Hamiltonian Torus Bifurcations Related to Simple Singularities

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26 August 2003

Abstract

Lower dimensional n–tori in integrable and nearly integrable Hamiltonian systems with $n+1$ degrees of freedom are considered. The one–degree–of–freedom dynamics normal to the invariant tori is governed by (the singularities of) a family of planar Hamiltonian functions. The ensuing bifurcation scenarios are shown to survive both integrable and non–integrable perturbations.

1 Introduction

While equilibria of a generic Hamiltonian system are isolated, the periodic orbits form 1–parameter families and may therefore undergo bifurcations of co–dimension one. Families of invariant n–tori are parametrised by the n actions conjugate to the toral angles, so the co–dimensions of occurring bifurcations range from n (at isolated parameter values) over $n-1$ (along 1–paremeter families) to 1 (where we recover the above "simplest" bifurcations of periodic orbits).

In $n+1$ degrees of freedom the invariant n–tori determine the distribution of the (maximal) $(n+1)$–tori in phase space. Indeed, the stable and unstable manifolds of normally hyperbolic n–tori separate different families of maximal tori, and the latter may shrink down to normally elliptic n–tori. At bifurcations the lower dimensional tori may cease to exist or change their normal linear behaviour — which can be normally parabolic at the bifurcation or vanish completely. Such bifurcations have been studied in [Hanßmann;98], [Broer,Hanßmann,You;03,04] and the present paper aims to review and extend these results.

We work on the phase space $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ with (standard) symplectic structure

$$\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \;+\; \mathrm{d}q \wedge \mathrm{d}p \quad . \tag{1}$$

Here T is an n–torus with toral angles $x = (x_1, \ldots, x_n)$, conjugate actions $y \in \mathsf{Y} \subseteq \mathbb{R}^n$ and symplectic normal space $\mathsf{S} \subseteq \mathbb{R}^2$. We consider (integrable) x–independent Hamiltonian functions which we write as

$$H(x, y, q, p) \;=\; h(y) \;+\; \mathcal{H}_y(q, p) \tag{2}$$

with $\mathcal{H}_y(0,0) = 0$ for all $y \in \mathsf{Y}$. Denoting the frequencies by $\omega_i(y) = \frac{\partial h}{\partial y_i}(y)$, $i = 1, \ldots, n$, the invariant n–tori of the Hamiltonian system

$$\begin{aligned} \dot{x} &= \omega(y) + \nabla_y \mathcal{H}_y(q,p) \quad , \qquad \dot{y} = 0 \\ \dot{q} &= \frac{\partial \mathcal{H}_y}{\partial p}(q,p) \quad , \qquad \dot{p} = -\frac{\partial \mathcal{H}_y}{\partial q}(q,p) \end{aligned}$$

are given by the *singularities* of the family $\mathcal{H}_y$ of planar Hamiltonians, *i.e.* the points $(q,p) \in \mathsf{S}$ with $D\mathcal{H}_{y_0}(q,p) = 0$ where $y_0 \in \mathsf{Y}$ is fixed. Under variation of the parameter $y - y_0$ these singularities are in turn unfolded.

We will always arrange the most degenerate singularity to occur at $y_0 = 0$. While the co–dimension k of the most degenerate singularity in an n–parameter family does not have to attain the maximal number n , we indeed exclude singularities with co–dimension $k > n$ — this is a genericity condition on our Hamiltonian (2). In this way we obtain results for "typical" Hamiltonian systems.

