MOMENT ANALYSIS FOR LOCALIZATION IN RANDOM SCHRÖDINGER OPERATORS

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Abstract. We study localization effects of disorder on the spectral and dynamical properties of Schrödinger-type operators with random potentials. The new results include a fractional-moment method for continuum operators, which enables us to establish exponentially decaying bounds for the mean values of transition amplitudes, and of related resolvent operator kernels, for energies throughout the localization regime. The obstacles which have up to now prevented an extension to the continuum of this method, initially developed in the discrete context, are traced to the lack of a uniform bound on the Lifshitz-Krein spectral shift associated with the local potential terms, for which in the discrete case there is a simple universal upper bound. The difficulties are resolved here through an analysis of the resonance-diffusing effects of the disorder.

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Contents

1. Introduction

The addition of disorder can have a profound effect on the spectral and dynamical properties of a self adjoint differential operator. We consider here such phenomena for a class of operators in $L^2(\mathbb{R}^d)$ of the form

$$H_\omega \ := \ H_o \ + \ \lambda V_\omega \, , \tag{1.1}$$

with the disorder expressed through a random potential V_ω. In the prototypical example H_o is the Schrödinger operator

$$H_o \ = \ -\Delta \ + \ V_o(q) \tag{1.2}$$

with Δ the Laplacian and $V_o(q)$ a bounded periodic background potential. The random term V_ω is given by a sum of local non-negative "bumps", $U_\alpha(q) = U(q - q_\alpha)$, centered at the lattice sites $q_\alpha \in \mathbb{Z}^d$,

$$V_\omega(q) \ := \ \sum_{\alpha \in \mathcal{I}} \eta_{\alpha;\omega} \, U_\alpha(q) \ , \tag{1.3}$$

with $\{\eta_\alpha\}_{\alpha \in \mathcal{I}}$ a collection of independent random variables uniformly distributed in $[0, 1]$. It will be assumed that the space $\mathbb{R}^d$ is covered by the supports of $\{U_\alpha(\cdot)\}$ so that $\inf_q \sum U_\alpha(q) \geq 1$, with the parameter $\lambda \geq 0$ controlling the strength of the disorder. The subscript ω indicates a point in a probability space $(\Omega, \mathrm{Prob}(d\omega))$ and often will be dropped when it is clear from context we are discussing a random variable.

More generally, the initial term H_o may incorporate a magnetic field, i.e., take the form

$$H_o \ := \ \boldsymbol{D_A} \cdot \boldsymbol{D_A} \ + \ V_o(q) \tag{1.4}$$

where $\boldsymbol{D_A} = \mathrm{i}\nabla - \boldsymbol{A}(q)$ with $\boldsymbol{A}(q)$ the magnetic vector potential, and the periodicity of V_o and of the bump potentials may be replaced by more relaxed assumptions. The required technical conditions are listed in §1.2.

Our objective is to present tools for the study of the phenomenon known as Anderson localization [8], which concerns the potentially drastic effect of the disorder on the dynamical and spectral properties of the perturbed operator. In general terms, the effect is that in certain energy ranges the absolutely continuous spectrum of $H_0 = -\Delta$ may be modified to consist of a random dense set of eigenvalues associated with localized eigenfunctions.

A convenient tool is provided by the Green function $G_E(x, y)$, which is the kernel of the resolvent operator $(H_\omega - E - \mathrm{i}0)^{-1}$. This kernel is well known to decay exponentially in $|x - y|$ when E is in the resolvent set. The hallmark of localization is rapid (even exponential) decay of $G_E(x, y)$ at energies in the spectrum, though in this case it occurs with pre-factors which are not uniform in space and diverge at a dense countable set of eigenvalues. Rapid decay of the Green function is related to the non-spreading of wave packets supported in the corresponding energy regimes and various other manifestations of localization whose physical implications have been extensively studied in regards to the conductive properties of metals [8, 48, 66, 1, 47] and in particular to the quantum Hall effect [30, 50, 10, 12, 3].

The related mathematical analysis of localization for random operators has been a very active field, with various analytical methods appearing first in the context of discrete versions of the random operators (1.1), i.e. operators on the space ℓ^2 of a graph such as $\mathbb{Z}^d$, with the differential operator $\boldsymbol{D_A} \cdot \boldsymbol{D_A}$ replaced by a "hopping matrix". The goal of this work is to formulate a continuum version of the fractional-moment method, which for discrete systems has provided a simple perspective on localization and has enabled exponentially decaying bounds on expectation values of various propagation kernels [4, 2, 3, 6]. The existing results on localization in the continuum have been based on the multiscale analysis, first obtained for discrete operators [26, 25, 67] and then extended to the continuum [43, 16, 11, 39, 28, 64]. (We do not attempt to give an exhaustive survey of the vast literature on localization. The reader is referred to the recent book of P. Stollmann [64] for a review of the history of the subject and a gentle introduction to the multi-scale analysis – which is not used here.)

An important component of the analysis is the understanding of the effect of an independent variation of a random coefficient. Such a variation can resolve a singularity which may be present due to the proximity of the given energy to an eigenvalue whose eigenfunction has significant support nearby. However, a change in the parameter can produce the opposite effect, through the creation of a resonance. In the discrete setup the latter possibility occurs at not more than a single value of the random parameter, since each coefficient affects a rank-one term and the number of levels which can be moved past a given energy is bounded by the rank of the perturbation. The source of the difficulty in the extension of the discrete analysis – aside from the fact that the rank-one argument is not applicable here – can be traced to the fact that in the continuum setup there is no uniform bound on the corresponding "spectral shift". To circumvent these difficulties we employ the Birman-Schwinger principle in place of rank-one analysis and control the Lebesgue measure of the set of nearly-singular values of a coupling parameter by means of the "weak L^1" bound

$$\left|\left\{\eta : \left\|T(\eta + A + \mathrm{i}0)^{-1}T\right\|_{HS} > t\right\}\right| \;\leq\; \frac{C}{t}\,\|T\|_{HS}^2\ , \tag{1.5}$$

which is valid for any maximally dissipative operator A and Hilbert-Schmidt operator T.

1.1. **Statement of the main results.** Our analysis of the localization regime consists of three distinct, but inter-related, parts. The goal is to shed light on the spectral and dynamical localization exhibited by the random operators. We go beyond the past results by establishing, for suitable energy ranges, exponential decay of the *mean values* (which conveys more information than decay of the typical values) of the related kernels. The analysis involves also finite-volume "constructive criteria" which in principle may be used to map the energy range at which there is a finite localization length. However, all this relies on an essential step, which is the identification -and proof of preliminary bounds for- a pertinent quantity whose expectation value is finite and indicative of the behavior of various other quantities of interest.

A key to the properties of an operator is provided by the family of resolvent operators $(H - z)^{-1}$, and their finite volume versions $(H^{(\Lambda)} - z)^{-1}$ for $\Lambda \subset \mathbb{R}^d$. However, even in the simpler lattice case the expectation value of the Green function does not provide the optimal diagnostic; the mean value $\mathbb{E}\left(G_{E+\mathrm{i}0}(x, y)\right)$ may decay exponentially due to cancellations even in the absence of localization whereas $\mathbb{E}\left(|G_E(x, y)|\right) = \infty$ for E in the regime of localization due to the presence of non-integrable singularities. However a useful diagnostic is provided by the fractional moment $\mathbb{E}\left(|G(x,y)|^s\right)$, which is finite for $s < 1$ under suitable regularity assumptions on the distribution of the random coefficients (the value of s does not play an important role in the discussion) [4].

For the random Schrödinger operators in the continuum it has been noted [16, 38] that a useful counterpart of the discrete Green function $|G(x,y)|$ need not be the integral kernel of $(H-E-i0)^{-1}$, but rather the operator norm $\|\chi_x(H-E-i0)^{-1}\chi_y\|$ for suitable compactly supported functions χ_x and χ_y. A preliminary result for our discussion is that the fractional moments of these quantities are finite also for the continuum operators. This is not a minor step, as it requires to prove that the randomness present in H_ω suffices to diffuse the singularities which are present when the above quantity is evaluated for energies in the spectrum. The statement requires some regularity assumptions, whose detailed presentation is postponed to section 1.2.

We shall be interested in restrictions of H to open sets $\Lambda \subset \mathbb{R}^d$, for which we denote the Dirichlet boundary conditions operator on $L^2(\Lambda)$ by $H^{(\Lambda)}$. However, all that we say is rather insensitive to the boundary conditions. In particular, the methods used are applicable to operators with Neumann, periodic, or quasi-periodic boundary conditions, which will play a role in our discussion of the application of density of states bounds (section 6.3).

Throughout we denote the characteristic function of a set Λ by $\mathbf{1}_\Lambda$. It is convenient to set the distance unit as r - the size of the "bumps" U_α, as described in assumption **A2** below. Thus, for $x \in \mathbb{R}^d$ we let $\chi_x = \mathbf{1}_{B_x^r}$, where B_x^r is the ball of radius r centered at x.

Decay rates will be expressed here through a distance function $\mathrm{dist}(x,y) = |x-y|$ for which the choice of the norm on $\mathbb{R}^d$ does not affect our analysis. It is convenient to interpret it as $|x| = \sup_j |x_j|$, in which case "balls" B_x^r are hypercubes. We shall also use the domain-adapted:

$$\mathrm{dist}_\Lambda(x,y) \;=\; \min\left\{|x-y|, \mathrm{dist}(x,\Lambda^c) + \mathrm{dist}(y,\Lambda^c)\right\} \;, \tag{1.6}$$

for which the boundary of $\Lambda \subset \mathbb{R}^d$ is in effect regarded as a single point. The (optional) use of the modified distance, as in [6], enables the analysis to cover also the cases where exponential localization in the bulk may possibly conicide with the occurance of extended boundary states in certain subdomains.

Following is the important preliminary result. (Regardless of its preliminary and fundamental nature, its proof is placed here after that of the theorem stated next). Here, and in the following, E_o denotes the minimum of the spectrum of H_o.

Lemma 1.1. *Let H_ω be a random Schrödinger operator of the type (1.1) which satisfies the regularity assumptions* **A** *(see below). Then, for each $s \in (0,1)$, there exists $C_{s,\lambda} < \infty$ such that*

$$\sup_{\varepsilon>0} \mathbb{E}\left(\left\|\chi_x \,\frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon}\,\chi_y\right\|^s\right) \;\le\; C_{s,\lambda}\,(1+1/\lambda)^s(1+|E-E_0|)^{s(d+2)} \tag{1.7}$$

for any open $\Omega \subset \mathbb{R}^d$, $x, y \in \Omega$, and $E \in \mathbb{R}$, with

$$C_{s,\lambda} \;\le\; \text{Const.}\,\frac{(1+\lambda)}{1-s}^{s(d+2)} \;. \tag{1.8}$$

For any bounded region Ω, (1.7) holds also with $\varepsilon = 0$, and if H satisfies the (stronger) condition $\mathbf{A3}'$ *then it holds with*

$$C_{s,\lambda} \ \leq \ \text{Const.} \ \frac{1}{1-s} \ . \tag{1.9}$$

Our next statement is that the above resolvent-related fractional moment provides information on a host of other quantities of interest:

Theorem 1.1. *Let H be a random Schrödinger operator which satisfies the regularity assumptions* $\mathbf{A}$, *let Ω be an open subset of $\mathbb{R}^d$, and Λ_n an increasing sequence of bounded open subsets of Ω with $\cup\Lambda_n = \Omega$. Suppose that for some $0 < s < 1$ and $\mathcal{J}$ an open bounded interval there are constants $A < \infty$ and $\mu > 0$ such that*

$$\int_{\mathcal{J}} \mathbb{E}\left(\|\chi_x \frac{1}{H^{(\Lambda_n)} - E} \chi_y\|^s \right) \mathrm{d}E \ \leq \ A e^{-\mu \operatorname{dist}_{\Lambda_n}(x,y)} \tag{1.10}$$

for all $n \in \mathbb{N}$, $x, y \in \Lambda_n$. Then for every $r < 1/(2-s)$ there exists $A_r < \infty$ such that, for all f $x, y \in \Omega$,

$$\mathbb{E}\left(\sup_{g:\,|g|\leq 1} \|\chi_x g(H^{(\Omega)}) P_{\mathcal{J}}(H^{(\Omega)}) \chi_y\|_1 \right) \ \leq A_r e^{-r\mu \operatorname{dist}_\Omega(x,y)} \ , \tag{1.11}$$

where the supremum is taken over all Borel measurable functions g which satisfy $|g| \leq 1$ pointwise, and $\|\cdot\|_1$ is the trace norm."

In case $\Omega = \mathbb{R}^d$, one may further conclude that the spectrum of H in $\mathcal{J}$ is almost surely pure point, with eigenfunctions $\{\psi_{n;\omega}(x)\}$ which for any $\nu \in (0, \mu/(2-s))$ satisfy

$$|\psi_{n;\omega}(x)| \ \leq \ B_{n,\nu;\omega} \, \mathrm{e}^{-\nu \operatorname{dist}(x, x_{n;\omega})} \ . \tag{1.12}$$

with $B_{n,\nu;\omega} < \infty$ (almost surely).

Remarks: *i.* The bound (1.11) implies *dynamical localization*, with $g(H) = \mathrm{e}^{-\mathrm{i}tH}$.
ii. Eq. (1.12) is equivalent to $\limsup_{x\to\infty} \ln|\psi_{n;\omega}(x)|/|x| \leq -\nu$, and $x_{n;\omega}$ may be omitted there (replaced by 0) - at the expense of a somewhat worse bound on B_n.
iii. The proof, given in Section 3, yields more explicit information on the constant A_r. In particular, for given r, s and λ, it depends polynomially on $E_+ = \sup \mathcal{J}$ with degree slightly larger than $d/2$.

Our next result deals with finite volume sufficiency criteria for the localization bounds (1.10). We define the boundary layer of a set Λ to be the (open) set

$$\delta\Lambda := \{q \ : \ r < \operatorname{dist}(q, \Lambda^c) < 23r\}, \tag{1.13}$$

where the choice of the depth is somewhat arbitrary, but convenient for our argument. The basic idea is that if in some ball Λ the fractional moments from the center to $\partial\Lambda$ are "small enough" then the input criteria of Theorem 1.1 are satisfied. In the following statement r_o is the (independence) length introduced in the below-stated assumption **IAD**.

Theorem 1.2. *Let H be a random Schrödinger operator which satisfies the assumptions* $\mathbf{A}$ *as well as* **IAD**. *For each $s \in (0, 1/3)$, there exists $M_{s,\lambda} < \infty$, such that if for some $E \in \mathbb{R}$ and $L > r_0 + 23r$,*

$$\begin{aligned} \mathrm{e}^{-\gamma} \ &:= \ M_{s,\lambda}(1+1/\lambda)^{2s}(1+|E-E_0|)^{5s(d+2)}(1+L)^{2(d-1)} \\ &\qquad \times \sup_{\alpha\in\mathcal{I}} \limsup_{\varepsilon\downarrow 0} \mathbb{E}\left(\left\| \chi_\alpha \, \frac{1}{H^{(B^L_\alpha)} - E - \mathrm{i}\varepsilon} \mathbf{1}_{\delta B^L_\alpha} \right\|^s \right) \ < \ 1 \ , \end{aligned} \tag{1.14}$$

then for any open $\Omega \subset \mathbb{R}^d$, and any $x, y \in \Omega$

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon} \chi_y \right\|^s \right) \leq \mathrm{e}^{\gamma} A(s, \lambda, E)\, \mathrm{e}^{-\gamma\, \mathrm{dist}_\Omega(x,y)/2L} , \tag{1.15}$$

with $A(s, \lambda, E)$ the fractional moment bound provided by Lemma 1.1 – i.e, the right hand side of eq. (1.7).

Furthermore, for any bounded region Ω (1.15) holds also with $\varepsilon = 0$, and if H satisfies also **A3′** *then $M_{s,\lambda}$ can be chosen to depend on s and λ as*

$$M_{s,\lambda} \;=\; \frac{\text{Const.}}{1 - 3s} . \tag{1.16}$$

whereas without the stronger assumption we need to allow an additional factor of $(1 + \lambda)^{5s(d+4)}$.

To complete the discussion it is useful to observe that exponential decay of the Green function in $\mathbb{R}^d$ at a single energy implies the localization bound eq. (1.10) in all sufficiently large volumes. Namely a necessary and sufficient condition for eq. (1.15), and thus eq. (1.11), to hold in a neighborhood of an energy E is eq. (1.17) below. Thus, in principle the entire regime of localization in the sense of eq. (1.15) may be mapped out using the criterion provided by Theorem 1.2.

Theorem 1.3. *Let H be a random Schrödinger operator which satisfies* **A** *and* **IAD**. *Suppose that for some $A < \infty$, $\mu > 0$ and $E \in \mathbb{R}$*

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\|\chi_\alpha \frac{1}{H - E - \mathrm{i}\varepsilon} \chi_\beta\|^s \right) \leq A e^{-\mu|\alpha - \beta|} \tag{1.17}$$

for all $\alpha, \beta \in \mathcal{I}$. Then, for sufficiently large L, eq. (1.14) is satisfied uniformly for all E' in an open neighborhood of E.

1.2. **Assumptions.** Following are the regularity conditions required for our results. The condition **A** refers to the collection **A1-A3**.

- **A1** The components of the vector potential $\boldsymbol{A}$, its first derivatives $\partial_i \boldsymbol{A}$ for $i = 1, ..., d$, and $V_{o,+}$ (the positive part of V_o) are locally bounded on $\mathbb{R}^d$. The negative part $V_{o,-}$ of V_o is bounded.
- **A2** Each function U_α is bounded, non-negative, supported in a ball of radius r around α for fixed $r > 0$ and some $\alpha \in \mathcal{I}$, with $\mathcal{I}$ a discrete set of points in $\mathbb{R}^d$. The number of points α falling within any unit cube is uniformly bounded by some $N < \infty$, and the function

$$F(q) \;:=\; \sum_{\alpha \in \mathcal{I}} U_\alpha(q) \tag{1.18}$$

satisfies uniform bounds

$$1 \;\leq\; \inf_q F(q) \;\leq\; \sup_q F(q) \;=:\; b_+ \;<\; \infty . \tag{1.19}$$

Moreover, $|\partial(\operatorname{supp} U_\alpha)| = 0$.

- **A3** The random variables η_α, $\alpha \in \mathcal{I}$ take values in $[0, 1]$ and the conditional distribution of η_α at specified values of $\{\eta_\zeta\}_{\zeta \neq \alpha}$ has a density, denoted $\rho_\alpha(\eta \,|\, \omega)$, which is uniformly bounded:

$$D \;:=\; \sup_\alpha \|\rho_\alpha(\cdot|\cdot)\|_\infty \;<\; \infty \tag{1.20}$$

where $\|\cdot\|_\infty$ indicates the essential supremum over η and ω.

Assumption **A** is sufficiently general to cover many important examples, such as the two-dimensional Landau hamiltonian with random potential, where $\boldsymbol{A}(q_1, q_2) = \frac{1}{2}(-Bq_2, Bq_1)$. For both $\boldsymbol{A}$ and V_o one could allow suitable, dimension-dependent, L^p-type singularities, but we prefer to avoid the additional technicalities which are caused by this.

A key requirement for Theorem 1.2 is "independence at a distance," namely the random functions obtained by restricting V_ω to well separated regions are pairwise independent.

IAD: There exists $r_0 > 0$ such that if $\Lambda, \Lambda' \subset \mathbb{R}^d$ with $\mathrm{dist}(\Lambda, \Lambda') > r_0$ then the collections of random variables $\eta_\Lambda := \{\eta_\zeta : \zeta \in \Lambda \cap \mathcal{I}\}$ and $\eta_{\Lambda'} := \{\eta_\zeta : \zeta \in \Lambda' \cap \mathcal{I}\}$ are independent , i.e.,

$$\mathbb{E}\left(f(\eta_\Lambda) g(\eta_{\Lambda'})\right) \ = \ \mathbb{E}\left(f(\eta_\Lambda)\right) \mathbb{E}\left(g(\eta_{\Lambda'})\right) \ , \tag{1.21}$$

for arbitrary bounded measurable functions f, g on $\mathbb{R}^{\Lambda \cap \mathcal{I}}$ and $\mathbb{R}^{\Lambda' \cap \mathcal{I}}$, respectively. Without loss of generality we assume that $r_0 \geq 2r$.

We note, however, that this assumption is not required for the proof of the boundedness of the fractional moments (Lemma 1.1) and also not for the derivation of localization from the Green function decay (Theorem 1.1).

The restriction of η_α to range over $[0, 1]$ is not essential; through the adjustment of the background potential and the disorder parameter that range can be replaced by any other bounded interval. However, the normalization becomes relevant when one considers the strong disorder regime.

Some of the bounds derived below – c.f. Lemmas 4.3 and 4.4 – exhibit coefficients which grow with increasing λ. For applications of these bounds to the "large disorder regime" ($\lambda >> 1$) it is useful to break the coupling $\lambda\eta_\alpha$ into a sum of variables $\eta_{1;\lambda} + \eta_{2;\lambda}$ such that $\eta_{1;\lambda}$ is of order one and obeys **A3**. This could be accomplished in a number of ways – e.g., let $\eta_{1;\lambda}$ be the fractional part of $\lambda\eta_\alpha$. However, without an additional assumption these decompositions might become more and more singular as λ increases. Following is a useful notion.

Definition 1.1. A real valued random variable X, with an absolutely continuous probability measure of density $\rho(\cdot)$ on $\mathbb{R}$, is *blow-up regular* if there exist two sequences of real valued random variables, $\{X^{(n)} : n \geq 1\}$ and $\{Y^{(n)} : n \geq 1\}$ such that

(1) $X^{(n)}$ takes values in $[0, 1]$.
(2) For each $n \geq 1$,

$$nX \ = \ Y^{(n)} \ + \ X^{(n)} \ . \tag{1.22}$$

(3) The conditional distribution of $X^{(n)}$, at a specified value $Y^{(n)} = y$, has a bounded density, $\rho_n(\cdot|y)$, and

$$D \equiv \sup_{n\geq 1;\, x,y\in\mathbb{R}} |\rho_n(x|y)| \ < \ \infty \ . \tag{1.23}$$

The *blow-up norm* of the probability density ρ, denoted D_ρ, is the infimum of the above quantity D taken over all sequences $\left\{X^{(n)}, Y^{(n)} : n \geq 1\right\}$ satisfying (1) and (2).

