Hamiltonian Bifurcations of Invariant Tori with a Vanishing Floquet Exponent

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27 August 2003

Abstract

Local bifurcations of invariant tori are induced by the normal behaviour and occur where the latter changes from elliptic to hyperbolic. For invariant tori in Floquet form this is described by the Floquet exponents. When one of the Floquet exponents vanishes even the persistence of the bifurcating tori themselves is in question. With the actions conjugate to the toral angles serving as unfolding parameters, one can look for persistence of the pertinent bifurcation scenario instead. Such a study lies at the intersection of KAM theory and singularity theory. This paper presents the results of current research and ends with a speculation on how far the borders may be pushed.

1 Introduction

The distribution in phase space of maximal invariant tori of (nearly) integrable Hamiltonian systems (as obtained from KAM theory) is governed by lower dimensional invariant tori. Different families of maximal tori are separated by the stable and unstable manifolds of normally hyperbolic tori and may shrink down to normally elliptic tori. In this way the lower dimensional tori determine the dynamics of integrable Hamiltonian systems, and one is therefore interested in persistence of lower dimensional invariant tori under (small) non–integrable perturbations.

In [Mel'nikov;65] a first such theorem had been formulated. Next to a strong form of non–resonance of the internal frequencies, for lower dimensional tori the symplectic normal behaviour becomes important as well. For the unperturbed integrable system the dynamics near the torus can be put in *Floquet form*

$$\begin{aligned} \dot{x} &= \omega(y) + \mathcal{O}(y,z) \\ \dot{y} &= 0 \\ \dot{z} &= \Omega z + \mathcal{O}(yz,z^2) \end{aligned} \tag{1}$$

with an x–independent matrix $\Omega \in \mathfrak{sp}(2m,\mathbb{R})$. The m eigenvalues Λ_j of Ω (where the eigenvalues $-\Lambda_j$ have been omitted) are called *Floquet exponents* and in [Mel'nikov;65] the Diophantine conditions

$$|2\pi \mathrm{i}(k \mid \omega) + (\ell \mid \Lambda)| \;\geq\; \frac{\gamma}{|k|^\tau} \qquad \forall_{k\in\mathbb{Z}^n\setminus\{0\}} \quad \forall_{\ell\in\mathbb{N}_0^m\, ,\, |\ell|\leq 2} \tag{2}$$

are used. While (2) does put restrictions also on exponents $\Lambda_j \notin \mathrm{i}\mathbb{R}$, such hyperbolic normal linear behaviour can be dealt with by means of a centre manifold. This makes (2) a non–resonance condition involving internal and normal frequencies.

The n actions y_i do not suffice to completely control occurring normal frequencies next to the n internal frequencies. For a single normal frequency $\omega_0 := \frac{\Lambda_1}{2\pi\mathrm{i}}$ the control of the freqency ratio $[\omega_0 : \omega_1 : \ldots : \omega_n]$ was used in [Moser;67] to prove persistence of lower dimensional tori. When there are more normal frequencies the control of internal and normal frequencies by the actions $y \in \mathsf{Y} \subseteq \mathbb{R}^n$ has to be relaxed even further. The condition on the frequency mapping, similar to (5) below, amounts to imposing that the image of that mapping be sufficiently curved within the space of all frequencies. This prevents that the occurring frequencies become "trapped" in a resonance gap and allows to conclude that "most" invariant tori satisfy (2). For more details the reader is referred to [Broer,Huitema,Sevryuk;96] and references therein.

In the periodic case $n = 1$ the Diophantine conditions (2) prevent that the periodic orbit undergoes a centre–saddle, period–doubling or Hamiltonian Hopf bifurcation. During the latter two bifurcations the periodic orbit merely loses (dynamical) stability, but as the Floquet multiplier at the bifurcation is (still) different from 1 one has structural stability of the periodic orbit as such. Correspondingly, persistence has been shown in [Xu,You;01] for tori that satisfy the inequalities in (2) only with $|\ell| \leq 1$. Thus, next to the inevitable Diophantine condition with $\ell = 0$ in (2) the frequencies have to be bounded away from the resonances

$$\Lambda_j \;=\; 2\pi\mathrm{i}(k \mid \omega) \quad , \quad j = 1, \ldots, m \quad .$$

The present paper addresses lower dimensional tori where one of these resonant equalities does hold, say for $j = 1$. After a better adapted choice of the toral angles x_i this resonance turns into

$$\Lambda_1 \;=\; 0 \quad ,$$

cf. [Xu,You;01]. Such a vanishing Floquet exponent leads to (quasi–periodic) bifurcations, the simplest being the centre–saddle bifurcation treated in [Hanßmann;98]. In particular one has to impose conditions on the nonlinear terms in the $\dot{z}$ equation in (1) as we shall do in the next section. From now on we restrict to $m = 1$ and write $z = (q, p)$ with conjugate co–ordinates $q, p \in \mathbb{R}$. Thus, the lower dimensional tori with vanishing Floquet exponent we consider are invariant n–tori in $n+1$ degrees of freedom.

