On Non–Degenerate Hamiltonian Hopf Bifurcations in 3DOF Systems

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Abstract

We give a short review of available methods to determine the non–degeneracy of Hamiltonian Hopf bifurcations in three–degree–of–freedom systems. We illustrate the geometric method to more detail, using the example of the Lagrange top.

1 Introduction

Hamiltonian Hopf bifurcations appear in various examples of Hamiltonian systems with more than two degrees of freedom such as the Lagrange top [3, 8, 2], the Kirchhoff top [1], the double spherical pendulum [6] and the $3D$ Hénon–Heiles family [4, 5].

Recall from [7] that the non–degenerate Hamiltonian Hopf bifurcation is the generic one–parameter bifurcation at which the equilibrium changes stability. The name finds its origin in the fact that it is the Hamiltonian analogue of the non–conservative Hopf bifurcation.

The presence of a Hamiltonian Hopf bifurcation is usually detected by computing the eigenvalue behaviour of the linearized system at the bifurcation value of the parameter the system depends upon. The eigenvalues change from purely imaginary to eigenvalues with non–zero real part.

There are two possibilities. The first is that two pairs of purely imaginary eigenvalues come together on the imaginary axis and split off into the complex plane. This is the typical behaviour to have in systems where the linear system consists of a semisimple part, which is in $1:-1$ resonance, plus a nilpotent part. The semisimple part can be used to normalize the Hamiltonian, thus introducing an S^1–symmetry. The second possibility is that two equal pairs of purely imaginary eigenvalues come together in the origin and split off along the real axis. This is the behaviour met in S^1–symmetric Hamiltonian systems with a nilpotent linear part. In this case one should have the standard S^1–symmetry, *i.e.* one coming from a $1:-1$ resonance.

Once the presence of a Hamiltonian Hopf bifurcation is established the next step is to show that it is non–degenerate and thus of co–dimension one. This is in general a more difficult problem because one has to deal with a nonlinear second order effect, *i.e.* one has to compute higher order terms in the normal form and check that it can be compared to the standard form. The latter comes actually down to showing equivalence with the standard form. In systems with more than two degrees of freedom these computations have in general to be performed after a first symmetry–reduction is completed. That is, computations have to be performed in a local chart describing the first reduced phase space which in general carries a nonlinear symplectic form.

Before proceeding on the methods by which this can be done we will first recall the methods by which in the two–degree–of–freedom case the standard form for the Hamiltonian Hopf bifurcation is established. There are several steps involved :

1. Normalize the Hamiltonian function in the neighbourhood of the bifurcating equilibrium with respect to its quadratic part up to some order k, k large enough [7].

2. Apply a version of Liapunov–Schmidt reduction (Moser–Weinstein reduction) to obtain an S^1–symmetric Hamiltonian system [7, 10].

3. Apply to the energy–momentum mapping of the obtained integrable system singularity theoretic methods to show equivalence to the standard form [7].

Note that the Hamiltonians of the first two steps are the same up to order k. The singularity theory now gives conditions on the higher order terms for co–dimension one, *i.e.* for the bifurcation to be non–degenerate. Some degenerate cases with low co–dimension are computed in [9].

2 The methods

To show the presence of a non–degenerate Hamiltonian Hopf bifurcation for the once reduced system, which has now two degrees of freedom but with a nonstandard symplectic form, one can step into this procedure at several stages resulting in different methods :

Flattening the Poisson structure. One steps into the above procedure at the beginning. The aim is to obtain a system with standard symplectic form. To this end one has to linearize the Poisson structure or symplectic structure within the local chart on the first reduced phase space. This has been done for the Lagrange top in [2].

Singularity theoretic equivalence. Here one steps into the above procedure at the second step. To be able to apply the singularity theory one needs to have invariance with respect to the standard S^1–action. The representation of this action depends on the co–ordinates chosen. This was applied in [3, 8] to the Lagrange top and in [5] to the $3D$ Hénon–Heiles family.

The geometric method. Here one replaces the third step with an analysis of the relative position of the phase space and the energy level sets of the one–degree–of–freedom system with the S^1–symmetry reduced as well. The second reduction is not regular and leads to a conical singularity. This has been performed for the $3D$ Hénon–Heiles family in [4] and will be carried out for the Lagrange top in the next section.

To apply the geometric method certain hypotheses should be checked.

H1. The stratification of the orbit space into reduced phase spaces should locally be equivalent to the standard case.