Singularities of functions and mappings have been thoroughly studied, cf. [Bröcker,Lander;75], [Arnol'd,Gusein–Zade,Varchenko;82] and references therein. Specifically, all isolated singularities of planar functions with non–vanishing 4–jet are classified in [Arnol'd, Gusein–Zade,Varchenko;82]. Here we content ourselves with the three series A_{k+1}, D_{k+1} and E_6, E_7, E_8 of so–called *simple* singularities. Together with their universal unfoldings these are given by

$$\begin{aligned}
A_{k+1} &: \quad \mathcal{H}_y(q,p) = \frac{a(y)}{2}p^2 + \frac{b(y)}{(k+2)!}q^{k+2} + \sum_{j=1}^{k} \frac{y_j}{j!}q^j \\
D_{k+1} &: \quad \mathcal{H}_y(q,p) = \frac{a(y)}{2}p^2 q + \frac{b(y)}{k!}q^k + \sum_{j=1}^{k-1} \frac{y_j}{j!}q^j + y_k p \\
E_6 &: \quad \mathcal{H}_y(q,p) = \frac{a(y)}{6}p^3 + \frac{b(y)}{24}q^4 + \sum_{j=1}^{2} \frac{y_j}{j!}q^j + \sum_{j=0}^{2} \frac{y_{j+3}}{j!}pq^j \\
E_7 &: \quad \mathcal{H}_y(q,p) = \frac{a(y)}{6}p^3 + \frac{b(y)}{6}pq^3 + \sum_{j=1}^{4} \frac{y_j}{j!}q^j + \sum_{j=0}^{1} \frac{y_{j+5}}{j!}pq^j \\
E_8 &: \quad \mathcal{H}_y(q,p) = \frac{a(y)}{6}p^3 + \frac{b(y)}{5!}q^5 + \sum_{j=1}^{3} \frac{y_j}{j!}q^j + \sum_{j=0}^{3} \frac{y_{j+4}}{j!}pq^j
\end{aligned} \tag{3}$$

with $a(0) \cdot b(0) \neq 0$, and Y is chosen so small that both coefficients are bounded away from zero. We also work local in (q,p) and let S be a sufficiently small neighbourhood of the origin.

The A, D, E singularities are called simple because they do not have *moduli*, *i.e.* it is possible to transform the coefficient functions $a(y)$ and $b(y)$ away. The reason we give these singularities in the above form is that in applications it is much easier to simply check the (relative) sign of $a(y)$ and $b(y)$ than to perform the actual transformations we sketch now. Reversing time by multiplying $\mathcal{H}_y$ with -1 , if necessary, we may assume $a(y) > 0$. Then the symplectic transformation

$$(x, y, q, p) \quad \mapsto \quad \left(x - \frac{\alpha'(y)}{\alpha(y)} pq \, , \, y \, , \, \frac{1}{\alpha(y)} q \, , \, \alpha(y) p \right)$$

with $\alpha(y) = \sqrt{a(y)}$ allows to put $a(y) \equiv 1$. For E_6, E_7, E_8 the initial multiplication by -1 is not necessary and $\alpha(y) = \sqrt[3]{a(y)}$. In this way the coefficient $b(y)$ gets replaced by $\beta(y) = \pm b(y) \cdot (\alpha(y))^\ell$ with a suitable $\ell \in \mathbb{N}$ depending on the singularity. If $\beta(0) < 0$ the π–rotation $(q, p) \mapsto (-q, -p)$ followed, if necessary, by another reversion of time $\mathcal{H}_y \mapsto -\mathcal{H}_y$ allows to achieve $\beta(0) > 0$ except for A_{k+1} with even k and D_{k+1} with odd k . Here we have indeed different cases that are accordingly labelled as $A^\pm_{2\ell+1}$ and $D^\pm_{2\ell}$, whence we arrive again at $\beta(0) > 0$ with the explicit sign put in front of the second term.

The final transformation that yields $\beta(y) \equiv 1$ does not preserve the symplectic form (1). For the A and E singularities we simply scale q , while we have to rescale p for the D singularities as well to retain $a(y) \equiv 1$. As shown in [Broer,Chow,Kim,Vegter;93] non–symplectic transformations provide equivalences between planar Hamiltonian systems like that defined by $\mathcal{H}_y$, see also Section 2 below.