2 The integrable dynamics

On the phase space $\mathcal{P} = \mathbb{T}^n \times \mathsf{Y} \times \mathbb{R}^2$ with symplectic structure $\sum \mathrm{d}x_i \wedge \mathrm{d}y_i + \mathrm{d}q \wedge \mathrm{d}p$ we consider a real analytic Hamiltonian function of the form

$$H(x, y, q, p) \quad = \quad h(y) \; + \; \mathcal{H}_y(q, p) \quad . \tag{3}$$

As $H, y_1, \ldots, y_n$ are $n+1$ conserved quantities in involution the Hamiltonian system defined by (3) is clearly integrable. In fact, the integrals $y_1, \ldots, y_n$ are the components of the *momentum mapping* of the n–torus symmetry

$$\begin{array}{ccc} \mathbb{T}^n \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ (\xi, (x, y, q, p)) & \mapsto & (x + \xi, y, q, p) \end{array}$$

of (3). Reducing this symmetry yields a one–degree–of–freedom problem with a family of planar Hamiltonians $\mathcal{H}_y$ depending on the value y of the momentum mapping. The equilibria of the reduced system correspond precisely to the invariant n–tori in $\mathcal{P}$.

The local study of such families of planar Hamiltonian systems amounts to studying planar singularities and their versal unfoldings. While the dynamical behaviour of saddles and centres is governed by the quadratic part of the Hamiltonian, the higher order terms of $\mathcal{H}_y(q, p)$ become decisive where bifurcations occur. We first consider the *simple singularities*.

Normally parabolic tori lead to the one–degree–of–freedom problem of a 1–dimensional particle in a potential. This is described by the unfolding of the singularity A_k , $k \geq 1$:

$$\mathcal{H}_y(q, p) \quad = \quad \frac{a(y)}{2} p^2 \; + \; \frac{b(y)}{(k+1)!} q^{k+1} \; + \; \sum_{j=1}^{k-1} \frac{c_j(y)}{j!} q^j$$

with $a(0), b(0) \neq 0$ and $c(0) = 0$ — we can always arrange the most degenerate singularity to occur for $y = 0$ and at the origin $(q, p) = (0, 0)$. The index k labels the *multiplicity* of the singularity, whence the co–dimension is $k - 1$. In particular, we get for $k = 2$ the centre–saddle bifurcation, with $c_1(y)$ playing the rôle of the bifurcation parameter.

When there are no quadratic terms the matrix Ω in (1) vanishes completely. This is a co–dimension three phenomenon, and the (unfolded) singularities of type D_k , $k \geq 4$:

$$\mathcal{H}_y(q, p) \quad = \quad \frac{a(y)}{2} p^2 q \; + \; \frac{b(y)}{(k-1)!} q^{k-1} \; + \; \sum_{j=1}^{k-2} \frac{c_j(y)}{j!} q^j \; + \; c_{k-1}(y) p$$

are related to the umbilic catastrophes, cf. [Bröcker,Lander;75]. The phase portrait does not have an immediate physical interpretation as in the parabolic case.

Next to these two infinite series there is a third (finite) series of three simple singularities E_6, E_7, E_8 , see [Arnol'd,Gusein–Zade,Varchenko;82], [Hanßmann;03]. The singularities A, D, E are simple because only finitely many "types" occur. It is immediate to scale $a(y) \equiv 1$ and one can also achieve at least $b(y) \equiv \pm 1$ — there are dynamically different cases $A^{\pm}_{2\ell+1}$ and $D^{\pm}_{2\ell}$. The latter transformation is not symplectic, and the full freedom of singularity theory to use general (not area–preserving) diffeomorphisms of the plane is also needed to transform away higher order terms. As shown in [Broer,Chow,Kim,Vegter;93] this use of non–canonical transformations in one degree of freedom merely results in a reparametrisation of time — the two flows before and after the transformation are no longer conjugate, but only equivalent.

The reconstruction of the flow defined by (3) in $n+1$ degrees of freedom is straightforward. While planar periodic orbits give rise to maximal tori, the equilibria (q_0, p_0) of $\mathcal{H}_{y=y_0}$ lead to lower dimensional tori

$$\mathbb{T}^n \times \{y_0\} \times \{(q_0, p_0)\} \quad \subseteq \quad \mathcal{P}$$

with frequency vector $\omega = \frac{\partial H}{\partial y}(y_0, q_0, p_0)$. In particular, the distribution of lower dimensional tori is governed by the universal unfolding of the central (planar) singularity $\mathcal{H}_0$, provided that the mapping

$$\begin{array}{rccc} c: & \mathsf{Y} & \longrightarrow & \mathbb{R}^{k-1} \\ & y & \mapsto & c(y) \end{array} \tag{4}$$

is a submersion.