H2. As the bifurcation parameter varies, the energy level set through the vertex of the cone should change "with non–zero speed" from passing outside the cone to intersecting the interior of the cone.

H3. The energy level set should have second order contact at the vertex of the cone at the bifurcation value of the parameter.

The first hypothesis ensures that at the critical value of the integral one locally has a cone deforming into a hyperboloid when the value of the integral is varied. This means that locally the invariants form a Lie algebra isomorphic to $sl(2,\mathbb{R})$. The second hypothesis ensures that the unfolding of the linear part is (uni)versal. The third hypothesis is the actual non–degeneracy condition.

3 The Lagrange top

The Lagrange top is a heavy symmetric rigid body with one point fixed, cf. [2, 6]. The system is completely integrable on its phase space $T^*SO(3)$ with canonical symplectic form. The two symmetries are a right S^1–action corresponding to rotation about the axis of symmetry of the body and a left S^1–action corresponding to rotation about the vertical axis in space. Reduction with respect to the left S^1–action gives the Euler–Poisson equations

$$\dot{x} \;=\; x \times \frac{\partial H}{\partial y}\,, \quad \dot{y} \;=\; x \times \frac{\partial H}{\partial x} + y \times \frac{\partial H}{\partial y} \tag{1}$$

for the top. Here $x \in \mathbb{R}^3$ is the so–called Poisson vector, the direction of gravity measured in a body set of axes, whence the Casimir $C_1 = x_1^2 + x_2^2 + x_3^2$ of (1) takes the value 1. Similarly $y \in \mathbb{R}^3$ is the angular momentum vector measured in the body set of axes. The Casimir $C_2 = x_1y_1 + x_2y_2 + x_3y_3$ therefore takes the value a of the angular momentum along the gravity axis. This defines the reduced phase space

$$T_aS^2 \;=\; \{\, (x,y) \in \mathbb{R}^6 \;\mid\; C_1 = 1 \,,\; C_2 = a \,\} \;\subseteq\; \mathbb{R}^6$$

which is a 4–dimensional smooth submanifold invariant under the flow of the vector field given by (1). Rescaling time and length as in [8] the Hamiltonian reads

$$H \;=\; \frac{1}{2}(y_1^2 + y_2^2) \;+\; \frac{\gamma}{2}y_3^2 \;+\; x_3$$

where $\gamma \geq \frac{1}{2}$ is the quotient of the principal moments $I_1 = I_2$ and I_3 of inertia. The equilibrium point $(x,y) = (0,0,1,0,0,a)$ is the upright standing position of the top, which becomes gyroscopically stabilized ("sleeping") if the top spins sufficiently fast.

The reduced system still has the integral of motion $L(x,y) = y_3$, which is the angular momentum corresponding to rotation about the symmetry axis of the top. The corresponding action is

$$\rho :\; (x,y) \;\mapsto\; (R_tx, R_ty)$$

with

$$R_t \;=\; \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \;.$$

Therefore we perform a second reduction, cf. [2, 3]. The invariants of ρ are

$$\begin{aligned} \pi_1 &= x_1^2 + x_2^2 \,, & \pi_3 &= x_1y_1 + x_2y_2 \,, & \pi_5 &= x_3 \,, \\ \pi_2 &= y_1^2 + y_2^2 \,, & \pi_4 &= x_1y_2 - x_2y_1 \,, & \pi_6 &= y_3 \end{aligned}$$

with relations

$$\pi_3^2 \;+\; \pi_4^2 \;=\; \pi_1\pi_2 \,, \quad \pi_1 \geq 0 \,, \quad \pi_2 \geq 0 \;.$$

Furthermore we have

$$C_1 \;=\; \pi_1 + \pi_5^2 \;=\; 1 \,, \quad C_2 \;=\; \pi_3 + \pi_5\pi_6 \;=\; a \,, \quad L \;=\; \pi_6 \;=\; b$$

where b is the value of the third integral of motion. Consequently we can eliminate π_1, π_3, π_6 and the twice reduced phase space is given by

$$V_a^b \;=\; \{\, (\pi_2, \pi_4, \pi_5) \in \mathbb{R}^3 \;\mid\; R_a^b(\pi) = 0 \,,\; \pi_2 \geq 0 \,,\; |\pi_5| \leq 1 \,\}$$

with

$$R_a^b(\pi) \;=\; \pi_4^2 \;+\; (a - b\pi_5)^2 \;-\; (1 - \pi_5^2)\pi_2 \;.$$

After skipping constant terms the reduced Hamiltonian on V_a^b is

$$\mathcal{H} \;=\; \frac{1}{2}\pi_2 \;+\; \pi_5$$

and the trajectories are given by the intersections $\{\mathcal{H} = h\} \cap V_a^b$. Note that the energy level sets $\{\mathcal{H} = h\} \subseteq \mathbb{R}^3$ are planes perpendicular to

$$\begin{pmatrix} \pi_2 \\ \pi_4 \\ \pi_5 \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \;.$$