The following assumption, in lieu **A3**, allows better bounds for the strong disorder regime.

A3′ The random variables η_α, $\alpha \in \mathcal{I}$ take values in $[0,1]$, and for each α the conditional distribution of η_α at specified values of $\{\eta_\zeta\}_{\zeta \neq \alpha}$ is blow-up regular, with the blow-up regularity norms bounded by a common $D < \infty$.

The above condition is satisfied in case of the uniform distribution in [01], and also if $\ln \rho$ is Lipshitz-continuous in $[0,1]$ (see Appendix A).

Naturally, both the assumptions and the results outlined above, deserve a number of more technical comments. In order to keep the introduction relatively free of technicalities, further discussion is postponed to the Appendix.

The derivation of the results presented above is organized as follows. Section 2 contains a short description of the Birman-Schwinger relation, which is used throughout this work. In Section 3 we discuss the relation of resolvent moment bounds with eigenvalue correlators and dynamical localization. We derive there Theorem 1.1 and some related results. In Section 4 we turn to the main technical issue of finiteness of the resolvent's fractional moments. The technical part of Section 4, which is §4.4, is essential for the progress presented here, however the reader interested only in the general outline of our method may wish to skip that portion at first reading since those arguments are not used elsewhere in this work. In Section 5 we concentrate on the sufficiency criteria of Theorem 1.2. Finally, we present a proof of Theorem 1.3 and several applications of our method -to specific energy and disorder regimes- in Section 6. Further discussion and related results are found in Appendices A-C.

2. The Birman-Schwinger relation

Central to our analysis is the consideration of one-parameter operator families of the form

$$A_\xi = A_0 - \xi V \tag{2.1}$$

with a non-negative operator V and $\xi \in \mathbb{R}$. Such families arise here when all but one of the random couplings in the random operator (1.1) are considered fixed. Their study replaces the rank one perturbation arguments which have played a key role in the analysis of discrete operators, e.g. in [62, 4].

The family (2.1) formally satisfies the *Birman-Schwinger relation*:

$$V^{1/2} A_\xi^{-1} V^{1/2} = \Big((V^{1/2} A_0^{-1} V^{1/2})^{-1} - \xi \mathbf{1} \Big)^{-1} \tag{2.2}$$

to be interpreted as equality of operators in $(\ker V)^\perp$, subject to issues of invertibility. In the "classical" version of this relation [14, 56], $A_0 = -\Delta + \gamma \mathbf{1}$ for some $\gamma > 0$, and V is a non-negative relatively compact potential perturbation. It is shown, e.g. in [57, Sect.8], that the number of eigenvalues of $-\Delta - V$ less than $-\gamma < 0$ equals the number of eigenvalues of $V^{1/2}(-\Delta + \gamma \mathbf{1})^{-1} V^{1/2}$ greater than one. This can be understood through the observation that the two sides of (2.2) become singular for the same values of the parameters γ and ξ.

We shall need two non-classical, though certainly not new, versions of the Birman-Schwinger relation which are described below. In our applications they arise through the two standard procedures for regularizing the Green function at energies in the infinite volume spectrum: (1) adding a small imaginary part to the energy or (2) using finite volume approximations.

The case of complex energy is covered by the following.

Lemma 2.1. *Let $A_0 = B + iC$ be an operator on a separable Hilbert space $\mathcal{H}$, where B is self-adjoint and C is bounded with $C \geq \delta \mathbf{1}$ for some $\delta > 0$. Also, let V be a bounded non-negative operator in $\mathcal{H}$. Then the Birman-Schwinger operator*

$$A_{BS} := (V^{1/2} A_0^{-1} V^{1/2})^{-1} \tag{2.3}$$

with $\mathcal{D}(A_{BS}) = \mathcal{R}(V^{1/2} A_0^{-1} V^{1/2})$ is maximally dissipative in $(ker V)^\perp$. Moreover, its resolven set $\rho(A_{BS})$ includes the closed lower half plane $\overline{\mathbb{C}}_-$, and the Birman-Schwinger relation

$$(A_{BS} - \xi \mathbf{1})^{-1} = V^{1/2} (A_0 - \xi V)^{-1} V^{1/2} \tag{2.4}$$

holds in $(ker V)^\perp$ for every $\xi \in \overline{\mathbb{C}}_-$.

Remarks:

(1) Recall that a densely defined operator A is called *dissipative* if $\mathrm{Im}\,\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}(A)$. It is called *maximally dissipative* if it has no proper dissipative extension, which is equivalent to contractivity of the semi-group $e^{\mathrm{i}tA}$ generated by $\mathrm{i}A$.

(2) In our applications we V appears as a non-negative potential and $A_0 = z - H$ with $\mathrm{Im}\, z > 0$ and H a self-adjoint Schrödinger operator on $L^2(\Omega)$. In this case $(\ker V)^\perp = L^2(\{x : V(x) > 0\})$.

Proof. Note that A_0 is boundedly invertible with $\|A_0^{-1}\| \leq 1/\delta$. To show that the restriction of $V^{1/2} A_0^{-1} V^{1/2}$ to $(\ker V)^\perp$ is invertible, suppose we are given $f \in (\ker V)^\perp$ such that $V^{1/2} A_0^{-1} V^{1/2} f = 0$. Then $g := A_0^{-1} V^{1/2} f \in \mathcal{D}(A_0)$ with

$V^{1/2}g = 0$. Thus $0 = \langle V^{1/2}g, f\rangle = \langle g, V^{1/2}f\rangle = \langle g, A_0 g\rangle$. Taking imaginary parts and using $C \geq \delta\mathbf{1}$ we find that $g = 0$. Thus $V^{1/2}f = A_0 g = 0$ and therefore $f \in (\ker V) \cap (\ker V)^\perp = \{0\}$.

We conclude that A_{BS} exists as an operator in $(\ker V)^\perp$. It is densely defined, since if $\langle f, V^{1/2}A_0^{-1}V^{1/2}g\rangle = 0$ for all $g \in (\ker V)^\perp$, then $V^{1/2}(A_0^{-1})^* V^{1/2} f = 0$ and, by the same argument as above, $f = 0$. It is easily checked that A_{BS} is dissipative.

For $\xi \in \overline{\mathbb{C}}_-$ one verifies explicitly that

$$(A_{BS} - \xi\mathbf{1})V^{1/2}A_0^{-1}V^{1/2} = \mathbf{1} - \xi V^{1/2}A_0^{-1}V^{1/2}$$

is the inverse of $I + \xi V^{1/2}(A_0 - \xi V)^{-1}V^{1/2}$ in $(\ker V)^\perp$. In particular, for $\xi = -i$, this shows that $(A_{BS} + iI)\mathcal{D}(A_{BS}) = (\ker V)^\perp$, proving that A_{BS} is maximally dissipative (see [65]). Using the resolvent identity we also get (2.4). $\square$

The argument in the above proof which showed that $V^{1/2}A_0^{-1}V^{1/2}$ is invertible does not generally carry over to the case where A_0 is an invertible self-adjoint operator. If A_0 has fixed sign (as in the classical BS-principle) it does, but not if 0 is in a spectral gap for A. However, in the case which is of interest to us, namely that $A_0 = H - E$ with H a finite volume Schrödinger operator, we can make use of the fact that H is *local* in the sense that $H\varphi$ vanishes on an open set O if $\varphi \in \mathcal{D}(H)$ vanishes on O.

Thus, let $\Lambda \subset \mathbb{R}^d$ be open and bounded and let H_0 be the Dirichlet restriction of a Schrödinger operator $(i\nabla - A)^2 + V_0$ onto $L^2(\Lambda)$, where A and V_0 satisfy the general assumptions of Section 1. Let $V \geq 0$ be a bounded non-zero potential of compact support with $|\partial(\operatorname{supp} V)| = 0$, and for $\xi \in \mathbb{R}$ let

$$H_\xi = H_0 - \xi V. \tag{2.5}$$

Then H_ξ is self-adjoint with compact resolvent. Thus, by Theorem VII.3.9 of [36], its repeated eigenvalues $E_n(\xi)$, $n \in \mathbb{N}$, and corresponding complete set of orthonormal eigenfunctions $\psi_n(\xi)$ can be labeled such that $E_n(\cdot)$ and $\psi_n(\cdot)$ are holomorphic (note that crossings are possible, that is, the E_n may be degenerate and are not necessarily in increasing order). By the Feynman-Hellmann Theorem and the unique continuation property of eigenfunctions,

$$E_n'(\xi) = -\langle V\psi_n(\xi), \psi_n(\xi)\rangle < 0. \tag{2.6}$$

Thus $\Gamma_n := E_n^{-1}$ exists on the range of E_n and

$$\Gamma_n'(E) = -\frac{1}{\langle V\psi_n(\Gamma_n(E)), \psi_n(\Gamma_n(E))\rangle}. \tag{2.7}$$

For real $E \notin \sigma(H_\xi)$, we define

$$K_{\xi,E} = V^{1/2}(H_\xi - E)^{-1}V^{1/2} \tag{2.8}$$

as an operator in $L^2(\operatorname{supp} V)$, where it is compact and self-adjoint. We claim that $\ker K_{\xi,E} = \{0\}$. Indeed, if $V^{1/2}(H_\xi - E)^{-1}V^{1/2}f = 0$ for $f \in L^2(\operatorname{supp} V)$, then $(H_\xi - E)^{-1}V^{1/2}f = 0$ on $\operatorname{supp} V$ and, since H_ξ is local, $V^{1/2}f = 0$ in the interior of $\operatorname{supp} V$. This implies that $f = 0$ since $|\partial(\operatorname{supp} V)| = 0$.

We conclude that $K_{\xi,E}^{-1}$ exists in $L^2(\operatorname{supp} V)$ as an unbounded self-adjoint operator with discrete spectrum. By the same arguments used in the proof of Lemma 2.1 it may be shown that for $E \notin \sigma(H_\xi) \cup \sigma(H_0)$,

$$K_{\xi,E} = (K_{0,E}^{-1} - \xi\mathbf{1})^{-1}. \tag{2.9}$$

Lemma 2.2. *Let $E \notin \sigma(H_0)$ and $\xi \neq 0$. Then*

(1) *ϕ is a normalized eigenfunction of $K_{0,E}^{-1}$ with eigenvalue ξ if and only if $\psi := \xi(H_0 - E)^{-1}V^{1/2}\phi$ is an eigenfunction of H_ξ with eigenvalue E and $\langle \psi, V\psi \rangle = 1$.*

(2) *the repeated eigenvalues of $K_{0,E}^{-1}$ are given by $\Gamma_n(E)$, $n \in \mathbb{N}$, with (non-normalized) complete eigenvectors $V^{1/2}\psi_n(E)$.*

Proof. The second claim follows from the first and the definition of $\Gamma_n(E)$.

To prove the first claim, first suppose that that $K_{0,E}^{-1}\phi = \xi\phi$ and let $\psi := \xi(H_0 - E)^{-1}V^{1/2}\phi$. Then $V^{1/2}\psi = \xi K_{0,E}\phi = \phi$ and $(H_\xi - E)\psi = \xi(H_\xi - E)(H_0 - E)^{-1}V^{1/2}\phi = \xi V^{1/2}\phi - \xi V\psi = 0$.

Conversely, if $H_\xi\psi = E\psi$ with $\langle V\psi, \psi \rangle = 1$, then $\psi = \xi(H_0 - E)^{-1}V\psi$ and thus $\phi := V^{1/2}\psi \in \mathcal{R}(V^{1/2}(H_0 - E)^{-1}V^{1/2}) = \mathcal{D}(K_{0,E}^{-1})$ and is normalized. Moreover, $K_{0,E}^{-1}\phi = K_{0,E}^{-1}V^{1/2}\phi = \xi K_{0,E}^{-1}K_{0,E}V^{1/2}\psi = \xi\phi$. □

3. From resolvent bounds to eigenfunction correlators and dynamical localization

In this section we prove Theorem 1.1. For discrete models such results were derived in ref. [2, 6] employing the observation that the eigenfunctions of the operator H play the role of Green functions for a re-sampled operator $\hat{H}$, at other values of the coupling variables. Key in that analysis were properties of rank one perturbations. Use was also made of a very convenient interpolation argument which permits to extract bounds on the off-diagonal matrix elements of the spectral projections of H from fractional moment bounds on the Green function of $\hat{H}$.

We find that the approach of ref. [2, 6] can also be applied to continuum operators, with the arguments which were based on rank one perturbation replaced, or generalized, by considerations of the Birman-Schwinger operator.

3.1. **Correlators – eigenfunction and other.** In the discussion of dynamical localization for discrete random operators in ref. [6], estimates were developed for the spectral measures $\mu_{x,y}$ which are defined through the Riesz theorem by

$$\int f(E)\mu_{x,y}(\mathrm{d}E) \;=\; \langle x|f(H)|y\rangle\,, \qquad f \in C_o(\mathbb{R}) \tag{3.1}$$

Exponential decay (in $|x-y|$) of the *total variation* of these measures provided a strong description of dynamical localization. For analogous bounds in the present context, it is convenient to work with the "operator valued measures,"

$$f \mapsto \chi_x f(H^{(\Omega)})\chi_y \qquad f \in C_c(\mathbb{R})\ . \tag{3.2}$$

We introduce also the "total variation" of these measures,

$$Y_\Omega(\mathcal{J};x,y) \;:=\; \sup_{\substack{f\in C_c(\mathcal{J})\\ \|f\|_\infty \le 1}} \left\|\chi_x f(H^{(\Omega)})\chi_y\right\|_1\ , \tag{3.3}$$

defined for bounded open intervals $\mathcal{J}$ where $C_c(\mathcal{J})$ denotes the continuous functions compactly supported inside $\mathcal{J}$. (For the main applications it would suffice to estimate the operator-norm analog of Y_Ω, however our methods yield at no extra effort the stronger *trace norm* bounds, c.f., (3.6) below.)

For a finite region Λ and fixed $\alpha \in \mathcal{I}$, we use the Birman-Schwiner relation of Section 2 to study the dependence of $H^{(\Lambda)}$ on the single random parameter η_α, keeping $\{\eta_\beta\}_{\beta\neq\alpha}$ fixed. We express $H_{\eta_\alpha} \equiv H^{(\Lambda)}$ in terms of a re-sampled reference operator $H_{\hat{\eta}_\alpha}$ as

$$H_{\eta_\alpha} = H_{\hat{\eta}_\alpha} - \lambda(\hat{\eta}_\alpha - \eta_\alpha)U_\alpha, \tag{3.4}$$

where we will take the re-sampled variable $\hat{\eta}_\alpha$ to have the same conditional distribution as η_α. The family H_{η_α} has the form of the one-parameter family eq. (2.5) of Section 2, with $H_0 = H_{\hat{\eta}_\alpha}$, $V = U_\alpha$ and $\xi = \lambda(\hat{\eta}_\alpha - \eta_\alpha)$.

For $0 \le r \le 2$ we define the fractional "eigenfunction correlators" as

$$Q_r(\mathcal{J};x,\alpha) \;=\; \sum_{n:\,E_n\in\mathcal{J}} \langle \chi_x\psi_n,\psi_n\rangle^{r/2}\ \langle U_\alpha\psi_n,\psi_n\rangle^{1-r/2}\ , \tag{3.5}$$

where n labels the eigenvalues $E_n = E_n(\xi)$ and the corresponding orthonormal eigenfunctions $\psi_n = \psi_n(\xi)$ of H_{η_α}, choosing the labeling so that these are holomorphic in ξ, as in Section 2.

For $r = 1$ the eigenfunction correlators provide bounds for $Y_\Lambda(\mathcal{J}; x, y)$: If $f \in C_c(\mathcal{J})$, then $f(H^{(\Lambda)}) = \sum_{n: E_n \in \mathcal{J}} f(E_n) P_{\psi_n}$, where P_{ψ_n} is the orthogonal projector onto ψ_n. Thus, by the "covering condition" (1.19),

$$\begin{aligned} Y_\Lambda(\mathcal{J}; x, y) &\leq \sum_{\substack{\alpha \in \mathcal{I} \\ |y-\alpha| \leq 2r}} \sup_{\substack{f \in C_c(\mathcal{J}) \\ |f| \leq 1}} \left\| \chi_x f(H^{(\Lambda)}) U_\alpha^{1/2} \right\|_1 \\ &\leq \sum_{\substack{\alpha \in \mathcal{I} \\ |y-\alpha| \leq 2r}} \sum_{n: E_n \in \mathcal{J}} \left\| \chi_x P_{\psi_n} U_\alpha^{1/2} \right\|_1 \\ &= \sum_{\substack{\alpha \in \mathcal{I} \\ |y-\alpha| \leq 2r}} Q_1(\mathcal{J}; x, \alpha) \ . \end{aligned} \tag{3.6}$$

Here it was used that $\left\| \chi_x P_{\psi_n} U_\alpha^{1/2} \right\|_1 = \|\chi_x \psi_n\| \left\| U_\alpha^{1/2} \psi_n \right\|$.

For all $0 < r < 2$ the quantity $Q_r(\mathcal{J}; x, \alpha)$ reflects the overlap between the eigenfunctions at x and α. That, however, is not the case at the end-points $r = 0, 2$ for which the corresponding values of Q_r depend only on the density of states:

$$\begin{aligned} Q_0(\mathcal{J}; x, \alpha) &= \operatorname{Tr} U_\alpha \, P_{\mathcal{J}}(H^{(\Lambda)}) \\ Q_2(\mathcal{J}; x, \alpha) &= \operatorname{Tr} \chi_x \, P_{\mathcal{J}}(H^{(\Lambda)}) \end{aligned} \tag{3.7}$$

with $P_{\mathcal{J}}(H^{(\Lambda)})$ the spectral projection operator. For the Schrödinger operators considered here, both $Q_0(\mathcal{J}; x, \alpha)$ and $Q_2(\mathcal{J}; x, \alpha)$ are of order one, for a finite interval $\mathcal{J}$, in the sense that they are finite and do not decay for increasing $\operatorname{dist}(x, \alpha)$. In particular, if $\mathcal{J} \subset (-\infty, E)$ and $p > d/2$ we have

$$\begin{aligned} Q_2(\mathcal{J}; x, \alpha) &\leq \operatorname{Tr} \chi_x P_{\leq E}(H^{(\Lambda)}) \\ &\leq (|E - E_o| + 1)^p \operatorname{Tr} \chi_x (H^{(\Lambda)} + E_o + 1)^{-p} \\ &\leq C(|E - E_o| + 1)^p \ . \end{aligned} \tag{3.8}$$

This bound is deterministic, and thus holds also for $\mathbb{E}(Q_2(\mathcal{J}; x, \alpha))$.

Lemma 3.1. *$Q_r(\mathcal{J}; x, \alpha)$ is log convex in r, and for any $r \in (0, 1)$:*

$$\mathbb{E}\left(Q_1(\mathcal{J})\right) \ \leq \ \mathbb{E}\left(Q_r(\mathcal{J})\right)^{1/(2-r)} \ \mathbb{E}\left(Q_2(\mathcal{J})\right)^{(1-r)/(2-r)} \ . \tag{3.9}$$

at any value of the (omitted) argument (x, α) of Q_r.

Proof. The log convexity of Q_r in r is a standard observation for a function of the form $F(r) = \sum_n A_n B_n^r$. In particular, for $r < 1 < 2$, writing 1 as a convex combination of the other values: $1 = ar + (1-a)2$, one gets via the Hölder inequality: $F(1) \leq F(r)^a \, F(2)^{1-a}$ (with $a = 1/(2-r)$). In the present context that yields

$$Q_1(\mathcal{J}) \ \leq \ Q_r(\mathcal{J})^{1/(2-r)} \ Q_2(\mathcal{J})^{(1-r)/(2-r)} \, . \tag{3.10}$$

One more application of the Hölder inequality, this time to the average over the randomness ($\mathbb{E}(\cdot)$), yields eq. (3.9). □

3.2. **Eigenfunction correlators and resolvent moments.** Here we will relate the Q_r to fractional resolvent moments by applying the results of Lemma 2.2 to the family eq. (3.4). By eq. (2.6) and eq. (2.7) we have

$$\frac{d}{d\xi} E_n = -\langle U_\alpha \psi_n, \psi_n \rangle \tag{3.11}$$

and for $\Gamma_n(E)$, the inverse function of E_n,

$$\frac{d}{dE}\Gamma_n(E) = -\frac{1}{\langle U_\alpha \psi_n(\Gamma_n(E)), \psi_n(\Gamma_n(E))\rangle} \; . \tag{3.12}$$

For $E \notin \sigma(H_{\eta_\alpha})$ define $K_{\eta_\alpha,E} := U_\alpha^{1/2}(H_{\eta_\alpha} - E)^{-1}U_\alpha^{1/2}$. If $E \notin \sigma(H_{\hat\eta_\alpha})$ then, by Lemma 2.2, $\{\Gamma_n(E)\}$ are the repeated eigenvalues and $\phi_n(E) := U_\alpha^{1/2}\psi_n(\Gamma_n(E))$ corresponding complete (non-normalized) eigenvectors for the unbounded self-adjoint operator $K^{-1}_{\hat\eta_\alpha,E}$. With this notation we have

Theorem 3.1. *If $\eta_\alpha \neq \hat\eta_\alpha$ and $\sigma(H_{\eta_\alpha}) \cap \sigma(H_{\hat\eta_\alpha}) \cap \mathcal{J} = \emptyset$, then the eigenfunction correlator $Q_r(\mathcal{J}; x, \alpha)$ for $H^{(\Lambda)}$ admits the following representation:*

$$\begin{aligned} Q_r(\mathcal{J}; x, \alpha) \; = \; & \sum_n \int_{\mathcal{J}} dE \; \delta(\Gamma_n(E) + \lambda(\eta_\alpha - \hat\eta_\alpha)) \; |\Gamma_n(E)|^r \\ & \times \left\| \chi_x \, (H_{\hat\eta_\alpha} - E)^{-1} \, U_\alpha^{1/2} \, \phi_n(E) \right\|^r / \, \|\phi_n(E)\|^r \; . \end{aligned} \tag{3.13}$$

Furthermore, for any E and $a < b$ such that E is not an eigenvalue of H_a or H_b,

$$\int_a^b \mathrm{d}\eta_\alpha \; \sum_n \; \delta(\Gamma_n(E) + \lambda(\eta_\alpha - \hat\eta_\alpha)) \; = \; [\mathrm{Tr}\, P_{\le E}(H_a) \; - \; \mathrm{Tr}\, P_{\le E}(H_b)] \; / \; \lambda \; . \tag{3.14}$$

Remark: Using Lemma 2.2 it is easy to see that the condition $\sigma(H_{\eta_\alpha}) \cap \sigma(H_{\hat\eta_\alpha}) \cap \mathcal{J} = \emptyset$ holds for Lebesgue almost every η_α.