The above transformations affect the unfolding coefficients y_j and one might indeed more generally consider unfolding coefficients $c_j(y)$ to start with. One would then require the function $c : \mathsf{Y} \longrightarrow \mathbb{R}^k$ to be submersive, see [Broer,Hanßmann,You;03,04]. The necessary changes are easily performed and left to the reader.

Theorem 1.1 *On the phase space* $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *with symplectic structure* (1) *let* $G = G(x, y, q, p)$ *be a small perturbation of the integrable Hamiltonian* (2)*, where* $\mathcal{H}_y$ *is one of the simple singularities* (3)*. Then one has persistence of the lower dimensional tori and their bifurcations.*

The statement of this theorem is deliberately vague and the aim of this paper is to give it a precise meaning. In the next section we assume G to be x–independent as well. In this simplified setting the rôle of the necessary assumptions becomes very transparent. Section 3 addresses the general case, with emphasis given to the singularities E_6, E_7, E_8 . Possible generalizations, in particular to torus bifurcations related to non–simple singularities, are pointed out in the last section.

2 The case of integrable perturbations

In this section we are given an x–independent perturbation $G(x, y, q, p) = G(y, q, p)$ of (2). Both H and G are invariant under the n–torus action

$$x \mapsto x + \xi \tag{4}$$

and can therefore be reduced to one degree of freedom. For H this results in the n–parameter family $\mathcal{H}_y(q, p)$, versely unfolding one of the simple singularities (3). For G this similarly yields an n–parameter family $\mathcal{G}_y(q, p)$ of planar Hamiltonians.

From singularity theory [Bröcker,Lander;75] we obtain a family of local diffeomorphisms $\Phi_y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and a reparametrisation $c : \mathsf{Y} \longrightarrow \mathbb{R}^k$ such that

$$\mathcal{G}_y \circ \Phi_y = \mathcal{H}_{c(y)} \qquad \forall_{y \in \mathsf{Y}} \quad . \tag{5}$$

Since G is a small perturbation of H we have that Φ is close to the identity and c is close to the projection onto the first k components of $y \in \mathsf{Y} \subseteq \mathbb{R}^n$, *i.e.* we can complete c to a local diffeomorphism $\tilde{c} : \mathsf{Y} \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ by putting $\tilde{c}_j(y) = y_j$ for $j = k+1, \ldots, n$.

In general Φ will not preserve the symplectic structure (*i.e.* the area element), but let us assume for the moment that we do have $\det D\Phi_y \equiv 1$. Then the co–ordinate transformation

$$(x, y, q, p) \mapsto (\gamma(x, y, q, p), \tilde{c}(y), \Phi_y(q, p)) \tag{6}$$

with

$$\gamma(x, y, q, p) = (D\tilde{c}(y))^{-1} \left(x - \int_0^q \frac{\partial \Phi_y^1}{\partial q} \frac{\partial \Phi_y^2}{\partial y} \, \mathrm{d}\tilde{q} + \int_0^p \frac{\partial \Phi_y^1}{\partial y} \frac{\partial \Phi_y^2}{\partial p} \, \mathrm{d}\tilde{p} \right)$$

is symplectic and provides a local conjugation between the Hamiltonian vector fields X_H and X_G .

A slight caveat is that (6) is global with respect to the toral angles x only when $(D\tilde{c}(y))^{-1}$ maps a fundamental domain of T onto a fundamental domain. This can be remedied by ignoring slight rescalings in y , or in fact by considering more general coefficient functions in the universal unfoldings (3) from the beginning, as already discussed in the introduction.

