Quasi–Periodic Response Solutions at Normal–Internal Resonances*

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Abstract

In the conservative dynamics of certain quasi–periodically forced oscillators, normal–internal resonances are considered in a bifurcational setting. The unforced system is a one degree of freedom oscillator, under forcing the system becomes a skew–product flow with a quasi–periodic motion on an n–dimensional torus as driving system. In this work, we investigate the persistence and the bifurcations of quasi–periodic n–dimensional tori (so–called "response solutions") in the averaged system, filling normal–internal resonance 'gaps' that had been excluded in previous analyses.

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1 Introduction

This note summarizes some recent results [3] on the dynamics of families of quasi–periodically forced nonlinear Hamiltonian oscillators at normal–internal resonances. The motivating example, which is quite representative for the general case, is the quasi–periodically forced mechanical pendulum

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{x} &= y, \\ \dot{y} &= -\alpha^2 \sin x + \varepsilon G(\theta), \end{aligned} \tag{1}$$

where $G : \mathbb{T}^n \to \mathbb{R}$, $x, y \in \mathbb{R}$, $\alpha > 0$ and $\varepsilon \geq 0$. The components of the frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ are assumed to be rationally independent. The oscillator is said to be at a normal–internal $1 : \ell$ resonance, if α satisfies $\langle k, \omega \rangle + \ell\alpha = 0$, for some $k \in \mathbb{Z}^n \backslash \{0\}$, some $\ell \in \{1, 2, \ldots\}$, and if ℓ is the smallest positive integer with this property. Throughout this article, the frequency vector ω is fixed and Diophantine, while the parameter α ranges over a compact interval not containing 0.

A classical question [16] asks for solutions of (1) that are quasi–periodic with the same frequency vector as the forcing: the so–called *response solutions*. The central question of this article is whether response solutions exist, and if so, how these solutions behave as the parameter α takes values near a normal–internal resonance value.

Note that any invariant torus that can be represented as the graph of a function $\tau = (\tau_1, \tau_2) : \mathbb{T}^n \to \mathbb{R}^2$ corresponds to a family of response solutions $x(t) = \tau_1(\omega t + \theta_0)$, parametrised by θ_0, and *vice versa*. Hence, in a geometrical language, we are asking for invariant tori of the system (1) in graph form. The forcing strength ε will be regarded as a perturbation parameter. For $\varepsilon = 0$, the torus $\tau = 0$ is invariant; we are interested in the fate of this torus if ε is small and positive.

The unperturbed torus $\tau = 0$ is normally elliptic, and its normal frequency is equal to α. Kolmogorov–Arnold–Moser (KAM) theory shows that for small $\varepsilon > 0$, this torus will persist for the parameter α taking values in a Cantor set of positive Lebesgue measure [5, 6, 11, 12, 14, 15]. The "holes" that conform the Cantor structure correspond to normal–internal resonances; in their presence standard KAM theory cannot be applied directly. Note that in the well–known periodic analogue of this problem, that is in the case $n = 1$, response solutions are known to persist for all α ranging over some compact interval. The main goal of this work is hence to analyse what happens in a given normal–internal resonance gap.