Proof. As $\Gamma_n(E) = E_n^{-1}(E)$ we have for arbitrary η_α that $E \in \sigma(H_{\eta_\alpha})$ if and only if $\Gamma_n(E) = \xi = \lambda(\hat\eta_\alpha - \eta_\alpha)$ for some n. Thus one may express sums over eigenvalues as integrals with suitably weighted δ-functions:

$$\sum_{n:\, E_n \in \mathcal{J}} \cdots \; = \; \int_{\mathcal{J}} \mathrm{d}E \sum_n \delta\left(\Gamma_n(E) + \lambda\, \Delta\eta_\alpha\right) \left| \frac{\mathrm{d}}{\mathrm{d}E} \Gamma_n(E) \right| \cdots \, , \tag{3.15}$$

where $\Delta\eta_\alpha = \eta_\alpha - \hat\eta_\alpha$. In particular, eq. (3.12) implies that

$$\begin{aligned} Q_o(\mathcal{J}; x, \alpha) \; &= \; \sum_{n:\, E_n \in \mathcal{J}} \langle U_\alpha \psi_n(E), \psi_n(E) \rangle \\ &= \; \int_{\mathcal{J}} \mathrm{d}E \sum_n \delta\left(\Gamma_n(E) + \lambda\, \Delta\eta_\alpha\right) \; , \end{aligned} \tag{3.16}$$

and for other values of r:

$$\begin{aligned} Q_r(\mathcal{J}; x, \alpha) \; &= \; \sum_{n:\, E_n \in \mathcal{J}} \langle U_\alpha \psi_n, \psi_n \rangle \left(\frac{\langle \chi_x \psi_n, \psi_n \rangle}{\langle U_\alpha \psi_n, \psi_n \rangle} \right)^{r/2} \\ &= \; \int_{\mathcal{J}} \mathrm{d}E \sum_n \delta\left(\Gamma_n(E) + \lambda\, \Delta\eta_\alpha\right) \; \frac{\langle \chi_x \psi_n, \psi_n \rangle^{r/2}}{\|\phi_n(E)\|^r} \; . \end{aligned} \tag{3.17}$$

For $\Gamma_n(E) = -\lambda\Delta\eta_\alpha = \xi$, i.e. $E = E_n(\xi)$, it follows that

$$\psi_n(\xi) = \xi (H_{\hat\eta_\alpha} - E)^{-1} U_\alpha^{1/2} \phi_n(E) \; , \tag{3.18}$$

since $K^{-1}_{\hat\eta_\alpha}\phi_n(E) = \Gamma_n(E) U_\alpha^{1/2} \psi_n(\xi)$. Thus eq. (3.13) follows from eq. (3.17).

For fixed E, the left-hand side of eq. (3.14), up to a factor $1/\lambda$, counts the number of n for which $\Gamma_n(E) = \lambda(\hat\eta_\alpha - \eta_\alpha)$ has a solution with $a < \eta_\alpha < b$. This number is exactly the decrease in the number of eigenvalues of H_{η_α} below E as η_α is moved from a to b. That yields eq. (3.14). □

3.3. **Spectral shift bounds.** In applying the interpolation argument seen in Lemma 3.1, we shall need bounds on the *spectral shift function*, defined by

$$S_{\alpha,\lambda}(H^{(\Lambda)};E) \ := \ \mathrm{Tr}\left[P_{\leq E}(H^{(\Lambda)}_{\eta_\alpha=0}) - P_{\leq E}(H^{(\Lambda)}_{\eta_\alpha=1})\right] , \tag{3.19}$$

and expressing how many energy levels are pushed over E when the value of the parameter η_α is increased by 1. Note that $S_{\alpha,\lambda}(H^{(\Lambda)};E)$ is non-negative and bounded (since $|\Lambda| < \infty$):

$$0 \ \leq \ S_{\alpha,\lambda}(H^{(\Lambda)};E) \ \leq \ \mathrm{Tr}\, P_{\leq E}(H^{(\Lambda)}_{\eta_\alpha=0}) \ \leq \ C_r\,(1+|E-E_o|)^r|\Lambda| , \tag{3.20}$$

for any $r > d/2$.

In the discrete setup, where the role of U_α is taken by a rank-one operator, the shift is at most 1. For continuum operators there is no such uniform bound independent of the volume (see ref. [37]), however $S_{\alpha,\lambda}(H^{(\Lambda)};E)$ has locally bounded L^p norms as a function of the energy E, as was shown in ref. [17].

Lemma 3.2 (L^p boundedness of the spectral shift). *Let $1 \leq p < \infty$ and $n \in \mathbb{N}$ such that $n > dp/2$. Then there exists a constant $C_{p,\lambda,n} < \infty$ such that*

$$\int_{-\infty}^{E_+} S_{\alpha,\lambda}(H^{(\Lambda)};E)^p \mathrm{d}E \ \leq \ C_{p,\lambda,n}(1+|E_+-E_o|)^{n+1} \tag{3.21}$$

uniformly in the domain Λ as well as in the choice of α and the random parameters η_β ($\beta \neq \alpha$).

3.4. **From fractional moment bounds to localization.** We shall now put together the elements introduced in the previous sections and prove that rapid decay of the resolvent moments implies localization in its various manifestations.

Theorem 3.2. *Let H_ω be a random Schrödinger operator which satisfies* **A**. *Let $0 < s < 1$ and $\mathcal{J} \subset (-\infty, E_+]$ be a bounded open interval. Suppose that for some $C < \infty$, $\eta > 0$ and a bounded region Λ,*

$$\mathbb{E}\left(\int_{\mathcal{J}} \mathrm{d}E \left\|\chi_x \frac{1}{H^{(\Lambda)}-E}\chi_y\right\|^s\right) \ \leq Ce^{-\eta\,\mathrm{dist}_\Lambda(x,y)} , \tag{3.22}$$

for all $x, y \in \Lambda$. Then, for any $r < 1/(2-s)$, there exists a volume independent constant $C_{s,r}(E_+,\lambda) < \infty$ such that

$$\mathbb{E}\left(\sup_{f\in C_c(\mathcal{J}):\,|f|\leq 1}\left\|\chi_x\, f(H^{(\Lambda)})\,\chi_y\right\|_1\right) \ \leq C_{s,r}(E_+,\lambda)\ e^{-r\eta\,\mathrm{dist}_\Lambda(x,y)} , \tag{3.23}$$

for all $x, y \in \Lambda$. Here $C_c(\mathcal{J})$ denotes the continuous functions compactly supported inside $\mathcal{J}$.

Remark: The operators $H^{(\Lambda)}$ have finite spectrum in $\mathcal{J}$. Thus, in eq. (3.23) one could equivalently take the supremum over *all* functions on $\mathcal{J}$ as long as they pointwise satisfy $|f| \leq 1$. The only reason for stating eq. (3.23) in terms of functions in $C_c(\mathcal{J})$ is to prepare for a limiting argument in the proof of Thm 1.1.

Proof. Assume that for some open interval $\mathcal{J}$ the fractional moment bound eq. (3.22) holds. By eq. (3.6), to prove Theorem 3.2 it suffices to establish a related bound on $\mathbb{E}(Q_1(\mathcal{J};x,\alpha))$ for $\alpha \in \mathcal{I}$ with $\operatorname{dist}(\alpha, y) \le 2r$. This in turn is controlled through Lemma 3.1 by an interpolating product of $\mathbb{E}(Q_2(\mathcal{J};x,\alpha))$ and $\mathbb{E}(Q_r(\mathcal{J};x,\alpha))$, with $r<1$. Leaving room for one more interpolation, we choose $r<s$.

To estimate $\mathbb{E}(Q_r(\mathcal{J};x,\alpha))$, we start from the representation eq. (3.13) and average first over η_α at specified values of η_ζ for $\zeta \ne \alpha$. For almost every choice of η_ζ ($\zeta \ne \alpha$), E is neither an eigenvalue of $H^{(\Lambda)}_{\eta_\alpha=0}$ nor of $H^{(\Lambda)}_{\eta_\alpha=1}$. Also, for fixed $\hat\eta_\alpha$ and almost every η_α we have $\sigma(H^{(\Lambda)}_{\eta_\alpha}) \cap \sigma(H^{(\Lambda)}_{\hat\eta_\alpha}) \cap \mathcal{J} = \emptyset$. Thus we can apply eq. (3.13) and eq. (3.14) to conclude

$$\begin{aligned}&\mathbb{E}(Q_r(\mathcal{J};x,\alpha)) \\ &\qquad\le\ 2^r\lambda^{r-1}D\,\mathbb{E}\left(\int_{\mathcal{J}} \mathrm{d}E\ S_{\alpha,\lambda}(H^{(\Lambda)};E)\,\|\chi_x(H^{(\Lambda)}-E)^{-1}U_\alpha^{1/2}\|^r\right)\ . \qquad (3.24)\end{aligned}$$

Here we have used **A3**, the bound $|\Gamma_n(E)| \le 2\lambda$ (imposed by the δ functions in eq. (3.13)), and we chose the re-sampled variable $\hat\eta_\alpha$ to have the same conditional distribution as η_α.

Using Hölder's inequality we can further estimate the right hand side of eq. (3.24) to get

$$\begin{aligned}\mathbb{E}(Q_r(\mathcal{J};x,\alpha)) &\le 2^r\lambda^{r-1}D\left(\mathbb{E}\int_{\mathcal{J}} \mathrm{d}E\ S_{\alpha,\lambda}(H^{(\Lambda)};E)^{\frac{s}{s-r}}\right)^{\frac{s-r}{s}} \\ &\qquad\times\left(\mathbb{E}\int_{\mathcal{J}}\mathrm{d}E\ \|\chi_x(H^{(\Lambda)}-E)^{-1}U_\alpha^{1/2}\|^s\right)^{r/s} \\ &\le C_{s,r,\lambda,n}(1+|E_+-E_o|)^{(n+1)(s-r)/s}e^{-\eta r\operatorname{dist}_\Lambda(x,\alpha)/s}\end{aligned} \qquad (3.25)$$

for any integer $n > \frac{ds}{2(s-r)}$. In the final estimate we have used the spectral shift bound from Lemma 3.2 and the assumption eq. (3.22).

Collecting all our bounds as well as the uniform bound for $\mathbb{E}(Q_2(\mathcal{J};x,\alpha))$ provided by eq. (3.8), we arrive at

$$\mathbb{E}(Y_\Lambda(\mathcal{J};x,y)) \le C_{r,s}(E_+,\lambda)e^{-\frac{\eta r\operatorname{dist}_\Lambda(x,\alpha)}{s(2-r)}}\ . \qquad (3.26)$$

After reorganizing the exponent this yields eq. (3.23). $\square$

3.5. **Infinite volumes – proof of Theorem 1.1.** The eigenfunction correlator methods used above are most easily implemented for the finite volume operators $H^{(\Lambda)}$. However, it is important to note that the bounds obtained in this way do *not* depend on the size of the volume Λ (except for the presence of the modified distance $\operatorname{dist}_\Lambda(x,\alpha)$, which for fixed x,α is equal to the usual distance $|x-\alpha|$ for sufficiently large Λ). In this section we prove Theorem 1.1, stating that uniform exponential bounds in *finite* volumes imply exponential localization in the infinite system. The fundamental analytic tools are 1) strong resolvent convergence and 2) the RAGE theorem. The infinite region Ω being fixed throughout this section, we write $H = H^{(\Omega)}$.

Proof of eq. (1.11): For $g \in C_c(\mathcal{J})$, $g(H^{(\Lambda_n)})$ converges strongly to $g(H)$ since $H^{(\Lambda_n)}$ converges to H in the strong resolvent sense. By lower semi-continuity of the trace norm in the strong operator topology [58, Theorem 2.7(d)] one sees that

$$\|\chi_x g(H)\chi_y\|_1 \le \liminf_{n\to\infty} \|\chi_x g(H^{(\Lambda_n)})\chi_y\|_1 \,. \tag{3.27}$$

Taking suprema and applying Fubini's theorem yields

$$\mathbb{E}\left(\sup_{g:\,|g|\le 1} \|\chi_x g(H)P_{\mathcal{J}}(H)\chi_y\|_1\right) \;\le\; \liminf_{n\to\infty}\mathbb{E}\left(\sup_{f\in C_c(\mathcal{J}):\,|f|\le 1} \|\chi_x f(H^{(\Lambda_n)})\chi_y\|_1\right)\,. \tag{3.28}$$

The supremum on the left-hand side is initially only taken over $g \in C_c(\mathcal{J})$, but can be extended to all Borel functions g without changing its value. To see this, let $T_g := \chi_x g(H)P_{\mathcal{J}}(H)\chi_y$ for a fixed Borel function g with $|g| \le 1$. Recall that

$$\|T_g\|_1 = \sup |\operatorname{Tr} T_g G| \,, \tag{3.29}$$

where the supremum is taken over all finite rank operators G with $\|G\| \le 1$. Thus, for any $\varepsilon > 0$ there is a finite rank operator $G = \sum_{k=1}^N \langle\cdot,\psi_k\rangle\phi_k$ with $\|G\| \le 1$ and

$$\|T_g\|_1 \le \varepsilon + |\operatorname{Tr} T_g G| \,. \tag{3.30}$$

One calculuates

$$\operatorname{Tr} T_g G = \sum_{k=1}^N \langle g(H)P_{\mathcal{J}}(H)\chi_y\phi_k, \chi_x\psi_k\rangle = \sum_{k=1}^N \int_{\mathcal{J}} g(\lambda)\, d\mu_k(\lambda) \,, \tag{3.31}$$

with the spectral measures $d\mu_k(\lambda) = d\langle E(\lambda)\chi_y\phi_k, \chi_x\psi_k\rangle$ associated to H. The μ_k are complex Borel measures on $\mathbb{R}$ and thus regular (e.g. [54]). Thus it follows from elementary considerations that there exist $g_n \in C_c(\mathcal{J})$ with $|g_n| \le 1$ and $\int_{\mathcal{J}} g_n\, d\mu_k \to \int_{\mathcal{J}} g\, d\mu_k$ as $n \to \infty$ simultaneously for $k = 1, \ldots, N$.

In particular, using eqs. (3.31), (3.30) and (3.29) we see that there is $\tilde g \in C_c(\mathcal{J})$ with $|\tilde g| \le 1$ and $\|T_g\|_1 \le 2\varepsilon + \|T_{\tilde g}\|_1$. As $\varepsilon > 0$ was arbitrary, we conclude that the left-hand side of eq. 3.28 does not change if the supremum is taken over all Borel functions with $|g| \le 1$.

Equation (1.11) now follows through the uniform bounds eq. (3.23) provided by Theorem 3.2 under the assumption eq. (1.10). □

We now turn to the spectral type of H, where $\Omega = \mathbb{R}^d$ is assumed, and therefore $\operatorname{dist}_\Omega(x, y) = \operatorname{dist}(x, y)$. The absence of continuous spectrum can be demonstrated from our estimates using the RAGE theorem (e.g., see ref. [19] for discussion and references) which implies that the projection onto the continuous spectrum satisfies

$$\|P_c(H)\psi\|^2 \;=\; \lim_{R\to\infty}\;\lim_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\;\left\|\chi_{\{|x-x_o|\ge R\}}\,\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2 \tag{3.32}$$

As a consequence,

Theorem 3.3 (RAGE theorem for random operators)**.** *For the random operators considered here, if for some open interval $\mathcal{J}$*

$$\mathbb{E}\left(\left\|\chi_{\{|x-x_o|\ge R\}}\,\mathrm{e}^{-\mathrm{i}tH}\,P_{\mathcal{J}}(H)\,\chi_{x_o}\right\|\right) \;\le\; g(R) \tag{3.33}$$

with $g(R) \to 0$ as $R \to \infty$ uniformly in t and x_o, then $P_c(H)P_{\mathcal{J}}(H) = 0$ almost surely.

Proof. If $\phi \in L^2(\mathbb{R}^d)$ is compactly supported with $\operatorname{supp}\phi \subset B^r_{x_0}$ for some $x_0 \in \mathbb{R}^d$, then the RAGE theorem implies

$$\|P_c(H)P_{\mathcal{J}}(H)\phi\|^2 \le \liminf_{R\to\infty}\liminf_{T\to\infty}\int_0^T \frac{\mathrm{d}t}{T}\|\chi_{|x-x_0|\ge R}e^{-\mathrm{i}tH}P_{\mathcal{J}}(H)\chi_{x_0}\|^2\|\phi\|^2 \, . \tag{3.34}$$

Upon taking expectations, Fubini's theorem and Fatou's lemma yield

$$\mathbb{E}\left(\|P_c(H)P_{\mathcal{J}}(H)\phi\|^2\right) \le \liminf_{R\to\infty}\liminf_{T\to\infty}\int_0^T \frac{dt}{T} g(R)\|\phi\|^2 = 0 \, . \tag{3.35}$$

Thus $P_c(H)P_{\mathcal{J}}(H)\phi = 0$ almost surely. This implies the theorem since there exists a *countable* total set of ϕ with each ϕ supported in $\subset B^r_{x_0}$ for suitable x_0. $\square$

To complete the proof of Theorem 1.1 we note that eq. (1.11), proven above, yields the assumption of Theorem 3.3: Covering $\{|x-x_0| \ge R\}$ with balls B^r_α, $\alpha \in \mathcal{I}$, we get for every $\nu \in (0, \mu/(2-s))$ that $\mathbb{E}\left(\|\chi_{\{|x-x_0|\ge R\}}e^{-\mathrm{i}tH}P_{\mathcal{J}}(H)\chi_{x_0}\|\right) \le Ce^{-\nu R}$. Thus the spectrum of H is almost surely pure point in $\mathcal{J}$.

Another consequence of eq. (1.11) is that

$$\mathbb{E}\left(\sum_{x,y}\frac{\mathrm{e}^{\nu|x-y|}}{1+|y|^{d+1}}\sup_{g:|g|\le 1}\|\chi_x g(H)P_{\mathcal{J}}(H)\chi_y\|\right) \;<\; \infty \, , \tag{3.36}$$

which implies

$$\sup_{g:|g|\le 1}\|\chi_x g(H)P_{\mathcal{J}}(H)\chi_y\| \;\le\; \text{const.}\,(1+|y|^{d+1})\,\mathrm{e}^{-\nu|x-y|} \qquad \text{a.s.}. \tag{3.37}$$

Since the *Kronecker* delta functions $\delta_E(x)$ ($= 1$ if $x = E$ and 0 otherwise) are Borel measurable, this implies with probability one all eigen-projections of H satisfy

$$\ln\|\chi_x\delta_E(H)\chi_y\| \;\le\; -\nu|x-y| \;+\; \mathrm{o}(|x-y|) \, . \tag{3.38}$$

If it were known that H almost surely has non-degenerate spectrum, one could readily conclude exponential decay of eigenfunctions from eq. (3.38). Although non-degeneracy has been shown for lattice random operators [61], it is not clear whether that result can be extended to the continuum Schrödinger operators considered here. However, it can be shown that almost surely all eigenvalues of H in $\mathcal{J}$ are finitely degenerate. An argument for this (based on compactness considerations and spectral averaging) which applies to our model was provided in the proof of Theorem 3.2 of ref. [16].

Given this, we can proceed as follows. Setting $y = 0$ in eq. (3.37) we have, for almost every configuration ω, (1) The range of $\delta_E(H)$, denoted $\mathcal{R}(\delta_E(H))$, is finite dimensional for every $E \in \mathcal{J}$ and (2) $\|\chi_x\delta_E(H)\chi_0 f\| \le Ce^{-\nu|x|}\,\|f\|$ for every $x \in \mathbb{R}^d$, every $f \in L^2(\mathbb{R}^d)$ and every $E \in \mathcal{J}$. Now (2) implies

$$\|\chi_x\phi\| \le C_\phi e^{-\nu|x|} \tag{3.39}$$

for every $\phi \in \mathcal{R}(\delta_E(H)\chi_0)$, the range of $\delta_E(H)\chi_0$. If we can show that

$$\mathcal{R}(\delta_E(H)\chi_0) = \mathcal{R}(\delta_E(H)) \, , \tag{3.40}$$

then eq. (3.39), which is an integral version of eq. (1.12), holds for all eigenfunctions with eigenvalue E.

Clearly $\mathcal{R}(\delta_E(H)\chi_0) \subset \mathcal{R}(\delta_E(H))$. Suppose these two sets are not equal. Then there is a non-zero eigenfunction $\tilde{\phi} \in \mathcal{R}(\delta_E(H))$ which is orthogonal to $\mathcal{R}(\delta_E(H)\chi_0)$, since the latter space is finite dimensional and therefore closed by (1). Thus

$$0 = \langle \tilde{\phi}, \delta_E(H)\chi_0\psi\rangle = \langle \tilde{\phi}, \chi_0\psi\rangle = \langle \chi_0\tilde{\phi}, \psi\rangle \tag{3.41}$$

for every $\psi \in L^2(\mathbb{R}^d)$. We conclude $\chi_0\tilde{\phi} = 0$, which contradicts unique continuation. Therefore eq. (3.40) holds and eq. (3.39) follows for all $\phi \in \mathcal{R}(\delta_E(H)\chi_0)$.

We have shown that almost surely eq. (3.39) holds for every eigenfunction with eigenvalue $E \in \mathcal{J}$. The L^∞ version of exponential decay stated in Theorem 1.1 follows from well know facts regarding the smoothness of eigenfunctions of Schrödinger operators ("elliptic regularity", e.g. see ref. [60]).

This completes the proof of Theorem 1.1. □

4. Finiteness of the fractional-moments

In this section we prove two technical results, Lemmas 4.3 and 4.4, which underly our analysis. In particular, Lemma 4.3 yields the boundedness of fractional moments of the resolvent norms stated in Lemma 1.1, in the Introduction. An improved bound under the stronger assumption **A3′** is presented in Proposition 5.3 below.