We now return to the general case where Φ is not necessarily area–preserving. The above discussion shows that we may concentrate on the relation between the one–degree–of–freedom problems defined by $\mathcal{H}_y(q, p)$ and $\mathcal{G}_y(q, p)$. The equality (5) implies

$$X_{\mathcal{G}_y} \circ \Phi_y = \det D\Phi_y \cdot X_{\mathcal{H}_{c(y)}} \qquad \forall_{y \in \mathsf{Y}} \quad ,$$

cf. [Broer,Chow,Kim,Vegter;93]. Thus Φ_y provides for all $y \in \mathsf{Y}$ an equivalence of the two Hamiltonian vector fields $X_{\mathcal{G}_y}$ and $X_{\mathcal{H}_{c(y)}}$, where the deviation $\det D\Phi_y$ from area–preservation determines the reparametrisation of time. Superposing the quasi–periodic flows of $\dot{x} = \frac{\partial H}{\partial y}$ and $\dot{x} = \frac{\partial G}{\partial y}$ and re–arranging the parametrising actions according to c yields the desired persistence result.

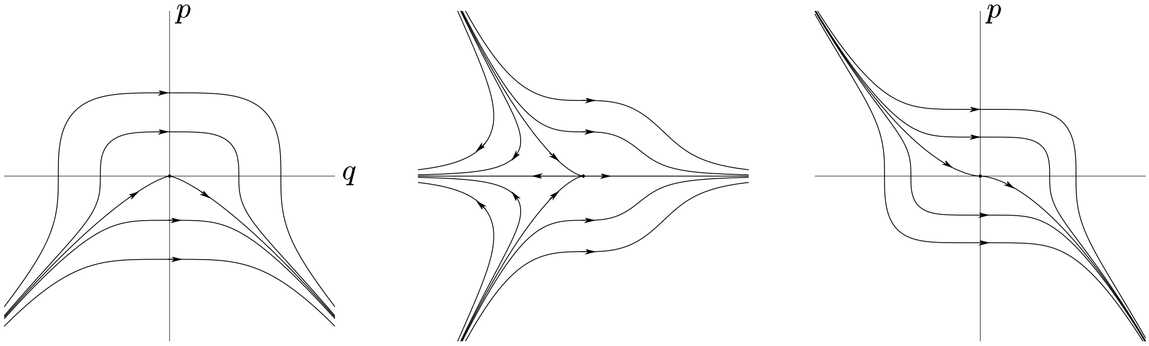

Figure 1: The planar phase flows defined by the central singularities E_6, E_7 and E_8 .

The above approach shows that our main interest does not lie in the singularities (3) as functions, but only in their (energy) level sets. For instance, the phase portraits of *Figure 1* are obtained by plotting level sets.

3 Persistence under non–integrable perturbations

In this section we consider a general perturbation $G = G(x, y, q, p)$ of (2) and again ask for "persistence" of the lower dimensional tori and their bifurcations. The dependence of G on x breaks the n–torus symmetry (4). As a consequence, those invariant tori

$$\mathsf{T} \times \{y_0\} \times \{(q_0, p_0)\} \quad \subseteq \quad \mathsf{T} \times \mathsf{Y} \times \mathsf{S} \tag{7}$$

that are resonant may break up. To obtain persistence of (7) when passing from the integrable Hamiltonian system X_H to X_G one therefore requires *Diophantine conditions* to hold, a strong form of non–resonance. The simplest case is that the equilibrium (q_0, p_0) of the one–degree–of–freedom system $X_{\mathcal{H}_{y_0}}$ is a saddle. Here the Diophantine conditions only concern the internal frequency vector $\omega = Dh(y_0)$ and read

$$|(k \mid \omega)| \quad \geq \quad \frac{\gamma}{|k|^\tau} \qquad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \tag{8}$$

with $\tau > n - 1$ and $\gamma > 0$. By $(.. \,|\, ..)$ we denote the standard inner product. If (q_0, p_0) is a centre of $X_{\mathcal{H}_{y_0}}$ the normal linear behaviour enters the Diophantine conditions as well. Denoting by Ω the *normal frequency*, *i.e.* the imaginary part of the eigenvalues $\pm \mathrm{i}\Omega$ of $X_{\mathcal{H}_{y_0}}(q_0, p_0)$, one replaces (8) by

$$|(k \mid \omega) + \ell\, \Omega| \quad \geq \quad \frac{\gamma}{|k|^\tau} \qquad \forall_{k \in \mathbb{Z}^n \setminus \{0\}} \quad \forall_{\ell \in \{0, \pm 1, \pm 2\}} \tag{9}$$

