a core for H_0. The assumptions on the support of $\mathbb{P}_0$ and $\mathbb{P}_1$ and the decay assumptions on $\wp$ imply, that there is a $|\mathcal{V}_n| < \infty$, such that $\|V_n(\omega, \cdot)\|_{L^\infty} \le |\mathcal{V}_n| < \infty$, $\mathbb{P}$-almost everywhere. Hence $H(\omega) = H_0 - V_n(\omega)$ is bounded from below, selfadjoint on $\mathcal{D}(-\triangle)$ and $C_0^\infty(\mathbb{R}^{nd})$ is a common core for $\mathbb{P}$ almost every $\omega \in \Omega$. □

The same theorem and proof applies for the potentials V_I from (1.2), when we assume that V_I is infinitesimally $-\triangle$-bounded.

Theorem 2.2. *There are closed sets $\Sigma = \Sigma_{\mathrm{ess}}$, Σ_{ac}, Σ_{sc}, Σ_{pp}, Σ_{c}, $\Sigma_{\mathrm{s}} \subseteq \mathbb{R}$, such that $\Sigma = \sigma(H(\omega)) = \sigma_{\mathrm{ess}}(H(\omega))$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H(\omega))$, $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H(\omega))$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H(\omega))$, $\Sigma_{\mathrm{c}} = \sigma_{\mathrm{c}}(H(\omega))$ and $\Sigma_{\mathrm{s}} = \sigma_{\mathrm{s}}(H(\omega))$ for $\mathbb{P}$-almost every $\omega \in \Omega$.*

Proof. The result of Theorem 2.1 establishes measurability of the operator H by [CaLa] Proposition V.3.1 or [St] Example 1.2.7. The next ingredient is a family of ergodic transformations on Ω, which we choose in the following way: The product structure of Ω immediately shows

$$\lim_{k \to \infty} \mathbb{P}(\theta_j^{-k} A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \tag{2.3}$$

for all $j \in \mathbb{Z}^d \backslash \{0\}$ and all cylinder sets A, B in $\mathcal{B}(\Omega)$, where $\theta_j : \Omega \longrightarrow \Omega$ is the measure preserving transformation defined by

$$\theta_j((\omega_k)_{k\in\mathbb{Z}^d}) = (\omega_{j+k})_{k\in\mathbb{Z}^d}. \tag{2.4}$$

Equality (2.3) extends to all $A, B \in \mathcal{B}(\Omega)$, cf. [Wa] Theorem 1.17, hence it is the mixing property, which implies ergodicity of the system $(\Omega, \mathbb{P}, (\theta_j)_{j\in\mathbb{Z}^d})$. Apart from the ergodic transformations $(\theta_j)_{j\in\mathbb{Z}^d}$, we have to find a family $(\mathcal{T}_j)_{j\in\mathbb{Z}^d}$ of unitary operators in $L^2(\mathbb{R}^{nd})$, such that the "covariance condition"

$$H(\theta_j\omega)f = \mathcal{T}_j^* H(\omega)\mathcal{T}_j f \tag{2.5}$$

holds for each element f in the common core $C_0^\infty(\mathbb{R}^{nd})$ of the operators $H(\omega)$, $\omega \in \Omega$, see [CaLa] Remark V.2.6 and Proposition V.2.7. We choose $\mathcal{T}_j$ as the unitary operators on $L^2(\mathbb{R}^{nd})$, which are given by

$$(\mathcal{T}_j f)(x_1, ..., x_n) := f(x_1 - j, ..., x_n - j) \tag{2.6}$$

for each $f \in C_0^\infty(\mathbb{R}^{nd})$. Thus $(\mathcal{T}_j)_{j\in\mathbb{Z}^d}$ is a unitary representation of $\mathbb{Z}^d$ in $L^2(\mathbb{R}^{nd})$ which is total, see [CaLa] page 247. The operators $\mathcal{T}_j$ are tensor products of the translation operators T_j in $L^2(\mathbb{R}^d)$ given by $(T_j g)(x) := g(x - j)$ for $x \in \mathbb{R}^d$ and $g \in C_0^\infty(\mathbb{R}^d)$. So

$$\begin{aligned}
\mathcal{T}_j^* V_n(\omega)\mathcal{T}_j &= (T_j^*)^{\otimes n}\left[\sum_{k=1}^{n} \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(k-1)} \otimes V_1(\omega) \otimes \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(n-k)}\right] T_j^{\otimes n} \\
&= \sum_{k=1}^{n} \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(k-1)} \otimes T_j^* V_1(\omega) T_j \otimes \mathbf{1}_{L^2(\mathbb{R}^d)}^{\otimes(n-k)}
\end{aligned} \tag{2.7}$$

and the standard calculation

$$\begin{aligned}
(V_1(\theta_j\omega)g)(x) = V_1(\theta_j\omega, x)g(x) &= \sum_{k\in\mathbb{Z}^d} \omega'_{k+j}\wp(x - k - \omega''_{k+j})g(x) = \\
&= \sum_{k\in\mathbb{Z}^d} \omega'_{k+j}\wp((x+j) - (k+j) - \omega''_{k+j})g(x) = V_1(\omega, x+j)g(x) = \\
&= (V_1(\omega)T_j g)(x+j) = (T_j^* V_1(\omega) T_j g)(x)
\end{aligned}$$

for each $x \in \mathbb{R}^d$ and $g \in C_0^\infty(\mathbb{R}^d)$ in the one electron case proves

$$\mathcal{T}_j^* V_n(\omega)\mathcal{T}_j f = V_n(\theta_j\omega) f \tag{2.8}$$

for $f \in C_0^\infty(\mathbb{R}^{nd})$. From the invariance of the pair interaction potential with respect to simultaneous translations, i.e. $V_I(x_1, ..., x_n) = V_I(x_1 - x, ..., x_n - x)$ for each $x \in \mathbb{R}^d$ we conclude

$$\begin{aligned}
(\mathcal{T}_j^* V_I \mathcal{T}_j f)(x_1, ..., x_n) &= V_I(x_1 + j, ..., x_n + j)(\mathcal{T}_j f)(x_1 + j, ..., x_n + j) = \\
&= V_I(x_1, ..., x_n) f(x_1, ..., x_n).
\end{aligned}$$

Translation invariance of the Laplacian now finally implies the covariance condition (2.5), so we may apply ergodic operator family theory; this finishes the proof, see [CaLa] chap. V.2 or [St] chap. 1.2. $\square$

Theorem 2.2 allows us to speak of Σ as the deterministic spectrum of H. In the same way the essential, continuous, absolute continuous, singular continuous, singular and pure point spectrum are deterministic. Note that there are no isolated eigenvalues of finite multiplicity and that the eigenvalues of H depend on the realizations ω, but the pure point spectrum of H, which is the closure of the set of eigenvalues, is deterministic.

3 The Wegner estimate

Let $\Lambda \subseteq \mathbb{R}^{nd}$ be an open bounded cube, let $H_\Lambda(\omega)$ denote the restriction of $H(\omega)$ to Λ with Dirichlet boundary conditions and $(\pi_\Lambda(A,\omega))_{A\in\mathcal{B}(\mathbb{R})}$ the associated spectral measure. Then Wegner's estimate

$$\mathbb{E}(\operatorname{Trace}\pi_\Lambda(]E_0-\eta,E_0+\eta[)) \leq C_W\eta|\Lambda|$$

bounds the expectation value of the trace of the spectral projection of H_Λ onto the interval $]E_0-\eta,E_0+\eta[$ by the length of the interval and the volume $|\Lambda|$ of the cube. The proof in [CoHi] in the one electron model we want to adapt uses a spectral averaging procedure plus Dirichlet Neumann bracketing. So we have to assume, that there is a density $g \in L^\infty(\mathbb{R})$ for the probability distribution of $\mathbb{P}_0$, i.e.

$$\mathbb{P}_0(]-\infty,t[) = \int_{-\infty}^{t} g(x)dx. \tag{3.1}$$

This assumption for $\mathbb{P}_0$, which seems rather strange from physical point of view is the price one has to pay - even in the single electron Anderson model- for a proof of Wegner's estimate with a volume dependent bound proportional to $|\Lambda|$. This type of spectral averaging works with a family of operators $H(\mu) = H_0 - \mu V, \mu \in \mathbb{C}$, with a selfadjoint operator H_0 and a bounded $V \geq 0$. We use the spectral averaging from section 3.2 by putting kinetic and electron-electron interaction terms of $H_\Lambda(\omega)$ into the operator H_0 of spectral averaging plus a ω-dependent decomposition of the Anderson random potential $V_n(\omega)$ into $-\mu V$ and H_0 parts. To be sure that we are dealing with well defined operators, we look first at the restrictions of $-\triangle + W$ to an open bounded cube Λ with Dirichlet and Neumann boundary conditions, where W is an infinitesimally $-\triangle$ bounded perturbation.

