Boundary conditions of Sturm–Liouville operators with mixed spectra

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Abstract

We study bounds on averages of spectral functions corresponding to Sturm–Liouville operators on the half line for different boundary conditions. As a consequence constraints are obtained which imply existence of singular spectrum embedded in a.c. spectrum for sets of boundary conditions with positive measure and potentials vanishing in an interval $[0, N]$. These constraints are related to estimates on the measure of sets where the spectral density is positive.

Key words: Sturm–Liouville Operator, Mixed Spectra, Spectral Measure.
2000 Mathematics Subject Classification: 34L40, 34L05, 34B24, 34B40.

1 Introduction

In this paper we study spectral properties of Sturm–Liouville operators on the half line, see (1), (2) below. We are particularly interested on the behavior of different parts of the spectrum, when selfadjoint boundary conditions vary.

[1] Partially supported by Project CONACYT No. 37444-E
[2] Supported by Project CONACYT No. G-35458-A

These conditions depend on a real parameter θ and it is known, that we cannot have a set of eigenvalues dense in an interval for all $\theta \in [0, \pi)$ see [1], [7]. In a way the essential spectrum hinders the existence of dense point spectra for many θ's. On the other hand it is possible to have singular continuous spectra for all $\theta \in [0, \pi)$. It is natural in this context to study how the existence of a.c. spectrum affects the possibility of singular spectra and to look for a description of the set of θ's where coexistence of different spectral types is possible. Some results in this direction may be found in [3], [4], [2].

In [8] the author gave an explicit example of a potential which generates s.c. spectrum embedded in a.c. spectrum for a set of θ's of positive Lebesgue measure. The present note was motivated by the attempt to give a more precise description of this set of boundary conditions. We were able to obtain conditions on $\alpha, \beta \in [0, \pi)$ and on the measure of the support of the singular part, which guarantee the existence of a set $B \subset (\alpha, \beta)$ of positive Lebesgue measure, such that for $\theta \in B$ there is mixed spectra in Remling's example. In fact we shall prove more general results which involve the set where the spectral density is bounded.

Our main tools will be bounds from above and below of averages of spectral functions of Sturm–Liouville operators for different boundary conditions. Particularly for the bound from below, the condition that the potential vanishes in $[0, N]$ will be needed. Besides the results on embedded singular spectra, these bounds will allow us to give estimates on the set where the spectral density is larger than some constant for operators with a potential vanishing in $[0, N]$.

The paper is organized as follows. In Section 2 we introduce notation and prove two estimates on the integral of the spectral function with respect to the boundary condition. One important ingredient will be a result of [9] on bounds of spectral functions. In Section 3 we prove our main results. These concern the set where the derivative of the spectral function is greater than some constant. Examples are given where we can have some control on the set of boundary conditions where coexistence is possible, particularly in the example of Remling mentioned above.

2 Auxiliary results concerning the spectral function.

We consider one dimensional Schrödinger equations

$$\begin{gathered} ly = -y''(x) + v(x)\,y(x) = Ey(x) \\ 0 \le x < \infty \end{gathered} \tag{1}$$

and the associated selfadjoint operators

$$H_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) \text{ on } L_2(0, \infty)$$

generated by the boundary condition

$$\begin{gathered} y(0)\cos\alpha - y'(0)\sin\alpha = 0 \\ \alpha \in [0, \pi) \end{gathered} \tag{2}$$

Let $u_1(x,z)$ and $u_2(x,z)$ be solutions of

$$lu = zu \tag{3}$$

which satisfy

$$\begin{aligned} u_1(0,z) &= \sin\alpha & u_2(0,z) &= -\cos\alpha \\ u_1'(0,z) &= \cos\alpha & u_2'(0,z) &= \sin\alpha \end{aligned}$$

For every non real z the exists a function

$$\varphi_\alpha(x,z) = u_2(x,z) + m_\alpha(z)u_1(x,z)$$

which is solution of (3) and belongs to $L_2(0,\infty)$.

Note that u_1 satisfies the boundary conditions (2). The function $m_\alpha(z)$ is called the Weyl m function and has an integral representation of the form

$$m_\alpha(z) = c + \int_{\mathbb{R}} \left(\frac{1}{\mu - z} - \frac{\mu}{\mu^2+1} \right) \mathrm{d}\rho_\alpha(\mu)$$

where ρ_α is a Lebesgue–Stieltjes measure uniquely determined by m_α. The measure ρ_α is called the spectral function of H_α.