(again with $\tau > n-1$ and $\gamma > 0$). For $\ell = 0$ this rephrases (8). For $\ell = \pm 1$ this excludes normal–internal resonances $\Omega = (k \mid \omega)$, which after a better adapted choice of co–ordinates $x_1, \ldots, x_n$ on T can be written as $\Omega = 0$, cf. [Xu,You;01]. This is exactly what happens for $y_0 = 0$ at $(q_0, p_0) = (0,0)$. The Diophantine conditions (9) with $\ell = \pm 2$ are necessary to ensure that the perturbed torus inherits the normally elliptic behaviour from the unperturbed torus. For the mere persistence of (7), without any control of the normal linear behaviour, it is sufficient to require (9) only with $\ell \in \{0, \pm 1\}$, see [Xu,You;01].

The equilibrium $(0,0)$ of $X_{\mathcal{H}_0}$ is neither a saddle nor a centre, so a mere control of the linear normal behaviour can no longer guarantee persistence and we have to take the nonlinear normal behaviour into account. We have seen in the previous section that the universal unfoldings (3) are sufficiently rich to yield persistence of all lower dimensional tori under integrable perturbations. The aim of this section is to obtain a similar result for non–integrable perturbations — provided that we are sufficiently far from resonance. The proof relies on a KAM iteration scheme and it turns out that (8) are again the Diophantine conditions we need.

To pull the collection of Diophantine tori from frequency space back into phase space one often uses a Kolmogorov–like condition

$$\det D\omega(y) \quad = \quad \det D^2 h(y) \quad \neq \quad 0 \tag{10}$$

to ensure that the frequency mapping $\omega : \mathsf{Y} \longrightarrow \mathbb{R}^n$ is a (local) diffeomorphism. Since the set of Diophantine frequency vectors $\omega \in \mathbb{R}^n$ is of large (relative) measure, this ensures that the Cantor family of Diophantine tori forms a subset of large relative (Hausdorff) measure within the set of all lower dimensional tori. This is in fact how one proves persistence of normally hyperbolic tori, after having solved the corresponding problem with the internal frequencies $\omega_1, \ldots, \omega_n$ acting as (external) parameters.

However, for the central tori $(y_0, q_0, p_0) = (0,0,0)$ unfolded in (3) we have only the last $n-k$ actions $y_{k+1}, \ldots, y_n$ left to control the internal frequencies. In particular, the central torus $(y_0, q_0, p_0) = (0,0,0)$ is isolated and may fall into a resonance gap if $k = n$. Therefore we sharpen our genericity condition $k \leq n$ and assume $k \leq n-1$ from now on. We will eventually overcome our "lack of parameter" problem by replacing (10) with a Rüssmann–like condition, see (12) below. But for the moment we completely decouple these questions from the KAM iteration and consider

$$H(x,y,q,p;\lambda,\omega) \quad = \quad (\omega \mid y) \ + \ \mathcal{H}_\lambda(q,p) \tag{11}$$

with (external) unfolding parameters $\lambda_1, \ldots \lambda_n$ and frequency parameters $\omega_1, \ldots \omega_n$. The key observation is that (11) is a *quasi–homogeneous polynomial* in (y,q,p,λ) of order μ with weight $(\alpha_y, \alpha_q, \alpha_p; \alpha_{\lambda_1}, \ldots, \alpha_{\lambda_k})$, cf. [Arnol'd,Gusein–Zade,Varchenko;82], [Broer,Hanßmann,You;03]. This notion amounts to

$$H(x, \mathrm{e}^{\varsigma\alpha_y} y, \mathrm{e}^{\varsigma\alpha_q} q, \mathrm{e}^{\varsigma\alpha_p} p; \mathrm{e}^{\varsigma\alpha_{\lambda_1}} \lambda_1, \ldots, \mathrm{e}^{\varsigma\alpha_{\lambda_k}} \lambda_k, \omega) \quad \equiv \quad \mathrm{e}^{\varsigma\mu} H(x,y,q,p;\lambda,\omega)$$

where the integers $\mu = \alpha_y, \alpha_q, \alpha_p, \alpha_{\lambda_1}, \ldots, \alpha_{\lambda_k}$ depend on the singularity (3) at hand.