Our aim is to use the geometry of V_a^b to check the hypotheses H1–H3, thereby proving that a non–degenerate Hamiltonian Hopf bifurcation takes place at $a = b = \pm 2$. The twice reduced phase space is smooth for $|a| \neq |b|$ and degenerates into a cone at $a = b$ (and also at $a = -b$, but this "hanging top" is always stable). Thus hypothesis H1 is satisfied.

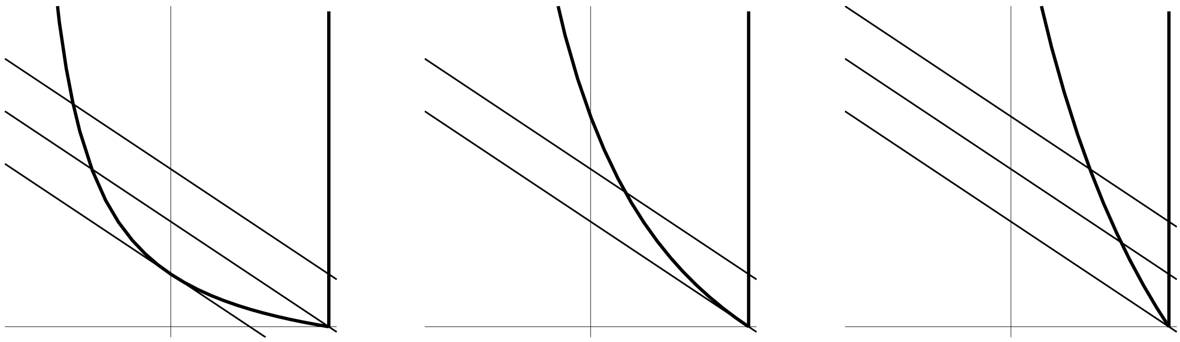

Figure 1: Relative position of $\{\mathcal{H} = h\}$ (thin) and V_a^a (thick) within $\{\pi_4 = 0\}$ for $|a| < 2$, $|a| = 2$ and $|a| > 2$. The horizontal axis is π_5 and the vertical axis is $\frac{1}{3}\pi_2$.

While the actual eigenvalue movement is lost through the second reduction, the bifurcation value can also be deduced from the one–degree–of–freedom system — for this value of $a = b$ the energy level set $\{\mathcal{H} = 1\}$ enters the conical singularity with the same "slope" as V_a^a at some direction of the cone. Note that for tangency of $\{\mathcal{H} = 1\}$ to the reduced phase space V_a^a one must have $\pi_4 = 0$. Consequently we have to compute the behaviour of $\mathcal{H}$ with respect to V_a^a within the (π_5, π_2)–plane. Both on $\{\mathcal{H} = 1\}$ and on $V_a^a \cap \{\pi_4 = 0\}$ we can write π_2 as a function of π_5, which leads us to considering the difference function

$$f_a(\pi_5) \;=\; a^2 \frac{1 - \pi_5}{1 + \pi_5} \;-\; 2(1 - \pi_5) \;.$$

The equilibrium $(\pi_2, \pi_4, \pi_5) = (0, 0, 1)$ bifurcates if and only if the derivative $f_a'(1) = -\frac{1}{2}a^2 + 2$ vanishes, that is at $a = \pm 2$.

The function f_a contains the necessary information on the relative position of $\{\mathcal{H} = 1\}$ to V_a^a within $\{\pi_4 = 0\}$. Checking hypotheses H2 and H3 is now a straightforward computation, see also Fig. 1. Hypothesis H2 is satisfied since $\frac{\mathrm{d}}{\mathrm{d}a} f_a'(1) = -a$ is non–zero at $a = \pm 2$, which shows that we have linear versality. Similarly, we need for hypothesis H3 that $f_{\pm 2}''(1) = 2$ does not vanish. Consequently there is quadratic contact from the outside and we conclude that the Lagrange top undergoes a supercritical Hamiltonian Hopf bifurcation.

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