The question of why the fractional moments are finite requires some discussion and offers an instructive introduction to some of the techniques which underly this work. We begin with an overview of the argument and then present the two lemmas. Finally we give a proof of the two main technical results. The details of the proof are not used in the derivation of Theorem 1.1 or 1.2 or the other arguments in this work.

4.1. **Why are the moments finite?** The analysis presented below has its genesis in the discrete setup, where the finiteness of the fractional moments of the Green function is quickly implied by a rank-one perturbation argument. However, for the continuum operators considered here, that short argument requires thorough remaking since the local potential term (U_α) is now an operator of infinite rank. Nonetheless before discussing the more general analysis, which starts from the Birman-Schwinger relation, it will be instructive to stop and consider the analysis in the discrete case.

The Birman-Schwinger relation for an operator of the form $H = \widehat{H} + \lambda\eta U$, with $\lambda > 0$ a disorder parameter, $\eta \in \mathbb{R}$ a variable coefficient, and U a positive local potential term, states that

$$U^{1/2}\frac{1}{\widehat{H} + \lambda\eta U - z}U^{1/2} \;=\; \left[\left[\widehat{K}_z\right]^{-1} \;+\; \lambda\eta\right]^{-1} \tag{4.1}$$

where

$$\widehat{K}_z \;=\; U^{1/2}\frac{1}{\widehat{H} - z}U^{1/2}\,. \tag{4.2}$$

The relation holds in the space $L^2(\operatorname{supp} U)$, in which the inverses denoted by $[\quad]^{-1}$ are defined (see eq. (2.4) and eq. (2.9)).

In the discrete case, with $U = |o\rangle\langle o|$ a rank-one operator, $\widehat{K}_z^{-1}$ acts in a one-dimensional space, i.e., it reduces to a complex number. The relation eq. (4.1) then directly implies the finiteness of fractional moments:

$$\int \left|\langle o|\,\frac{1}{\widehat{H} + \lambda\eta\,|o\rangle\,\langle o| - z}\,|o\rangle\right|^s \rho(\eta)\,\mathrm{d}\eta \;\le\; \sup_{\gamma\in\mathbb{C}}\int \frac{1}{|\lambda\eta - \gamma|^s}\,\rho(\eta)\,\mathrm{d}\eta \;\le\; C_{s,\rho}\lambda^{-s} \;<\; \infty \tag{4.3}$$

whenever $s < 1$ and ρ is bounded and integrable. Another manifestation of this boundedness is the weak L^1 bound:

$$\left|\left\{\eta \;:\; \left|\langle o|\,\frac{1}{\widehat{H} + \lambda\eta\,|o\rangle\,\langle o| - z}\,|o\rangle\right| > t\right\}\right| \;\le\; \frac{2}{\lambda t} \tag{4.4}$$

for any z and $\widehat{H}$ (with $|\cdot|$ denoting the Lebesgue measure).

For general U, we may use the Birman-Schwinger relation eq. (2.9) to show that, for $z = E \in \mathbb{R}$,

$$\left| \left\{ \eta \in [0,1] \ : \ \left\| U^{1/2} \frac{1}{\widehat{H} + \lambda \eta U - E} U^{1/2} \right\| > t \right\} \right| \ \leq \ \frac{2(1+\xi_{E,\lambda})}{\lambda t} \tag{4.5}$$

where $\xi_{E,\lambda}$ denotes the number of eigenvalues of $\widehat{K}_E^{-1}$ in the interval $(-\lambda, 0)$. For perturbations U which are finite rank, or even trace class, the simple bound $\xi_{E,\lambda} \leq \mathrm{Tr}\ U$ allows the analysis to proceed from eq. (4.5) much as in the rank-one case.

However, for the continuum Schrödinger operators considered here, with U one of the "bumps" U_α, the perturbation $\lambda \eta U$ is of infinite rank. Nonetheless, it is *relatively compact* with respect to $\widehat{H}$ due to the kinetic term $\boldsymbol{D}^{\boldsymbol{A}} \cdot \boldsymbol{D}^{\boldsymbol{A}}$. It follows that for $E \notin \sigma(\widehat{H})$ the operator $\widehat{K}_E$ is *compact* and therefore $\xi_{E,\lambda}$ is *finite.* However, there is no reason to expect a bound on $\xi_{E,\lambda}$ which is uniform in E and in the various parameters implicit in $\widehat{H}$.

To examine the factor $\xi_{E,\lambda}$ more closely, let us recall that E is an eigenvalue of $H = \widehat{H} + \lambda \eta U$ precisely when $\lambda \eta$ is an eigenvalue of $(-\widehat{K}_E^{-1})$. Hence, we may equate $\xi_{E,\lambda}$ with the number of times that E becomes an eigenvalue of H as η is moved from 0 to 1. By the monotonicity of H in η (implied by the positivity of U), we get

$$\xi_{E,\lambda} \ \leq \ \mathrm{Tr}\left[P(\widehat{H} < E) - P(\widehat{H} + \lambda U < E) \right], \tag{4.6}$$

with equality unless E is a degenerate eigenvalue for some $\eta \in (0,1)$.

The quantity on the right side of eq. (4.6) is the 'Krein spectral shift' at energy E between $\widehat{H}$ and $\widehat{H} + \lambda U$. For a comparison with the discrete case, note that this quantity is trivially bounded by $\operatorname{rank} U$, and in particular by 1 for a rank one operator.

The spectral shift has recently been the subject of several of studies [17, 33]. A key fact is that for Schrödinger operators the Krein spectral shift is locally integrable as a function of E – indeed, it has been shown to be locally L^p for every $p \in [1, \infty)$ with explicit bounds [17], a fact which we use below in the proof of Theorem 1.1. However, in general the spectral shift is not locally bounded; examples exist of Schrödinger operators for which it is arbitrarily large at certain energies [37].

Thus, the moments of continuum "Green functions" differ from their discrete counterparts in an essential way – after averaging over a *single* coupling one still does not get a uniform bound. Nevertheless, it is natural to guess that for random operators the spectral shift has a finite expectation value, since averaging over disorder may play a role somewhat similar to averaging over energy.

In the derivation of our fractional moment bounds (see Lemma 4.3 below), we do not consider directly the average of the spectral shift. Instead, we find it more convenient to derive an analogue of eq. (4.4) for continuum operators (from which fractional moment bounds follow quite easily). Nonetheless, the notion that a disorder averaged spectral shift is bounded is one of the driving ideas behind this work, and we find that the techniques developed in this section imply a result of this type (Theorem C.1, stated and proved in Section C below). In the end however, the fractional moment bounds obtained here are somewhat stronger than those which follow from Theorem C.1.

4.2. **The $1/t$-tails.** A useful result for moment bounds is a general weak-L_1 type bound for the boundary values of resolvents of maximally dissipative operators.

We state this as the second part of the following Lemma, which is essentially a special case of results proven in [49]. For completeness, we give a proof based on properties of the Hilbert transform in Appendix B. The first part of the Lemma is a well known result in scattering theory, e.g. [22].

Lemma 4.1 (Weak L_1 bound)**.** *Let $\mathcal{H}$ and $\mathcal{H}_1$ be separable Hilbert spaces, A a maximally dissipative operator in $\mathcal{H}$, as well as $M_1 : \mathcal{H} \to \mathcal{H}_1$ and $M_2 : \mathcal{H}_1 \to \mathcal{H}$ Hilbert-Schmidt operators. Then*

(1) *The "boundary values"*

$$M_1 \frac{1}{A - v + \mathrm{i}0} M_2 \;=\; \lim_{\epsilon \downarrow 0} M_1 \frac{1}{A - v + \mathrm{i}\epsilon} M_2 \tag{4.7}$$

exist as Hilbert-Schmidt operators for almost every $v \in \mathbb{R}$, with convergence in the Hilbert-Schmidt norm.

(2) *There exists $C_W < \infty$ (independent of A, M_1, M_2) such that*

$$\left| \left\{ v \;:\; \left\| M_1 \frac{1}{A - v + \mathrm{i}0} M_2 \right\|_{HS} > t \right\} \right| \;\le\; C_W \; \|M_1\|_{HS} \; \|M_2\|_{HS} \; \frac{1}{t} \; . \tag{4.8}$$

Remark: If a dissipative operator A has strictly positive imaginary part, such as a Birman-Schwinger operator A_{BS} of the type consider in Lemma 2.1, then the $i0$ in (4.8) is not needed since $(\xi - A_{BS})^{-1}$ is norm-continuous for ξ in the closed lower half plane.

In our applications of Lemma 4.1 we will use the following "off-diagonal" version which follows from the lemma via the Birman-Schwinger identity and a simple change of variables. For ease of presentation, we state this version with the additional assumption that the operator A has strictly positive imaginary part, thus avoiding the issue of whether certain limits exist.

Proposition 4.2. *Let A be a maximally dissipative operator with strictly positive imaginary part on a Hilbert space $\mathcal{H}$, let M_1, M_2 be Hilbert-Schmidt operators, and let U_1, U_2 be non-negative operators. Then*

$$\begin{aligned} &\left| \left\{ \langle v_1, v_2 \rangle \in [0,1]^2 \;:\; \left\| M_1 U_1^{1/2} \frac{1}{A - v_1 U_1 - v_2 U_2} U_2^{1/2} M_2 \right\| > t \right\} \right| \\ &\qquad\qquad\qquad\qquad \le\; 2\, C_W \, \|M_1\|_{HS} \, \|M_2\|_{HS} \, \frac{1}{t} \; . \end{aligned} \tag{4.9}$$

Proof. The key to the proof is the change of variables $v_\pm = \frac{1}{2}(v_1 \pm v_2)$, so that $v_1 U_1 + v_2 U_2 = v_+(U_1 + U_2) + v_-(U_1 - U_2)$. From the Birman-Schwinger identity, eq. (2.4),

$$\left[(U_1 + U_2)^{1/2} \frac{1}{A - v_1 U_1 - v_2 U_2} (U_1 + U_2)^{1/2} \right] \;=\; \left[[K(v_-)]^{-1} \;-\; v_+ \right]^{-1} \; , \tag{4.10}$$

where $[\cdot]$ denotes the restriction of an operator to $\ker(U_1 + U_2)^{\perp}$ and

$$K(v_-) \;=\; (U_1 + U_2)^{1/2} \frac{1}{A - v_-(U_1 - U_2)} (U_1 + U_2)^{1/2} \; . \tag{4.11}$$

Thus,

$$M_1 U_1^{1/2} \frac{1}{A - v_1 U_1 - v_2 U_2} U_2^{1/2} M_2 \;=\; \widetilde{M_1} \left[[K(v_-)]^{-1} \;-\; v_+ \right]^{-1} \widetilde{M_2} \tag{4.12}$$

with $\widetilde{M_1} = M_1 U_1^{1/2} P[U_1+U_2]^{-1/2}P$ and $\widetilde{M_2} = P[U_1+U_2]^{-1/2}PU_2^{1/2}M_2$ where P denotes orthogonal projection onto $\ker(U_1+U_2)^{\perp}$.

Since U_1 and U_2 are positive, $\left\| U_j^{1/2}P[U_1+U_2]^{-1/2}P \right\| \le 1$ and therefore

$$\left\| \widetilde{M_j} \right\|_{HS} \le \|M_j\|_{HS} . \tag{4.13}$$

Furthermore $[K(v_-)]^{-1}$ is maximally dissipative as shown in Lemma 2.1. Thus for each fixed v_- we are in the situation governed by Lemma 4.1, and Prop. 4.2 follows via the Fubini Theorem. The factor of 2 on the right hand side results from the Jacobian of the transformation $\langle v_1, v_2\rangle \mapsto \langle v_+, v_-\rangle$. □

We will use Prop. 4.2 to conclude that for the random operator H,

$$\left| \left\{ \langle \eta_\alpha, \eta_\beta \rangle \in [0,1]^2 \ : \left\| M_1 U_\alpha^{1/2} \frac{1}{H-z} U_\beta^{1/2} M_2 \right\|_{HS} > t \right\} \right| \le 2\, C_W\, \|M_1\|_{HS}\, \|M_2\|_{HS}\, \frac{1}{\lambda t} . \tag{4.14}$$

Several comments are in order:

(1) When the energy z lies in the lower half plane (4.14) follows directly from Prop. 4.2 with $v_1 = \eta_\alpha, v_2 = \eta_\beta$ and $U_1 = \lambda U_\alpha, U_2 = \lambda U_\beta$.
(2) Eq. (4.14) also holds for z in the upper half plane, as can be seen by taking conjugates.
(3) For real energies ($z = E \in \mathbb{R}$), eq. (4.14) holds also in the limits $z \to E \pm \mathrm{i}0$, as follows from the bound at complex energies and Fatou's lemma (compare the argument for proving Lemma 4.1 in Appendix B).

The above results display that once the resolvent of the continuum operator is bracketed with a pair of Hilbert-Schmidt operators M_1, M_2 its fractional moments can be handled similarly to the discrete case. In the proofs of Lemmas 4.3 and 4.4 we use an argument which shows that $\chi_x(H-z)^{-1}\chi_y$ can be presented as a sum of a *bounded* operator and one of the form

$$M_x \chi_{B_x} (H-z)^{-1} \chi_{B_y} M_y , \tag{4.15}$$

with $M_\sharp$ Hilbert-Schmidt operators and B_x, B_y sets somewhat larger than the balls of radius r around x and y. This will allow us to conclude finiteness of fractional moments after "averaging over the local environment," i.e., averaging over all η_α with U_α non-zero in B_x or B_y.

4.3. **A pair of fractional-moment lemmas.** We now present the two basic technical results which give finiteness of the fractional moments, as well as a "decoupling argument" to be used in the proof of Theorem 1.2.

Definition 4.1. For $\alpha \in \mathcal{I}$, we denote

$$\mathcal{I}_\alpha := \{\zeta \in \mathcal{I} : \mathrm{dist}(\alpha, \zeta) < 3r\} \tag{4.16}$$

and set $\mathcal{I}_{\alpha,\beta} = \mathcal{I}_\alpha \cup \mathcal{I}_\beta$. Likewise, for any subset $\mathcal{L} \subset \mathcal{I}$, $\mathcal{I}_{\mathcal{L}} := \cup_{\alpha \in \mathcal{L}} \mathcal{I}_\alpha$. By $\mathcal{F}^c_{\mathcal{L}}$ we denote the σ-algebra generated by all η_α with α *not* in $\mathcal{I}_{\mathcal{L}}$. Thus, $\mathbb{E}(\cdot|\mathcal{F}^c_{\mathcal{L}})$ represents averaging over the "local environment" of $\mathcal{L}$.

Our first bound yields the finiteness of the s-moments after averaging over the local-environment.

Lemma 4.3. *Let H_ω be a random Schrödinger operator which satisfies* **A** *with disorder strength $\lambda > 0$. Then there exists $C_\lambda < \infty$ such that any restriction of H_ω to a region Ω obeys*

$$\mathrm{Prob}\left(\left\| U_\alpha \frac{1}{H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon} U_\beta \right\| > t \,\middle|\, \mathcal{F}^c_{\alpha,\beta} \right) \\ \leq \; C_\lambda \; (1 + 1/\lambda)(1 + |E - E_o|)^{d+2} \, \frac{D^2}{t} \; , \quad (4.17)$$

for any $\alpha, \beta \in \mathcal{I}$, any $E \in \mathbb{R}$ and $\varepsilon > 0$, where the coefficient C_λ obeys

$$C_\lambda \; \leq \; \text{const.} \; (1 + \lambda)^{d+2} \; . \quad (4.18)$$

Remarks:

(1) Recall that *RecallthatE*$_o = \inf \sigma(H_o)$ and that D is a bound on the conditional densities for η_α – see eq. (1.20).
(2) In Prop. 5.3 below we show that if H satisfies **A3′** then Lemma 4.3 can be improved so that eq. (4.17) holds with $\sup_\lambda C_\lambda < \infty$.
(3) For fractional moments we use the "layer-cake" representation – $\mathbb{E}\,(X^s) = \int_0^\infty \mathrm{Prob}(X > t^{1/s})\mathrm{d}t$ – to conclude that

$$\mathbb{E}\left(\left\| U_\alpha \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon} U_\beta \right\|^s \,\middle|\, \mathcal{F}^c_{\alpha,\beta} \right) \\ \leq \; \frac{C_\lambda^s}{1-s} \; (1 + 1/\lambda)^s \; (1 + |E - E_0|)^{s(d+2)} \; D^{2s} \; . \quad (4.19)$$

Of course, this bound implies a similar estimate for the average over all variables.
(4) Using condition **A2**, we obtain the following bound from eq. (4.19): For all measurable $\Lambda, \Lambda' \subset \mathbb{R}^d$,

$$\mathbb{E}\left(\left\| \mathbf{1}_\Lambda \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon} \mathbf{1}_{\Lambda'} \right\|^s \,\middle|\, \mathcal{F}^c_{\mathcal{L}(\Lambda \cup \Lambda')} \right) \\ \leq \; \frac{C_\lambda^s}{1-s} \; (1 + 1/\lambda)^s \; b_-^{-2s} \; (1 + |E - E_0|)^{s(d+2)} \; D^{2s} \; N_\Lambda \, N_{\Lambda'} \quad (4.20)$$

where $\mathcal{L}(\Lambda) := \{\gamma \in \mathcal{I} : \mathbf{1}_\Lambda U_\gamma \neq 0\}$ and N_Λ indicates the number of points in $\mathcal{L}(\Lambda)$. In particular, this proves eq. (1.7) of Lemma 1.1 under assumption **A**. The improvement under **A3′** follows from Prop. 5.3 below.

The second lemma (Lemma 4.4) focuses on the average of $\|U_\alpha (H - z)^{-1} U_\beta\|$ with respect to the local environment of *one* of the sites α, β. The idea underlying this result is "re-sampling:" we compare the distribution of $(H - z)^{-1}$ with that of a reference operator $(\widehat{H} - z)^{-1}$, where $\widehat{H}$ is obtained from H by redrawing the coupling variables η_ζ for ζ "near" β. The basic result is an estimate of the form

$$\mathbb{E}\left(\left\| U_\alpha \frac{1}{H_\omega - z} U_\beta \right\|^s \,\middle|\, \mathcal{F}^c_\beta \right) \; \leq \; \text{const.} \left\| \chi_\alpha \frac{1}{\widehat{H}_\omega - z} \mathbf{1}_S \right\|^s \; , \quad (4.21)$$

with S an appropriate neighborhood of β. However, the result permits a number of variations, and the estimate (4.22) stated below is slightly complicated because it is tailored to the required application in the proof of Theorem 1.2.

For that application it is convenient to use a smooth function in place of U_β. Thus we fix a choice of a partition of unity, $\{\Theta_\alpha : \alpha \in \mathcal{I}\}$, with the following properties:

(1) Each function Θ_α is non-negative and smooth with compact support in the ball of radius $4r/3$ centered at α.
(2) The collection $\{\Theta_\alpha\}$ is a partition of unity: $\sum_\alpha \Theta_\alpha(q) = 1$.
(3) $\sup_\alpha(\|\nabla\Theta_\alpha\|_\infty, \|\Delta\Theta_\alpha\|_\infty) < \infty$.

Any choice with these properties will do, although the constant $\widetilde{C}$ in eq. (4.22) below will depend on the supremum in (3).

Lemma 4.4. *Let H_ω be a random Schrödinger operator which satisfies* $\mathbf{A}$ *with disorder strength $\lambda > 0$. Then there exists $\widetilde{C}_\lambda < \infty$ such that any restriction of H to a region Ω obeys*

$$\operatorname{Prob}\left(\left\| \chi_x \frac{1}{H_\omega^{(\Omega)} - z} \Theta_\beta (1 + H_o - E_o)^{1/2} \right\| > t \,\middle|\, \mathcal{F}^c_{\beta,\gamma} \right) \leq \widetilde{C}_\lambda \, (1 + |z - E_0|)^{(d+3)} \left\| \chi_x \frac{1}{\widehat{H}_\omega^{(\Omega)} - z} \mathbf{1}_{S_{\beta,\gamma}} \right\| \frac{D^2}{t} \, , \tag{4.22}$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$, $x \in \mathbb{R}^d$, and $\beta, \gamma \in \mathcal{I}$ with $\operatorname{dist}(x, \beta)$, $\operatorname{dist}(x, \gamma) > 6r$. *The coefficient $\widetilde{C}_\lambda$ obeys*

$$\widetilde{C}_\lambda \leq \text{const.} \, (1 + \lambda)^{d+4} \, . \tag{4.23}$$

Here $S_{\beta,\gamma} = \{q : \operatorname{dist}(q, \{\beta, \gamma\}) \leq 5r\}$ and $\widehat{H}$ is obtained from H by replacing $\{\eta_\zeta : \zeta \in \mathcal{I}_{\beta,\gamma}\}$ with arbitrary values $\hat{\eta}_\zeta \in [0, 1]$:

$$\widehat{H} = H + \lambda \sum_{\zeta \in \mathcal{I}_{\beta,\gamma}} (\hat{\eta}_\zeta - \eta_\zeta) U_\zeta(q) \, . \tag{4.24}$$

Remark: *i.* Note that we have resampled H at a second site $\gamma \in \mathcal{I}$ as well as β, with the result that Θ_β is replaced by the characteristic function of a neighborhood of γ and β. This additional resampling is required in the application of this lemma in the proof of Theorem 1.2, but does not play a key role in the present section. *ii.* The factor $(1 + H_o - E_o)^{1/2}$ is included here to control various commutators $[H, \Theta]$ which appear in applications of the lemma. It is to bound this factor that we choose to work with the smooth function Θ_β.