3 The perturbed dynamics

The condition that the mapping (4) be a submersion puts an upper bound $k-1 \leq n$ on the co–dimension of occurring lower dimensional invariant n–tori. Where this bound is attained the n–torus in question is isolated and under perturbation the torus may fall into a resonance gap. Therefore we sharpen our assumption on occurring co–dimensions for the singularities $\mathcal{H}_y(q,p)$ in (3) to $k-1 \leq n-1$. Then we have the following result.

Theorem 3.1 *On the phase space $\mathcal{P}$ consider the Hamiltonian* (3) *where $\mathcal{H}_y(q,p)$ is one of the singularities A_k, D_k, E_k with $k \leq n$. Let* (4) *be a submersion and let the $\binom{n+2}{n}$ vectors*

$$\frac{\partial^{|\ell|}}{\partial y^\ell} \begin{pmatrix} c \\ \omega \end{pmatrix} (y) \quad , \quad |\ell| \leq 2 \tag{5}$$

span $\mathbb{R}^{k-1} \times \mathbb{R}^n$ for all $y \in \mathsf{Y}$, where $\omega(y) := Dh(y)$. Then the dynamics defined by a sufficiently small perturbation K of H on $\mathcal{P}$ admit a Cantor stratification into families of maximal and lower dimensional tori induced by the stratification of the unperturbed dynamics described in the previous section, with all continuous parametrisations replaced by parametrisations of Cantor sets with large relative Hausdorff measure.

The proof relies on keeping track of the most degenerate lower dimensional torus, the one corresponding to the central singularity $(y, q, p) = (0, 0, 0)$. To this end a KAM iteration scheme is used performing two operations at each iteration step. First the "lower order terms" are made x–independent. Here one has to deal with small denominators to solve a homological equation. Then these "lower order terms" are transformed into the universal unfolding of the central singularity. This is achieved by explicit shear transformations $p \mapsto p + \tau(q)$ and translations $q \mapsto q + c$. For more details the reader is referred to [Broer,Hanßmann,You;03,04], [Hanßmann;03].

4 Non–simple planar singularities

Further singularities $\mathcal{H}_y(q, p)$ have *moduli*. This notion refers to a coefficient $m(y)$ of a monomial $p^k q^l$ that provides a continuous labelling of non–equivalent central singularities. Recall that we were able to reduce the coefficients $a(y)$ and $b(y)$ in the simple singularities A_k, D_k, E_k to a discrete set $a(y) \equiv 1$, $b(y) \equiv \pm 1$. For singularities with moduli we have to distinguish the co–dimension of the individual singularities from the co–dimension of the whole family — it is this latter number that determines how many parameters are necessary for a family "in general position" to encounter such singularities. The difference between these two co–dimensions is precisely the number of moduli. The singularity with lowest co–dimension that has a modulus is $J_{2,0}$:

$$\mathcal{H}_y(q, p) \;=\; \frac{a(y)}{6} p^3 \;+\; \frac{m(y)}{4} p^2 q^2 \;+\; \frac{b(y)}{24} p q^4 \;+\; \sum_{j=1}^{5} \frac{c_j(y)}{j!} q^j \;+\; \sum_{j=0}^{2} \frac{c_{j+6}(y)}{j!} p q^j$$

where one can scale $a(y) \equiv 1$ and achieve $b(y) \equiv \pm 1$, while $m(y)$ essentially acts as the 9th unfolding coefficient. The reason that one cannot find a smooth transformation that gets rid of the term $p^2 q^2$ is that it is impossible to simultaneously transform three given parabolas to three predescribed normal form parabolas.

The proof sketched in the previous section can be adapted to show that under the conditions of Theorem 3.1 with, say, ten degrees of freedom and singularity $\mathcal{H}_y(q, p)$ given by $J_{2,0}$, a sufficiently small perturbation K of (3) has a 1–dimensional Cantor family of lower dimensional 9–tori with normal behaviour described by $J_{2,0}$. The situation is in fact even better since there are only finitely many non–equivalent one–degree–of–freedom flows with Hamiltonians $\mathcal{H}_{y=0}$ of type $J_{2,0}$ if one allows the equivalence between the flows to be a homeomorphism instead of a diffeomorphism.

In the proof of Theorem 3.1 the full power of singularity theory is not needed as it is only applied to the "lower order terms" where one can use explicit transformations. The "higher order terms" are always relegated to the next iteration step and thus dealt with by the rapid convergence of the KAM iteration scheme. For singularities with high co–dimension there also appear moduli as coefficients of "higher order terms", an example (of co–dimension twelve) is given by

$$\frac{a(y)}{5!}p^5 + \frac{m_1(y)}{12}p^3q^2 + \frac{m_2(y)}{12}p^2q^3 + \frac{b(y)}{5!}q^5 + \frac{m_3(y)}{36}p^3q^3 \quad ,$$

cf. [Arnol'd,Gusein–Zade,Varchenko;82]. During the iteration process one loses control on this third modulus $m_3(y)$.

Acknowledgements. I thank Henk Broer, Àngel Jorba, Jordi Villanueva, Florian Wagener and Jiangong You for helpful discussions. Support from the FOM grant MFGBb1 is gratefully acknowledged.

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