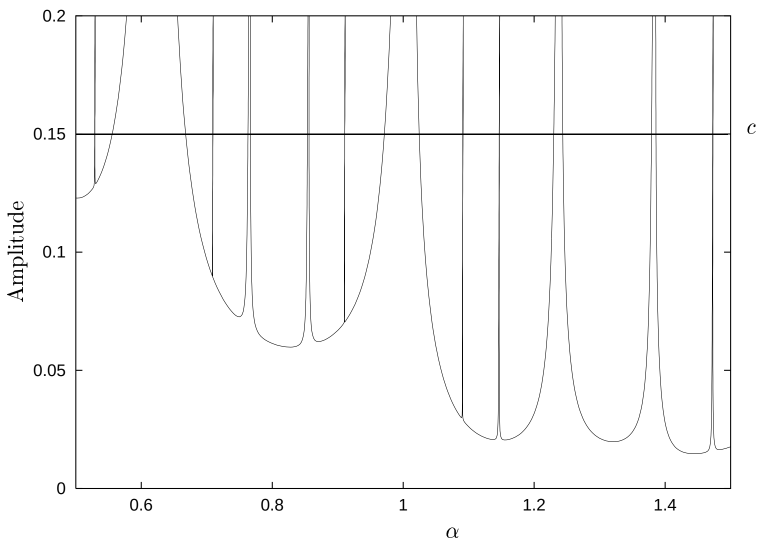

Figure 1: Response diagram of (1). Shown are the parameter α on the horizontal axis, against the L^2 norm of the response solution on the vertical axis. Classical KAM–theory can only prove the existence of invariant tori whose L^2–norm is below some bound c, which is of the same order $O(\varepsilon)$ as the forcing, as $\varepsilon \to 0$. Moreover the normal dynamics of these invariant tori should be elliptic, which fails to be the case for $1:2$ resonances (the corresponding gaps are not shown in this figure).

2 Phenomenology

To illustrate this dynamical phenomenon, numerical simulations of (1) have been performed for the case $n = 2$, $\omega = (1, -\frac{1}{2} + \frac{1}{2}\sqrt{5})$, cf. [7]. We have taken $G(\theta) = (3 + \cos\theta_1 + \cos\theta_2)^{-1}$, whose Fourier coefficients are all non–zero. The perturbation strength has been set to $\varepsilon = 0.05$.

Response solutions have been computed numerically [8, 10] for a range of values of the parameter α. The results are summarised in Figure 1, where on the vertical axis the L^2 norm of the Fourier series is plotted against the parameter α. In the figure, it can be seen clearly how invariant tori are 'pushed away' by $1:1$ resonances; however, these resonances do not destroy the tori. As remarked before, 'classical' KAM theory handles only the case that response solutions have an amplitude of order $O(\varepsilon)$: this corresponds to the lower part of the curves. The 'spikes' are cut off, and hence correspond to the 'gaps' in the Cantor sets for which classical KAM theory fails to find invariant tori. These gaps will be almost completely 'filled' by the results of the present paper.

Another case for which gaps occur in the KAM Cantor sets is for $1:2$ resonances. Although in that case invariant tori are not pushed away, their normal linear behaviour changes from elliptic to hyperbolic and back again. Classical KAM theory also fails for

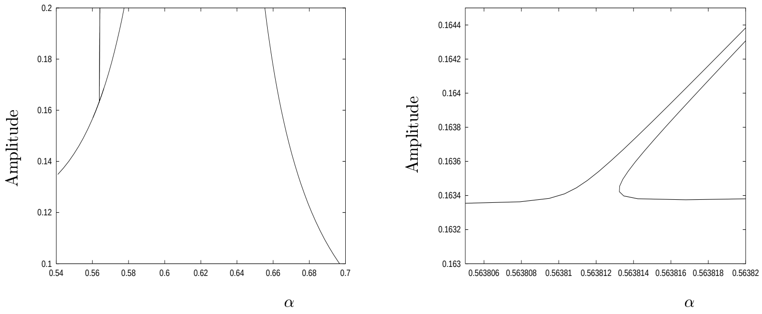

Figure 2: Two response diagrams associated to the $1:1$ normal–internal resonance. The left plot shows the big resonance for α near the golden number $-\frac{1}{2}+\frac{1}{2}\sqrt{5}=0.618\ldots$, and the right plot is a magnification of the 'secondary' gap that appears in the left plot.

these tori: this gives rise to another kind of 'gap' in the KAM Cantor sets, which will be filled completely by the present paper. Note that this kind of gap is not present in schemes that do not compute the normal behaviour of the tori [1, 2].

Both kind of gaps are already present in the case of periodic forcing [13] $n=1$ where the resonance gaps are isolated. In our quasi–periodic case, these resonance phenomena reappear, but now forming a dense set. In fact, in the case $n=1$ the parameter values of α for which $k\omega+\ell\alpha=0$ for some fixed ℓ — the resonances — form a set of isolated points; the corresponding set for $n\geq 2$ is dense.

