THERE IS NO AHARONOV-BOHM EFFECT IN DIMENSION THREE

D. Yafaev
Department of Mathematics, University of Rennes,
Campus Beaulieu, 35042, Rennes, France,
yafaev@univ-rennes1.fr

Abstract

Consider the scattering amplitude $s(\omega, \omega'; \lambda)$, $\omega, \omega' \in \mathbb{S}^2$, $\lambda > 0$, corresponding to an arbitrary three-dimensional short-range magnetic field $B(x)$. The magnetic potential $A^{(tr)}(x)$ such that $\operatorname{curl} A^{(tr)}(x) = B(x)$ and $< A^{(tr)}(x), x >= 0$ decays at infinity as $|x|^{-1}$ only. Nevertheless, we show that the structure of $s(\omega, \omega'; \lambda)$ is the same as for short-range magnetic potentials. In particular, the leading diagonal singularity $s_0(\omega, \omega')$ of $s(\omega, \omega'; \lambda)$ is the Dirac function. Thus, up to the diagonal Dirac function, the scattering amplitude has only a weak singularity in the forward direction and hence scattering is essentially of short-range nature. This is qualitatively different from the two-dimensional case where $s_0(\omega, \omega')$ is a linear combination of the Dirac function and of a singular denominator, that is the Aharonov-Bohm effect occurs. Our approach relies on a construction of a special gauge, adapted to a given magnetic field $B(x)$, such that the corresponding magnetic potential $A(x)$ is also short-range.

1. INTRODUCTION

There is no precise definition of the Aharonov-Bohm effect in the literature. In the original paper [1] (see also [2, 5]) the Schrödinger operator H with magnetic potential

$$A(x) = \alpha(-x_2, x_1)|x|^{-2}, \quad \alpha \in \mathbb{R}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

was considered. For such potentials, variables in the Schrödinger equation can be separated in polar coordinates (r, θ). This allows to show that the scattering matrix (SM) $S(\lambda)$, $\lambda > 0$ is the energy, for the operator H has two eigenvalues $e^{i\alpha\pi}$ and $e^{-i\alpha\pi}$ with corresponding eigenfunctions $e^{im\theta}$ ($m = 0, \pm 1, \pm 2, \dots$ is the angular momentum) for $m \leq \alpha$ and $m \geq \alpha$, respectively. This implies of course that the SM is non-trivial, that is, $S(\lambda) \neq I$ (I is the identity operator). Since the magnetic field $B(x) = \operatorname{curl} A(x) = 0$ for $x \neq 0$, the fact that $S(\lambda) \neq I$ appeared to be surprising. We do not share this point of view because, for the Aharonov-Bohm potential, the field is not zero but equals $2\pi\alpha\delta(x)$ where $\delta(x)$ is the Dirac function, and a quantum particle interacts always with this field. Nevertheless the results of [1, 2, 5] are quite interesting. In fact, the Aharonov-Bohm potential is one of rare examples where the SM can be calculated explicitly. Moreover, its properties are quite different from the case of short-range potentials satisfying the condition

$$|A(x)| \leq C(1+|x|)^{-\rho}, \quad \rho > 1. \tag{1.1}$$

Let us discuss properties of the SM in a more general framework. Suppose that a magnetic field $B(x) = (0, 0, B(x))$, $x \in \mathbb{R}^2$, satisfies the condition

$$|B(x)| \leq C(1+|x|)^{-r}, \quad r > 2, \tag{1.2}$$

(for simplicity, one may assume for a moment that $B(x)$ has compact support). Then the magnetic potential $A^{(tr)}(x) = (A_1^{(tr)}(x), A_2^{(tr)}(x), 0)$, $\operatorname{curl} A^{(tr)}(x) = B(x)$, obeying the transversal gauge condition $< A^{(tr)}(x), x >= 0$ has asymptotics as $|x| \to \infty$

$$A^{(\infty)}(x) = a(\hat{x})(-x_2, x_1)|x|^{-2}, \quad \hat{x} = x/|x|, \tag{1.3}$$

where a is a smooth function on the unit circle. Potentials with asymptotics (1.3) were considered in [4, 8]. For any $\omega \in \mathbb{S}$, we denote by $\omega^{(\pm)}$ the points obtained from ω by rotation at the angle $\pm\pi/2$ in the positive (counter-clockwise) direction. Let

$$f(\omega) = \int_{\mathbb{S}(\omega^{(-)}, \omega^{(+)})} a(\theta) d\theta = \int_{<x,\omega>\geq 0} B(x) dx,$$

where the first integral is taken in the positive direction over the half-circle $\mathbb{S}(\omega^{(-)}, \omega^{(+)})$ between the points $\omega^{(-)}$ and $\omega^{(+)}$, and let

$$\Phi = \int_{\mathbb{S}} a(\theta) d\theta = \int_{\mathbb{R}^2} B(x) dx$$

be the total magnetic flux. The kernel $s(\omega, \omega'; \lambda)$ of the SM regarded as an integral operator (the variables $\omega, \omega' \in \mathbb{S}$ and $\lambda > 0$) is smooth away from the diagonal $\omega = \omega'$ and its leading diagonal singularity is given by the function

$$s_0(\omega, \omega') = e^{i(f(\omega^{(-)}) - f(\omega^{(+)}))/2} \Big(\cos(\Phi/2)\, \delta(\omega, \omega') + \pi^{-1} \sin(\Phi/2)\, \mathrm{P.V.} \{\omega, \omega'\}^{-1} \Big). \tag{1.4}$$