For the singularity A_{k+1} we have $(\alpha_y, \alpha_q, \alpha_p; \alpha_{\lambda_j}) = (2k+2, 2, k+1; 2k-2j+4)$. When proving a persistence theorem the difficult part is to keep track of the most degenerate torus in the perturbed system. Lower dimensional tori that undergo subordinate bifurcations can then be obtained in an inductive manner. In this way persistence is proven in [Broer,Hanßmann,You;03] of Diophantine n–tori of the Hamiltonian system defined by (11) when $\mathcal{H}_\lambda$ is the unfolding of A_{k+1} .

For the singularity D_{k+1} we have $(\alpha_y, \alpha_q, \alpha_p; \alpha_{\lambda_j}, \alpha_{\lambda_d}) = (2k, 2, k-1; 2k-2j, k+1)$. Adapting the proof in [Broer,Hanßmann,You;03] a similar result is proven in [Broer,Hanßmann,You;04] for Diophantine n–tori in the unfolding related to D_{k+1} . Here one has, next to normally hyperbolic and elliptic tori, both normally parabolic and normally umbilic tori (of co–dimensions $k-1, k-2, \ldots, 3$) subordinate to the central normally umbilic tori. However, even for $k \geq 9$ no subordinate n–tori with normal behaviour E_6, E_7 or E_8 occur.

The difference between the proofs in [Broer,Hanßmann,You;03,04] is induced by the difference between the weights α of the parabolic and the umbilic singularities. For the singularities of type E we have

	μ	α_y	α_q	α_p	α_{λ_1}	α_{λ_2}	α_{λ_3}	α_{λ_4}	α_{λ_5}	α_{λ_6}	α_{λ_7}
E_6	12	12	3	4	9	6	8	5	2	–	–
E_7	27	27	6	9	21	15	9	3	18	12	–
E_8	15	15	3	5	12	9	6	10	7	4	1

and performing the necessary adaptions yields the following result.

Theorem 3.1 *Fix* $k \in \{5, 6, 7\}$ *, let* T *be an* n*–torus and* $\mathsf{Y} \subseteq \mathbb{R}^n$ *,* $\mathsf{S} \subseteq \mathbb{R}^2$ *,* $\Lambda \subseteq \mathbb{R}^k$ *be neighbourhoods of the respective origins. By* $\mathcal{O}'_\gamma$ *denote the set of frequency vectors* $\omega \in \mathcal{O} \subseteq \mathbb{R}^n$ *that satisfy the Diophantine condition* (8) *and have distance* γ *to* $\partial\mathcal{O}$ *.*

On the phase space $\mathsf{T} \times \mathsf{Y} \times \mathsf{S}$ *, with symplectic structure* (1)*, consider the analytic Hamiltonian function* $H : \mathsf{T} \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \mathcal{O} \longrightarrow \mathbb{R}$ *defined by* (11) *and* E_{k+1} *in* (3)*. Let the coefficient functions* $a, b : \mathcal{O} \longrightarrow \mathbb{R}$ *satisfy*

$$|a|_{\mathcal{O}} \, , \; |b|_{\mathcal{O}} \, , \; \left|\frac{1}{a}\right|_{\mathcal{O}} \, , \; \left|\frac{1}{b}\right|_{\mathcal{O}} \, , \; |Da|_{\mathcal{O}} \, , \; |Db|_{\mathcal{O}} \;\; < \;\; C$$

for some constant $C > 0$ *. Then there exists a small positive constant* ε *, independent of* $\mathcal{O}$ *, with the following property. For any analytic perturbation* G *of* H *with*

$$|G - H|_{\mathsf{T}\times\mathsf{Y}\times\mathsf{S}\times\Lambda\times\mathcal{O}} \; < \; \varepsilon$$

there exists a C^∞*–diffeomorphism* Φ *on* $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^k \times \mathcal{O}$ *such that*