3.1 Restricting the nonrandom Hamiltonian H_0

The potential

$$V_C = \sum_{1\le k<l\le n} \frac{1}{|x_k - x_l|} \tag{3.2}$$

acting in $L^2(\mathbb{R}^{nd})$ describes the repulsive Coulomb interaction of n electrons. In Theorem 2.1 we have defined $H_0 = -\triangle + V_C$ as a selfadjoint operator on $\mathcal{D}(-\triangle) = H^2(\mathbb{R}^{nd})$, then we remarked, that the same can be done for all infinitesimally $-\triangle$ bounded potentials W. Now we want to restrict the configuration space from $\mathbb{R}^{nd}$ to a bounded open cube $\Lambda \subseteq \mathbb{R}^{nd}$ and define $H_{0,\Lambda}$ and $H_{0,\Lambda,N}$, the restrictions of H_0 to Λ with Dirichlet and Neumann boundary conditions.

Lemma 3.1. *Let W be a real valued function on $\mathbb{R}^{nd}$, such that the corresponding multiplication operator is infinitesimally $-\triangle$-bounded in $L^2(\mathbb{R}^{nd})$. Then the form domain $Q(W)$ of W contains $H^1(\mathbb{R}^{nd})$ and for each $\alpha > 0$ there is a $\gamma(\alpha) < \infty$, such that*

$$|\langle \Psi, W\Psi\rangle| \le \alpha \|\nabla\Psi\|^2 + \gamma(\alpha)\|\Psi\|^2 \tag{3.3}$$

for all $\Psi \in H^1(\mathbb{R}^{nd})$.

Proof. According to [RS2] Theorem X.18 the infinitesimally $-\triangle$-bounded selfadjoint perturbation W of the positive operator $-\triangle$ is infinitesimally form bounded, which is by definition, see again [RS2] p. 168, the claim of the lemma. □

Theorem 3.2. *Let W be a infinitesimally $-\triangle$-bounded potential acting in $L^2(\mathbb{R}^{nd})$ and let $\Lambda \subseteq \mathbb{R}^{nd}$ be an open bounded cube. The restriction $H_{0,\Lambda}$ of $H_0 = -\triangle + W$ to Λ with Dirichlet boundary conditions, given by the closure of the quadratic form*

$$\begin{array}{rcl} q_\Lambda : C_0^\infty(\Lambda) & \longrightarrow & \mathbb{R} \\ \Psi & \longmapsto & \langle \nabla\Psi, \nabla\Psi\rangle + \langle \Psi, W\Psi\rangle \end{array} \tag{3.4}$$

is a selfadjoint operator, bounded from below with compact resolvent.

Proof. From the construction of the Dirichlet Laplacian $-\triangle^\Lambda$ we know that $\begin{array}{rcl} H_0^1(\Lambda) & \longrightarrow & \mathbb{R} \\ \Psi & \longmapsto & \|\nabla\Psi\|^2 \end{array}$ is the closure of $\begin{array}{rcl} C_0^\infty(\Lambda) & \longrightarrow & \mathbb{R} \\ \Psi & \longmapsto & \|\nabla\Psi\|^2 \end{array}$. The form bound (3.3) is true for all $\Psi \in C_0^\infty(\Lambda) \subseteq H^1(\mathbb{R}^{nd})$, hence q_Λ is closable, see [Ka] VI. Theorem 1.33. So the closure $\overline{q_\Lambda}$ of q_Λ is a closed form bounded from below on a dense domain of $L^2(\Lambda)$, so the representation theorems, see

[Ka] VI. §2, apply and give a selfadjoint operator $H_{0,\Lambda}$ bounded from below. The form domain $H_0^1(\Lambda)$ of $H_{0,\Lambda}$ is a subset of $H^1(\mathbb{R}^{nd})$, so due to (3.3)

$$\langle\Psi, -\triangle^\Lambda\Psi\rangle(1-\alpha)-\gamma(\alpha)\|\Psi\|^2 \leq \langle\Psi, H_{0,\Lambda}\Psi\rangle \leq \langle\Psi, -\triangle^\Lambda\Psi\rangle(1+\alpha)+\gamma(\alpha)\|\Psi\|^2$$

for all $\Psi \in H_0^1(\Lambda)$ hence the numbers

$$\begin{aligned}\mu_k(H_{0,\Lambda}) \quad := \quad & \sup\{\inf\{\langle\Psi, H_{0,\Lambda}\Psi\rangle : \Psi \in H_0^1(\Lambda), \Psi \in \mathrm{lin}\{\varphi_1, ..., \varphi_{k-1}\}^\perp \\ & \qquad\qquad\qquad \|\Psi\| = 1\} : \varphi_1, ..., \varphi_{k-1} \in L^2(\Lambda)\}\end{aligned}$$

in the min-max principle, [RS4] Theorem XIII.2 are bounded by

$$\mu_k(-\triangle^\Lambda)(1-\alpha) - \gamma(\alpha) \leq \mu_k(H_{0,\Lambda}) \leq \mu_k(-\triangle^\Lambda)(1+\alpha) + \gamma(\alpha). \tag{3.5}$$

Compact resolvent of $-\triangle^\Lambda$ is equivalent to $\mu_k(-\triangle^\Lambda) \stackrel{k\to\infty}{\longrightarrow} \infty$, see [RS4] Theorem XIII.64, so $\mu_k(H_{0,\Lambda}) \stackrel{k\to\infty}{\longrightarrow} \infty$ by choosing $\alpha < 1$ in (3.5), hence $H_{0,\Lambda}$ has compact resolvent. □

For the definition of $H_{0,\Lambda,N}$, the restriction of H_0 with Neumann boundary conditions the shape of Λ comes into play.

Lemma 3.3. *Fix a bounded open cube $\Lambda \subseteq \mathbb{R}^{nd}$ and an infinitesimally $-\triangle$-bounded potential $W \geq 0$ in $L^2(\mathbb{R}^{nd})$. Then for all $\alpha > 0$ there is a $\eta(\alpha) < \infty$, such that*

$$\langle\Psi, W\Psi\rangle \leq \alpha\|\nabla\Psi\|^2 + \eta(\alpha)\|\Psi\|^2 \tag{3.6}$$

for all $\Psi \in H^1(\Lambda)$.