The spectral density $\frac{\mathrm{d}\rho_\alpha}{\mathrm{d}\lambda}$ is given almost everywhere by

$$\begin{aligned} \frac{\mathrm{d}\rho_\alpha(\lambda)}{\mathrm{d}\lambda} &= \lim_{E\to 0+} \frac{1}{\pi} \operatorname{Im} m_\alpha(\lambda + iE) \\ &=: \frac{1}{\pi} \operatorname{Im} m_\alpha(\lambda + i0) \end{aligned}$$

We may think of it as a local probability density for the energy of the system.

The proof of next Lemma is similar to the one of [10, Theorem 1.12].

Lemma 1

$$\int_{\alpha}^{\beta} \rho_\theta(A)\mathrm{d}\theta = \frac{1}{\pi}\int_A \arg\left[\frac{\cos\beta + \sin\beta\, m_0(E+i0)}{\cos\alpha + \sin\alpha\, m_0(E+i0)}\right]\mathrm{d}E$$

PROOF.

$$\int_{\alpha}^{\beta}\left[\int_{\mathbb{R}} \frac{\mathrm{d}\rho_\theta(E)}{(E-z)^2}\right]\mathrm{d}\theta = \int_{\alpha}^{\beta}\frac{\mathrm{d}}{\mathrm{d}z}\Big[m_\theta(z)\Big]\mathrm{d}\theta = \frac{\mathrm{d}}{\mathrm{d}z}\int_{\alpha}^{\beta}\left[\frac{m_0(z)\cos\theta - \sin\theta}{m_0(z)\sin\theta + \cos\theta}\right]\mathrm{d}\theta$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\log\Big(\cos\beta + \sin\beta\, m_0(z)\Big) - \frac{\mathrm{d}}{\mathrm{d}z}\log\Big(\cos\alpha + \sin\alpha\, m_0(z)\Big) \tag{4}$$

The first equality above follows from the integral representation of m_α and the second is a consequence of a well known relation between m_θ and m_0, see for example [4]

Now using the Herglotz integral representation of log, we get:

$$\log\Big(\cos\theta + \sin\theta m_0(z)\Big) = c + \int_{\mathbb{R}}\left[\frac{1}{x-z} - \frac{x}{x^2+1}\right] f_\theta(x)\mathrm{d}x$$

where

$$\begin{aligned} f_\theta(x) &= \frac{1}{\pi}\,\mathrm{Im}\log\Big(\cos\theta + \sin\theta\, m_0(x+i0)\Big) \\ &= \frac{1}{\pi}\arg\Big(\cos\theta + \sin\theta\, m_0(x+i0)\Big) \end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\Big(\cos\theta+\sin\theta\, m_0(z)\Big)=\int\limits_{\mathbb{R}}\frac{f_\theta(x)}{(x-z)^2}\mathrm{d}x$$

and from (4) we obtain

$$\int\limits_{\alpha}^{\beta}\left[\int\limits_{\mathbb{R}}\frac{\mathrm{d}\rho_\theta(E)}{(E-z)^2}\right]\mathrm{d}\theta=\int\limits_{\mathbb{R}}\frac{f_\beta(E)-f_\alpha(E)}{(E-z)^2}\mathrm{d}E$$

Since the functions $(E-z)^2$ have linear combinations which are dense in a space big enough to imply

$$\int\limits_{\alpha}^{\beta}\mathrm{d}\rho_\theta(E)\mathrm{d}\theta=\Big(f_\beta(E)-f_\alpha(E)\Big)\mathrm{d}E$$

as an equality of measures, the statement of the Lemma follows.

□

Observe that if we take $\beta=\alpha+\frac{\pi}{2}$ we get

$$\int\limits_{\alpha}^{\alpha+\frac{\pi}{2}}\rho_\theta(A)\mathrm{d}\theta=\frac{1}{\pi}\int\limits_{A}\arg\Big(m_\alpha(E+i0)\Big)\mathrm{d}E$$

Let

$$\Lambda_M:=\{E/\operatorname{Im}\, m_0(E+i0)>M\} \tag{5}$$

Recall that Λ_0 is a support of the absolutely continuous part of the spectral measure, see [6] for example.