Here $\delta(\omega, \omega')$ is the Dirac function on the unit circle, $\{\omega, \omega'\}$ is the oriented angle between an initial vector ω and a final vector ω' and the second term is understood in the sense of the principal value. Singularity (1.4) is qualitatively different from the short-range case where

$$s(\omega, \omega'; \lambda) = \delta(\omega, \omega') + O(|\omega - \omega'|^{-2+\rho})$$

as $\omega - \omega' \to 0$. Thus, in the dimension two, even for magnetic fields of compact support with $\Phi \notin 2\pi\mathbb{Z}$, the SM contains the singular integral operator. Hence the forward singularity $\pi^{-2} \sin^2(\Phi/2) |\omega - \omega'|^{-2}$ of the scattering cross section $|s(\omega, \omega'; \lambda)|^2$ is stronger than for short-range magnetic potentials where it is $O(|\omega - \omega'|^{-4+2\rho})$. This fact can be taken for a definition of the Aharonov-Bohm effect. We emphasize that the Coulomb decay of $A(x)$ cannot be gauged away even for fields $B(x)$ of compact support. Indeed, if $A(x) = o(|x|^{-1})$ as $|x| \to \infty$, then the circulation of $A(x)$ over the circle $|x| = r$ tends to 0 as $|x| \to \infty$. The Stokes theorem implies that in this case necessarily $\Phi = 0$.

In the present paper we show that in the three-dimensional case the scattering amplitude corresponding to a short-range magnetic field $B(x)$ satisfying condition (1.2) has also the short-range structure, that is the Aharonov-Bohm effect does not occur. The difference between dimensions $d = 3$ and $d = 2$ is of topological nature: if $d = 3$, then the set $\mathbb{R}^d \setminus \{0\}$ is simply connected, whereas it is not true for $d = 2$. This allows us to construct for $d = 3$ a special gauge, adapted to a given short-range magnetic field $B(x)$, such that the corresponding magnetic potential $A(x)$,

$$\operatorname{curl} A(x) = B(x), \tag{1.5}$$

is also short-range, i.e., it satisfies condition (1.1) with $\rho = r - 1 > 1$. If $B(x)$ has compact support, then $A(x)$ is also of compact support. Of course, our results remain true for all dimensions $d \geq 3$ (in the general case one has to consider A as a 1-form and $B = dA$ as a 2-form).

In the next section we recall some basic facts of scattering theory and discuss the behaviour of the SM with respect to gauge transformations. In Section 3 we prove our main result announced above. Finally, in Section 4 we show that condition (1.2) is precise, that is the Aharonov-Bohm effect occurs if $B(x)$ decays as $|x|^{-2}$ only.

2. SCATTERING BY MAGNETIC FIELDS

2.1. Let us first discuss briefly some basic facts of scattering theory. Consider the Schrödinger operator

$$H = (i\nabla + A(x))^2, \quad x \in \mathbb{R}^d, , \quad d \geq 2,$$

with a real magnetic potential $A(x) = (A_1(x), \dots, A_d(x))$ satisfying condition (1.1). We do not assume that the function $A(x)$ is differentiable so that, strictly speaking, H is correctly defined as a self-adjoint operator in the space $L_2(\mathbb{R}^d)$ in terms of the corresponding quadratic form. In general, equality (1.5) should be understood in the sense of distributions.

Let $H_0 = -\Delta$ be the "free" operator. Under assumption (1.1) the wave operators

$$W_\pm = W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$$

exist, are unitary and possess the intertwining property $HW_\pm = W_\pm H_0$. The scattering operator $\mathbf{S} = W_+^* W_-$ is unitary and commutes with H_0. Let $\hat{f} = \mathcal{F} f$ be the Fourier transform of $f \in L_2(\mathbb{R}^d)$, and let the unitary operator $F : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}_+; L_2(\mathbb{S}^{d-1}))$ be defined by the formula

$$(Ff)(\lambda; \omega) = 2^{-1/2} \lambda^{(d-2)/4} \hat{f}(\lambda^{1/2}\omega), \quad \lambda > 0, \quad \omega \in \mathbb{S}^2.$$

Since $\mathbf{S} H_0 = H_0 \mathbf{S}$, we have that

$$(F\mathbf{S}f)(\lambda) = S(\lambda)(Ff)(\lambda)$$

where the unitary operator $S(\lambda) : L_2(\mathbb{S}^{d-1}) \to L_2(\mathbb{S}^{d-1})$ is known as the scattering matrix (SM). The scattering amplitude $s(\omega, \omega'; \lambda)$, $\omega, \omega' \in \mathbb{S}^{d-1}$, is kernel of $S(\lambda)$ regarded as integral operator. The following assertion is well known (see, e.g., [7, 8]).

Proposition 2.1 *Let condition* (1.1) *hold. Then the operator* $T(\lambda) = S(\lambda) - I$ *is compact and it belongs to the trace class if* $\rho > d$. *If* $\rho > d+n$, $n = 0, 1, 2, \ldots$, *then* $T(\lambda)$ *is integral operator with kernel from the class* $C^n(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$.

Let the condition

$$|\partial^\alpha A(x)| \leq C_\alpha (1 + |x|)^{-\rho-|\alpha|}$$

hold for some $\rho \in (1, d)$ *and all multi-indices* α. *Then the operator* $T(\lambda)$ *has integral kernel which is a* C^∞*-function off the diagonal* $\omega = \omega'$ *and is bounded by* $C|\omega - \omega'|^{-d+\rho}$ *as* $\omega' \to \omega$.