Proof. Regularity of the boundary of Λ allows the construction of a linear operator

$$F_{\Lambda_\varepsilon} : H^1(\Lambda) \longrightarrow H_0^1(\Lambda_\varepsilon)$$

from the Sobolev space $H^1(\Lambda)$ into the H^1 closure $H_0^1(\Lambda_\varepsilon)$ of C^∞-functions with compact support inside the ε-ball $\Lambda_\varepsilon := \{x \in \mathbb{R}^{nd} : \mathrm{dist}(x,\Lambda) < \varepsilon\}$ around Λ with the following properties, see [Wl] Satz 5.6 and Folgerung 5.2:

1. For each $\Psi \in H^1(\Lambda)$ the restriction $F_{\Lambda_\varepsilon}\Psi|_\Lambda$ of $F_{\Lambda_\varepsilon}\Psi$ to Λ coincides with Ψ: $F_{\Lambda_\varepsilon}\Psi|_\Lambda = \Psi$.
2. F_{Λ_ε} is continuous with respect to the H^1-norms and the L^2-norms, i. e. there are $c_{H^1}, c_{L^2} < \infty$, such that $\|F_{\Lambda_\varepsilon}\Psi\|_{H^1(\Lambda_\varepsilon)} \leq c_{H^1}\|\Psi\|_{H^1(\Lambda)}$ and $\|F_{\Lambda_\varepsilon}\Psi\|_{L^2(\Lambda_\varepsilon)} \leq c_{H^1}\|\Psi\|_{L^2(\Lambda)}$.

From the orthogonal decomposition $F_{\Lambda_\varepsilon}\Psi = \Psi + (F_{\Lambda_\varepsilon}\Psi - \Psi)$ in $L^2(\mathbb{R}^{nd})$ and the fact, that the positive multiplication operator W in $L^2(\mathbb{R}^{nd})$ does not change the support, we conclude

$$\langle F_{\Lambda_\varepsilon}\Psi, W F_{\Lambda_\varepsilon}\Psi\rangle = \langle \Psi, W\Psi\rangle + \langle (F_{\Lambda_\varepsilon}\Psi - \Psi), W(F_{\Lambda_\varepsilon}\Psi - \Psi)\rangle \geq \langle \Psi, W\Psi\rangle. \tag{3.7}$$

For $F_{\Lambda_\varepsilon}\Psi \in H_0^1(\Lambda_\varepsilon) \subseteq H^1(\mathbb{R}^{nd})$ estimate (3.3) applies, so for all $\Psi \in H^1(\Lambda)$ and $\alpha > 0$ we get:

$$\begin{aligned} 0 &\leq \langle \Psi, W\Psi\rangle \leq \langle F_{\Lambda_\varepsilon}\Psi, W F_{\Lambda_\varepsilon}\Psi\rangle \leq \alpha\|\nabla F_{\Lambda_\varepsilon}\Psi\|_{L^2}^2 + \gamma(\alpha)\|F_{\Lambda_\varepsilon}\Psi\|_{L^2}^2 \\ &= \alpha\|F_{\Lambda_\varepsilon}\Psi\|_{H^1}^2 + (\gamma(\alpha) - \alpha)\|F_{\Lambda_\varepsilon}\Psi\|_{L^2}^2 \leq \\ &\leq c_{H^1}^2\alpha\|\Psi\|_{H^1}^2 + c_{L^2}^2(\gamma(\alpha) - \alpha)\|\Psi\|_{L^2}^2 \leq \\ &\leq \alpha c_{H^1}^2\|\nabla\psi\|_{L^2}^2 + (c_{L^2}^2(\gamma(\alpha) - \alpha) + \alpha c_{H^1}^2)\|\Psi\|_{L^2}^2, \end{aligned} \tag{3.8}$$

which is the desired bound. $\square$

Theorem 3.4. *Let $\Lambda \subseteq \mathbb{R}^{nd}$ be an open bounded cube and let $W \geq 0$ be an infinitesimally $-\triangle$-bounded operator in $L^2(\mathbb{R}^{nd})$. The restriction $H_{0,\Lambda,N}$ of $H_0 = -\triangle + W$ to Λ with Neumann boundary conditions, given by the quadratic form*

$$\begin{aligned} q_{\Lambda,N} : H^1(\Lambda) &\longrightarrow \mathbb{R} \\ \Psi &\longmapsto \langle \nabla\Psi, \nabla\Psi\rangle + \langle \Psi, W\Psi\rangle \end{aligned} \tag{3.9}$$

is a positive selfadjoint operator with compact resolvent.

Proof. From Lemma 3.3 we conclude, that $\begin{aligned} H^1(\Lambda) &\longrightarrow \mathbb{R} \\ \Psi &\longmapsto \langle \Psi, W\Psi\rangle \end{aligned}$ is an infinitesimally form bounded perturbation of the Neumann Laplacian $-\triangle_N^\Lambda$ on Λ. Hence $q_{\Lambda,N}$ defines a closed positive form on a dense domain and the corresponding operator $H_{0,\Lambda,N}$ is selfadjoint and positive. By (3.6)

$$\langle \Psi, -\triangle_N^\Lambda \Psi\rangle \leq \langle \Psi, H_{0,\Lambda,N}\Psi\rangle \leq \langle \Psi, -\triangle_N^\Lambda \Psi\rangle(1+\alpha) + \eta(\alpha)\|\Psi\|^2$$

for all $\Psi \in H^1(\Lambda)$, hence by the min-max principle

$$\mu_k(-\triangle_N^\Lambda) \leq \mu_k(H_{0,\Lambda,N}) \leq \mu_k(-\triangle_N^\Lambda)(1+\alpha) + \eta(\alpha), \tag{3.10}$$

so $\mu_k(H_{0,\Lambda,N}) \stackrel{k\to\infty}{\longrightarrow} \infty$, which is equivalent to $H_{0,\Lambda,N}$ being an operator with compact resolvent. $\square$

3.2 Spectral Averaging

In this section we have a look at the following quite general situation, which is already sketched in [CoHi], chapter 4: $\mathcal{H}$ is a separable complex Hilbert space and H_0 a selfadjoint operator in $\mathcal{H}$ with domain $\mathcal{D}(H_0)$. $V \geq 0$ is a bounded Operator on $\mathcal{H}$ and for $\mu \in \mathbb{C}$, we define the operator

$$H(\mu) := H_0 - \mu V \tag{3.11}$$

on the domain $\mathcal{D}(H(\mu)) = \mathcal{D}(H_0)$. The operators $H(\mu)$ are not selfadjoint for $\mu \notin \mathbb{R}$, so we first list some properties:

Lemma 3.5. *$H(\mu)$ is closed, $H(\mu)^* = H(\bar{\mu})$ and for $z = x + iy \in \mathbb{C}$, $y = \Im z \neq 0$ and $\mu = \alpha + i\beta \in \mathbb{C}$ with $\beta y \geq 0$*

$$W(\mu) := \{\langle H(\mu)u, u\rangle : u \in \mathcal{D}(H_0), \|u\| = 1\} \subseteq \{w \in \mathbb{C} : \beta\Im w \leq 0\}$$

is the numerical range of $H(\mu)$ and $H(\mu) - z$ has a bounded inverse.

Proof. Let $u, v \in \mathcal{H}$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(H_0)$ such that $u_n \to u$ and $H(\mu)u_n \to v$. Continuity of V implies $Vu_n \to Vu$, hence $H_0u_n = H(\mu)u_n + \mu Vu_n \to v + \mu Vu \in \mathcal{H}$. As H_0 is closed, we get $u \in \mathcal{D}(H_0)$ and $H_0u = v + \mu Vu$ or $H(\mu)u = H_0u - \mu Vu = v$, so $H(\mu)$ is closed. For $u, v \in \mathcal{D}(H_0)$

$$\langle H(\mu)u, v\rangle = \langle (H_0-\mu V)u, v\rangle = \langle u, (H_0-\bar{\mu}V)v\rangle = \langle u, H(\bar{\mu})v\rangle = \langle u, H(\mu)^*v\rangle,$$

so $H(\mu)^* = H(\bar{\mu})$. In the rest of the proof we may assume $\beta \neq 0$. For each $u \in \mathcal{D}(H_0)$

$$\langle H(\mu)u, u\rangle = \langle H_0u, u\rangle - \alpha\langle Vu, u\rangle - i\beta\langle Vu, u\rangle \in \{w \in \mathbb{C} : \beta\Im w \leq 0\},$$