1). Φ *is real analytic for fixed* ω *.*

2). Φ *is symplectic for fixed* (λ, ω) *.*

3). Φ *is* C^∞*–close to the identity.*

4). On $\mathsf{T} \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^k \times \mathcal{O}'_\gamma \;\cap\; \Phi^{-1}(\mathsf{T} \times \mathsf{Y} \times \mathsf{S} \times \Lambda \times \mathcal{O})$ *one can split* $G \circ \Phi = H_\infty + P_\infty$ *into an integrable part* H_∞ *and higher order terms* P_∞ *. Here* H_∞ *has the same form as* H *. The* x*–dependence is pushed into the higher order terms, i.e.* $\dfrac{\partial^{|j|+k+l+|m|} P_\infty}{\partial y^j \partial p^k \partial q^l \partial \lambda^m}(x,0,0,0,0,\omega) = 0$ *for all* $(x,\omega) \in \mathsf{T} \times \mathcal{O}'_\gamma$ *and all* j,k,l,m *satisfying* $\alpha_y |j| + \alpha_q k + \alpha_p l + \alpha_{\lambda_1} m_1 + \ldots + \alpha_{\lambda_k} m_k \le \mu$ *.*

The Hamiltonian vector field X_G *admits a Cantor stratification of a Cantor subset of* $\Lambda \times \mathcal{O}$ *of large measure into* $(n+k-j)$*–dimensional Cantor sets* C_j *such that* C_0 *parametrises Cantor families of normally hyperbolic and elliptic tori and* C_j *parametrises Cantor families of* n*–tori with normal behaviour governed by* A_{j+1}, D_{j+1} *or* E_{j+1} *.*

It remains to explain how the n actions $y_1, \ldots, y_n$ can replace the $n+k$ parameters $\lambda_1, \ldots, \lambda_k, \omega_1, \ldots, \omega_n$. First of all we need $\lambda : \mathsf{Y} \longrightarrow \mathbb{R}^k$ to be a submersion to pull back the entire bifurcation scenario. The remaining $n-k$ directions in Y do not allow for a full control of the frequencies. To obtain Cantor families of invariant n–tori (without trying to further connect the perturbed tori to the unperturbed ones) we require that the collection of $\binom{n+2}{n}$ vectors

$$\frac{\partial^{|\ell|}}{\partial y^\ell} \begin{pmatrix} c \\ \omega \end{pmatrix} (y) \quad , \quad |\ell| \le 2 \tag{12}$$

spans $\mathbb{R}^k \times \mathbb{R}^n$, cf. [Broer,Hanßmann,You;03,04]. Where $\omega(y)$ enters a resonance zone and fails to satisfy (8), this time with $\gamma > 0$ and $\tau > 2n - 1$, the curvature of the image $\operatorname{im}\omega$ of the frequency mapping ω (or even the remaining $n-k$ linear directions) ensure(s) that only a small portion of $\operatorname{im}\omega$ lies in that resonance zone.