In the applications of Lemma 4.4 we will choose $\widehat{\eta}_\zeta$ to have the same probability distribution as η_ζ. This will provide us with a "decoupling argument," as follows: If X is a quantity for which $\mathbb{E}\left(X^s | \mathcal{F}^c_{\beta,\gamma}\right) \leq A_s < \infty$, then eq. (4.22) and the Cauchy-Schwarz inequality imply that

$$\begin{aligned} \mathbb{E}&\left(X^{s/2} \left\| \chi_x \frac{1}{H^{(\Omega)} - z} \Theta_\beta (1 + H_o - E_o)^{1/2} \right\|^{s/2} \middle| \mathcal{F}^c_{\beta,\gamma} \right) \\ &\leq A_s^{1/2} \, \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \Theta_\beta (1 + H_o - E_o)^{1/2} \right\|^s \middle| \mathcal{F}^c_{\beta,\gamma} \right)^{1/2} \\ &\leq \text{const.} \, A_s^{1/2} \left\| \chi_x \frac{1}{\widehat{H}^{(\Omega)} - z} \mathbf{1}_S \right\|^{s/2} \, . \end{aligned} \tag{4.25}$$

If $\widehat{\eta}_\zeta$ is distributed identically to η_ζ then, upon taking expectations, we obtain

$$\mathbb{E}\left(X^{s/2} \left\| \chi_x \frac{1}{H^{(\Omega)} - z} \Theta_\beta (1 + H_o - E_o)^{1/2} \right\|^{s/2} \right) \\ \leq \text{ const. } A_s^{1/2} \, \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \mathbf{1}_S \right\|^{s/2} \right) . \quad (4.26)$$

In the final expression, we have replaced $\widehat{H}$ by H since the two are identically distributed.

4.4. **Averaging over local environments.** We now derive the two technical lemmas stated above. This subsection is essential for our analysis, but the reader may wish to skip it at first reading, as the arguments here are not required elsewhere in the article.

Proof of Lemma 4.3. We will write $z = E + i\varepsilon$ and assume without loss that $\varepsilon < 1$. Thus $1 + |E - E_0| \sim 1 + |z - E_0|$. Before applying Prop. 4.2 and the associated eq. (4.14), we must introduce Hilbert-Schmidt operators to the left and the right of $U_\alpha (H - z)^{-1} U_\beta$. The procedure we use to insert these operators is somewhat involved, so let us first outline the argument:

(1) We replace U_α, U_β by the upper bounds $b_+ \Theta^2$, $b_+ \Psi^2$,
$$\left\| U_\alpha \frac{1}{H - z} U_\beta \right\| \ \leq \ b_+^2 \left\| \Theta^2 \frac{1}{H - z} \Psi^2 \right\| , \quad (4.27)$$
where Θ, Ψ denote smoothed characteristic functions for the balls B_α^r, B_β^r respectively.

(2) The smoothness of Θ allows us to prove an *identity*
$$\Theta^2 \frac{1}{H - z} \ = \ B \ + \ T \widetilde{\Theta}^2 \frac{1}{H - z} , \quad (4.28)$$
where $\widetilde{\Theta}$ is a "fattened" version of Θ and T is Hilbert-Schmidt with HS-norm which is uniformly bounded with respect to the disorder. The operator B is norm bounded (uniformly with respect to the disorder).

(3) We repeat this procedure to the right of the resolvent to obtain
$$\Theta^2 \frac{1}{H - z} \Psi^2 \ = \ T_\alpha \widetilde{\Theta}^2 \frac{1}{H - z} \widetilde{\Psi}^2 T_\beta \ + \ B' , \quad (4.29)$$
where B' is norm bounded.

(4) We introduce the partition of unity $\sum U_\alpha / F$ between $\widetilde{\Theta}^2, \widetilde{\Psi}^2$ and the resolvent, and then apply Prop. 4.2 to each term in the resulting sum (the actual argument is complicated somewhat by the fact that T_α, T_β carry some dependence on the randomness).

We now turn to specifics. A good deal rests on the proof of eq. (4.28). This identity is a special case of the following: let Θ_j be any sequence of smooth functions such that Θ_j takes value 1 on the support of Θ_{j-1}, then for each $n \geq 1$

$$\Theta_1^2 \frac{1}{H - z} \ = \ T_n \, \Theta_{n+1}^2 \frac{1}{H - z} \ + \ B_n \quad (4.30)$$

where

$$T_n \ = \ A_1 \cdots A_n \quad (4.31)$$

with

$$A_j \;=\; \Theta_j \frac{1}{H_o + a} \left(\Theta_j (a + z - \lambda V_\omega) \;-\; [\Theta_j, H_o] \right) \;, \tag{4.32}$$

and

$$B_n \;=\; \Theta_1 \frac{1}{H_o + a} \Theta_1 \;+\; T_1 \Theta_2 \frac{1}{H_o + a} \Theta_2 \;+\; \cdots \;+\; T_{n-1} \Theta_n \frac{1}{H_o + a} \Theta_n \;. \tag{4.33}$$

Here $a := 1 - E_0$, so $H_o + a \geq 1$.

To verify (4.30), use induction on n with the induction step provided by the following "commutator argument":

$$\begin{aligned}
\Theta_j^2 \frac{1}{H - z} \; &= \Theta_j \frac{1}{H_o + a} (H_o + a) \Theta_j \frac{1}{H - z} \\
&= \Theta_j \frac{1}{H_o + a} \Theta_j (H_o + a) \frac{1}{H - z} \;-\; \Theta_j \frac{1}{H_o + a} [\Theta_j, H_o] \frac{1}{H - z} \\
&= \Theta_j \frac{1}{H_o + a} \Theta_j \\
&\qquad +\; \Theta_j \frac{1}{H_o + a} \left(\Theta_j (a + z - \lambda V_\omega) \;-\; [\Theta_j, H_o] \right) \frac{1}{H - z} \;.
\end{aligned} \tag{4.34}$$

The crucial point here is that

$$\Theta_j (a + z - \lambda V_\omega) \;-\; [\Theta_j, H_o] \;=\; \left(\Theta_j (a + z - \lambda V_\omega) \;-\; [\Theta_j, H_o] \right) \, \Theta_{j+1}^2 \;, \tag{4.35}$$

since Θ_{j+1} is identically one throughout the support of Θ_j and $\Theta_j (a + z - \lambda V_\omega) - [\Theta_j, H_o]$ is a differential – hence local – operator.

The representation provided by eq. (4.30) has two key features:

(1) The norm of B_n is bounded uniformly in ω and locally uniformly in energy.
(2) T_n is in $\mathcal{I}_p$ for arbitrary $p > d/n$ with $\| \cdot \|_p$-norm bounded uniformly in ω and locally uniformly in energy.

To see this we first observe that $A_j \in \mathcal{I}_p$ for $p > d$ with uniform $\| \cdot \|_p$-bounds. This follows since $\theta_j (H_0 + a)^{-1} \in \mathcal{I}_p$ for $p > d/2$, $\theta_j (H_0 + a)^{-1/2} \in \mathcal{I}_p$ for $p > d$ and $(H_0 + a)^{-1/2} [\theta_j, H_0] = (H_0 + a)^{-1/2} (-2\mathrm{i} \boldsymbol{D_A} \cdot (\nabla \theta_j) - (\Delta \theta_j))$ is bounded. Thus (2) follows from the Hölder property of Schatten classes and (1) from

$$B_n = B_{n-1} + T_{n-1} \theta_n \frac{1}{H_0 + a} \theta_n . \tag{4.36}$$

Once $n > d/2$, property (2) above implies that the operator T_n is Hilbert-Schmidt. Henceforth, we fix n to be the least integer greater than $d/2$ – in any case $n \leq (d+2)/2$. This provides the representation eq. (4.28) with $\widetilde{\Theta} = \Theta_n$.

There is much flexibility in the definition of Θ_j which, in turn, affects the norm estimates for T_n and B_n. For our purposes, it is sufficient to let Θ_j be supported in $\{q : \mathrm{dist}(q, \alpha) \;\leq\; r + [j/n] r\}$ so that $\widetilde{\Theta} = \Theta_n$ is supported in $\{q : \mathrm{dist}(q, \alpha) \;\leq\; 2r\}$. The specific choice of Θ_j is not so important. We note, however, that the choice may be made so that

$$\| \boldsymbol{\nabla} \Theta_j \|_\infty \;\leq\; \mathrm{O}(d) \;, \; \| \Delta \Theta_j \|_\infty \;\leq\; \mathrm{O}(d^2) \;, \tag{4.37}$$

since the gradient of Θ_j is supported on a set of width $r/n \approx 2r/d$.

Having fixed the sequence Θ_j we obtain a Hilbert-Schmidt operator T_n. More precisely, T_n is a *random* Hilbert-Schmidt operator since A_j depend on the random potential through the terms $\Theta_j V_\omega$. However, T_n is a product of n terms each of which is linear in $\Theta_j V_\omega$. Since

$$\Theta_j V_\omega = \sum_{\zeta \in \mathcal{I}_\alpha} \eta_\zeta \Theta_j U_\zeta \ , \tag{4.38}$$

we conclude that T_n is a polynomial of degree n in the variables $\{\eta_\zeta : \zeta \in \mathcal{I}_\alpha\}$ with Hilbert-Schmidt valued coefficients. Recall that $\mathcal{I}_\alpha$ is the set of lattice sites within distance $3r$ of α.

We now repeat this procedure to the right of the resolvent to obtain the two sided representation:

$$\Theta \frac{1}{H-z} \Psi \ = \ B \ + \ T_\alpha \widetilde{\Theta}^2 \frac{1}{H-z} \widetilde{\Psi}^2 T_\beta \ . \tag{4.39}$$

The above discussion shows that we may obtain the following properties for the terms in this identity:

(1) There is $C_d < \infty$, which depends on the parameters of the model and the choice of Θ_j (but not on the coupling variables η_ζ), such that

$$\begin{aligned} \|B\| \ &\le C_d \ (1 + \lambda + |z - E_0|)^{2n} \\ &\le C_d \ (1+\lambda)^{2n} (1 + |z - E_0|)^{2n} \ . \end{aligned} \tag{4.40}$$

(2) Each $T_\sharp$ is a polynomial of degree $n \le (d{+}2)/2$ in the variables η_ζ for $\zeta \in \mathcal{I}_\sharp$ with coefficients which are (non-random) Hilbert-Schmidt operators:

$$T_\sharp \ = \ \sum_{(k)} T_\sharp^{(k)} \prod_{\zeta \in \mathcal{I}_\sharp} \lambda^{|k|} \eta_\zeta^{k_\zeta} \ , \tag{4.41}$$

where the summation is over multi-indices $(k) \in \mathbb{N}^{\mathcal{I}_\sharp}$ with $|k| := \sum_\zeta k_\zeta \le n$.

(3) There is $\widetilde{C}_d < \infty$ such that

$$\left\| T_\sharp^{(k)} \right\|_{HS} \ \le \ \widetilde{C}_d (1 + |z - E_0|)^n \ , \tag{4.42}$$

for $\sharp = \alpha, \beta$ and each $(k) \in \mathbb{N}^{\mathcal{I}_\sharp}$ with $|k| \le n$.

(4) $\widetilde{\Theta}$ and $\widetilde{\Psi}$ are bounded by one and supported in $\{q : \mathrm{dist}(q, \alpha) < 2r\}$ and $\{q : \mathrm{dist}(q, \beta) < 2r\}$ respectively.

With this representation in hand we insert the partition of unity $\sum U_\zeta / F$ between each factor $\widetilde{\Theta}$, $\widetilde{\Psi}$ and the resolvent $(H-z)^{-1}$. Upon taking norms and applying the triangle inequality this yields

$$\left\| U_\alpha \frac{1}{H-z} U_\beta \right\| \le \ \|B\| \ + \sum_{\substack{\zeta \in \mathcal{I}_\alpha \\ \zeta' \in \mathcal{I}_\beta}} \ \sum_{\substack{(k) \in \mathbb{N}^{\mathcal{I}_\alpha} \\ (l) \in \mathbb{N}^{\mathcal{I}_\beta}}} \lambda^{|k|+|l|} \left\| T_\alpha^{(k)} \widetilde{\Theta} \frac{U_\zeta}{F} \frac{1}{H-z} \frac{U_{\zeta'}}{F} \widetilde{\Psi} T_\beta^{(l)} \right\| \ , \tag{4.43}$$

where we have used that $|\eta_\zeta| \le 1$.

Consider now the probability, conditioned on $\mathcal{F}^c_{\alpha,\beta}$, that $\left\| U_\alpha \frac{1}{H-z} U_\beta \right\| > t$. This may be bounded from above by the probability that *one* of the terms on the right hand side of eq. (4.43) is greater than t/M, where M is the number of terms. In turn this may be bounded by the sum of the individual probabilities:

$$\begin{aligned} \text{Prob}\left(\left\| U_\alpha \frac{1}{H-z} U_\beta \right\| > t \middle| \mathcal{F}^c_{\alpha,\beta} \right) \; &\leq \; \text{Prob}\left(\|B\| > t/M \middle| \mathcal{F}^c_{\alpha,\beta} \right) \\ + \sum_{\substack{\zeta\in\mathcal{I}_\alpha \\ \zeta'\in\mathcal{I}_\beta}} \sum_{\substack{(k)\in\mathbb{N}^{\mathcal{I}_\alpha} \\ (l)\in\mathbb{N}^{\mathcal{I}_\beta}}} &\text{Prob}\left(\lambda^{|k|+|l|} \left\| T^{(k)}_\alpha \widetilde{\Theta} \frac{U_\zeta}{F} \frac{1}{H-z} \frac{U_{\zeta'}}{F} \widetilde{\Psi} T^{(l)}_\beta \right\| > t/M \middle| \mathcal{F}^c_{\alpha,\beta} \right) . \end{aligned} \tag{4.44}$$

Applying Prop. 4.2 – via eq. (4.14) – and the bound on $\left\| T^{(\cdot)}_\sharp \right\|_{HS}$ provided by eq. (4.42), we see that each term of the summation is bounded:

$$\begin{aligned} \text{Prob}&\left(\lambda^{|k|+|l|} \left\| T^{(k)}_\alpha \widetilde{\Theta} \frac{U_\zeta}{F} \frac{1}{H-z} \frac{U_{\zeta'}}{F} \widetilde{\Psi} T^{(l)}_\beta \right\| > t/M \middle| \mathcal{F}^c_{\alpha,\beta} \right) \\ &\leq \; 2\, C_W\, b_+\, \widetilde{C}^2_d (1+|z-E_0|)^{2n} D^2 \frac{(1+\lambda)^{2n} M}{\lambda t} , \end{aligned} \tag{4.45}$$

while by eq. (4.40)

$$\text{Prob}\left(\|B\| > t/M \middle| \mathcal{F}^c_{\alpha,\beta} \right) \; \leq \; C_d \; (1+\lambda)^{2n} (1+|z-E_0|)^{2n} \, \frac{M}{t} \, . \tag{4.46}$$

The factor D^2 in eq. (4.45) is an upper bound for the joint density of η_ζ and $\eta_{\zeta'}$.

Putting this all together we find that

$$\begin{aligned} \text{Prob}&\left(\left\| U_\alpha \frac{1}{H-z} U_\beta \right\| > t \middle| \mathcal{F}^c_{\alpha,\beta} \right) \\ &\leq M \; \left(C_d \; + \; (M-1) \times \left[2\, C_W\, \widetilde{C}^2_d\, b_+ \right] \times \frac{D^2}{\lambda} \right) \\ &\qquad \times (1+\lambda)^{2n} \; (1+|z-E_0|)^{2n} \; \frac{1}{t} \\ &\leq M \; \left(C_d \; + \; (M-1) \times \left[2\, C_W\, \widetilde{C}^2_d\, b_+ \right] \right) \\ &\qquad \times (1+\lambda)^{2n} \; (1+|z-E_0|)^{2n} \; (1+1/\lambda) \; \frac{D^2}{t} \, , \end{aligned} \tag{4.47}$$

since $D \geq 1$. By **A2** the number of terms M is bounded independent of α, β. This completes the proof of Lemma 4.3. □

Proof of Lemma 4.4. This proof follows rather closely that of Lemma 4.3. The main new ingredients are (1) controlling the (un-bounded) factor $(a + H_o)^{1/2}$ and (2) the "re-sampling."

The unbounded factor is controlled by a one-step "commutator argument," similar to that used to generate the Hilbert-Schmidt operators in the proof of Lemma 4.3. The key is the following identity

$$\begin{aligned} \chi_x &\frac{1}{H-z} \Theta_\beta (a+H_o)^{1/2} \\ &= \; \chi_x \frac{1}{H-z} \; \widetilde{\Theta}^2_\beta \; \left((z+a-\lambda V_\omega)\Theta_\beta + [\Theta_\beta, H_o] \right) \; \frac{1}{(a+H_o)^{1/2}} \, , \end{aligned} \tag{4.48}$$

where we have used that $\chi_x \Theta_\beta = 0$. Here $\widetilde{\Theta}_\beta$ may be any function which is one throughout the support of Θ_β. As a result,

$$\left\| \chi_x \frac{1}{H-z} \Theta_\beta (a+H_o)^{1/2} \right\| \;\le\; c\,(1+\lambda)(1+|z-E_0|)\, \left\| \chi_x \frac{1}{H-z}\, \widetilde{\Theta}_\beta^2 \right\| \,, \tag{4.49}$$

with $c<\infty$ (depending on the size of $\boldsymbol{\nabla}\Theta_\beta$). We choose $\widetilde{\Theta}_\beta$ to have support in the ball of radius $5r/3$ centered at β.

Thus it suffices to prove eq. (4.22) with $\Theta_\beta(a+H_o)^{1/2}$ replaced by $\widetilde{\Theta}_\beta^2$ provided the power of $(1+|z-E_0|)$ on the right hand side and the permitted power growth of $\widetilde{C}_\lambda$ are each reduced by one. I.e., we must show

$$\begin{aligned} &\text{Prob}\left(\left\| \chi_x \frac{1}{H_\omega^{(\Omega)}-z} \widetilde{\Theta}_\beta^2 \right\| > t \,\middle|\, \mathcal{F}^c_{\beta,\gamma} \right) \\ &\qquad \le\; C(1+\lambda)^{d+3}\, (1+|z-E_0|)^{(d+2)} \left\| \chi_x \frac{1}{\widehat{H}_\omega^{(\Omega)}-z} \mathbf{1}_{S_{\beta,\gamma}} \right\| \,\frac{D^2}{t}\,. \end{aligned} \tag{4.50}$$

We want to average over $\zeta\in\mathcal{I}_{\beta,\gamma}$, using the Birman Schwinger identity together with the weak 1-1 inequality, and compare the result to the "re-sampled" operator:

$$\chi_x \frac{1}{H-z} \widetilde{\Theta}_\beta^2 \;=\; \chi_x \frac{1}{\widehat{H}-z} \widetilde{\Theta}_\beta^2 \;+\; \chi_x \frac{1}{\widehat{H}-z} (\widehat{H} \;-\; H) \frac{1}{H-z} \widetilde{\Theta}_\beta^2 \,, \tag{4.51}$$

where

$$\widehat{H} \;=\; H \;+\; \lambda \sum_{\zeta\in\mathcal{I}_{\beta,\gamma}} (\widehat{\eta}_\zeta - \eta_\zeta) U_\zeta(q)\,. \tag{4.52}$$

We use the construction presented in the proof of Lemma 4.3 – see eq. (4.30) – to introduce the necessary Hilbert-Schmidt operators.

We begin by noting that

$$\chi_x \frac{1}{\widehat{H}-z} (\widehat{H}-H) \;=\; \chi_x \frac{1}{\widehat{H}-z} \Psi^2 (\widehat{H}-H) \tag{4.53}$$

where Ψ is any smooth function identically one throughout $\{q : \text{dist}(q,\{\beta,\gamma\}) < 4r\}$. Thus the "commutator argument" in the proof of Lemma 4.3 yields the identity

$$\chi_x \frac{1}{\widehat{H}-z} (\widehat{H}-H) \;=\; \chi_x \frac{1}{\widehat{H}-z} \widetilde{\Psi}^2 \widehat{T} (\widehat{H}-H) \tag{4.54}$$

where $\widetilde{\Psi}$ is one on the support of Ψ and supported in the set $\{q : \text{dist}(q,\{\beta,\gamma\}) < 5r\}$, and $\widehat{T}$ is Hilbert-Schmidt with uniformly bounded norm: $\|\widehat{T}\|_2 \le \widetilde{C}_d(1+\lambda)^n(1+|z-E_0|)^n$ (n the smallest integer greater than $d/2$). The bounded term, "B", drops out of this representation because $\chi_x\widetilde{\Psi}^2 = 0$ and multipliers of support no larger than that of $\widetilde{\Psi}$ appear on the left of the terms of B. Note that $\widehat{T}$ is independent of $\{\eta_\zeta : \zeta\in\mathcal{I}_{\beta,\gamma}\}$ since it is constructed from $\widehat{H}$.

Next, we apply the "commutator argument" to the left of $(H-z)^{-1}$ to obtain

$$\frac{1}{H-z} \widetilde{\Theta}_\beta^2 \;=\; \widehat{\Theta}_\beta B \;+\; \frac{1}{H-z} \widehat{\Theta}_\beta^2 T \tag{4.55}$$

with B bounded, $\|B\| \le C_d(1+\lambda+|z-E_0|)^{2n}$, and T a polynomial in $\{\eta_\zeta : \zeta\in\mathcal{I}_\beta\}$,

$$T \;=\; \sum_{\substack{(k)\in\mathbb{N}^{\mathcal{I}_\beta} \\ |k|\le n}} \lambda^{|k|} T^{(k)} \prod_{\zeta\in\mathcal{I}_\beta} \eta_\zeta^{k_\zeta} \,, \tag{4.56}$$

with Hilbert-Schmidt coefficients,

$$\left\| T^{(k)} \right\|_{HS} \leq \widetilde{C}_d (1 + |z - E_0|)^n \, . \tag{4.57}$$

Here the function $\widehat{\Theta}_\beta$ is bounded by one and supported in $\{q : \mathrm{dist}(q, \beta) < 2r\}$. Note that $U_\zeta \widehat{\Theta}_\beta = 0$ for $\zeta \notin \mathcal{I}_\beta$.