2.2. Let us now discuss the behaviour of the SM with respect to gauge transformations defined by the formula

$$\tilde{H} = e^{i\phi} H e^{-i\phi} = (i\nabla + \tilde{A}(x))^2 \tag{2.1}$$

where ϕ is a differentiable function and the magnetic potential

$$\tilde{A}(x) = A(x) + \operatorname{grad} \phi(x). \tag{2.2}$$

Of course $\operatorname{curl} \tilde{A}(x) = \operatorname{curl} A(x)$. We are particularly interested in functions $\phi(x)$ which are asymptotically homogeneous of degree zero. More precisely, we assume that $\phi(x) = \phi_0(x) + \phi_1(x)$ where $\phi_0(x) = \phi_0(\hat{x})$ and $\phi_1(x) = o(1)$ as $|x| \to \infty$.

Let us find a relation between the wave operators $W(H, H_0)$ and $W(\tilde{H}, H_0)$.

Proposition 2.2 *Let the wave operators* $W_\pm(H, H_0)$ *exist, and let* ϕ *be an asymptotically homogeneous function of degree zero with homogeneous part* ϕ_0. *Then the wave operators* $W_\pm(\tilde{H}, H_0)$ *also exist and*

$$W_\pm(\tilde{H}, H_0) = e^{i\phi(x)} W_\pm(H, H_0) \mathcal{F}^* e^{-i\phi_0(\pm\xi)} \mathcal{F}. \tag{2.3}$$

Indeed, since

$$(\exp(-iH_0 t) f)(x) = e^{i|x|^2/(4t)} (2it)^{-d/2} \hat{f}(x/(2t)) + o(1),$$

we have that

$$\begin{aligned}
(e^{-i\phi} \exp(-iH_0 t) f)(x) &= e^{i|x|^2/(4t)} (2it)^{-d/2} e^{-i\phi_0(\pm x/(2t))} \hat{f}(x/(2t)) + o(1) \\
&= e^{i|x|^2/(4t)} (2it)^{-d/2} \hat{f}^{(\pm)}(x/(2t)) + o(1), \quad t \to \pm\infty,
\end{aligned}$$

where $\hat{f}^{(\pm)}(\xi) = e^{-i\phi_0(\pm\xi)} \hat{f}(\xi)$ and the remainder $o(1)$ tends to 0 in $L_2(\mathbb{R}^d)$ as $t \to \pm\infty$. This is equivalent to the relation

$$e^{-i\phi} \exp(-iH_0 t) f = \exp(-iH_0 t) f^{(\pm)} + o(1),$$

which, in view of definition (2.1), implies that

$$\begin{aligned}
W_\pm(\tilde{H}, H_0) f &= \lim_{t \to \pm\infty} e^{i\tilde{H}t} e^{-iH_0 t} f = \lim_{t \to \pm\infty} e^{i\phi} e^{iHt} e^{-i\phi} e^{-iH_0 t} f \\
&= \lim_{t \to \pm\infty} e^{i\phi} e^{iHt} e^{-iH_0 t} f^{(\pm)} = e^{i\phi} W_\pm(H, H_0) f^{(\pm)}.
\end{aligned}$$

This proves (2.3).

As an immediate consequence of Proposition 2.2, we obtain a relation between the corresponding scattering operators and matrices.

Proposition 2.3 *Under the assumptions of Proposition* 2.2*, the scattering operators for the pairs* H_0, H *and* $H_0, \tilde{H}$ *are related by the equation*

$$\mathcal{F}\mathbf{S}(\tilde{H}, H_0)\mathcal{F}^* = e^{i\phi_0(\xi)}\mathcal{F}\mathbf{S}(H, H_0)\mathcal{F}^* e^{-i\phi_0(-\xi)}. \tag{2.4}$$

The SM $S(\lambda) = S(H, H_0; \lambda)$ *and* $\tilde{S}(\lambda) = S(\tilde{H}, H_0; \lambda)$ *satisfy for all* $\lambda > 0$ *the relations*

$$\tilde{S}(\lambda) = e^{i\phi_0(\omega)} S(\lambda) e^{-i\phi_0(-\omega)} \tag{2.5}$$

or, in terms of the scattering amplitudes,

$$\tilde{s}(\omega, \omega'; \lambda) = e^{i\phi_0(\omega) - i\phi_0(-\omega')} s(\omega, \omega'; \lambda). \tag{2.6}$$

We emphasize that relations (2.4)-(2.6) for the scattering operators and matrices (but not (2.3) for the wave operators) depend only on the asymptotics ϕ_0 of the function ϕ. Probably, formulas (2.3), (2.4) appeared first in the paper [5] in the case $d = 2$ under some assumptions on A. Actually, these formulas do not require any assumptions at all.

Formulas (2.5) or (2.6) show that the SM is not determined by the magnetic field $B(x) = \operatorname{curl} A(x)$ only although we have an explicit formula which connects the SM in different gauges. This seems to contradict the following mental experiment. Suppose that a quantum particle interacts with a magnetic field. Note that it is exactly a field but not a potential which can be created by our hands. However, to calculate the SM, we have to introduce a magnetic potential and then solve the Schrödinger equation. Thus, the SM depends on a potential. So it appears that a particle itself chooses a gauge convenient for it. There could be (at least) two possible explanations of this seeming contradiction. The first is that the scattering amplitude $s(\omega, \omega'; \lambda)$ cannot be measured experimentally although it is widely believed to be possible. From this point of view only the scattering cross section $|s(\omega, \omega'; \lambda)|^2$ can be practically found which is compatible with (2.6). Another point of view is that experimental devices used for observation of a quantum particle are not harmless and fix some specific gauge.