The next result is about an upper bound that will be used later. For the examples in the next section where we analyze singular spectrum it will be enough to assume $M = 0$. In this case the statement of the following Lemma is just the well known bound

$$\int_\alpha^\beta \rho_\theta\Big(I \cap \Lambda_0\Big)\mathrm{d}\theta \le \Big|I \cap \Lambda_0\Big|$$

where $|\cdot|$ denotes Lebesgue measure.

Lemma 2

$$\int_\alpha^\beta \rho_\theta\Big(I \cap \Lambda_M\Big)\mathrm{d}\theta \le \frac{2}{\pi}\arctan\Big(\frac{1}{2M}(\cot\alpha - \cot\beta)\Big)\Big|I \cap \Lambda_M\Big|$$

where I is an arbitrary interval.

PROOF. According to Lemma 1 we have

$$\int_\alpha^\beta \rho_\theta(A)\mathrm{d}\theta = \frac{1}{\pi}\int_A \arg\left[\frac{\cos\beta + \sin\beta\, m_0(E+i0)}{\cos\alpha + \sin\alpha\, m_0(E+i0)}\right]\mathrm{d}E \tag{6}$$

for every Borel set A.

Let

$$w = Tz = \frac{z\sin\beta + \cos\beta}{z\sin\alpha + \cos\alpha}$$

For each $M > 0, T$ maps the half–plane Im $z > M$ onto the disk

$$\left(x - \frac{\sin\beta}{\sin\alpha}\right)^2 + \left(y - \frac{\sin(\beta-\alpha)}{2M\sin^2\alpha}\right)^2 < \left(\frac{\sin(\beta-\alpha)}{2M\sin^2\alpha}\right)^2$$

From here it follows that if Im $z > M$ then $\arg w \leq 2\arctan\Big(\frac{1}{2M}(\cot\alpha - \cot\beta)\Big)$.

Therefore, if Im $m_0(E+i0) > M$ using (6) we get

$$\begin{aligned}\int_\alpha^\beta \rho_\theta\big(I\cap\Lambda_M\big) &= \frac{1}{\pi}\int_{I\cap\Lambda_M} \arg\Big(T(m_0(E+i0)\Big)\mathrm{d}E \\ &\leq \frac{2}{\pi}\int_{I\cap\Lambda_M} \arctan\Big(\frac{1}{2M}(\cot\alpha - \cot\beta)\Big)\mathrm{d}E \\ &= \frac{2}{\pi}\arctan\Big(\frac{1}{2M}(\cot\alpha - \cot\beta)\Big)\big|I\cap\Lambda_M\big|\end{aligned}$$

□

In [9] C. Remling proved the following result. The set M_N below, denotes certain family of measures which has the following property: given v on $[0,N]$, for an arbitrary (locally integrable) extension of v to $[0,\infty)$ the spectral measure of the corresponding half–line problem belongs to M_N.

Theorem 1 *(Corollary 1.2 in [9]) Let* λ_i, λ_j *be both eigenvalues of (1) on an interval* $[0, N]$ *with boundary conditions (2) and similar conditions in* N*. Let* ρ_0 *be the spectral measure of this problem.*

Then

$$\rho_0\Big([\lambda_i, \lambda_j]\Big) = \max_{\rho \in M_N} \rho\Big([\lambda_i, \lambda_j]\Big)$$
$$\rho_0\Big((\lambda_i, \lambda_j)\Big) = \min_{\rho \in M_N} \rho\Big((\lambda_i, \lambda_j)\Big)$$

The following Lemma is an application of this theorem.

Lemma 3 *Assume* $v(x) \equiv 0$ *for* $x \in [0, N]$ *where* N *is an arbitrary positive real number*

a) Let $I_1 = \left(\left(\frac{\pi k}{N}\right)^2, \left(\frac{\pi(k+2)}{N}\right)^2\right)$*, where* $k \in \mathbb{N}$

then

$$\int_\alpha^\beta \rho_\theta(I_1) \mathrm{d}\theta \geq \frac{2\pi(k+1)}{N^2} \int_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{\mathrm{d}x}{1+x^2}$$