so the numerical range satisfies $W(\mu) \subseteq \{w \in \mathbb{C} : \beta\Im w \leq 0\}$. The two scalarproducts $\langle \beta Vu, u\rangle$ and $\langle yu, u\rangle$ are of the same sign for each $u \in \mathcal{D}(H_0)$, hence

$$\begin{aligned} |\langle (H(\mu) - z)u, u\rangle| &= \sqrt{\langle (H_0 - \alpha V - x)u, u\rangle^2 + \langle (\beta V + y)u, u\rangle^2} \geq \\ &\geq |\langle (\beta V + y)u, u\rangle| \geq |y|\,\|u\|^2, \end{aligned}$$

so $\mathrm{Kern}(H(\mu) - z) = \mathrm{Kern}(H(\bar{\mu}) - \bar{z}) = \{0\}$ and $\mathrm{Kern}((H(\mu) - z)^*) = \mathrm{Kern}(H(\bar{\mu}) - \bar{z}) = \mathrm{Ran}(H(\mu) - z)^\perp = \{0\}$. This means that $\mathrm{Ran}(H(\mu) - z)$ is dense in $\mathcal{H}$, so according to [HiSi] Proposition 19.7 the resolvent set $\rho(H(\mu))$ of $H(\mu)$ satisfies $\mathbb{C}\backslash\overline{W(\mu)} \subseteq \rho(H(\mu))$, which proves $\{z \in \mathbb{C} : \beta\Im z > 0\} \subseteq \rho(H(\mu))$. ∎

Proposition 3.6. *Let B be a bounded selfadjoint operator on $\mathcal{H}$, such that there is a $c_2 > 0$ satisfying $0 \le c_2 B^2 \le V$. For $t > 0$ and $z \in \mathbb{C} \backslash \mathbb{R}$ define*

$$F_t(z) := \int_{\mathbb{R}} \frac{1}{1+t\alpha^2} B(H(\alpha) - z)^{-1} B d\alpha, \tag{3.12}$$

then

$$\|F_t(z)\| \le \frac{\pi}{c_2} \tag{3.13}$$

Proof. According to Lemma 3.5 and [HiSi] Proposition 19.7 the operator $K(\mu, z) := B(H(\mu) - z)^{-1}B$ is bounded for $z = x + iy \in \mathbb{C}$, $\Im z > 0$ and $\mu = \alpha + i\beta \in \mathbb{C}, \Im\mu \ge 0$ by

$$\begin{aligned} \|K(\mu,z)\| &= \|B(H(\mu)-z)^{-1}B\| \le \|B\|^2 \|(H(\mu)-z)^{-1}\| \le \\ \le \ & \|B\|^2 \frac{1}{\operatorname{dist}(z, \overline{W(\mu)})} \le \|B\|^2 \frac{1}{\operatorname{dist}(z, \{w \in \mathbb{C} : \Im w \le 0\})} = \|B\|^2 \frac{1}{|\Im z|}. \end{aligned} \tag{3.14}$$

Thus for $\mu = \alpha + i\beta, \beta \in [0, \frac{1}{2\sqrt{t}}]$ the integral

$$F_t(\beta, z) := \int_{-\infty}^{\infty} \frac{1}{1 + t(\alpha + i\beta)^2} K(\alpha + i\beta, z) d\alpha$$

exists, because of the estimate

$$\left| \frac{1}{1+t\mu^2} \right| = \frac{1}{\sqrt{(1+t\alpha^2 - t\beta^2)^2 + (2t\alpha\beta)^2}} \le \frac{1}{\frac{3}{4} + t\alpha^2}. \tag{3.15}$$

The strategy of the proof is a calculation of $F_t(\beta, z)$ for $\beta > 0$ plus limiting process $\beta \searrow 0$: For $\mu, \nu \in \mathbb{C}$, $\Im\mu, \Im\nu \ge 0$ the second resolvent equation for $z \in \mathbb{C}$, $\Im z > 0$ yields:

$$K(\mu, z) - K(\nu, z) = (\mu - \nu) B(H(\nu) - z)^{-1} V (H(\mu) - z)^{-1} B, \tag{3.16}$$

so $\|(H(\mu) - z)^{-1}\| \le \frac{1}{|\Im z|}$ implies the operator norm bound

$$\|K(\mu, z) - K(\nu, z)\| \le |\mu - \nu| \, \|V\| \frac{\|B\|^2}{(\Im z)^2}. \tag{3.17}$$

Equation (3.16) shows the existence of

$$K'(\mu, z) = \lim_{\nu \to \mu} \frac{K(\nu, z) - K(\mu, z)}{\nu - \mu} = B(H(\mu) - z)^{-1} V (H(\mu) - z)^{-1} B,$$

for all $\mu \in \mathbb{C}$ with $\Im\mu > 0$. So given $t > 0$ and $z \in \mathbb{C}$, $\Im z > 0$ the function

$$\begin{array}{rcl} f_t : \{\mu \in \mathbb{C} : \Im\mu > 0\} & \longrightarrow & L(\mathcal{H}) \\ \mu & \longmapsto & \frac{1}{1+t\mu^2} K(\mu, z) \end{array}$$

has a pole of order 1 at $\frac{i}{\sqrt{t}}$ and is analytic else. The residue of f_t at $\frac{i}{\sqrt{t}}$ is

$$\mathrm{Res}(f_t, \frac{i}{\sqrt{t}}) = \lim_{\mu \to \frac{i}{\sqrt{t}}} \frac{(\mu - \frac{i}{\sqrt{t}})K(\mu, z)}{(1 + i\sqrt{t}\mu)(1 - i\sqrt{t}\mu)} = \frac{-i}{2\sqrt{t}} K(\frac{i}{\sqrt{t}}, z).$$

Choose $\beta \in]0, \frac{1}{2\sqrt{t}}]$ and $r > \frac{1}{\sqrt{t}}$ and integrate f_t along the complex paths $\begin{array}{rcl} C_1(\beta, r) : [-r, r] & \longrightarrow & \mathbb{C} \\ s & \longmapsto & s + i\beta \end{array}$ and $\begin{array}{rcl} C_2(\beta, r) : [0, \pi] & \longrightarrow & \mathbb{C} \\ s & \longmapsto & i\beta + re^{is} \end{array}$ where f_t is analytic. Using the bound $\|K(\mu, z)\| \leq \frac{\|B\|^2}{|\Im z|}$, which is independent of r,

$$\lim_{r \to \infty} \int\limits_{C_2(\beta, r)} \frac{1}{1+t\mu^2} K(\mu, z) d\mu = 0,$$

so for $\beta \in]0, \frac{1}{2\sqrt{t}}]$

$$\begin{array}{rcl} F_t(\beta, z) & = & \lim\limits_{r \to \infty} \left(\int\limits_{C_1(\beta, r)} f_t(\mu) d\mu + \int\limits_{C_2(\beta, r)} f_t(\mu) d\mu \right) = \\ & = & 2\pi i \mathrm{Res}(f_t, \frac{i}{\sqrt{t}}) = \frac{\pi}{\sqrt{t}} K(\frac{i}{\sqrt{t}}, z). \end{array}$$

The estimate $\|\frac{1}{1+t(\alpha+i\beta)^2} K(\mu, z)\| \leq |\frac{1}{\frac{3}{4}+t\alpha^2}| \frac{\|B\|^2}{|\Im z|}$ together with convergence $K(\alpha + i\beta, z) \overset{\beta \searrow 0}{\longrightarrow} K(\alpha, z)$ in operator norm resulting from (3.14), (3.15) respectively. (3.17) allow us to apply Lebesgue dominated convergence theorem, which yields

$$F_t(z) = \int\limits_{\mathbb{R}} \frac{1}{1+t\alpha^2} K(\alpha, z) d\alpha = \lim_{\beta \searrow 0} F_t(\beta, z) = \frac{\pi}{\sqrt{t}} K(\frac{i}{\sqrt{t}}, z). \tag{3.18}$$