Putting this all together, we obtain

$$\begin{aligned} \left\| \chi_x \frac{1}{H - z} \widetilde{\Theta}_\beta^2 \right\| \leq \left\| \chi_x \frac{1}{\widehat{H} - z} \widetilde{\Psi} \right\| \times \Bigg(1 \ + \ & \left\| (\widehat{H} - H) \widehat{\Theta}_\beta B \right\| \\ & + \sum_{(k) \in \mathbb{N}^{\mathcal{I}_\beta}} \lambda^{|k|} \left\| \widetilde{\Psi} \, \widehat{T} \, (\widehat{H} \ - \ H) \frac{1}{H - z} \widehat{\Theta}_\beta^2 T^{(k)} \right\| \Bigg) \, . \end{aligned} \tag{4.58}$$

Thus,

$$\begin{aligned} \left\| \chi_x \frac{1}{H - z} \widetilde{\Theta}_\beta^2 \right\| \leq \ \left\| \chi_x \frac{1}{\widehat{H} - z} \widetilde{\Psi} \right\| \times \Bigg(1 \ + \ 2 \, b_+ \lambda \, \| B \| \ + & \\ + \sum_{\substack{\zeta \in \mathcal{I}_{\beta, \gamma}, \zeta' \in \mathcal{I}_\beta \\ (k) \in \mathbb{N}^{\mathcal{I}_\beta}}} 2 \lambda^{1 + |k|} \left\| \widetilde{\Psi} \, \widehat{T} U_\zeta \frac{1}{H - z} \frac{U_{\zeta'}}{F} \widehat{\Theta}_\beta^2 T^{(k)} \right\| \Bigg) \, . & \end{aligned} \tag{4.59}$$

¿From (4.59), using Prop. 4.2 and the arguments from the proof of Lemma 4.3, it follows that

$$\begin{aligned} & \mathrm{Prob} \left(\left\| \chi_x \frac{1}{H - z} \widetilde{\Theta}_\beta^2 \right\| > t \, \middle| \, \mathcal{F}_{\beta, \gamma}^c \right) \\ & \quad \leq \left\| \chi_x \frac{1}{\widehat{H} - z} \widetilde{\Psi} \right\| \times \Big(1 \ + \ 2 b_+ C_d \lambda (1 + \lambda)^{2n} (1 + |z - E_0|)^{2n} \\ & \qquad\qquad + (M - 2) \, 4 C_W \widetilde{C}_d^2 (1 + \lambda)^{2n} (1 + |z - E_0|)^{2n} \, b_+ \, D^2 \Big) \ \frac{M}{t} \\ & \quad \leq \left\| \chi_x \frac{1}{\widehat{H} - z} \widetilde{\Psi} \right\| \times M \left(1 \ + \ 2 b_+ C_d (M - 2) \, 4 C_W \widetilde{C}_d^2 \right) \\ & \qquad\qquad\qquad \times (1 + \lambda)^{2n+1} (1 + |z - E_0|)^{2n} \frac{D^2}{t} \, , \end{aligned} \tag{4.60}$$

where M is the number of terms between the brackets in (4.59). By **A2**, M is bounded uniformly in x, β, γ. This completes the proof of Lemma 4.4. □

5. Finite-volume criteria

5.1. **The finite-volume inequality.** With the bounds provided by Lemmas 4.3 and 4.4, the proof of Theorem 1.2 proceeds according to a set of arguments familiar from the fractional moment method for discrete operators [6], and related to the multi-scale analysis of random Schrödinger operators, e.g. [67, 16], as well as the analysis of a number of lattice models in statistical mechanics, e.g. [31, 59, 45, 7, 5, 23]. We begin by proving a "correlation inequality" – eq. (5.2) below – and then iterate this inequality to obtain exponential decay of the bulk Green-function.

Lemma 5.1. *Let H_ω be a random Schrödinger operator with disorder parameter $\lambda > 0$ which obeys* **A** *and* **IAD***. Then, for each $s < 1/3$ there exists $\widetilde{C}_{\lambda,s} < \infty$ such that if we define, for $L > 23r$,*

$$a(x;L) \;:=\; L^{d-1} \sum_{\zeta \in \mathcal{S}_{x,L}} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(B_x^L)} - z} \chi_\zeta \right\|^s \right) \tag{5.1}$$

where $\mathcal{S}_{x,L} = \{\zeta \in \mathcal{I} : L - 23r < |\zeta - x| < L - 3r\}$, then for any region $\Omega \supset B_x^L$,

$$\begin{aligned} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^s \right) \leq\; & \widetilde{C}_{\lambda,s} (1 + 1/\lambda)^{2s} D^{10s} \, (1 + |z - E_0|)^{5s(d+2)} \\ & \times a(x;L) \sum_{\zeta \in \mathcal{S}'_{x,L}} \mathbb{E}\left(\left\| \chi_\zeta \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^s \right) \end{aligned} \tag{5.2}$$

for any y with $\mathrm{dist}(x,y) > L + r_o + 23r$, where $\mathcal{S}'_{x,L} = \{\zeta \in \mathcal{I} : L + r_o - 13r < |x - \zeta| < L + r_o + 23r\}$ (recall that r_o is the length scale appearing in **IAD***), with*

$$\widetilde{C}_{\lambda,s} \;\leq\; \text{Const.}\, \frac{(1+\lambda)}{1 - 3s}^{5s(d+4)} \, . \tag{5.3}$$

If H satisfies the stronger condition **A3′** *then*

$$\widetilde{C}_{\lambda,s} \;\leq\; \text{Const.}\, \frac{1}{1 - 3s} \, . \tag{5.4}$$

The main tools in the proof are the moment bounds presented above and a well known analog of the resolvent expansion commonly used for discrete Schrödinger operators, sometimes called the *geometric resolvent identity.*

Lemma 5.2. *Let H be a Schrödinger operator. Consider a sequence of three open sets $\Lambda_o \subset \Lambda \subset \Omega$ with $\mathrm{dist}(\Lambda_o, \Lambda^c) > 0$ and let Θ be a smooth function which is identically 1 in a neighborhood of Λ_o and identically zero in a neighborhood of Λ^c. Given any restrictions $H^{(\Omega)}$ and $H^{(\Lambda)}$ of H to Ω and Λ respectively,*

$$\mathbf{1}_{\Lambda_o} \frac{1}{H^{(\Omega)} - z} \;=\; \mathbf{1}_{\Lambda_o} \frac{1}{H^{(\Lambda)} - z} \Theta \;+\; \mathbf{1}_{\Lambda_o} \frac{1}{H^{(\Lambda)} - z} \, [H, \Theta] \, \frac{1}{H^{(\Omega)} - z} \tag{5.5}$$

for any z at which both resolvents exist.

If furthermore, $\Lambda'_o \subset \Lambda' \subset \Omega$ with $\mathrm{dist}(\Lambda'_o, \Lambda'^c) > 0$ and $\mathrm{dist}(\Lambda', \Lambda) > 0$, and Θ' is a function which is identically 1 in a neighborhood of Λ'_o and identically 0 in a neighborhood of Λ'^c then

$$\mathbf{1}_{\Lambda_o} \frac{1}{H^{(\Omega)} - z} \mathbf{1}_{\Lambda'_o} \;=\; -\mathbf{1}_{\Lambda_o} \frac{1}{H^{(\Lambda)} - z} \, [H, \Theta] \, \frac{1}{H^{(\Omega)} - z} \, [H, \Theta'] \, \frac{1}{H^{(\Lambda')} - z} \mathbf{1}_{\Lambda'_o} \, , \tag{5.6}$$

with $H^{(\Lambda')}$ any restriction of H to Λ' and z such that all three inverses exist.

Proof. The second identity, eq. (5.6), is a consequence of the first, eq. (5.5), and its transpose (eq. (5.8) below). A number of terms drop out because $\Lambda \cap \Lambda' = \emptyset$. To verify eq. (5.5) use the identity

$$[H, \Theta] \;=\; (H^{(\Lambda)} - z)\Theta - \Theta(H^{(\Omega)} - z) \tag{5.7}$$

on $\mathcal{D}(H^{(\Omega)})$, which follows from the fact that $H^{(\Lambda)} f = H^{(\Omega)} f$ if the support of f is strictly contained in Λ. Multiplying on the left by $\mathbf{1}_{\Lambda_o}(H^{(\Lambda)} - z)^{-1}$ and on the right by $(H^{(\Omega)} - z)^{-1}$ yields eq. (5.5). $\square$

Remarks:

(1) The identity,

$$\frac{1}{H^{(\Omega)} - z}\mathbf{1}_{\Lambda_o} \;=\; \Theta\frac{1}{H^{(\Lambda)} - z}\mathbf{1}_{\Lambda_o} \;-\; \frac{1}{H^{(\Omega)} - z}\,[H, \Theta]\,\frac{1}{H^{(\Lambda)} - z}\mathbf{1}_{\Lambda_o}\;, \tag{5.8}$$

follows from the transpose of eq. (5.5) (at conjugate z).

(2) Eq. (5.6) holds in a number of other contexts. In particular, it is true for discrete Schrödinger operators and in that case gives the usual geometric resolvent expansion (see [6, eq. (2.16)]).

Proof of Lemma 5.1. We shall assume **A3**. For the extension of the argument to couplings which satisfy **A3′** see Prop. 5.4 below.

We start by using the geometric resolvent identity eq. (5.6) of Lemma 5.2 with the sets

$$\begin{array}{rclcrcl} \Lambda_o & = & B_x^{L-11r}\;, & & \Lambda & = & B_x^{L}\;, \\ \Lambda_o' & = & \Omega \setminus B_x^{L+r_o+12r}\;, & & \Lambda' & = & \Omega \setminus B_x^{L+r_o}\;, \end{array} \tag{5.9}$$

and $\boldsymbol{\nabla}\Theta$, $\boldsymbol{\nabla}\Theta'$ supported in $\{q : L - 11r < |q - x| < L - 10r\}$, $\{q : L + r_o + 11r < |q - x| < L + r_o + 12r\}$ respectively.

The operators $\left[H, \Theta^\sharp\right]$ are *local* and supported in the set where $\boldsymbol{\nabla}\Theta$ is non-zero. A particular consequence of this observation is that

$$\begin{aligned} [H, \Theta] \;&=\; \sum_{\zeta_1 \in \mathcal{S}_o} \; \sum_{\substack{\zeta_2 \\ U_{\zeta_2}\Theta_{\zeta_1} \not\equiv 0}} \Theta_{\zeta_1}\,[H, \Theta]\,\frac{U_{\zeta_2}}{F} \\ [H, \Theta'] \;&=\; \sum_{\zeta_1' \in \mathcal{S}_o'} \; \sum_{\substack{\zeta_2' \\ U_{\zeta_2'}\Theta_{\zeta_1'} \not\equiv 0}} \frac{U_{\zeta_2'}}{F}\,[H, \Theta']\,\Theta_{\zeta_1'}\;, \end{aligned} \tag{5.10}$$

with $\mathcal{S}_o = \{q \in \mathcal{I} \;:\; L - 13r < |q - x| < L - 8r\}$ and $\mathcal{S}_o' = \{q \in \mathcal{I} \;:\; L + r_o + 9r < |q - x| < L + r_o + 14r\}$. Here Θ_ζ is the smooth partition of unity used in Lemma 4.4.

Recall that

$$\left\| \frac{1}{(a + H_o)^{1/2}}\left[H, \Theta^\sharp\right] \right\| \;\le\; \left(\left\|\Delta\Theta^\sharp\right\|_\infty \;+\; \sqrt{2}\left\|\boldsymbol{\nabla}\Theta^\sharp\right\|_\infty\right)\,, \tag{5.11}$$

which is bounded for smooth Θ. We may choose Θ so that the right hand side is bounded uniformly in L, being no larger than (const.) d^2/r^2.

With these observations, we obtain from eq. (5.6) the inequality:

$$\begin{aligned}
&\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \chi_y \right\| \\
&\quad \leq \text{ (const.)} \sum_{\substack{\zeta_1 \in \mathcal{S}_o \\ \zeta_2 : |\zeta_2 - \zeta_1| < 3r}} \sum_{\substack{\zeta_1' \in \mathcal{S}_o' \\ \zeta_2' : |\zeta_2' - \zeta_1'| < 3r}} \left\| \chi_x \frac{1}{H^{(B_x^L)} - z} \Theta_{\zeta_1} (a + H_o)^{1/2} \right\| \\
&\qquad \times \left\| U_{\zeta_2} \frac{1}{H^{(\Omega)} - z} U_{\zeta_2'} \right\| \\
&\qquad \times \left\| (a + H_o)^{1/2} \Theta_{\zeta_1'} \frac{1}{H^{(\Omega \setminus B_x^{L + r_o})} - z} \chi_y \right\| .
\end{aligned} \tag{5.12}$$

We now raise eq. (5.12) to the power $s < 1/3$ and take expectation values, using the inequality $(\sum a_n)^s \leq \sum a_n^s$. Consider each term on the right-hand side separately: first estimate the expectation conditioned on $\mathcal{F}^c_{\zeta_1,\zeta_2,\zeta_1',\zeta_2'}$, using the Hölder inequality to separate factors. The central factor may be estimated with Lemma 4.3 (see eq. (4.19)), to yield

$$\begin{aligned}
\mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^s \right) &\leq \text{ (const.)} \frac{C_\lambda^s}{(1-3s)^{1/3}} (1 + 1/\lambda)^s (1 + |z - E_0|)^{s(d+2)} D^{2s} \\
&\times \sum_{\substack{\langle \zeta_1, \zeta_2 \rangle \\ \langle \zeta_1', \zeta_2' \rangle}} \mathbb{E} \left[\mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(B_x^L)} - z} \Theta_{\zeta_1} (a + H_o)^{1/2} \right\|^{3s} \middle| \mathcal{F}^c_{\zeta_1,\zeta_2} \right)^{1/3} \right. \\
&\times \left. \mathbb{E}\left(\left\| (a + H_o)^{1/2} \Theta_{\zeta_1'} \frac{1}{H^{(\Omega \setminus B_x^{L + r_o})} - z} \chi_y \right\|^{3s} \middle| \mathcal{F}^c_{\zeta_1',\zeta_2'} \right)^{1/3} \right] .
\end{aligned} \tag{5.13}$$

Here we have noted that $H^{(B_x^L)}$ ($H^{(\Omega \setminus B_x^{L+r_o})}$) does not depend on the variables η_α with $\alpha \in \mathcal{I}_{\zeta_1',\zeta_2'}$ ($\alpha \in \mathcal{I}_{\zeta_1,\zeta_2}$).

The remaining two factors in each term are i) independent by **IAD** and ii) of the correct form to be estimated using Lemma 4.4. Thus,

$$\begin{aligned}
&\mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^s \right) \\
&\quad \leq \text{ (const.)} \left[\frac{C_\lambda^s}{(1-3s)^{1/3}} (1 + 1/\lambda)^s (1 + |z - E_0|)^{s(d+2)} D^{2s} \right] \\
&\qquad \times \left[\frac{\widetilde{C}_\lambda^s}{(1-3s)^{1/3}} (1 + |z - E_0|)^{s(d+3)} D^{2s} \right]^2 \\
&\qquad \times \sum_{\substack{\langle \zeta_1, \zeta_2 \rangle \\ \langle \zeta_1', \zeta_2' \rangle}} \mathbb{E}\left(\left\| \chi_x \frac{1}{\widehat{H}^{(B_x^L)}_{\zeta_1,\zeta_2} - z} \mathbf{1}_{S_{\zeta_1,\zeta_2}} \right\|^s \right) \\
&\qquad \times \mathbb{E}\left(\left\| \mathbf{1}_{S_{\zeta_1',\zeta_2'}} \frac{1}{\widehat{H}^{(\Omega \setminus B_x^{L+r_o})}_{\zeta_1',\zeta_2'} - z} \chi_y \right\|^s \right) .
\end{aligned} \tag{5.14}$$

Here $\widehat{H}_{\zeta_1,\zeta_2}$ ($\widehat{H}_{\zeta_1',\zeta_2'}$) denotes a version of H which is re-sampled over $\zeta \in \mathcal{I}_{\zeta_1,\zeta_2}$ ($\zeta \in \mathcal{I}_{\zeta_1',\zeta_2'}$) in the sense of eq. (4.24). The assumptions $L > 23r$ and $\mathrm{dist}(x,y) > L+r_o+23r$ in Lemma 5.1 are used to satisfy the distance requirements of Lemma 4.4.

We now pick the re-sampled variables with the same distribution as the $\{\eta_\zeta\}$, and include averaging over these variables in the expectations. Since $S_{\zeta_1,\zeta_2} \subset B^{8r}_{\zeta_1}$, $S_{\zeta_1',\zeta_2'} \subset B^{8r}_{\zeta_1'}$ and for each ζ_1, ζ_1' there is only a fixed finite number of values for ζ_2, ζ_2' (by **A2**), we infer from eq. (5.14), after adjusting the constant,

$$\begin{aligned}\mathbb{E}\left(\left\|\chi_x \frac{1}{H^{(\Omega)}-z}\chi_y\right\|^s\right) \le\ & (\mathrm{const.})\ \frac{C_\lambda^s \widetilde{C}_\lambda^{2s}}{1-3s}(1+1/\lambda)^s(1+|z-E_o|)^{3s(d+3)}\ D^{6s} \\ & \times \sum_{\substack{\zeta_1\in S_o\\ \zeta_1'\in S_o'}} \mathbb{E}\left(\left\|\chi_x \frac{1}{H^{(B_x^L)}-z}\mathbf{1}_{B^{8r}_{\zeta_1}}\right\|^s\right) \\ & \times \mathbb{E}\left(\left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}\chi_y\right\|^s\right) . \end{aligned} \tag{5.15}$$

To complete the proof of eq. (5.2) we need a bound for the resolvent of $H^{(\Omega\setminus B_x^{L+r_o})}$ in terms of the resolvent of $H^{(\Omega)}$. For this we will apply eq. (5.5) of Lemma 5.2 with

$$\Lambda = \Omega\setminus B_x^{L+r_o}, \qquad \Lambda_o = \Omega\setminus B_x^{L+r_o+r} \tag{5.16}$$

and a smooth function Θ'' such that $\nabla\Theta''$ is supported in $\{q : L+r_o < |q-x| < L+r_o+r\}$. Note that $\Theta''\chi_y = \chi_y$ and $B^{8r}_{\zeta_1'} \subset \Lambda_o$ for every $\zeta_1' \in \mathcal{S}_o'$. In place of (5.12) we obtain

$$\begin{aligned}\left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}\chi_y\right\| \le & \left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega)}-z}\chi_y\right\| \\ & + (\mathrm{const.}) \sum_{\substack{\zeta_1''\in\mathcal{S}_o''\\ \zeta_2'':|\zeta_2''-\zeta_1''|<3r}} \left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}U_{\zeta_2''}\right\| \\ & \times \left\|(a+H_o)^{1/2}\Theta_{\zeta_1''}\frac{1}{H^{(\Omega)}-z}\chi_y\right\| , \end{aligned} \tag{5.17}$$

where $\mathcal{S}_o'' := \{q\in\mathcal{I} :\ L+r_o-2r < |q-x| < L+r_o+3r\}$.

This yields the expectation bound

$$\begin{aligned}\mathbb{E}\left(\left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}\chi_y\right\|^s\right) \le & \ \mathbb{E}\left(\left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega)}-z}\chi_y\right\|^s\right) \\ & + (\mathrm{const.}) \sum_{\langle\zeta_1'',\zeta_2''\rangle} \sum_{\substack{\zeta\in\mathcal{I}\\ |\zeta-\zeta_1'|<9r}} \mathbb{E}\left(\left\|U_\zeta \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}U_{\zeta_2''}\right\|^s\right. \\ & \left.\times \left\|(a+H_o)^{1/2}\Theta_{\zeta_1''}\frac{1}{H^{(\Omega)}-z}\chi_y\right\|^s\right) . \end{aligned} \tag{5.18}$$

For the terms in the sum of eq. (5.18) we first take the expectation conditioned on $\mathcal{F}^c_{\zeta_1'',\zeta_2'',\zeta}$ and use the Cauchy-Schwarz inequality on the product. The resulting term

$$\mathbb{E}\left(\left\| U_\zeta \frac{1}{H^{(\Omega \setminus B_x^{L+r_o})} - z} U_{\zeta_2''} \right\|^{2s} \middle| \mathcal{F}^c_{\zeta_1'',\zeta_2'',\zeta} \right) , \tag{5.19}$$

is treated by eq. (4.19), while

$$\mathbb{E}\left(\left\| (a+H_o)^{1/2} \Theta_{\zeta_1''} \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^{2s} \middle| \mathcal{F}^c_{\zeta_1'',\zeta_2'',\zeta} \right) \tag{5.20}$$

can be estimated through a straightforward variant of Lemma 4.4, with re-sampling done over all variables in $\mathcal{I}_{\zeta_1'',\zeta_2'',\zeta}$.

Averaging over the re-sampled variables yields

$$\begin{aligned}\mathbb{E}\left(\left\| \mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega \setminus B_x^{L+r_o})} - z} \chi_y \right\|^{s} \right) &\leq \mathbb{E}\left(\left\| \mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^{s} \right) \\ &+ \ (\text{const.}) \frac{C_\lambda^s \widetilde{C}_\lambda^s}{1-2s} (1+1/\lambda)^s (1+|z-E_0|)^{2s(d+3)} D^{4s} \\ &\times \sum_{\langle \zeta_1'',\zeta_2'',\zeta \rangle} \mathbb{E}\left(\left\| \mathbf{1}_{S_{\zeta_1'',\zeta_2'',\zeta}} \frac{1}{H^{(\Omega)} - z} \chi_y \right\|^{s} \right) .\end{aligned} \tag{5.21}$$

After inserting this into eq. (5.15) we arrive at eq. (5.2) by covering all the sets $B^{8r}_{\zeta_1}$, $B^{8r}_{\zeta_1'}$ and $S_{\zeta_1'',\zeta_2'',\zeta}$ with balls B^r_α, $\alpha \in \mathcal{I}$. The factor L^{d-1} appears in this calculation since by **A2** the sets S_o' and S_o'' contain CL^{d-1} points in $\mathcal{I}$. This completes the proof of eq. (5.2).

The resulting expression for the coefficient $\widetilde{C}_{s,\lambda}$,

$$\widetilde{C}_{s,\lambda} \ = \ (\text{const.}) \frac{C_\lambda^{2s} \widetilde{C}_\lambda^{3s}}{(1-2s)(1-3s)} , \tag{5.22}$$

and the growth bounds for C_λ and $\widetilde{C}_\lambda$ provided by Lemmas 4.3 and 4.4 easily yield eq. (5.3). For the proof of eq. (5.4), we refer to Prop. 5.4 in the following section. □

5.2. **Large disorder and blow-up regularity.** In this section we consider the large disorder regime ($\lambda >> 1$) and discuss the use of assumption **A3′** to improve the bounds provided by Lems. 4.3 and 5.1 – i.e., eq. (4.19) and eq. (5.2). The basic idea is to apply Lemmas 4.3 and 4.4 to fluctuations of $\lambda\eta_\zeta$ which are of order one.