$\alpha, \beta \in (0, \pi)$

b) Let $I_2 = \left[\left(\frac{\pi k}{N}\right)^2, \left(\frac{\pi(k+1)}{N}\right)^2\right]$, $k \in \mathbb{N}$, $N \in \mathbb{R}^+$

then

$$\int_\alpha^\beta \rho_\theta(I_2) \mathrm{d}\theta \leq \frac{2\pi k}{N^2} \int_{\frac{N}{\pi k}\cot\beta}^{\frac{N}{\pi k}\cot\alpha} \frac{\mathrm{d}x}{1+x^2} + \frac{2\pi(k+1)}{N^2} \int_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{\mathrm{d}x}{1+x^2}$$

$\alpha, \beta \in (0, \pi)$

Proof of a)

The function

$$\psi_\alpha(x, \frac{\pi k}{N}) \;=\; \frac{1}{2}\left[\sin\alpha + \frac{N}{i\pi k}\cos\alpha\right]\mathrm{e}^{\frac{i\pi k}{N}x} + \frac{1}{2}\left[\sin\alpha - \frac{N}{i\pi k}\cos\alpha\right]\mathrm{e}^{\frac{-i\pi k}{N}x}$$

satisfies

$$\begin{aligned}\psi(0) &= \sin\alpha\\ \psi'(0) &= \cos\alpha\end{aligned}$$

and

$$-\psi''\Big(x, \frac{\pi k}{N}\Big) = \Big(\frac{\pi k}{N}\Big)^2 \psi\Big(x, \frac{\pi k}{N}\Big)$$

for $x \in [0, N]$ with boundary conditions

$$\begin{aligned}\psi(N)\cos\alpha - \psi'(N)\sin\alpha = 0\\ \psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0\end{aligned}$$

We have moreover, the same eigenvalues $\left(\frac{\pi k}{N}\right)^2$ for all $\alpha \in [0, \pi)$.

After some elementary calculations we get

$$\int_0^N \Big|\psi_\alpha\Big(x, \frac{\pi k}{N}\Big)\Big|^2 \mathrm{d}x = \Big\|\psi_\alpha\Big(x, \frac{\pi k}{N}\Big)\Big\|^2_{L_2(0,N)} = \frac{(\sin\alpha)^2}{2}N + \frac{1}{2}N\Big(\frac{N\cos\alpha}{\pi k}\Big)^2$$

Applying Theorem 1 we get

$$\rho_\theta\left(\Big(\frac{\pi k}{N}\Big)^2, \Big(\frac{\pi(k+2)}{N}\Big)^2\right) \geq \Big\|\psi\Big(x, \frac{\pi(k+1)}{N}\Big)\Big\|_{L_2(0,N)}^{-2} = \rho_0\left(\Big(\frac{\pi k}{N}\Big)^2, \Big(\frac{\pi(k+2)}{N}\Big)^2\right)$$

Therefore

$$\int\limits_\alpha^\beta \rho_\theta(I)\mathrm{d}\theta \geq \int\limits_\alpha^\beta \frac{\mathrm{d}\theta}{\frac{N}{2}(\sin\theta)^2 + \frac{N}{2}\left(\frac{N\cos\theta}{\pi(k+1)}\right)^2} = \frac{2\pi(k+1)}{N^2} \int\limits_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{\mathrm{d}x}{1+x^2}$$

and a) is proved

□

Proof of b) Using again Theorem 1 we see that

$$\int\limits_\alpha^\beta \mathrm{d}\theta\left[\frac{(\sin\theta)^2}{2}N + \frac{1}{2}N\Big(\frac{N\cos\theta}{\pi k}\Big)^2\right]^{-1} + \int\limits_\alpha^\beta \mathrm{d}\theta\left[\frac{(\sin\theta)^2}{2}N + \frac{1}{2}N\Big(\frac{N\cos\theta}{\pi(k+1)}\Big)^2\right]^{-1}$$
$$\geq \int\limits_\alpha^\beta \rho_\theta\left(\left[\Big(\frac{\pi k}{N}\Big)^2, \Big(\frac{\pi(k+1)}{N}\Big)^2\right]\right)\mathrm{d}\theta$$

and therefore b) follows.