On the other hand, for a given field, the SM is stable with respect to short-range perturbations of a potential.

Proposition 2.4 *Let the wave operators* $W_\pm(H, H_0)$ *exist. Suppose that* $\operatorname{curl} A(x) = \operatorname{curl} \tilde{A}(x)$ *and*

$$\tilde{A}(x) - A(x) = O(|x|^{-\rho}), \quad \rho > 1, \tag{2.7}$$

as $|x| \to \infty$*. Then the wave operators* $W_\pm(\tilde{H}, H_0)$ *also exist and the scattering operators and matrices for the pairs* H_0, H *and* $H_0, \tilde{H}$ *coincide.*

Indeed, according to Propositions 2.2 and 2.3, it suffices to show that $A(x)$ and $\tilde{A}(x)$ are related by equality (2.2) where the function $\phi(x)$ has a limit as $|x| \to \infty$. Let us define $\phi(x)$ as a curvilinear integral

$$\phi(x) = \int_{\Gamma_x} < \tilde{A}(y) - A(y), dy >$$

taken between 0 and a variable point x. By the Stokes theorem, this integral does not depend on a choice of Γ_x so that equality (2.2) holds. Moreover, choosing Γ_x as the piece of straight line connecting 0 and $x = r\omega$, $\omega \in \mathbb{S}^{d-1}$, and using (2.7), we see that the limit of $\phi(r\omega)$ as $r \to \infty$ exists. It remains to show that this limit does not depend on $\omega \in \mathbb{S}^{d-1}$. Again by the Stokes theorem,

$$\phi(r\omega_2) - \phi(r\omega_1) = \int_{\mathbb{S}_r[\omega_1,\omega_2]} < \tilde{A}(y) - A(y), dy >, \tag{2.8}$$

where $\mathbb{S}_r[\omega_1, \omega_2]$ is the arc of the circle centered at the origin and passing through the points $r\omega_1$ and $r\omega_2$. Condition (2.7) implies that integral (2.8) tends to 0 as $r \to \infty$.

3. MAIN RESULT

3.1. In this section we suppose that the dimension $d = 3$. Our results remain true for all $d \geq 3$ but not for $d = 2$. Let $B(x) = (B_1(x), B_2(x), B_3(x))$ be a magnetic field such that $\operatorname{div} B(x) = 0$ and estimate (1.2) is satisfied. Recall that a magnetic potential $A^{(tr)}(x) = (A_1^{(tr)}(x), A_2^{(tr)}(x), A_3^{(tr)}(x))$ satisfying the conditions

$$\operatorname{curl} A^{(tr)}(x) = B(x), \quad < A^{(tr)}(x), x >= 0 \tag{3.1}$$

can be constructed by the formula

$$A_1^{(tr)}(x) = \int_0^1 \Big(B_2(sx)x_3 - B_3(sx)x_2\Big) s ds. \tag{3.2}$$

Expressions for components $A_2^{(tr)}(x)$ and $A_3^{(tr)}(x)$ are obtained by cyclic permutations of indices in (3.2). It follows that $A^{(tr)}(x)$ admits the representation

$$A^{(tr)}(x) = A^{(\infty)}(x) + A^{(reg)}(x), \tag{3.3}$$

where

$$A_1^{(\infty)}(x) = |x|^{-2} \int_0^\infty \Big(B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2\Big) s ds, \tag{3.4}$$

$$A_1^{(reg)}(x) = -|x|^{-2} \int_{|x|}^\infty \Big(B_2(s\hat{x})x_3 - B_3(s\hat{x})x_2\Big) s ds. \tag{3.5}$$

Thus, $A^{(\infty)}(x)$ is a homogeneous function of degree -1 and $A^{(reg)}(x) = O(|x|^{-\rho})$ with $\rho = r - 1 > 1$ as $|x| \to \infty$. According to formulas (3.1) and (3.3) $\operatorname{curl} A^{(\infty)}(x) = O(|x|^{-r})$. Since $r > 2$ and $\operatorname{curl} A^{(\infty)}(x)$ is a homogeneous function of degree -2, this implies that

$$\operatorname{curl} A^{(\infty)}(x) = 0, \quad x \neq 0. \tag{3.6}$$

Similarly,

$$< A^{(\infty)}(x), x >= 0. \tag{3.7}$$

3.2. Given a magnetic field satisfying condition (1.2), we construct now a magnetic potential satisfying condition (1.1). We proceed from the magnetic potential $A^{(tr)}$ in the transversal gauge. Let $A^{(\infty)}$ be function (3.4). Then we define (cf. the proof of Proposition 2.4) the function $U(x)$ for $x \neq 0$ as a curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} < A^{(\infty)}(y), dy > \tag{3.8}$$

taken between some fixed point $x_0 \neq 0$ and a variable point x. It is required that $0 \notin \Gamma_{x_0,x}$, so that, in view of (3.6) and the Stokes theorem, $U(x)$ does not depend on a choice of $\Gamma_{x_0,x}$. Here it is used that the set $\mathbb{R}^3 \setminus \{0\}$ (and $\mathbb{R}^d \setminus \{0\}$ for all $d \geq 3$) is simply connected. Clearly,