The last equality reduces the proof to a clever estimate of $\|K(\mu, z)\|$:
Using selfadjointness of H_0, we see $[(H(\mu) - z)^{-1}]^* = ((H(\mu) - z)^*)^{-1} = (H(\bar{\mu}) - \bar{z})^{-1}$, so together with the resolvent equation, the imaginary part of

$K(\mu, z)$ is

$$\begin{aligned}
\Im K(\mu, z) &= \frac{1}{2i}(K(\mu, z) - K(\mu, z)^*) = \qquad (3.19)\\
&= \frac{1}{2i}B\left[(H(\mu) - z)^{-1} - (H(\bar{\mu}) - \bar{z})^{-1}\right]B = \\
&= \frac{1}{2i}B\left[(H(\mu) - z)^{-1}(2i\Im\mu V + 2i\Im z)(H(\bar{\mu}) - \bar{z})^{-1}\right]B = \\
&= B\left[(H(\mu) - z)^{-1}(y + \beta V)(H(\bar{\mu}) - \bar{z})^{-1}\right]B
\end{aligned}$$

So for $u \in \mathcal{H}$ we estimate:

$$\begin{aligned}
\langle(\Im K(\mu, z))u, u\rangle &= \langle B(H(\mu) - z)^{-1}(y + \beta V)(H(\bar{\mu}) - \bar{z})^{-1}Bu, u\rangle = \\
&= \langle(y + \beta V)(H(\bar{\mu}) - \bar{z})^{-1}Bu, (H(\bar{\mu}) - \bar{z})^{-1}Bu\rangle = \\
&= \beta\langle V(H(\bar{\mu}) - \bar{z})^{-1}Bu, (H(\bar{\mu}) - \bar{z})^{-1}Bu\rangle + \\
&\quad + y\|(H(\bar{\mu}) - \bar{z})^{-1}Bu\|^2 \\
&\geq \beta c_2\langle B^2(H(\bar{\mu}) - \bar{z})^{-1}Bu, (H(\bar{\mu}) - \bar{z})^{-1}Bu\rangle = \\
&= \beta c_2\langle B(H(\mu) - z)^{-1}BB(H(\bar{\mu}) - \bar{z})^{-1}Bu, u\rangle = \\
&= \beta c_2\langle K(\mu, z)K(\bar{\mu}, \bar{z})u, u\rangle. \qquad (3.20)
\end{aligned}$$

This estimate leads to

$$\begin{aligned}
\|K(\mu, z)^*u\|^2 &= \langle K(\mu, z)^*u, K(\mu, z)^*u\rangle = \langle K(\mu, z)K(\mu, z)^*u, u\rangle \leq \\
&\leq \frac{1}{|\beta|c_2}\langle\Im K(\mu, z)u, u\rangle = \frac{1}{|\beta|c_2 2i}\langle(K(\mu, z) - K(\mu, z)^*)u, u\rangle \\
&\leq \frac{\|u\|^2}{2|\beta|c_2}(\|K(\mu, z)\| + \|K(\mu, z)^*\|) = \frac{\|u\|^2}{|\beta|c_2}\|K(\mu, z)\|
\end{aligned}$$

for each $u \in \mathcal{H}$, hence

$$\|K(\mu, z)\|^2 = \|K(\mu, z)^*\|^2 = \left(\sup_{u\neq 0}\frac{\|K(\mu, z)^*u\|}{\|u\|}\right)^2 \leq \frac{\|K(\mu, z)\|}{|\beta|c_2}$$

or

$$\|K(\mu, z)\| \leq \frac{1}{|\beta|c_2}. \qquad (3.21)$$

Combining (3.21) with (3.18) we conclude

$$\|F_t(z)\| = \frac{\pi}{\sqrt{t}}\|K(\frac{i}{\sqrt{t}}, z)\| \leq \frac{\pi}{\sqrt{t}}\frac{1}{c_2\frac{1}{\sqrt{t}}} = \frac{\pi}{c_2},$$

which completes the proof. □

Lemma 3.7. *For $\alpha \in \mathbb{R}$ let π_α denote the spectral measure corresponding to the selfadjoint operator $H(\alpha) = H_0 + \alpha V$. Let g be a bounded nonnegative function on $\mathbb{R}$ with compact support and $M \in \mathcal{B}(\mathbb{R})$ a Borel set of finite Lebesgue measure $\lambda(M) < \infty$. Then*

$$\| \int_{\mathbb{R}} g(\alpha) B \pi_\alpha(M) B d\alpha \| \leq \frac{\|g\|_{L^\infty} \lambda(M)}{c_2}. \tag{3.22}$$

Proof. For $s, x \in \mathbb{R}$ and $y \in \mathbb{R}_+$, $\frac{y}{(s-x)^2+y^2} = \Im(\frac{1}{s-x-iy}) \geq 0$, hence functional calculus implies $\Im((H(\alpha) - x - iy)^{-1}) \geq 0$. Choosing $a, b \in \mathbb{R}$, $a < b$ Stone's formula, see [RS1] Theorem VII.13 together with $\pi_\alpha(]a, b[) \leq \pi_\alpha([a, b])$ implies:

$$\langle u, \pi_\alpha(]a, b[)u\rangle \leq \frac{1}{\pi} \liminf_{y \searrow 0} \int_a^b \Im\langle u, (H(\alpha) - x - iy)^{-1} u\rangle dx \tag{3.23}$$

Writing an open set $U \subseteq \mathbb{R}$ as a countably union of open pairwise disjoint intervals, $U = \bigcup_{k\in\mathbb{N}}]a_k, b_k[$, we conclude:

$$\begin{aligned}
\langle u, \pi_\alpha(U)u\rangle \quad \nwarrow \quad & \langle u, \pi_\alpha(\bigcup_{k=1}^m]a_k, b_k[)u\rangle = \sum_{k=1}^m \langle u, \pi_\alpha(]a_k, b_k[)u\rangle \leq \\
\leq \quad & \frac{1}{\pi} \sum_{k=1}^m \liminf_{y \searrow 0} \int_{a_k}^{b_k} \Im\langle u, (H(\alpha) - x - iy)^{-1} u\rangle dx \leq \\
\leq \quad & \frac{1}{\pi} \liminf_{y \searrow 0} \int_U \Im\langle u, (H(\alpha) - x - iy)^{-1} u\rangle dx,
\end{aligned}$$

so for each open set $U \subseteq \mathbb{R}$ we obtain the estimate:

$$\langle u, \pi_\alpha(U)u\rangle \leq \liminf_{y \searrow 0} \frac{1}{\pi} \int_U \Im\langle u, (H(\alpha) - x - iy)^{-1} u\rangle dx \tag{3.24}$$

Regularity of the Lebesgue measure allows us to choose for each $k \in \mathbb{N}$ an open set U_k, such that $M \subseteq U_k$ and $\lambda(U_k) \leq \lambda(M) + \frac{1}{k}$, hence $\pi_\alpha(M) \leq \pi_\alpha(U_k)$, too. Due to $\Im\left(B(H(\alpha) - x - iy)^{-1} B\right) = \Im K(\alpha, x + iy) \geq 0$ we may use Fatou's lemma and Fubini's theorem for nonnegative functions, so together with Proposition 3.6 this yields:

$$\int_{\mathbb{R}} \frac{1}{1 + t\alpha^2} \langle u, B\pi_\alpha(M)Bu\rangle d\alpha \leq \int_{\mathbb{R}} \frac{1}{1 + t\alpha^2} \langle u, B\pi_\alpha(U_k)Bu\rangle d\alpha \leq$$