□

The Lemma above, give us the lower bound that we need for next theorem

3 Main results and examples.

Using the upper and lower bounds of previous section we obtain the following

Theorem 1 *Assume $v(x) \equiv 0$ for $x \in [0, N], N \in \mathbb{R}^+,\ k \in \mathbb{N}$.*

If

$$\frac{2\pi(k+1)}{N^2} \int\limits_{\frac{N}{\pi(k+1)}\cot\beta}^{\frac{N}{\pi(k+1)}\cot\alpha} \frac{\mathrm{d}x}{1+x^2} > \frac{2|\Lambda_M \cap I|}{\pi} \int\limits_0^{\frac{1}{M}\frac{\cot\alpha - \cot\beta}{2}} \frac{\mathrm{d}x}{1+x^2}$$

$\alpha, \beta \in (0, \pi), \alpha < \beta$

then

$$\int\limits_\alpha^\beta \rho_\theta(I \cap \Lambda_M^c)\mathrm{d}\theta > 0$$

where

$$I = \left(\left(\frac{\pi k}{N}\right)^2, \left(\frac{\pi(k+2)}{N}\right)^2\right)$$

PROOF. The lower bound given by Lemma 3 a) together with the upper bound given by Lemma 2 and the hypotheses of the theorem give us

$$\int\limits_\alpha^\beta \rho_\theta(I)\mathrm{d}\theta > \int\limits_\alpha^\beta \rho_\theta(I \cap \Lambda_M)\mathrm{d}\theta$$

and since

$$\int_{\alpha}^{\beta} \rho_\theta(I)\mathrm{d}\theta = \int_{\alpha}^{\beta} \rho_\theta(I \cap \Lambda_M^c)\mathrm{d}\theta + \int_{\alpha}^{\beta} \rho_\theta(I \cap \Lambda_M)\mathrm{d}\theta$$

we obtain

$$\int_{\alpha}^{\beta} \rho_\theta(I \cap \Lambda_M^c)\mathrm{d}\theta > 0$$

and the theorem is proved.

□

Remark In fact a bound from below for $\int_\alpha^\beta \rho_\theta(I \cap \Lambda_M^c)\mathrm{d}\theta$ can be given taking the difference between the left and right members of the inequality stated in the theorem.

Examples

a) Set $k = 1, N = 2\pi$ and $M = 0$. According to Theorem 1, if

$$\beta - \alpha > \pi|\Lambda_0 \cap I|$$

where $I = \left(\frac{1}{4}, \frac{9}{4}\right)$, then

$$\int_{\alpha}^{\beta} \rho_\theta(I \cap \Lambda_0^c)\mathrm{d}\theta > 0$$

Since Λ_0^c is the support for the singular part, we conclude that there exists $B \subset (\alpha, \beta)$, $|B| > 0$ such that if $\theta \in B$ then H_θ has some singular spectrum in I.

b) Analogously, if we set $k = 2, N = 3\pi, M = 0$ then a sufficient condition to have singular spectrum in I for boundary conditions θ between α and β is

$$\frac{2}{3}(\beta - \alpha) > \pi |\Lambda_0 \cap I|$$

where in this case $I = \left[\frac{4}{9}, \frac{16}{9}\right]$

c) In [8] potentials of the form

$$v(x) = \sum_{n=1}^{\infty} g_n v_n(x - a_n)$$

with $g_n > 0, v_n \in L_1[-B_n, B_n]$ are considered. The intervals $[a_n - B_n, a_n + B_n]$ are assumed disjoint and the barries v_n have the form

$$v_n(x) = \chi_{(-B_n, B_n)}(x)\, w(x)$$

where

$$w(x) = \int_F \cos 2kx \,\mathrm{d}k$$

and F is a Cantor type set in an interval $[a, b]$, with Lebesgue measure any positive number less than $b - a$.

Let $L_n = a_n - B_n - a_{n-1} - B_{n-1}$, with $a_0 = B_0 = 0$

In [8] it is proven that under minor assumptions on F the following holds:

Theorem *Let* $g_n = n^{-1/2}, B_n = n^\beta$ *with* $(2-4/\gamma)^{-1} < \beta < \gamma/8$*, where* $\gamma > 6$*, and assume* $n^{\beta/2\gamma} L_{n-1}/L_n \stackrel{n\to\infty}{\longrightarrow} 0$*. Then the half line Schrödinger operators* H_α *with potential* v *given as above satisfy* $\sigma_{ac}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0,\infty), \sigma_p(H_\alpha) \cap (0,\infty) = \emptyset$ *and for a set of boundary conditions* α *of positive measure* $\sigma_{sc}(H_\alpha) \cap (0,\infty) \neq \emptyset$*.*

In proving this result it is shown that the absolutely continuous part of the spectral measures ρ_α corresponding to H_α, give zero weight to $F^2 = \{k^2 : k \in F\}$.