$$A^{(\infty)}(x) = \operatorname{grad} U(x). \tag{3.9}$$

Moreover, the function $U(x)$ is homogeneous of degree 0. Indeed, if $x_2 = \gamma x_1$, $\gamma > 1$, then we can choose $\Gamma_{x_0,x_2} = \Gamma_{x_0,x_1} \cup (x_1, x_2)$ where (x_1, x_2) is the piece of straight line connecting x_1 and x_2. If $y \in (x_1, x_2)$, then according to (3.7) $< A^{(\infty)}(y), dy >= 0$. Hence $U(x_1) = U(x_2)$. We use definition (3.8) away from some neighbourhood of the point $x = 0$, say, for $|x| \geq 1/2$, and extend $U(x)$ as a differentiable function to all $x \in \mathbb{R}^3$. For example, we can choose a function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\eta(x) = 1$ for $|x| \geq 1/2$, $\eta(x) = 0$ in a neighbourhood of $x = 0$ and then replace $U(x)$ by $\eta(x)U(x)$. Let us now set

$$\begin{aligned} A(x) &= A^{(tr)}(x) - \operatorname{grad}(\eta(x)U(x)) \\ &= A^{(reg)}(x) + (1 - \eta(x))A^{(\infty)}(x) - U(x)\operatorname{grad}\eta(x), \end{aligned} \tag{3.10}$$

so that $A(x) = A^{(reg)}(x)$ for $|x| \geq 1/2$ and $A(x) = A^{(tr)}(x)$ in a neighbourhood of $x = 0$. Thus, we have the following result.

Proposition 3.1 *Suppose that* $\operatorname{div} B(x) = 0$ *and that condition* (1.2) *holds. Let the magnetic potential* $A(x)$ *be defined by formula* (3.10) *where* $A^{(\infty)}(x)$, $A^{(reg)}(x)$ *and* $U(x)$ *are functions* (3.4), (3.5) *and* (3.8), *respectively. Then* $A(x)$ *satisfies condition* (1.5) *and estimate* (1.1). *Moreover,* $A(x)$ *has compact support if* $B(x)$ *has compact support.*

By the proof of Proposition 3.1 we could have proceeded from the magnetic potential $A^{(c)}(x)$ satisfying the Coulomb gauge condition $\operatorname{div} A^{(c)}(x) = 0$. This is however less convenient. We emphasize that Proposition 3.1 is definitely not true for $d = 2$ if the magnetic flux $\Phi \neq 0$.

Let us denote by $\mathcal{A}(B)$ the class of magnetic potentials satisfying (1.5) and (1.1) for some $\rho > 1$. This class is non-empty according to Proposition 3.1. If $A \in \mathcal{A}(B)$, then, for an arbitrary function $\phi(x)$ such that $\operatorname{grad}\phi(x) = O(|x|^{-\rho})$, potential (2.2) also belongs to this class. However according to Proposition 2.4 the SM for the pair $H_0 = -\Delta$, $H = (i\nabla + A(x))^2$ does not depend on the choice of $A \in \mathcal{A}(B)$ and, thus, is determined by the magnetic field $B(x)$ only. We say that this SM $S(\lambda) = S(\lambda; B)$ is the SM for the field $B(x)$.

Comparing Propositions 2.1 and 3.1, we arrive at our main result.

Theorem 3.2 *Let a magnetic field* $B(x)$ *be such that* $\operatorname{div} B(x) = 0$ *and condition* (1.2) *holds, and let a magnetic potential* $A \in \mathcal{A}(B)$. *Then the wave operators for the pair*

$H_0 = -\Delta$, $H = (i\nabla + A(x))^2$ *exist, are unitary and the SM* $S(\lambda)$ *for the magnetic field* $B(x)$ *is a unitary operator for all* $\lambda > 0$. *The operator* $T(\lambda) = S(\lambda) - I$ *is compact and it belongs to the trace class if* $r > 4$. *If* $r > 4 + n$, $n = 0, 1, 2, \ldots$, *then* $T(\lambda)$ *is integral operator with kernel from the class* $C^n(\mathbb{S}^2 \times \mathbb{S}^2)$. *If*

$$|\partial^\alpha B(x)| \leq C_\alpha (1 + |x|)^{-r-|\alpha|}$$

for some $r \in (2, 4)$ *and all multi-indices* α, *then the operator* $T(\lambda)$ *has integral kernel which is a* C^∞*-function off the diagonal* $\omega = \omega'$ *and is bounded by* $C|\omega - \omega'|^{-4+r}$ *as* $\omega' \to \omega$.

Let

$$\Sigma_{diff}(\omega; \omega_0, \lambda) = (2\pi)^2 \lambda^{-1} |s(\omega, \omega_0; \lambda)|^2, \quad \omega \neq \omega_0,$$

be the scattering cross section for an incident direction ω_0 of a beam of particles and a direction of observation ω.

Corollary 3.3 *If estimate* (1.2) *is satisfied for* $r > 4$, *then* $\Sigma_{diff}(\omega; \omega_0, \lambda)$ *is a bounded function of* $\omega \in \mathbb{S}^2$. *If condition* (1.2) *is satisfied for some* $r \in (2, 4)$ *and all multi-indices* α, *then*

$$\Sigma_{diff}(\omega; \omega_0, \lambda) = O(|\omega - \omega_0|^{-8+2r})$$

as $\omega \to \omega_0$.