This denseness of resonances leads us to expect 'secondary' resonances located on the 'primary' resonance curves. To illustrate these, the left–hand picture in figure 2 zooms in on the interval $[0.54, 0.70]\subset[0.5, 1.5]$ of Figure 1. In this way the 'primary' resonance at $\alpha=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ is magnified and a 'secondary' resonance at $\alpha\approx 0.563813$ becomes visible. Although resonances form a dense set, the width of the associated gaps decreases rapidly (in first approximation, this width is proportional to the corresponding Fourier

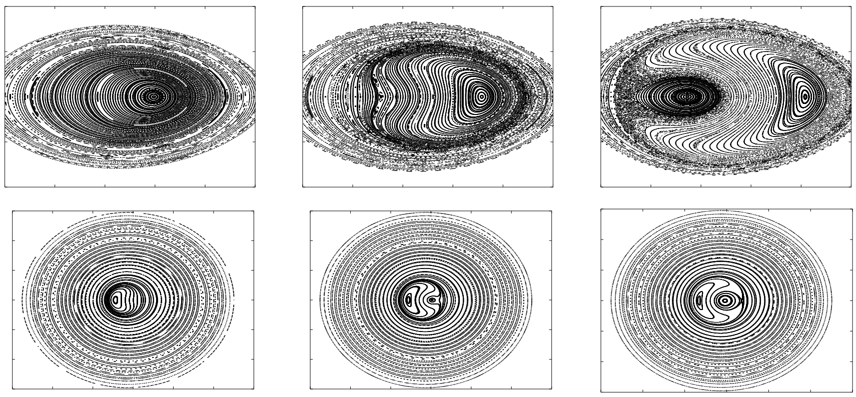

Figure 3: Phase portraits associated to the *1:1*–resonance, the first three for $\alpha\approx-\frac{1}{2}+\frac{1}{2}\sqrt{5}$, and the other three for 'a gap within that gap' ($\alpha\approx 0.563813$, see Figure *2*, left).

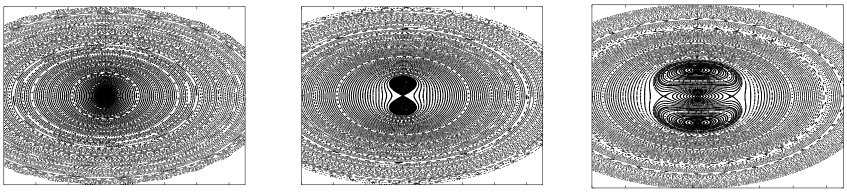

Figure 4: Phase portraits associated to a *1:2*–resonance, for $-3(-\frac{1}{2}+\frac{1}{2}\sqrt{5})+2\alpha \approx 0$.

coefficient of the forcing function $G(\theta)$) and the numerical detection of these phenomena becomes difficult.

In order to obtain a two–dimensional impression of the dynamics in phase space, 'quasi–periodic' Poincaré sections of system (1) are displayed in Figure 3. These are constructed by taking the first return map under the flow of (1) to the set $\Sigma = \{\theta_1 = 0,\ \theta_2 \in [a,b]\} \times \mathbb{R}^2$, where $[a,b]$ is a small interval, and then plotting the resulting points only using their $\mathbb{R}^2$–coordinates.

Figure 4 shows three slices of the phase space for the case $k=(0,-3)$, $\ell=2$, clearly showing a period doubling bifurcation.

3 On the proofs

The previous (numerical) results can be put into a rigorous setting [3]. Here we simply outline the scheme of the proof.

The Hamiltonian is simplified in two steps. First an integrable normal form plus a small remainder is computed. Second, after a better adapted choice of angles, the system is lifted (by a van der Pol transformation) to a covering space suited for the resonance at hand.

After truncation and reduction, a one degree of freedom system is obtained, which is the same as in the periodic case. Then, this one degree of freedom system is investigated for its own sake, since it forms the backbone of our perturbation problem. Our main interest is with the (relative) equilibria, since they correspond to response solutions in the full system. The analysis is simplified by suitable scalings, allowing us to retain only significant lowest order terms of the system.

The corresponding dynamics in $n+1$ degrees of freedom of the integrable part of the normal form is reconstructed using structural stability arguments. Finally, we re–incorporate the non–integrable terms neglected earlier on, and KAM techniques [4, 5, 6, 9, 11, 12, 14, 15] are applied to show persistence of response solutions, solving the original perturbation problem. Actually, in the analytic case, by using normal forms with exponentially small remainders, the new gaps in the parameter space are shown to be exponentially small in the perturbation parameter ε.

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