In Remling's theorem above, the potential $v(x)$ may be equal to zero in an interval $[0, N]$.

To apply Theorem 1 in this case we can take $k = 1, N = 2\pi$ and $M = 0$ as in example a). Then the condition to have singular continuous spectrum for a set boundary conditions of positive measure, in (α, β) is

$$\beta - \alpha > \pi|\Lambda_0 \cap I| = |(F^2)^c \cap I|$$

where $I = \left(\frac{1}{4}, \frac{9}{4}\right)$. Observe that in this case we can control the measure of F^2

In the next two theorems the full strength of Lemma 2 is used.

The restriction $v(x) \equiv 0$ on $[0, N]$ implies some restrictions on the measure of Λ_M if M is large, more precisely we have

Theorem 2 *Let Λ_M be as defined in (5) and assume $v(x) \equiv 0$ for $x \in [0, N]$. Then*

$$|\Lambda_M \cap I| \cdot M \le \frac{2\pi^3}{N^3}\Big(k^2 + (k+1)^2\Big)$$

where $I := \left[\left(\frac{\pi k}{N}\right)^2 \left(\frac{\pi(k+1)}{N}\right)^2\right]$, $k \in \mathbb{N},\ N \in \mathbb{R}^+$.

PROOF. We have the following chain of inequalities:

For $x \in [0, \frac{\pi}{2})$

$$\text{王}(x) := \frac{4\pi k}{N^2} \int\limits_0^{\frac{\pi k}{N}\tan x} \frac{1}{1+t^2}\mathrm{d}t + \frac{4\pi(k+1)}{N^2} \int\limits_0^{\frac{\pi(k+1)}{N}\tan x} \frac{1}{1+t^2}\mathrm{d}t$$

$$\overset{①}{\ge} \int\limits_0^x \rho_\theta(I)\mathrm{d}\theta + \int\limits_{\pi-x}^{\pi} \rho_\theta(I)\mathrm{d}\theta$$

$$\overset{②}{\ge} \int\limits_0^x \rho_\theta(\Lambda_M \cap I)\mathrm{d}\theta + \int\limits_{\pi-x}^{\pi} \rho_\theta(\Lambda_M \cap I)\mathrm{d}\theta$$

$$\overset{③}{\ge} \Big[1 - \frac{2}{\pi}\arctan\Big(\frac{1}{M} \cdot \tan\big(\frac{\pi}{2} - x\big)\Big)\Big]\,|\Lambda_M \cap I|$$

$$=: \text{亚}(x)$$

The inequality ① follows from Lemma 3 b), and ② is obvious. To prove ③ observe that as a consequence of Lemma 2 and the fact

$$\int_0^{\pi} \rho_\theta(I \cap \Lambda_M) \mathrm{d}\theta = |I \cap \Lambda_M|$$

we get the lower bound

$$\int_0^{x} \rho_\theta(\Lambda_M \cap I) \mathrm{d}\theta + \int_\beta^{\pi} \rho_\theta(\Lambda_M \cap I) \mathrm{d}\theta$$
$$\geq \left\{ 1 - \frac{2}{\pi} \arctan \left(\frac{1}{2M} \cdot (\cot x - \cot \beta) \right) \right\} |\Lambda_M \cap I|$$

This holds for an arbitrary interval I. Choosing $\beta = \pi - x$ inequality③ follows.

Therefore

$$\text{王}(x) \geq \text{亜}(x) \tag{7}$$

$x \in [0, \pi/2)$

Since $\text{王}(0) = \text{亜}(0) = 0$, (7) implies that

$$\text{王}'(0) \geq \text{亜}'(0)$$

But

$$\text{亜}'(0) = \frac{2|\Lambda_M \cap I| M}{\pi}$$
$$\text{王}'(0) = \frac{4\pi^2}{N^2} \left(k^2 + (k+1)^2 \right)$$

and the Theorem follows.

□

By the Inverse Spectral Theorem of Gelfand–Levitan we known that spectral functions of Sturm–Liouville operators may be arbitrary in a bounded interval. The theorem above give us restrictions on the possible spectral density when the potential vanishes in $[0, N]$

In next theorem the condition $v(x) \equiv 0$ in $[0, N]$ is not needed.