Using formula (3.10) and applying Proposition 2.3 to the function $\phi(x) = U(x)$, we obtain also the following

Proposition 3.4 *Suppose that a magnetic field* $B(x)$ *satisfies the assumptions of Theorem* 3.2. *Let* $S^{(tr)}(\lambda)$ *be the SM for the pair* $H_0 = -\Delta$, $H = (i\nabla + A^{(tr)}(x))^2$. *Set*

$$u(\omega) = U(\omega) - U(-\omega)$$

where the function $U(x)$ *is defined by formula* (3.8). *Let* S_0 *be the operator of multiplication by the function* $\exp(iu(\omega))$. *Then all the results of Theorem* 3.2 *are true for the operator* $T^{(tr)}(\lambda) = S^{(tr)}(\lambda) - S_0$.

Let us introduce the circulation

$$I(x, \omega) = \int_{-\infty}^{\infty} < A^{(\infty)}(x + t\omega), \omega > dt, \quad |\omega| = 1, \quad x \neq 0, \tag{3.11}$$

of $A^{(\infty)}(x)$ over the straight line $x + t\omega$, $t \in \mathbb{R}$. We shall consider this function only on the set $< x, \omega >= 0$. Integral (3.11) is convergent by virtue of transversal condition (3.7). It is easy to see that $I(x, \omega)$ is a homogeneous function of degree zero in the variable x. Under assumption (3.6) the function $I(x, \omega)$ does not actually depend on x and equals $u(\omega)$. Indeed, it follows from (3.9) that

$$\begin{aligned} I(x, \omega) = \int_{-\infty}^{\infty} < \operatorname{grad} U(x + t\omega), \omega > dt &= \lim_{T \to \infty} \int_{-T}^{T} \frac{d}{dt} U(x + t\omega) dt \\ &= \lim_{T \to \infty} (U(x + T\omega) - U(x - T\omega)) = U(\omega) - U(-\omega). \end{aligned} \tag{3.12}$$

3.3. As a concrete example, let us consider a toroidal solenoid $\mathbf{T}$ in the space $\mathbb{R}^3$ symmetric with respect to rotations around the x_3-axis (which does not intersect $\mathbf{T}$). We assume that the section of $\mathbf{T}$ by a half-plane passing through the x_3-axis is convex and has a smooth boundary $\partial\mathbf{T}$ but is not necessarily a disc. Suppose (which looks quite realistic) that

$$B(x_1, x_2, x_3) = \alpha(x_1^2 + x_2^2)^{-1}(-x_2, x_1, 0), \quad \alpha = \text{const},$$

inside of $\mathbf{T}$ and is zero outside. Then $\operatorname{div} B(x) = 0$ and the current $\operatorname{curl} B(x) = 0$ if $x \notin \partial\mathbf{T}$. Of course, Theorem 3.2 applies to this field and hence $S(\lambda) - I$ is integral operator with kernel from the class $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$.

Let us illustrate our construction on this example. In particular, we shall find the SM $S^{(tr)}(\lambda)$. First, we construct the potential $A^{(tr)}(x)$ by formula (3.2). Let the half-line L_z consist of points $s(1+z^2)^{-1/2}(1,0,z)$ for all $s \in \mathbb{R}_+$. Denote by z_1 and z_2 the values of z for which the half-line L_z is tangent to $\partial\mathbf{T}$ and, for $z \in [z_1, z_2]$, denote by $\varkappa_\pm(z)$, $\varkappa_+(z) \geq \varkappa_-(z)$, the values of s for which L_z intersects $\partial\mathbf{T}$. Set $z = z(x) = x_3(x_1^2+x_2^2)^{-1/2}$. Taking into account the rotational symmetry, we see that a point $s(x_1, x_2, x_3) \in \mathbf{T}$ if and only if $s|x|(1+z^2)^{-1/2}(1,0,z) \in \mathbf{T}$ or $\varkappa_-(z) \leq s|x| \leq \varkappa_+(z)$. Thus, integral (3.2) equals zero (and hence $A^{(tr)}(x) = 0$) if $z \notin (z_1, z_2)$ and it is actually taken over the set $[0,1] \cap [\varkappa_-(z)|x|^{-1}, \varkappa_+(z)|x|^{-1}]$ if $z \in (z_1, z_2)$. Therefore $A^{(tr)}(x) = 0$ if $|x| \leq \varkappa_-(z)$,

$$A_1^{(tr)}(x) = \alpha x_1 x_3 (x_1^2 + x_2^2)^{-1} \int_{\varkappa_-(z)/|x|}^{1} ds = \alpha x_1 x_3 (x_1^2 + x_2^2)^{-1}(1 - \varkappa_-(z)/|x|)$$

if $\varkappa_-(z) \leq |x| \leq \varkappa_+(z)$ and

$$A_1^{(tr)}(x) = \alpha x_1 x_3 (x_1^2 + x_2^2)^{-1} \int_{\varkappa_-(z)/|x|}^{\varkappa_+(z)/|x|} ds = \alpha x_1 x_3 (x_1^2 + x_2^2)^{-1} |x|^{-1} (\varkappa_+(z) - \varkappa_-(z))$$

if $|x| \geq \varkappa_+(z)$. Components $A_2^{(tr)}(x)$ and $A_3^{(tr)}(x)$ can be found quite similarly. In particular,

$$\left.\begin{array}{rcl} A_1^{(\infty)}(x) & = & x_1 x_3 (x_1^2 + x_2^2)^{-1} |x|^{-1} g(z), \\ A_2^{(\infty)}(x) & = & x_2 x_3 (x_1^2 + x_2^2)^{-1} |x|^{-1} g(z), \\ A_3^{(\infty)}(x) & = & -|x|^{-1} g(z), \end{array}\right\} \tag{3.13}$$

where

$$g(z) = \alpha(\varkappa_+(z) - \varkappa_-(z)).$$

Clearly, $g(z)$ is a bounded function, $\pm g(z) > 0$ if $\pm\alpha > 0$ for $z \in (z_1, z_2)$ and $g(z) = 0$ for $z \notin (z_1, z_2)$.