4 Extensions and open problems

In the previous two sections we have seen how bifurcations of lower dimensional tori that are related to simple singularities survive small perturbations. An obvious generalization is to consider bifurcation scenarios related to singularities that are not simple. The "first" such singularity is

$$J^{\pm}_{2,0} \quad : \qquad \mathcal{H}(q,p) \;=\; \frac{a}{6}p^3 \;+\; \frac{m}{4}p^2q^2 \;+\; \frac{b}{24}pq^4 \quad , \quad a,b,m \neq 0\,,\; 9m^2 \neq 4ab$$

which needs already 9 parameters for a universal unfolding. Here $\pm = \operatorname{sgn}(ab)$ and as seen in the introduction one could rescale to $a = 1$ and $b = \pm 1$. However, the coefficient m is a modulus that cannot be further simplified — the transformation $p \mapsto p + \alpha q^2$ has already been used to transform the term q^6 away. Correspondingly, $\frac{\lambda_9}{4}p^2q^2$ is one of the unfolding terms ; a singularity of type $J^{\pm}_{2,0}$ persistently occurs in (isolated points of) 8–parameter families. Thus, the approach of Section 2 still applies if we adopt from Section 3 the sharper inequality $k < n$ between the co–dimension k and the number n of parametrising actions. Under non–integrable perturbations we obtain a Cantor subset of moduli. The actual (normal) dynamics defined by $X_{\mathcal{H}}$ do not form a continuum, but fall into finitely many different cases. Indeed, we do not need the dynamical equivalences to be diffeomorphisms, a homeomorphism that maps level sets to level sets is sufficient.

Up to now we were concerned with resonances violating (9) with $\ell = \pm 1$. Normal–internal resonances violating (9) with $\ell = \pm 2$ can best be studied by passing to a 2 : 1 covering phase space. A suitable choice of toral angles then leads again to $\Omega = 0$, but now the normal dynamics is invariant under the $\mathbb{Z}_2$–symmetry $(q,p) \mapsto (-q,-p)$ which is induced by the deck transformation of the 2 : 1 covering. In this way one is led to $\mathbb{Z}_2$–symmetric planar singularities and their $\mathbb{Z}_2$–symmetric universal unfoldings.

If we consider invariant n–tori in, say, $n+2$ degrees of freedom, there are further phenomena. Indeed, there may be two normal frequencies in $1 : -1$ resonance, leading to a quasi–periodic Hamiltonian Hopf bifurcation. Even more complicated dynamics may be expected if (9) is violated by two different normal–internal resonances as this leads to bifurcations (of co–dimension as low as two) that are not even understood for equilibria.

For non–integrable perturbations we obtained results only for internal frequencies satisfying the Diophantine conditions (8). Where this condition is violated we are in fact again close to two simultaneous resonances : an internal resonance $(k \mid \omega) = 0$ and the normal–internal resonance that can be given the form $\Omega = 0$. In [Litvak–Hinenzon,Rom–Kedar;02a,02b] lower dimensional tori undergoing a Hamiltonian pitchfork bifurcation are studied when one or several (low order) internal resonances are present.

Hamiltonian systems are often reversible, for instance the universal unfolding of the singularity A_{k+1} is invariant under $(q,p) \mapsto (q,-p)$, and then (2) is reversible with respect to $(x,y,q,p) \mapsto (-x,y,q,-p)$. In many cases this property lowers the co–dimension, for instance the universal unfolding invariant under

$$(q,p) \quad \mapsto \quad (-q,p) \tag{13}$$

of A_4 consists of the single term $\frac{\lambda}{2}q^2$ — the Hamiltonian pitchfork bifurcation mentioned above. For the simple singularities of type E the occurring symmetries can be read off from *Figure 1.* For E_6 the reversing symmetry (13) cuts the co–dimension down to three. The singularity E_8 defines a Hamiltonian system that is reversible with respect to $(q,p) \mapsto (-q,-p)$; while this preserves the symplectic structure the Hamiltonian funtion is multiplied by -1 . Here the co–dimension becomes four. The Hamiltonian system defined by E_7 is not reversible, but in fact symmetric with respect to $(q,p) \mapsto (q,-p)$ as again the Hamiltonian is multiplied by -1 ; again only four unfolding terms share this symmetry.

Acknowledgments. I thank Henk Broer, Florian Wagener and Jiangong You for helpful discussions. Support from the Graduiertenkolleg Hierarchie und Symmetrie in der Mathematik is gratefully acknowledged.

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