Theorem 3 *Given $\alpha, \beta \in (0, \pi)$ and $1 > k > 0$, there exists $\infty > N(\alpha, \beta, k) > 0$ such that $R > N$ and*

$$\frac{|\Lambda_M \cap (-R, R)|}{R^{1/2}} < \frac{k(\cot\alpha - \cot\beta)}{\arctan\left(\frac{1}{M}\frac{\cot\alpha - \cot\beta}{2}\right)}$$

imply

$$\int_\alpha^\beta \rho_\theta\Big((-R, R) \cap \Lambda_M^c\Big)\mathrm{d}\theta > 0$$

for $R > N$.

PROOF. It is known that

$$\lim_{R\to\infty} \frac{1}{R^{1/2}} \int_{-R}^{R} \mathrm{d}\rho_\theta(x) = \frac{2(1 + \cot^2\theta)}{\pi}$$

see [5, formula A.9]

Using Fatou's Lemma we obtain

$$\int_\alpha^\beta \frac{2(1 + \cot^2\theta)}{\pi}\mathrm{d}\theta \leq \liminf_{R\to\infty} \frac{1}{R^{1/2}} \int_\alpha^\beta \rho_\theta\Big((-R, R)\Big)\mathrm{d}\theta$$

Therefore there exists $N(\alpha, \beta, k) > 0$ such that if $R > N$ then

$$kR^{1/2}\frac{2}{\pi}(\cot\alpha - \cot\beta) \leq \int\limits_{\alpha}^{\beta} \rho_\theta\Big((-R, R)\Big)\mathrm{d}\theta$$

where $0 < k < 1$.

Using the upper bound given by Lemma 2 we can conclude that, if the hypotheses of the theorem are satisfied, then

$$\int\limits_{\alpha}^{\beta} \rho_\theta\Big((-R, R) \cap \Lambda_M\Big)\mathrm{d}\theta < \int\limits_{\alpha}^{\beta} \rho_\theta\Big((-R, R)\Big)\mathrm{d}\theta$$

holds for every $R > N$. Hence

$$\int\limits_{\alpha}^{\beta} \rho_\theta\Big((-R, R) \cap \Lambda_M^c\Big)\mathrm{d}\theta > 0$$

$\square$

References

[1] del Rio, R., Makarov, N. and Simon, B., Operators with Singular Continuous Spectrum: II. Rank One Operators, *Commun. Math. Phys.* **165** (1994), 59–67.

[2] del Rio, R. and Simon, B., Point Spectrum and Mixed Spectral Types for Rank One Perturbations, *Proceedings of the AMS* **129** (1997), 3593–3599.

[3] del Rio, R., Fuentes, S. and Poltoratski, A., Coexistence of Spectra in Rank-One Perturbation Problems, *Bol. Soc. Mat. Mexicana* **3** Vol. 8 (2002), 49–61.

[4] del Rio, R., Fuentes, S. and Poltoratski, A., Families of Spectral Measures with Mixed Types, *Operator Theory: Advances and Applications* Vol. 132 (2002), 131–140.

[5] Gesztesy, F. and Simon, B., A new approach to inverse spectra theory, II. General real potentials and the connection to the spectral measure, *Annals of Mathematics* **152** (2000), 593–643.

[6] Gilbert, D.J. and Pearson B., On Subordinacy and Analysis of the Spectrum of One–Dimensional Schrödinger Operators, *Journal of Mathematical Analysis and Applications* Vol. 128 No. 1 (1987), 30–56.

[7] Gordon, A., Pure point spectrum under 1-parameter perturbations and instability of Anderson localization, *Commun. Math. Phys.* **164** (1994), 489–505.

[8] Remling, C., Embedded singular continuous spectrum for one–dimensional Schroedinger Operators, *Trans. Amer. Math. Soc.* **351** No. 6 (1999), 2479–2497.

[9] Remling, C., *Universal bounds on spectral measures of one–dimensional Schroedinger Operators* available electronically in `http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=02-394`.

[10] Simon, B., Spectral Analysis of Rank One Perturbations and Applications, *CMR Proceedings and Lecture Notes* Vol. 8 (1995) 109–149.