The function $g(z)$ can be calculated explicitly if the section of $\mathbf{T}$ by a half-plane passing through the x_3-axis is a disc. Suppose that this disc has radius r, its center belongs to the plane $x_3 = 0$ and the distance from the center to the x_3-axis is l, $l > r$. Then the equation of $\partial\mathbf{T}$ is

$$((x_1^2 + x_2^2)^{1/2} - l)^2 + x_3^2 = r^2.$$

The numbers $(1+z^2)^{-1/2}\varkappa_\pm(z)$ are defined as the roots of this equation for x_1 if $x_3 = zx_1$, $x_2 = 0$, so that

$$\varkappa_\pm(z) = (1+z^2)^{-1/2}(l \pm (r^2 - (l^2 - r^2)z^2)^{1/2})$$

and hence

$$g(z) = 2\alpha(1+z^2)^{-1/2}(r^2 - (l^2 - r^2)z^2)^{1/2}.$$

In particular, $-z_1 = z_2 = r(l^2 - r^2)^{-1/2}$ for this function.

Returning to the general case and taking into account (3.13), we see that a function $U(x)$ satisfying (3.9) can be constructed by the explicit formula

$$U(x) = G(x_3(x_1^2 + x_2^2)^{-1/2}),$$

where

$$G'(z) = -g(z)(z^2+1)^{-1/2}.$$

According to Proposition 3.4, up to an integral operator with C^∞-kernel, the SM $S^{(tr)}(\lambda)$ is the operator S_0 of multiplication by the function $\exp(iu(\omega))$, where

$$u(\omega) = q(\omega_3(1-\omega_3^2)^{-1/2}), \quad \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2, \tag{3.14}$$

and

$$q(z) = G(z) - G(-z) = -\int_{-z}^{z} g(t)(t^2+1)^{-1/2} dt.$$

Clearly, $q(-z) = -q(z)$, $q(z)$ is an increasing (decreasing) function if $\alpha < 0$ ($\alpha > 0$) and it is a constant, $q(z) = q_0$, if $z \geq \max\{-z_1, z_2\}$.

Let us finally show that $-q_0$ equals the magnetic flux Φ_s through a section of toroidal solenoid $\mathbf{T}$. Let $\omega_0 = (0,0,1)$, $< x_0, \omega_0 >= 0$, $x_0 \neq 0$, and let the function I be defined by formula (3.11). Then it follows from (3.12) and (3.14) that $q_0 = u(\omega_0) = I(x_0, \omega_0)$. By the Stokes theorem, Φ_s equals the circulation of the potential $A^{(tr)}(x)$ over the closed contour formed by the x_3-axis and the line $x_0 + \omega_0 t$, where $|x_0|$ is sufficiently large and t runs from ∞ to $-\infty$. We remark that the integrals over pieces of lines connecting these infinite parallel lines as well as the integral over the x_3-axis are equal to zero. Therefore, by (3.11),

$$\Phi_s = -I(x_0, \omega_0) = -q_0.$$

Note that smooth potentials satisfying (3.13) for $|x|$ large were considered in [3] by the method of [6].

4. MAGNETIC POTENTIALS WITH COULOMB DECAY

Here we consider arbitrary magnetic potentials $A(x)$, $x \in \mathbb{R}^2$, with Coulomb decay at infinity satisfying, at least asymptotically, the transversal gauge condition. For such potentials, the magnetic field $B(x) = \operatorname{curl} A(x)$ decays, in general, as $|x|^{-2}$ at infinity, so that the assumptions of the previous section are not satisfied. We shall show that in this case the SM contains a singular integral operator and hence the Aharonov-Bohm effect occurs.

Recall the following result obtained in [8].

Theorem 4.1 *Suppose that a magnetic potential $A(x)$ is a C^∞-function and admits the representation*
$$A(x) = A^{(\infty)}(x) + A^{(reg)}(x),$$
where $A^{(\infty)}(x)$ is a homogeneous function of degree -1 satisfying transversal condition (3.7) *and $\partial^\alpha A^{(reg)}(x) = O(|x|^{-\tilde{\rho}-|\alpha|})$, $\tilde{\rho} > 1$, as $|x| \to \infty$ for all α. Let Π_ω be the plane orthogonal to ω, and let the function $I(x,\omega)$ be defined by* (3.11)*. Set*

$$p^{(av)}(\omega) = (2\pi)^{-1} \int_{\mathbb{S}_\omega} \exp(iI(\psi,\omega))d\psi, \quad \mathbb{S}_\omega = \mathbb{S}^2 \cap \Pi_\omega,$$

$$q(\omega,\varphi) = -(2\pi)^{-2} \int_{\mathbb{S}_\omega} \Big(\exp(iI(\psi,\omega)) - p^{(av)}(\omega)\Big)(<\psi,\varphi> -i0)^{-2} d\psi \tag{4.1}$$

and
$$s_0(\omega,\omega') = p^{(av)}(\omega)\delta(\omega,\omega') + \mathrm{P.V.}q(\omega,\omega'-\omega).$$
Then for arbitrary $\lambda > 0$ the scattering amplitude satisfies the estimate

$$|s(\omega,\omega';\lambda) - s_0(\omega,\omega')| \leq C|\omega-\omega'|^{-2+\nu},$$

where $\nu = \tilde{\rho} - 1$ if $\tilde{\rho} \in (1,2)$ and $\nu = 1$ if $\tilde{\rho} \geq 2$.