$$
\begin{aligned}
&\le \frac{1}{\pi}\int\limits_{\mathbb{R}} \frac{1}{1+t\alpha^2}\liminf_{y\searrow 0}\left(\int\limits_{U_k} \Im\langle Bu,(H(\alpha)-x-iy)^{-1}Bu\rangle dx\right) d\alpha \le \\
&\le \frac{1}{\pi}\liminf_{y\searrow 0}\int\limits_{\mathbb{R}} \frac{1}{1+t\alpha^2}\left(\int\limits_{U_k} \Im\langle u,K(\alpha,x+iy)u\rangle dx\right) d\alpha = \\
&= \frac{1}{\pi}\liminf_{y\searrow 0}\Im\left(\int\limits_{U_k}\int\limits_{\mathbb{R}} \frac{1}{1+t\alpha^2}\langle u,K(\alpha,x+iy)u\rangle d\alpha dx\right) = \\
&= \frac{1}{\pi}\liminf_{y\searrow 0}\Im\int\limits_{U_k}\langle u,F_t(x+iy)u\rangle dx \le \frac{\|u\|^2\lambda(U_k)}{c_2} \le \frac{\|u\|^2}{c_2}(\lambda(M)+\frac{1}{k}).
\end{aligned}
$$

for each $k\in\mathbb{N}$, hence

$$
\int\limits_{\mathbb{R}} \frac{1}{1+t\alpha^2}\langle u,B\pi_\alpha(M)Bu\rangle d\alpha \le \frac{\lambda(M)\|u\|^2}{c_2}. \tag{3.25}
$$

Now we use the real function $h_t(\alpha):=g(\alpha)(1+t\alpha^2)$ to estimate the norm of the selfadjoint operator $\int\limits_{\mathbb{R}} g(\alpha)B\pi_\alpha(M)Bd\alpha$ by (3.25) as follows:

$$
\begin{aligned}
&\|\int\limits_{\mathbb{R}} g(\alpha)B\pi_\alpha(M)Bd\alpha\| = \sup\{|\int\limits_{\mathbb{R}} g(\alpha)\langle u,B\pi_\alpha(M)Bu\rangle d\alpha| : \|u\|=1\} = \\
&= \sup\{|\int\limits_{\mathbb{R}} h_t(\alpha)\frac{1}{1+t\alpha^2}\langle u,B\pi_\alpha(M)Bu\rangle d\alpha| : \|u\|=1\} \le \\
&\le \sup\{\inf_{t>0}\|h_t\|_{L^\infty}\frac{\|u\|^2\lambda(M)}{c_2} : \|u\|\le 1\} = \inf_{t>0}\|h_t\|_{L^\infty}\frac{\lambda(M)}{c_2}.
\end{aligned} \tag{3.26}
$$

Note that the compact support of g implies $\|g\|_{L^\infty} = \inf\limits_{t>0}\|h_t\|_{L^\infty}$, which finishes the proof. □

3.3 Proof of the Wegner estimate

Proposition 3.8. *Let $\Lambda\subseteq\mathbb{R}^{nd}$ be a bounded open cube and let*

$$
\begin{aligned}
H_\Lambda:\Omega &\longrightarrow SA(L^2(\Lambda)) \\
\omega &\longmapsto H_\Lambda(\omega)=H_{0,\Lambda}-V_n(\omega)|_\Lambda
\end{aligned} \tag{3.27}
$$

be the sum of the nonrandom operator $H_{0,\Lambda}$ constructed in Theorem 3.2 as $-\triangle+V_I$ restricted to Λ with Dirichlet boundary conditions and the restriction

$V_n(\omega)|_\Lambda$ of the random potential $V_n(\omega)$ for n electrons. Let π_Λ denote the spectral measure of H_Λ and $\mathbb{E}$ the expectation value on Ω with respect to $\mathbb{P}$ and suppose, that $\mathbb{P}_0$ is absolutely continuous with respect to Lebesgue measure with a bounded density $g \in L^\infty(\mathbb{R})$ of compact support. For each $j \in \mathbb{Z}^{nd}$ we write $1^j := 1_1^j \cdot 1_\Lambda$, then for each $M \in \mathcal{B}(\mathbb{R})$, $\lambda(M) < \infty$ the following estimate holds true:

$$\|\mathbb{E}(1^j \pi_\Lambda(M) 1^j)\| \leq \frac{\lambda(M)\|g\|_{L^\infty}}{c_0} \tag{3.28}$$

Proof. For $j = (j_1, ..., j_n) \in \mathbb{Z}^{nd}$ we write the probability space

$$(\Omega, \mathbb{P}) = (\mathbb{R}^{\mathbb{Z}^d} \times (\mathbb{R}^d)^{\mathbb{Z}^d}, \mathbb{P}_0^{\otimes \mathbb{Z}^d} \otimes \mathbb{P}_1^{\otimes \mathbb{Z}^d})$$

as the product of the probability spaces

$$\hat{\Omega}_{j_1} := \left(\mathbb{R}^{\mathbb{Z}^d \setminus \{j_1\}} \times \left(\mathbb{R}^d\right)^{\mathbb{Z}^d \setminus \{j_1\}}, \hat{\mathbb{P}} = \mathbb{P}_0^{\otimes \mathbb{Z}^d \setminus \{j_1\}} \otimes \mathbb{P}_1^{\otimes \mathbb{Z}^d \setminus \{j_1\}}\right),$$

$(\mathbb{R}^d, \mathbb{P}_1)$ and $(\mathbb{R}, \mathbb{P}_0)$ and according to this product decomposition we write $\omega = (\hat{\omega}, \omega''_{j_1}, \omega'_{j_1})$. We find this product structure also for the random potential, which can be written as

$$\begin{aligned} V_n(\omega) &= V_n(\hat{\omega}, \omega''_{j_1}, \omega'_{j_1}) = \\ &= \sum_{k=1}^n \left[\underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{k-1} \otimes (\sum_{\substack{i \in \mathbb{Z}^d \\ i \neq j_1}} \omega'_i T_{i+\omega''_i} \wp) \otimes \underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{n-k}\right] \\ &\quad + \omega'_{j_1} \sum_{k=1}^n \underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{k-1} \otimes T_{j_1+\omega''_{j_1}} \wp \otimes \underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{n-k} \\ &=: \widehat{V}^{j_1}(\hat{\omega}) + \omega'_{j_1} \tilde{V}(\omega''_{j_1}) \end{aligned} \tag{3.29}$$

The assumptions on $\wp$ and $\mathbb{P}_1$ imply $T_{j_1+\omega''_{j_1}} \wp \geq c_0(1^{j_1})^2$ as operators in $L^2(\mathbb{R}^d)$ for $\mathbb{P}_1$-almost every ω''_{j_1}, hence

$$T_{j_1+\omega''_{j_1}} \wp \otimes \underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{N-1} \geq c_0(1^j)^2 \tag{3.30}$$

as operators in $L^2(\mathbb{R}^{nd})$. From $\sum_{k=2}^n \underbrace{\mathbf{1} \otimes ... \otimes \mathbf{1}}_{k-1} \otimes T_{j_1+\omega''_{j_1}} \wp \underbrace{\otimes \mathbf{1} \otimes ... \otimes \mathbf{1}}_{n-k} \geq 0$ and (3.30) we conclude $\tilde{V}(\omega''_{j_1}) \geq c_0(1^j)^2$ in $L^2(\mathbb{R}^{nd})$ and

$$\tilde{V}(\omega''_{j_1})|_\Lambda \geq c_0(1^j)^2|_\Lambda \tag{3.31}$$

in $L^2(\Lambda)$. On the other hand

$$H_{\Lambda,j_1}(\hat{\omega}) := H_{0,\Lambda} - \widehat{V}^{j_1}(\hat{\omega})|_\Lambda \tag{3.32}$$

is for $\hat{\mathbb{P}}$-almost every $\hat{\omega} \in \hat{\Omega}_{j_1}$ a selfadjoint operator with

$$H_\Lambda(\omega) = H_{\Lambda,j_1}(\hat{\omega}) - \omega'_{j_1}\tilde{V}(\omega''_{j_1}). \tag{3.33}$$