It is instructive first to consider independent identically distributed random couplings η_ζ. Given $\lambda > 1$, we decompose the interval $[0,1]$ as a union of a finite number of intervals of length less than $1/\lambda$ with disjoint interiors, $I_j = [a_j, a_{j+1}]$, and consider η_ζ as a super-position of its "integer" and "fractional" parts with respect to this decomposition:

$$\lambda\eta_\zeta \ = \ \lambda a_\zeta \ + \ f_\zeta . \tag{5.23}$$

Here the random variable a_ζ takes value a_j when η_ζ falls in the interval I_j, and $f_\zeta = \lambda(\eta_\zeta - a_\zeta)$ is a random variable which takes values in the interval $[0,1]$. The conditional distribution of $f = f_\alpha$, at a specified value of a_α, is given by the following expression:

$$\frac{\rho(f/\lambda + a_\alpha)}{\lambda \int_{I_j} \rho(\eta) \mathrm{d}\eta} \ \mathrm{d}f , \tag{5.24}$$

where ρ is the common density for η_ζ. The denominator is just the probability that η_α falls in the interval I_j.

To apply Lemmas 4.3 and 4.4 to conditional averages with respect to f_ζ, we would need to use the following value for D:

$$D = D(\lambda) = \sup_j \operatorname*{ess-sup}_{f \in I_j} \frac{\rho(f/\lambda + a_j)}{\lambda \int_{I_j} \rho(\eta) \mathrm{d}\eta} . \tag{5.25}$$

For fixed $\lambda > 1$, this is certainly finite. However, $D(\lambda)$ depends on λ as well as the choice of I_j. Blow-up regularity of the distribution ρ, which is guaranteed by assumption **A3′**, is precisely the requirement that $I_j = I_j(\lambda)$ may be chosen so that $D(\lambda)$ remains bounded for large λ.

The situation for general couplings $\{\eta_\zeta\}$ obeying **A3′** is somewhat more complicated since we must admit *random* decompositions of the interval $[0, 1]$. Specifically, **A3′** guarantees that the conditional distributions $\rho_\alpha(\eta_\zeta|\omega)$ of η_α at specified values of the remaining couplings are almost-surely blow-up regular with uniformly bounded blow-up norm: $D_{\rho_\alpha(\cdot|\omega)} \le D$. Thus for each α we may choose $I_j^\alpha = I_j^\alpha(\omega) = [a_j^\alpha(\omega), a_{j+1}^\alpha(\omega)]$ which are measurable with respect to $\mathcal{F}_\alpha^c$, satisfy $|I_j^\alpha| \le 1/\lambda$, and such that

$$\sup_\alpha \operatorname*{ess-sup}_\omega \max_j \operatorname*{ess-sup}_{f \in I_j^\alpha(\omega)} \frac{\rho_\alpha(f/\lambda + a_j^\alpha(\omega)|\omega)}{\lambda \int_{I_j^\alpha(\omega)} \rho_\alpha(\eta|\omega) \mathrm{d}\eta} \le D . \tag{5.26}$$

We now define a_ζ and f_ζ as above: $a_\zeta = a_j^\alpha(\omega)$ if η_ζ falls in $I_j^\alpha(\omega)$ and $f_\zeta = \lambda\eta_\zeta - \lambda a_\zeta$. The density of the distribution of $f = f_\alpha$ conditioned on a_α and $\mathcal{F}_\alpha^c$ is

$$\frac{\rho_\alpha(f/\lambda + a_\alpha|\omega)}{\lambda \int_{I_j^\alpha(\omega)} \rho_\alpha(\eta|\omega) \mathrm{d}\eta} , \tag{5.27}$$

which is a density supported in $[0, 1]$ and bounded by D.

With these notions, it is an easy exercise in conditional expectations to prove the following extension of Lemma 4.3:

Proposition 5.3. *Let H_ω be a random Schrödinger operator which satisfies* **A1**, **A2**, *and* **A3′**. *Then eq. (4.17) of Lemma 4.3 holds with a coefficient C_λ which satisfies $\sup_\lambda C_\lambda < \infty$, provided we use for D the constant appearing in* **A3′** *– i.e, the blow-up norm of the distribution.*

Turning now to Lemma 4.4, we note that the disorder strength plays a role in the re-sampling procedure in addition to the averaging. Hence, it is most natural to use the lemma as stated and re-sample only the variables f_ζ with the a_ζ fixed. An argument along these lines will be used to complete the proof of eq. (5.4) from Lemma 5.1:

Proposition 5.4. *Let H_ω be a random Schrödinger operator which satisfies* **IAD**, **A1**, **A2**, *and* **A3′**. *Then eq. (5.2) of Lemma 5.1 holds with a coefficient $\widetilde{C}_{s,\lambda}$ which obeys eq. (5.4).*

Proof. The proof follows closely the derivation of eq. (5.2) given above with a few modifications which we now indicate.

Let $\mathcal{A}$ denote the sigma-algebra generated by a_ζ, $\zeta \in \mathcal{I}$. Conditioning on $\mathcal{A}$, we obtain a random Schrödinger operator with $\lambda = 1$ and couplings f_ζ, $\zeta \in \mathcal{I}$ which obey **IAD** and **A3**. Thus, following the proof of Lemma 5.1 we obtain the following analog of eq. (5.14):

$$\begin{aligned}\mathbb{E}\left(\left\|\chi_x \frac{1}{H^{(\Omega)}-z}\chi_y\right\|^s \middle| \mathcal{A}\right) \leq\ & (\text{const.})\ \frac{1}{1-3s}(1+|z-E_o|)^{3s(d+3)}\ D^{6s} \\ & \times \sum_{\substack{\zeta_1\in S_o \\ \zeta_1'\in S_o'}} \mathbb{E}\left(\left\|\chi_x \frac{1}{H^{(B_x^L)}-z}\mathbf{1}_{B^{8r}_{\zeta_1}}\right\|^s \middle| \mathcal{A}\right) \\ & \qquad \times \mathbb{E}\left(\left\|\mathbf{1}_{B^{8r}_{\zeta_1'}} \frac{1}{H^{(\Omega\setminus B_x^{L+r_o})}-z}\chi_y\right\|^s \middle| \mathcal{A}\right) . \end{aligned} \tag{5.28}$$

We now average with respect to $\mathcal{A}$, noting that $H^{(B_x^L)}$ and $H^{(\Omega\setminus B_x^{L+r_o})}$ are independent by **IAD**, and this independence is inherited by conditional averages over $\mathcal{A}$. The resulting bound is identical to eq. (5.14) with $\lambda = 1$. To complete the proof, use a similar argument to prove the analog of eq. (5.21), first conditioning on $\mathcal{A}$ to obtain coefficients with $\lambda = 1$. □

5.3. **The criteria – proof of Theorem 1.2.** Let

$$M_{s,\lambda} := \widetilde{C}_{\lambda,s} D^{10s} \times \sup_{\substack{\alpha\in\mathcal{I} \\ L>23r}} \frac{\#S'_{\alpha,L}}{L^{d-1}} , \tag{5.29}$$

with the sets $\mathcal{S}'_{\alpha,L}$ as in Lemma 5.1, and suppose eq. (1.14) holds for some $s \in (0, 1/3)$ and $E \in \mathbb{R}$.

For each open set $\Omega \subset \mathbb{R}^d$ and $\delta > 0$, we define

$$G^\delta_\Omega(\alpha,\beta) := \sup_{|\varepsilon|<\delta} \mathbb{E}\left(\left\|\chi_\alpha \frac{1}{H^{(\Omega)}-E-\mathrm{i}\varepsilon}\chi_\beta\right\|^s\right) \tag{5.30}$$

with $\alpha, \beta \in \mathcal{I}\cap\Omega$. Whenever $\alpha, \beta \in \mathcal{I}\cap\Omega$ with $B^L_\alpha \subset \Omega$ and $|\alpha-\beta| > L+r_0+23r$, Lemma 5.1 implies that

$$G^\delta_\Omega(\alpha,\beta) \ \leq\ \mathrm{e}^{-\gamma(\delta)} \frac{1}{\#S'_{\alpha,L}} \sum_{\zeta\in S'_{\alpha,L}} G^\delta_\Omega(\zeta,\beta) , \tag{5.31}$$

with

$$\begin{aligned}\mathrm{e}^{-\gamma(\delta)} \ :=\ & M_{s,\lambda}(1+1/\lambda)^{2s}(1+|E-E_0|)^{5s(d+2)}(1+L)^{2(d-1)} \\ & \times \sup_{\varepsilon<\delta}\sup_{\alpha\in\mathcal{I}} \mathbb{E}\left(\left\|\chi_\alpha\,(H^{(B^L_\alpha)}-E-i\varepsilon)^{-1}\,\mathbf{1}_{\delta B^L_\alpha}\right\|^s\right) . \end{aligned} \tag{5.32}$$

Note that $\gamma(\delta)$ is a decreasing function of δ and the assumed finite volume bound eq. (1.14) shows that $\lim_{\delta\to0}\gamma(\delta) > 0$. In particular, there is $\delta_0 > 0$ such that $\gamma(\delta) > 0$ for $\delta < \delta_o$.

Let us define,

$$F^\delta_\Omega(\nu) \ :=\ \sup_{\alpha,\beta\in\mathcal{I}\cap\Omega} \mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} G^\delta_\Omega(\alpha,\beta) . \tag{5.33}$$

The proof will proceed as follows. First, for $\nu < \gamma(\delta)$ and bounded Ω, we derive an Ω independent bound on $F^\delta_\Omega(\nu)$. Second, unbounded Ω are handled via finite volume approximations. Finally, we pass to the limits $\delta \to 0$ and $\nu \to \gamma$.

Now, consider a bounded region Ω and $\alpha, \beta \in \mathcal{I}\cap\Omega$ with $\mathrm{dist}_\Omega(\alpha,\beta) > 2L$. Observe that either eq. (5.31) or its conjugate holds for this pair, since $|\alpha-\beta| >$

$2L > L + r_0 + 23r$ and one of B^L_α or B^L_β is contained entirely in Ω. If, say $B^L_\alpha \subset \Omega$, then eq. (5.31) implies

$$\begin{aligned} \mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} G^\delta_\Omega(\alpha,\beta) \ &\le \mathrm{e}^{-\gamma(\delta)} \frac{1}{\# S'_{\alpha,L}} \sum_{\zeta\in S'_{\alpha,L}} \mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} G^\delta_\Omega(\zeta,\beta) \\ &\le \mathrm{e}^{\nu}\mathrm{e}^{-\gamma(\delta)} \frac{1}{\# S'_{\alpha,L}} \sum_{\zeta\in S'_{\alpha,L}} \mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\zeta,\beta)/2L} G^\delta_\Omega(\zeta,\beta)\ , \end{aligned} \tag{5.34}$$

where we have used the triangle inequality for dist_Ω and observed that $\mathrm{dist}_\Omega(\alpha,\zeta) \le 2L$ for $\zeta \in S'_{\alpha,L}$. If instead $B^L_\beta \subset \Omega$, the conjugate version of eq. (5.31) implies the conjugate version of this bound. Either way, the end result is that

$$\mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} G^\delta_\Omega(\alpha,\beta) \ \le \ \mathrm{e}^{\nu}\mathrm{e}^{-\gamma(\delta)} F^\delta_\Omega(\nu)\ . \tag{5.35}$$

If $\nu < \gamma(\delta)$, eq. (5.35) implies that $F^\delta_\Omega(\nu)$ may be found by restricting the supremum in eq. (5.33) to "nearby" pairs (here we use that $F^\delta_\Omega(\nu)$ is finite due to boundedness of Ω):

$$F^\delta_\Omega(\nu) \ = \sup_{\alpha,\beta:\mathrm{dist}_\Omega(\alpha,\beta)<2L} \mathrm{e}^{\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} G^\delta_\Omega(\alpha,\beta) \ \le \ \mathrm{e}^{\nu} A(s,\lambda,E)\ , \tag{5.36}$$

where $A(s,\lambda,E)$ is the fractional moment estimate provided by Lemma 1.1.

To complete the proof, we must extend eq. (5.36) to unbounded regions. For this purpose, fix an open set $\Omega \subset \mathbb{R}_d$ and let $\Omega_j = [-j,j]^d \cap \Omega$. One may verify, for example using the geometric resolvent identity eq. (5.5) and the Combes-Thomas estimate [18], that for any $\alpha,\beta \in \mathcal{I}\cap\Omega$ and $z \in \mathbb{C}\setminus\mathbb{R}$

$$\lim_{j\to\infty} \left\| \chi_\alpha \frac{1}{H^{(\Omega_j)} - z}\chi_\beta - \chi_\alpha \frac{1}{H^{(\Omega)} - z}\chi_\beta \right\| \ = \ 0\ . \tag{5.37}$$

Thus

$$\sup_{|\varepsilon|\le\delta} \left\| \chi_\alpha \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon}\chi_\beta \right\| \ \le \ \liminf_{j\to\infty} \sup_{|\varepsilon|\le\delta} \left\| \chi_\alpha \frac{1}{H^{(\Omega_j)} - E - \mathrm{i}\varepsilon}\chi_\beta \right\|\ , \tag{5.38}$$

and by Fatou's Lemma,

$$G^\delta_\Omega(\alpha,\beta) \ \le \ \liminf_{j\to\infty} G^\delta_{\Omega_j}(\alpha,\beta)\ , \tag{5.39}$$

Since $\mathrm{dist}_{\Omega_j}(\alpha,\beta) \to \mathrm{dist}_\Omega(\alpha,\beta)$, this implies eq. (5.36) holds for Ω when $\nu < \gamma(\delta)$.

In the limit $\delta \to 0$, eq. (5.36) implies that

$$\limsup_{\varepsilon\to 0} \mathbb{E}\left(\left\| \chi_\alpha \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon}\chi_\beta \right\|^s \right) \ \le \ \mathrm{e}^{\nu} A(s,\lambda,E)\mathrm{e}^{-\nu\,\mathrm{dist}_\Omega(\alpha,\beta)/2L} \tag{5.40}$$

for any $\nu < \gamma$. Finally, taking $\nu \to \gamma$, we obtain eq. (1.15). □

6. Applications: results for distinct energy ranges

To apply the above results on localization we need to verify the sufficiency criteria of Theorem 1.2 in specific disorder regimes and energy ranges. In this section we present several such results for regimes of interest, including the familiar large disorder and band edge (Lifshitz tail) regimes. The mechanisms which allow us to check eq. (1.14) are essentially those used to verify the initial length scale estimates for a multiscale analysis. As such, we do not attempt to give an exhaustive list of applications here, but rather try to illustrate how known methods may be combined with the arguments developed in this paper.

A useful feature of our results is that it suffices to check eq. (1.14) at a single energy $E \in \mathbb{R}$, since the following continuity result allows us to extend the obtained bounds onto an interval containing E (as well as off the real axis).

Lemma 6.1. *Let H satisfy* **A**. *Fix $0 < s < 1/2$, a bounded domain Λ, and subsets Λ_1 and Λ_2 of Λ. Let $G(z) := \mathbf{1}_{\Lambda_1}(H^{(\Lambda)} - z)^{-1}\mathbf{1}_{\Lambda_2}$. Then $z \mapsto \mathbb{E}\left(\|G(z)\|^s\right)$ is locally Hölder-continuous with exponent s for all $z \in \mathbb{C}$. More precisely, there exists a constant $A_{s,\lambda} < \infty$ such that*

$$|\mathbb{E}\left(\|G(z)\|^s\right) - \mathbb{E}\left(\|G(w)\|^s\right)| \leq A_{s,\lambda} N_\Lambda N_{\Lambda_1}^{1/2} N_{\Lambda_2}^{1/2} \times (1 + |z - E_0|)^{s(d+2)}(1 + |w - E_0|)^{s(d+2)}|z - w|^s \tag{6.1}$$

for all $z, w \in \mathbb{C}$. Here $N_\Lambda = \#(\mathcal{I} \cap \Lambda)$ is the "volume" defined in the discussion following Lemma 4.3 and

$$A_{s,\lambda} \;\leq\; \text{const.}\ \frac{(1+1/\lambda)^{2s}}{1-2s} \times \begin{cases} (1+\lambda)^{2s(d+2)} & \text{if } \mathbf{A3} \text{ holds,} \\ 1 & \text{if } \mathbf{A3'} \text{ holds.} \end{cases} \tag{6.2}$$

Proof. By the resolvent identity, noting that $\mathbf{1}_\Lambda = \mathbf{1}$ on $L^2(\Lambda)$,

$$G(z) - G(w) = (z - w)\mathbf{1}_{\Lambda_1}(H^{(\Lambda)} - z)^{-1}\mathbf{1}_\Lambda(H^{(\Lambda)} - w)^{-1}\mathbf{1}_{\Lambda_2}\,. \tag{6.3}$$

Thus

$$\begin{aligned} &|\mathbb{E}\left(\|G(z)\|^s\right) - \mathbb{E}\left(\|G(w)\|^s\right)| \leq \mathbb{E}\left(\|G(z) - G(w)\|^s\right) \\ &\leq |z-w|^s \mathbb{E}\left(\|\mathbf{1}_{\Lambda_1}\frac{1}{H^\Lambda - z}\mathbf{1}_\Lambda \frac{1}{H^{(\Lambda)} - w}\mathbf{1}_{\Lambda_2}\|^s\right) \\ &\leq |z-w|^s \mathbb{E}\left(\|\mathbf{1}_{\Lambda_1}\frac{1}{H^{(\Lambda)} - z}\mathbf{1}_\Lambda\|^{2s}\right)^{1/2} \mathbb{E}\left(\|\mathbf{1}_\Lambda\frac{1}{H^{(\Lambda)} - w}\mathbf{1}_{\Lambda_2}\|^{2s}\right)^{1/2}. \end{aligned} \tag{6.4}$$

The result now follows from Lemma 4.3 – see eq. (4.20) – combined with Prop. 5.3. □

A preliminary application of the continuity provided by Lemma 6.1 is the proof of Theorem 1.3.

Proof. Due to Lemma 6.1, we need only verify eq. (1.14) for the energy E appearing in eq. (1.17). For this we argue along the lines of the proof of Lemma 5.1, using the geometric resolvent identity eq. (5.5) to obtain for $L > 23r$,

$$\chi_\alpha \frac{1}{H^{(B^L_\alpha)} - z}\mathbf{1}_{\delta B^L_\alpha} \;=\; \chi_\alpha \frac{1}{H - z}\mathbf{1}_{\delta B^L_\alpha} - \chi_\alpha \frac{1}{H - z}\,[H, \Theta_L]\,\frac{1}{H^{(B^L_\alpha)} - z}\mathbf{1}_{\delta B^L_\alpha} \tag{6.5}$$

where Θ_L is any smooth function equal to one on B_α^{L-r} and zero on $\mathbb{R}^d \setminus B_\alpha^L$ – recall that $\delta B_\alpha^L = \{q : r < \mathrm{dist}(q, \partial\Lambda) < 23r\} \subset B_\alpha^{L-r}$. From this we conclude, as in the proof of eq. (5.21), that

$$\mathbb{E}\left(\left\| \chi_\alpha \frac{1}{H^{(B_\alpha^L)} - z} \mathbf{1}_{\delta B_\alpha^L} \right\|^s \right) \leq \text{ const. } L^{2(d-1)} \sum_{\substack{\zeta \\ |\zeta - \alpha| < 23r}} \mathbb{E}\left(\left\| \chi_\alpha \frac{1}{H - z} \chi_\zeta \right\|^s \right) . \tag{6.6}$$

By assumption the right hand side is $\mathcal{O}(L^{3(d-1)} \mathrm{e}^{-\mu L})$ uniformly in α. Thus eq. (1.14) is satisfied for sufficiently large L and the theorem follows. □

6.1. **The multi-scale analysis regime.** The region in which a multiscale analysis may be carried out provides another characterization of the localization regime. An important observation is that the "output" of the multiscale analysis implies the "input" eq. (1.14) for Theorem 1.2. Thus for the operators considered here the stronger results proved by our methods hold throughout the multiscale regime. A precise formulation of this statement is the following.

Theorem 6.1. *Let H be a random Schrödinger operator which satisfies* **A** *and* **IAD**. *Suppose that for some $A < \infty$, $\mu > 0$, $\xi > 2(d-1)$, $C < \infty$ and $E \in \mathbb{R}$ it holds for L sufficiently large that*

$$\sup_{\alpha \in \mathcal{I}} \mathrm{Prob}\left[\left\| \chi_\alpha \frac{1}{H^{(B_\alpha^L)} - E} \mathbf{1}_{\delta B_0^L} \right\| > Ae^{-\mu L} \right] \leq CL^{-\xi} . \tag{6.7}$$

Then, given $0 < s < 1/3$, there exist $A' < \infty$, $\mu' > 0$ and an open interval $\mathcal{J}$ containing E such that

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - E' - \mathrm{i}\varepsilon} \chi_y \right\|^s \right) \leq A' e^{-\mu' \, \mathrm{dist}_\Omega(x,y)} \tag{6.8}$$

for all open sets $\Omega \subset \mathbb{R}^d$, $x, y \in \Omega$ and $E' \in \mathcal{J}$.

Bounds on the probability of exceptional behavior of the type of eq. (6.7) are a characterization of the localized regime mapped by the multi-scale analysis, where ξ can be made arbitrarily large, e.g. [16]. Using a "bootstrap" approach in which the output of one multi-scale analysis serves as the input for another, Germinet and Klein have shown that this probability is $\mathcal{O}(\mathrm{e}^{-L^\alpha})$ with any $\alpha < 1$ throughout the multi-scale regime [28]. *A posteriori*, we conclude from Theorem 1.2 that this bound may be replaced by $\mathcal{O}(\mathrm{e}^{-\nu L})$ with $\nu > 0$ for the operators considered here.