We emphasize that the function $q(\omega,\varphi)$ is homogeneous in φ of degree -2 and satisfies the condition
$$\int_{\mathbb{S}_\omega} q(\omega,\varphi)d\varphi = 0.$$
Therefore the integral operator Q with kernel $q(\omega,\omega'-\omega)$ is well defined (as a bounded operator in $L_2(\mathbb{S}^2)$) in the sense of the principal value by the formula

$$(Qf)(\omega) = \lim_{\varepsilon \to 0} \int_{|\omega'-\omega|>\varepsilon} q(\omega,\omega'-\omega)f(\omega')d\omega'.$$

Theorem 4.1 implies

Corollary 4.2 *If $\omega \neq \omega'$ but $\omega - \omega' \to 0$, then*

$$s(\omega,\omega';\lambda) = q(\omega,\omega'-\omega) + O(|\omega-\omega'|^{-2+\nu}).$$

Corollary 4.3 *If $\omega \to \omega_0$, then*

$$\Sigma_{diff}(\omega;\omega_0,\lambda) = (2\pi)^2\lambda^{-1}|q(\omega_0,\omega_0-\omega)|^2 + O(|\omega-\omega_0|^{-4+\nu}).$$

Note that the order of singularity $|\omega-\omega_0|^{-4}$ here is the same as for electric Coulomb potentials.

If, under the assumptions of Theorem 4.1, $\operatorname{curl} A(x) = o(|x|^{-2})$ as $|x| \to \infty$, then necessarily condition (3.6) is satisfied and hence function (3.8) is correctly defined. It follows from equality (3.12) that in this case the function $I(x,\omega)$ does not depend on x and, thus, $q(\omega,\varphi) = 0$. This fact agrees of course with the results of the previous section for magnetic fields satisfying condition (1.2).

Nevertheless $q(\omega, \varphi)$ is non-trivial in the general case. Let us consider two concrete examples of potentials $A^{(\infty)}(x)$ homogeneous of degree -1 and satisfying transversal condition (3.7). The corresponding fields $\operatorname{curl} A(x)$ decay only as $|x|^{-2}$ at infinity, so that the conclusion of Theorem 3.2 is violated.

We define the first of these potentials by the equation

$$A^{(\infty)}(x) = |x|^{-3}(\alpha_1 x_2 x_3, \alpha_2 x_3 x_1, \alpha_3 x_1 x_2), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where α_j are constants and $\alpha_1 + \alpha_2 + \alpha_3 = 0$. An easy calculation shows that function (3.11) equals

$$I(x, \omega) = 2|x|^{-2}(\alpha_1 \omega_1 x_2 x_3 + \alpha_2 \omega_2 x_3 x_1 + \alpha_3 \omega_3 x_1 x_2). \tag{4.2}$$

As another example, we choose a modification of the Aharonov-Bohm potential

$$A^{(\infty)}(x) = \alpha |x|^{-2}(-x_2, x_1, 0).$$

In this case

$$I(x, \omega) = \pi \alpha |x|^{-1}(\omega_2 x_1 - \omega_1 x_2). \tag{4.3}$$

Functions (4.2) and (4.3) depend on the variables x_1, x_2, x_3 so that function (4.1) is non-trivial (see [8], for details). In both these examples the SM contain singular integral operators and hence the Aharonov-Bohm effect occurs.

Thus, a long-range behaviour (as $b(\hat{x})|x|^{-2}$) of a magnetic field in dimension 3 plays the same role as the topological obstruction in dimension 2 if the flux $\Phi \notin 2\pi\mathbb{Z}$.

REFERENCES

[1] Y. Aharonov and D. Bohm, Significance of electromagnetic potential in the quantum theory, Phys. Rev. **115** (1959), 485-491.

[2] W. C. Henneberger, Some aspects of the Aharonov-Bohm effect, Phys. Rev. A **22** (1980), 1383-1388.

[3] P. Roux, La matrice de diffusion pour l'opérateur de Schrödinger avec un potentiel électromagnétique à longue portée, Ph.D. thesis, University of Rennes-1, 2001.

[4] P. Roux and D. R. Yafaev, On the mathematical theory of the Aharonov-Bohm effect, J. Phys. A **35** (2002), 7481-7492.

[5] S. N. M. Ruijsenaars, The Aharonov-Bohm effect and scattering theory, Ann. Phys., **146** (1983), 1-34.

[6] D. R. Yafaev, The scattering amplitude for the Schrödinger equation with a long-range potential, Comm. Math. Phys. **191** (1998), 183-218.

[7] D. R. Yafaev, *Scattering theory: some old and new problems*, Lect. Notes Math. **1735** , Springer-Verlag, 2000.

[8] D. R. Yafaev, Scattering matrix for magnetic potentials with Coulomb decay at infinity, mp-arc 03-28, Int. Eq. Oper. Th. (2003), to be printed.