So we are in the position to use Lemma 3.7 with $\mathcal{H} = L^2(\Lambda)$, $H_0 = H_{\Lambda,j_1}(\hat{\omega})$ and $V = \tilde{V}^{j_1}(\omega''_{j_1})$ for the estimate of

$$\begin{aligned}
&\langle \phi, \mathbb{E}(1^j \pi_\Lambda(M) 1^j)\phi\rangle = && (3.34)\\
&= \int\limits_{\hat{\Omega}_{j_1}} d\hat{\mathbb{P}}(\hat{\omega}) \int\limits_{\mathbb{R}^d} d\mathbb{P}_1(\omega''_{j_1}) \int\limits_{\mathbb{R}} d\mathbb{P}_0(\alpha)\langle \phi, 1^j\pi_\Lambda(M,(\hat{\omega},\omega''_{j_1},\alpha))1^j\phi\rangle = \\
&= \int\limits_{\hat{\Omega}_{j_1}} d\hat{\mathbb{P}}(\hat{\omega}) \int\limits_{\mathbb{R}^d} d\mathbb{P}_1(\omega''_{j_1}) \int\limits_{\mathbb{R}} d\alpha g(\alpha)\langle \phi, 1^j\pi_\Lambda(M,(\hat{\omega},\omega''_{j_1},\alpha))1^j\phi\rangle \le \\
&\le \int\limits_{\hat{\Omega}_{j_1}} d\hat{\mathbb{P}}(\hat{\omega}) \int\limits_{\mathbb{R}^d} d\mathbb{P}_1(\omega''_{j_1}) \frac{\lambda(M)\|g\|_{L^\infty}}{c_0}\|\phi\|^2 = \frac{\lambda(M)\|g\|_{L^\infty}}{c_0}\|\phi\|^2,
\end{aligned}$$

hence $\|\mathbb{E}(1^j\pi_\Lambda(M)1^j)\| \le \frac{\lambda(M)\|g\|_{L^\infty}}{c_0}$. ∎

Theorem 3.9. *Let $\mathbb{P}_0$ have a density $g \in L^\infty(\mathbb{R})$ of compact support. For each bounded open cube $\Lambda \subseteq \mathbb{R}^{nd}$ of sidelength greater one, H_Λ, π_Λ and $\mathbb{E}$ as in Proposition 3.8 and each $E_0 \in \mathbb{R}$, there is a $C_W = C_W(E_0, \|g\|_{L^\infty}) < \infty$, such that*

$$\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(]E_0 - \eta, E_0 + \eta[))) \le C_W \eta \lambda(\Lambda) \tag{3.35}$$

for $\eta > 0$. Moreover the real function $E_0 \mapsto C_W(E_0, \|g\|_{L^\infty})$ remains bounded on each interval, which is bounded from above.

Proof. We start the proof with some remarks about $H_\Lambda(\omega)$. Let $J = J(\Lambda) := \{j \in \mathbb{Z}^{nd} : \Lambda(j,1) \cap \Lambda \neq \emptyset\}$ and for $j \in J$ define $\Lambda_j := \Lambda(j,1) \cap \Lambda$. According to [RS4] XIII.15 Proposition 3 and Proposition 4 the Dirichlet and Neumann Laplacians satisfy

$$-\triangle^\Lambda \ge -\triangle_N^\Lambda \ge -\triangle_N^{\Lambda\setminus\Lambda'} = -\bigoplus_{j\in J} \triangle_N^{\Lambda_j}.$$

$\Lambda' := \Lambda \setminus \bigcup_{j \in J} \Lambda_j$ has Lebesgue measure zero, so there is no restriction on the domain arising from the multiplication operator with the Coulomb potential

$V_C|_{\Lambda'}$ restricted to Λ', so by the $\mathbb{P}$ almost sure boundedness $\|V_n(\omega)\| \leq |\mathcal{V}_n|$, the operator inequality

$$H_\Lambda(\omega) \geq H_{N,\Lambda} := \bigoplus_{j\in J} H_{0,N,\Lambda_j} - |\mathcal{V}_n| \tag{3.36}$$

holds true, where H_{0,N,Λ_j} is the restriction of $-\triangle + V_I$ to Λ_j with Neumann boundary conditions. Fix $E_0 \in \mathbb{R}$ and set $I_\eta :=]E_0 - \eta, E_0 + \eta[$. Calculating the trace in an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\Lambda)$, from

$$\begin{aligned}
\langle e_k, \pi_\Lambda(I_\eta) e_k\rangle &= \int_{\mathbb{R}} 1_{I_\eta}(x) d\pi_{\Lambda,e_k,e_k}(x) = \int_{\mathbb{R}} 1_{I_\eta}(x) e^{-x} e^{x} d\pi_{\Lambda,e_k,e_k}(x) \leq \\
&\leq e^{E_0+\eta} \int_{\mathbb{R}} 1_{I_\eta}(x) e^{-x} d\pi_{\Lambda,e_k,e_k}(x) = e^{E_0+\eta}\langle e_k, \pi_\Lambda(I_\eta) e^{-H_\Lambda} e_k\rangle
\end{aligned}$$

we see

$$\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta))) \leq e^{E_0+\eta}\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta)e^{-H_\Lambda})). \tag{3.37}$$

Boundedness $\|V_n(\omega)\| \leq |\mathcal{V}_n|$ of the random potential $\mathbb{P}$-almost everywhere implies the estimate of $H_\Lambda(\omega)$ in terms of the restriction $H_{0,\Lambda}$ of $-\triangle + V_I$ to Λ with Dirichlet boundary conditions:

$$\langle \varphi, (H_{0,\Lambda} - |\mathcal{V}_n|)\varphi\rangle \leq \langle \varphi, H_\Lambda(\omega)\varphi\rangle \leq \langle \varphi, (H_{0,\Lambda} + |\mathcal{V}_n|)\varphi\rangle, \tag{3.38}$$

so the min-max principle gives an estimate for the eigenvalues of $H_\Lambda(\omega)$ as

$$\mu_k(H_{0,\Lambda}) - |\mathcal{V}_n| \leq \mu_k(H_\Lambda(\omega)) \leq \mu_k(H_{0,\Lambda}) + |\mathcal{V}_n| \tag{3.39}$$

and proves, that $H_\Lambda(\omega)$ has compact resolvent. Hence for almost every $\omega \in \Omega$ there is an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L^2(\Lambda)$ consisting of eigenvalues of $H_\Lambda(\omega)$ - thus depending on the realization ω. Let $M := \{k \in \mathbb{N} : \mu_k \in I_\eta\}$, then the Trace calculated in the orthonormal basis $(\varphi_k)_{k\in\mathbb{N}}$ is estimated by

$$\begin{aligned}
\mathrm{Trace}(\pi_\Lambda(I_\eta,\omega)e^{-H_\Lambda(\omega)}) &= \sum_{k\in\mathbb{N}} \langle \varphi_k, \pi_\Lambda(I_\eta,\omega) e^{-H_\Lambda(\omega)}\varphi_k\rangle = \qquad (3.40)\\
&= \sum_{k\in\mathbb{N}} \langle \pi_\Lambda(I_\eta,\omega)\varphi_k, e^{-H_\Lambda(\omega)}\varphi_k\rangle = \sum_{k\in M}\langle \varphi_k, e^{-H_\Lambda(\omega)}\varphi_k\rangle = \sum_{k\in M} e^{-\mu_k} = \\
&= \sum_{k\in M} e^{\langle \varphi_k, -H_\Lambda(\omega)\varphi_k\rangle} \leq \sum_{k\in M} e^{\langle \varphi_k, -H_{N,\Lambda}\varphi_k\rangle} \leq \sum_{k\in M}\langle \varphi_k, e^{-H_{N,\Lambda}}\varphi_k\rangle = \\
&= \sum_{k\in\mathbb{N}}\langle \pi_\Lambda(I_\eta,\omega)\varphi_k, e^{-H_{N,\Lambda}}\varphi_k\rangle = \sum_{k\in\mathbb{N}}\langle \varphi_k, \pi_\Lambda(I_\eta,\omega)e^{-H_{N,\Lambda}}\varphi_k\rangle = \\
&= \mathrm{Trace}(\pi_\Lambda(I_\eta,\omega)e^{-H_{N,\Lambda}}).
\end{aligned}$$

The last inequality of the third line above uses $e^{\langle\varphi,-H\varphi\rangle} \leq \langle\varphi, e^{-H}\varphi\rangle$ for each selfadjoint operator H bounded from below and for $\varphi \in \mathcal{D}(H)$, which is a result of spectral calculus and Jensen's inequality. For the calculation of $\mathrm{Trace}(\pi_\Lambda(I_\eta,\omega)e^{-H_{N,\Lambda}})$, we use the orthonormal basis $(\phi_{k,j})_{k\in\mathbb{N}}$ of $L^2(\Lambda_j)$ consisting out of eigenvectors for H_{0,N,Λ_j} with eigenvalues $E_{k,j}$. Then with the eigenvectors

$$\psi_{k,j} := (0,...,0,\phi_{k,j},0,...,0) \in \bigoplus_{j\in J} L^2(\Lambda_j)$$

of $H_{N,\Lambda}$ we obtain

$$\begin{aligned}
\mathrm{Trace}(\pi_\Lambda(I_\eta,\omega)e^{-H_{N,\Lambda}}) &= \sum_{k\in\mathbb{N}}\sum_{j\in J}\langle\psi_{k,j},\pi_\Lambda(I_\eta,\omega)e^{-H_{N,\Lambda}}\psi_{k,j}\rangle = \\
&= \sum_{k\in\mathbb{N}}\sum_{j\in J}\langle\psi_{k,j},\pi_\Lambda(I_\eta,\omega)\psi_{k,j}\rangle e^{-E_{k,j}+|\mathcal{V}_n|} = \\
&= \sum_{k\in\mathbb{N}}\sum_{j\in J}\langle 1^j\psi_{k,j},1^j\pi_\Lambda(I_\eta,\omega)1^j\psi_{k,j}\rangle e^{-E_{k,j}+|\mathcal{V}_n|} = \\
&= \sum_{k\in\mathbb{N}}\sum_{j\in J}\langle 1^j\phi_{k,j},1^j\pi_\Lambda(I_\eta,\omega)1^j\phi_{k,j}\rangle e^{-E_{k,j}+|\mathcal{V}_n|}
\end{aligned} \tag{3.41}$$

Taking expectation values on both sides of (3.41), by Proposition 3.8 we obtain

$$\begin{aligned}
\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta)e^{-H_{N,\Lambda}})) &= \mathbb{E}\left(\sum_{j\in J}\sum_{k\in\mathbb{N}}\langle\phi_{k,j},1^j\pi_\Lambda(I_\eta)1^j\phi_{k,j}\rangle e^{-E_{k,j}+|\mathcal{V}_n|}\right) = \\
&= \sum_{j\in J}\sum_{k\in\mathbb{N}}\langle\phi_{k,j},\mathbb{E}(1^j\pi_\Lambda(I_\eta)1^j)\phi_{k,j}\rangle e^{-E_{k,j}+|\mathcal{V}_n|} \leq \\
&\leq \sum_{j\in J}\sum_{k\in\mathbb{N}}\|\mathbb{E}(1^j\pi_\Lambda(I_\eta)1^j)\|e^{-E_{k,j}+|\mathcal{V}_n|} \leq \\
&\leq \sum_{j\in J}\sum_{k\in\mathbb{N}}\frac{\lambda(I_\eta)\|g\|_{L^\infty}}{c_0}e^{-E_{k,j}+|\mathcal{V}_n|} = \frac{2\eta\|g\|_{L^\infty}}{c_0}\sum_{j\in J}\sum_{k\in\mathbb{N}}e^{-E_{k,j}+|\mathcal{V}_n|},
\end{aligned} \tag{3.42}$$

the remaining problem is deterministic and we have to prove convergence of $\sum\limits_{j\in J}\sum\limits_{k\in\mathbb{N}} e^{-E_{k,j}}$. Suppose that for a given $j\in J$ the cube Λ_j has sidelength $0 < l_1^j, ..., l_{nd}^j \leq 1$, then

$$\pi^2 \leq a^j := \min\{\left(\frac{\pi}{l_k^j}\right)^2 : 1 \leq k \leq nd\} < \infty.$$

Due to [RS4] p. 266, the eigenvalues of the Neumann Laplacian are given by

$$\tilde{E}^j_{m_1,...,m_{nd}} = \sum_{k=1}^{nd} \left(\frac{\pi}{l^j_k}\right)^2 m_k^2 \geq a^j \sum_{k=1}^{nd} m_k,$$

indexed with $(m_1, ..., m_{nd}) \in \mathbb{N}_0^{nd}$. The corresponding eigenvalues $E^j_{m_1,...,m_{nd}}$ of H_{0,N,Λ_j} are estimated due to (3.10) by $\tilde{E}^j_{m_1,...,m_{nd}} \leq E^j_{m_1,...,m_{nd}}$, so

$$\begin{aligned} \sum_{j\in J} \sum_{(m_1,...,m_{nd})\in\mathbb{N}_0^{nd}} e^{-E^j_{m_1,...,m_d}} &\leq \sum_{j\in J} \sum_{(m_1,...,m_{nd})\in\mathbb{N}_0^{nd}} e^{-\tilde{E}^j_{m_1,...,m_d}} \leq \\ \leq \ \mathrm{Card}J \sum_{(m_1,...m_{nd})\in\mathbb{N}_0^{nd}} e^{-\pi^2 \sum\limits_{n=1}^{nd} m_n} &= \mathrm{Card}J \left(\sum_{n\in\mathbb{N}_0} e^{-\pi^2 n}\right)^{nd} \leq \\ \leq \ 3^{nd}\lambda(\Lambda) \left(\frac{e^{\pi^2}}{e^{\pi^2}-1}\right)^{nd}. & \end{aligned} \tag{3.43}$$

Pasting together (3.37), (3.40), (3.42) and (3.43) we obtain:

$$\begin{aligned} &\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta))) \leq e^{E_0+\eta}\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta)e^{-H_\Lambda})) \leq \\ &\leq \ e^{E_0+\eta}\mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta)e^{-H_{N,\Lambda}})) \leq e^{E_0+\eta+|\mathcal{V}_n|} \frac{2\|g\|_{L^\infty}}{c_0} \left(\frac{3e^{\pi^2}}{e^{\pi^2}-1}\right)^{nd} \eta\lambda(\Lambda) \\ &:= \ C_W(E_0, \|g\|_{L^\infty})\eta\lambda(\Lambda). \end{aligned} \tag{3.44}$$

From the explicit form of $C_W(E_0, \|g\|_{L^\infty})$ given in (3.44), we see that the real function $E_0 \mapsto C_W(E_0, \|g\|_{L^\infty})$ is bounded on intervals, which are bounded from above. □

Corollary 3.10. *In the situation of Theorem 3.9*

$$\mathbb{P}\{\omega \in \Omega : \mathrm{dist}(\sigma(H_\Lambda(\omega)), E_0) < \eta\} \leq C_W\eta\lambda(\Lambda). \tag{3.45}$$

Proof. $\{\omega \in \Omega : \mathrm{dist}(\sigma(H_\Lambda(\omega), E_0) < \eta\} = \{\omega \in \Omega : \mathrm{Trace}(\pi_\Lambda(I_\eta, \omega)) \geq 1\}$, so by Tschebyschev's inequality

$$\mathbb{P}\{\omega \in \Omega : \mathrm{dist}(\sigma(H_\Lambda(\omega), E_0) < \eta\} \leq \mathbb{E}(\mathrm{Trace}(\pi_\Lambda(I_\eta))). \tag{3.46}$$

□

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