For ergodic random Schrödinger operators (see **A4** in Section 6), there is a notion of "strong localization region," introduced by [29], analogous to Dobrushin and Shlosman's regime of complete analyticity in statistical mechanics [23]. The strong localization region may be characterized by any of a number of criteria, one of which is the applicability of the multi-scale analysis, and throughout this region all of those criteria hold (see [29, Theorem 4.4]). In the words of the authors of [29], it is a region "possessing every possible virtue we can imagine!" For the random operators considered here, one may add to that list of virtues exponential localization of Green function moments and dynamical localization in the (stronger) sense of Corollary 1.1.

Proof. Following the argument for the lattice case from Section 4.4 of ref. [6], we define complementary "good" and "bad" subsets of the probability space Ω by

$$\Omega_G \;:=\; \left\{\omega \mid \|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\| \le Ae^{-\mu L}\right\} \tag{6.9}$$

and $\Omega_B := \Omega_G^c$. Then

$$\begin{aligned} \mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^s\right) & \\ &= \mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^s I[\omega\in\Omega_G]\right) \\ &\quad + \mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^s I[\omega\in\Omega_B]\right) \;. \end{aligned} \tag{6.10}$$

The first term is bounded by $A^s e^{-s\mu L}$, while we may apply the Hölder inequality to bound the second term for any $s < t < 1$ by

$$\mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^t\right)^{s/t} \mathbb{E}\left(I[\omega\in\Omega_B]\right)^{1-s/t} \;. \tag{6.11}$$

We may further estimate this by $C(s,E,d)L^{(s(d-1)-\xi(t-s))/t}$ using (1) the fractional moment bound eq. (4.20) and (2) the assumed bound eq. (6.7) on $\mathrm{Prob}(\Omega_B)$. Here we have used that δB_0^L contains $\mathcal{O}(L^{d-1})$ points in $\mathcal{I}$.

In summary, we get the bound

$$\mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^s\right) \le A^s e^{-s\mu L} + C(t,E,\lambda)^{s/t}\, L^{(s(d-1)-\xi(t-s))/t} \;, \tag{6.12}$$

uniformly in α. Since $\xi > 2(d-1)$ we can choose s sufficiently close to 0 and t close to 1 to guarantee $(\xi(t-s) - s(d-1))/t > 2(d-1)$. Thus

$$\limsup_{L\to\infty} L^{2(d-1)} \sup_{\alpha\in\mathcal{I}} \mathbb{E}\left(\|\chi_\alpha(H^{(B_\alpha^L)} - E)^{-1}\mathbf{1}_{\delta B_\alpha^L}\|^s\right) \;=\; 0 \;, \tag{6.13}$$

and we may choose L large enough that eq. (1.14) holds at E.

Working in the finite volumes B_α^L, we can use the continuity given by Lemma 6.1 and the continuity of $b_s(\lambda, E)$ in E to conclude the existence of a complex neighborhood $\mathcal{U}$ of E such that eq. (1.14) holds for every $E' + i\varepsilon \in \mathcal{U}$. Then Theorem 1.2 applies at all real $E' \in \mathcal{U}$, which concludes the proof by Theorem 6.1. □

6.2. **Large disorder.** Perhaps the easiest localization regime to understand is that induced at the bottom of the spectrum by large disorder. If we fix an energy $E > E_0$ and a length scale L we may adjust λ so that for any $x \in \mathbb{R}^d$ it is overwhelmingly likely that E is far below the bottom of the spectrum of the "local Hamiltonian," i.e., $H^{(B_x^L)}$. Heuristically, this suggests that E lies in the localization regime, since the resolvent $(H^{(B_x^L)} - E)^{-1}$ is typically bounded with small norm.

In the discrete setting, the quantitative bound $\mathbb{E}(|G(x,x)|^s) \lesssim 1/\lambda^s$ provides the basis for the 'single-site' criterion of [4]. For the operators considered here, by taking λ large enough one may directly verify the localization condition eq. (1.14) at any fixed finite scale L allowed in Theorem 1.2. In fact, eq. (1.14) may be satisfied uniformly for all energies in an arbitrary finite interval. For this result, we assume blow-up regularity **A3′** to insure that the constants which appear in eq. (1.14) remain bounded as λ increases.

Theorem 6.2. *Let H satisfy* **IAD**, **A1**, **A2** *and* **A3′**. *Then to every $E_1 \in \mathbb{R}$ and $0 < s < 1/3$ there exists $\lambda_0 = \lambda_0(E_1, s)$ such that for every $\lambda > \lambda_0$ there are constants $A < \infty$ and $\mu > 0$ such that for any open set $\Omega \subset \mathbb{R}^d$*

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - E - \mathrm{i}\varepsilon} \chi_y \right\|^s \right) \leq A e^{-\mu \operatorname{dist}_\Omega(x,y)} \tag{6.14}$$

for all $E \in [E_0, E_1]$ and $x, y \in \Omega$.

Remark: The proof will show that we can take $\lambda_0 \propto (1 + |E_1 - E_0|)^{2+3d/2}$.

Proof. As mentioned in the introduction, we cannot use Lemma 1.1 to verify eq. (1.14), since the r.h.s. of eq. (1.7) does not approach 0 as $\lambda \to \infty$. Instead we fix any $L > 23r$ and use the method discussed in Section 3.1 to show that $\mathbb{E}(\|\chi_\alpha (H^{(B_\alpha^L)} - E)^{-1} \mathbf{1}_{\delta B_\alpha^L}\|^s)$ approaches 0 as $\lambda \to \infty$. This method gives volume dependent bounds on fractional moments, but as L is fixed in the present argument that is of no importance.

By the covering condition in **A2** it suffices to show that

$$\mathbb{E}\left(\left\| U_\beta \frac{1}{H^{(B_\alpha^L)} - E} U_\zeta \right\|^s \right) \to 0 \quad \text{as } \lambda \to \infty \tag{6.15}$$

for all $\beta, \zeta \in \mathcal{I}$. Convergence needs to be shown uniformly with respect to $\alpha, \beta, \zeta \in \mathcal{I}$ and $E \in [E_0, E_1]$. One may then switch to complex energy using Lemma 6.1 and complete the proof with Theorem 1.2.

To derive eq. (6.15), we write $H = H^{(B_\alpha^L)}$, understanding henceforth that all operators are restricted to $L^2(B_\alpha^L)$. Let $\widehat{H} = H - \lambda\eta_\beta U_\beta - \lambda\eta_\zeta U_\zeta$ and define $\eta_\pm = \frac{1}{2}(\eta_\beta \pm \eta_\zeta)$, so that $\eta_\beta U_\beta + \eta_\zeta U_\zeta = \eta_+(U_\beta + U_\zeta) + \eta_-(U_\beta - U_\zeta)$. Assumption **A2** and the Birman-Schwinger argument yield

$$\begin{aligned} \|U_\beta \frac{1}{H - E} U_\zeta\| &\leq b_+ \|(U_\beta + U_\zeta)^{1/2} \frac{1}{H - E} (U_\beta + U_\zeta)^{1/2}\| \\ &= b_+ \left\| [\widehat{K}_{E,\eta_-}^{-1} + \lambda\eta_+]^{-1} \right\| , \end{aligned} \tag{6.16}$$

where $\widehat{K}_{E,\eta_-} = (U_\beta + U_\zeta)^{1/2} (\hat{H} + \lambda\eta_-(U_\beta - U_\zeta) - E)^{-1} (U_\beta + U_\zeta)^{1/2}$ in $L^2((\operatorname{supp} U_\beta \cup \operatorname{supp} U_\zeta) \cap B_\alpha^L)$.

Now $\eta_+ \in [0, \infty)$ and with η_- fixed one can use the argument described in Section 3.1 to show that

$$\left| \left\{ \eta_+ \in [0, \infty) \ : \ \|[\widehat{K}_{E,\eta_-}^{-1} + \lambda\eta_+]^{-1}\| > t \right\} \right| \leq \frac{(1 + \xi_{E,\eta_-})}{\lambda t} , \tag{6.17}$$

where ξ_{E,η_-} may be expressed as a spectral shift function:

$$\begin{aligned} \xi_{E,\eta_-} \ \leq \ & \operatorname{Tr} \left[P(\widehat{H} + \lambda\eta_-(U_x - U_y) < E) \right] \\ & - \operatorname{Tr} \left[P(\widehat{H} + \lambda\eta_-(U_x - U_y) + \lambda(U_x + U_y) < E) \right] . \end{aligned} \tag{6.18}$$

Since both Schrödinger operators appearing in eq. (6.18) are positive perturbations of H_0, we may bound each trace by a Weyl-type bound (see eq. (3.8)) to yield

$$\xi_{E,\eta_-} \ \leq \ \operatorname{Tr} P(H_0 < E) \ \leq \ \text{const.}\ (1 + |E - E_0|)^{d/2} L^d . \tag{6.19}$$

Averaging over η_+, η_- as in the proof of Prop. 4.2 we obtain the following bound:

$$\mathrm{Prob}\left(\left\| U_\beta \frac{1}{H^{(B^L_\alpha)} - E} U_\zeta \right\| > t | \mathcal{F}^c_{\beta,\zeta}\right) \ \le\ \text{const.}\ D^2\ L^d\ \frac{(1 + |E_1 - E_0|)^{d/2}}{\lambda t}\ , \quad (6.20)$$

for any $\alpha, \beta, \zeta \in \mathcal{I}$ and $E \in [E_0, E_1]$, from which eq. (6.15) follows. □

6.3. **Localization via density of states bounds.** To state the bounds in this section, which are based on the notion of the *density of states*, it is necessary to assume that the random operator H_ω is *ergodic*, i.e. its distribution is invariant under a sufficiently large group of translations. In particular this implies that the spectrum of H_ω is a non-random set, c.f. [51, 64]. Hence, throughout we make the additional assumption

A4: $\mathcal{I}$ is a lattice containing the origin, $U_\alpha = U(\cdot - \alpha)$, and H_ω is ergodic with respect to shifts in $\mathcal{I}$.

We will drop the supremum over α in eq. (1.14) and work with $\alpha = 0$, since if **A4** holds eq. (1.14) is equivalent to

$$b_s(\lambda, E)(1 + L)^{2(d-1)} \limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\left\| \chi_0\, \frac{1}{H^{(B^L_0)} - E - i\varepsilon}\, \mathbf{1}_{\delta B^L_0} \right\|^s\right)\ <\ 1\ , \quad (6.21)$$

where

$$b_s(\lambda, E)\ :=\ M_{s,\lambda}(1 + 1/\lambda)^{2s}(1 + |E - E_0|)^{5s(d+2)}\ . \quad (6.22)$$

For simplicity, we shall assume that the lattice $\mathcal{I} = \mathbb{Z}^d$ and work throughout with the 'ℓ^∞-norm' on $\mathbb{R}^d$: $|x| = \max_j |x_j|$. Thus balls are cubes with sides parallel to the co-ordinate axes and unit balls are fundamental cells for the lattice $\mathcal{I}$. The reader should have no problem extending the results below to more general lattices, in which case it is natural to work with the 'ℓ^∞-norm' induced on $\mathbb{R}^d$ by the decomposition of a vector into components parallel to lattice generators.

The density of states measure for an ergodic random Schrödinger operator is a Borel measure κ on the real line defined through a limiting procedure via its action on compactly supported continuous functions:

$$\int f(t)\kappa(dt) := \lim_{L\to\infty} \frac{1}{(2L)^d}\,\mathrm{Tr}\, f(H^{(\Lambda_L)}) \quad (6.23)$$

where $\Lambda_L = B^L_0 = [-L, L]^d$. For the operators considered here, **A4** implies that this limit exists, is non-random [51], and is equal to

$$\int f(t)\kappa(\mathrm{d}t)\ =\ \lim_{L\to\infty} \frac{1}{(2L)^d}\,\mathrm{Tr}\,\mathbb{E}\left(f(H^{(\Lambda_L)})\right)\ =\ \frac{1}{|\mathcal{C}|}\mathbb{E}\left(\mathrm{Tr}\,\mathbf{1}_{\mathcal{C}} f(H)\right)\ , \quad (6.24)$$

where $\mathcal{C}$ is any unit cell of the lattice $\mathcal{I}$, e.g., $[0, 1]^d$. In addition, the random operators studied here satisfy a Wegner estimate which shows that the density of states measure is absolutely continuous with a density which is in L^p_{loc} for all finite p [17].

We also introduce the finite volume measures

$$\int_A \kappa_L(\mathrm{d}t)\ =\ \mathbb{E}\left(\mathrm{Tr}\, P_A(H^{(\Lambda_L)})\right)\ . \quad (6.25)$$

Again κ_L is absolutely continuous with a density in L^p_{loc} [17]. Note that $\lim_L \kappa_L = \kappa$ in the sense of weak convergence in the dual space of the compactly supported continuous functions.

If E is a band edge of the almost sure spectrum of H, e.g. if $E = \inf \sigma(H)$, then it is generally expected that $\kappa[E, E + \Delta E]$ vanishes to high very order in ΔE. This phenomenon was first observed by I. M. Lifshitz [46] and the regions where such estimates hold are called 'Lifshitz tails'.

In the Lifshitz tail regime one expects it is very rare to find an eigenvalue of a finite volume operator near the band edge E. With favorable estimates this suggests localization should hold near E since the Combes-Thomas estimate [18, 11] may typically be used to obtain resolvent decay across finite volumes.

One result in this vein shows that smallness bounds for the finite volume measures κ_L may be used to verify the localization criterion in Theorem 1.2 in a similar way as done for initial length scale estimates in proofs of band edge localization via multiscale analysis.

Theorem 6.3. *Let H satisfy* **IAD**, **A1-4**. *Suppose that for some $\beta \in (0, 2)$, $\xi > 2(d-1)$, $C_1 > 0$, $C_2 > 0$, and $E \in \mathbb{R}$ it holds for sufficiently large L that*

$$\kappa_L\left([E - C_1 L^{-\beta}, E + C_1 L^{-\beta}]\right) \ < \ C_2 L^{-\xi - d} \,. \tag{6.26}$$

Then the conclusion of Theorem 6.1 holds.

Proof. The proof is very similar to the proof of Theorem 6.1. This time, define $\Omega_G := \{\omega : \operatorname{dist}(\sigma(H^{(\Lambda_L)}), E) > C_1 L^{-\beta}\}$ and $\Omega_B = \Omega_G^c$. Then, using eq. (6.26)

$$\begin{aligned} \operatorname{Prob}(\Omega_B) \ &\leq \mathbb{E}\left(\operatorname{Tr} P_{[E - C_1 L^{-\beta}, E + C_1 L^{-\beta}]}(H^{(\Lambda_L)})\right) \\ &= (2L)^d \kappa_L\left([E - C_1 L^{-\beta}, E + C_1 L^{-\beta}]\right) \ \leq \ 2^d C_2 L^{-\xi} \end{aligned} \tag{6.27}$$

The improved Combes-Thomas estimate due to ref. [11] implies that there are $\eta > 0$ and $C < \infty$ such that for $\omega \in \Omega_G$,

$$\left\| \chi_0 \frac{1}{H^{(\Lambda_L)} - E} \mathbf{1}_{\delta \Lambda_L} \right\| \leq C L^{\beta} e^{-\eta L^{1-\beta/2}} \,. \tag{6.28}$$

This gives a bound for the analogue of the first term on the right hand side of eq. (6.10). From here on the proof is identical to the proof of Theorem 6.1. $\square$

There are two well-known situations in which bounds of the form (6.26) can be derived. One is the aforementioned Lifshitz-tail regime, which generally holds if the background operator H_0 is $\mathcal{I}$-periodic and E is the infimum of the almost sure spectrum. In this case one gets eq. (6.26) for arbitrary $\xi > 0$, see e.g. [16, 39]. Conditions on the periodic background operator for the appearance of this regime at more general band edges were given by F. Klopp [40].

It is not known if the conditions used in ref. [40] hold for general periodic H_0. But one always has the second option of "forcing" a bound like eq. (6.26) to hold by assuming the distribution density ρ of the random coupling constants η_α has small tails near the boundary of their support. For example, one gets eq. (6.26) with $\xi \in (0, 2\tau - d)$ at a lower band edge E_0 of the almost sure spectrum if $\int_0^h \rho(q)\, \mathrm{d}q = \mathcal{O}(h^\tau)$ for $h \to 0$, see [39, Prop. 4.1]. Thus Theorem 6.3 is applicable if $\tau > (3d-2)/2$.

To complement the above discussion, it is interesting to note that one may prove localization directly from an estimate on the infinite volume density of states.

Theorem 6.4. *Let H satisfy* **IAD**, **A1-4**. *For each $\xi > 3d - 1$ and $E \in \mathbb{R}$, there exists $C = C(E, \xi, \lambda) > 0$ and $\delta_0 = \delta(E, \xi, \lambda)$ such that if for some $\delta < \delta_0$*

$$\kappa([E - \delta, E + \delta]) \ \leq \ C\, \delta^{\xi} \,, \tag{6.29}$$

then there exist $s \in (0,1)$, $A < \infty$, and $\mu > 0$ such that for any open set $\Omega \subset \mathbb{R}^d$

$$\limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\left\| \chi_x \frac{1}{H^{(\Omega)} - E' - \mathrm{i}\varepsilon} \chi_y \right\|^s \right) \;\leq\; A\, \mathrm{e}^{-\mu\, \mathrm{dist}_\Omega(x,y)} \;, \tag{6.30}$$

for every $E' \in [E - \delta/4, E + \delta/4]$.

Proof. Klopp has shown that the densities of states for periodic approximations to a random Schrödinger operator approach the infinite volume density of states extremely rapidly, c.f. refs. [40, 41]. We shall adapt his argument to use eq. (6.29) to bound the probability that a certain finite volume operator with random quasi-periodic boundary conditions has spectrum in the interval $[E-\delta/2, E+\delta/2]$. Together with arguments in the proof of Thms. 6.1 and 6.3 this will imply the theorem via an analogue of Theorem 1.2 in which the smallness criterion is satisfied for a quasi-periodic finite volume Hamiltonian. That result may be proved in exactly the same way as Theorem 1.2 since, as remarked in the introduction, quasi-periodic boundary conditions preserve all the properties implied by **A1** which were needed in the proof.

We begin by introducing a sequence of periodic approximations to H. For each $\ell \in \mathbb{N}$ greater than r_o, define a random potential V_ℓ^P which is periodic under translations in $\ell\mathcal{I}$:

$$V_\ell^P(q) \;:=\; \sum_{\alpha \in \mathcal{I} \cap \Lambda_{\ell/2}} \eta_\alpha \sum_{\zeta \in \ell\mathcal{I}} U(q - \alpha - \zeta) \;, \tag{6.31}$$

and note that for any set $\Lambda \subset \mathbb{R}^d$ with diameter less than $\ell - r_o$ the random functions $V_\ell^P(q)\mathbf{1}_\Lambda(q)$ and $V(q)\mathbf{1}_\Lambda(q)$ are identically distributed. We define

$$H_\ell^P \;:=\; H_0 \;+\; V_\ell^P \;. \tag{6.32}$$

Then H_ℓ^P is periodic under shifts in $\ell\mathcal{I}$ and its distribution is invariant under shifts in $\mathcal{I}$.

The averaged density of states measure for H_ℓ^P, denoted κ_ℓ^P, is defined to be

$$\kappa_\ell^P(A) \;:=\; \frac{1}{\ell^d} \mathbb{E}\left(\mathrm{Tr}\, \mathbf{1}_{\Lambda_{\ell/2}} P_A(H_\ell^P) \right) \;. \tag{6.33}$$

By translation invariance of the distribution of H_ℓ^P, this is also given by

$$\kappa_\ell^P(A) \;=\; \mathbb{E}\left(\mathrm{Tr}\, \mathbf{1}_{\mathcal{C}} P_A(H_\ell^P) \right) \;, \tag{6.34}$$

with $\mathcal{C} = [0,1]^d$.

In the present situation, the arguments of ref. [41] may be adapted to show:

For each $n > 0$ there exists $C_n = C_n(E)$ such that

$$\kappa_\ell^P([E-\delta/2, E+\delta/2]) \;\leq\; \kappa([E-\delta, E+\delta]) \;+\; C_n\, \ell^{d+1-n} \delta^{-n} \;. \tag{6.35}$$

Since our assumptions differ somewhat from those ref. [41], let us describe the proof of eq. (6.35). In the following we use $C(E)$ to denote a generic energy and dimension dependent parameter whose value does not depend on the length scale ℓ but may change from line to line. Choose a C^∞ function f with $\mathbf{1}_{\mathcal{J}_{\delta/2}} \leq f \leq \mathbf{1}_{\mathcal{J}_\delta}$ where $\mathcal{J}_t := [E-t, E+t]$. Then

$$\kappa_\ell^P(\mathcal{J}_{\delta/2}) \;\leq\; \mathbb{E}\left(\mathrm{Tr}\, \mathbf{1}_{\mathcal{C}} f(H_\ell^P) \right) \;\leq\; \kappa(\mathcal{J}_\delta) + \mathbb{E}\left(\mathrm{Tr}\, \mathbf{1}_{\mathcal{C}} (f(H_\ell^P) - f(H)) \mathbf{1}_{\mathcal{C}} \right) \;. \tag{6.36}$$

Using the Helffer-Sjöstrand formula [32, 20], write

$$\begin{aligned}\mathbf{1}_{\mathcal{C}}(f(H_\ell^P) - f(H))\mathbf{1}_{\mathcal{C}} \\ = \int_{\mathcal{J}_\delta\times[-1,1]} \left(\partial_{\bar z}\widetilde{f}_n(z)\right) \mathbf{1}_{\mathcal{C}} \left(\frac{1}{z-H_\ell^P} - \frac{1}{z-H}\right)\mathbf{1}_{\mathcal{C}}\, \mathrm{d}x\mathrm{d}y \ , \end{aligned} \tag{6.37}$$

where $z = x + \mathrm{i}y$ and $\widetilde{f}_n$ is an "almost-analytic" extension of f which vanishes to order n at $y = 0$:

$$\widetilde{f}_n(x+\mathrm{i}y) \ = \ \left[\sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(x)(\mathrm{i}y)^j\right] \sigma(y) \ . \tag{6.38}$$

Here σ is a fixed cut-off function supported in $[-1, 1]$ and identically one in a neighborhood of zero. The key property here is that $\partial_{\bar z}\widetilde{f}(x + \mathrm{i}y) = \mathcal{O}\left(|y|^n/\delta^{n+1}\right)$ as may be verified directly.

The difference of resolvents appearing in eq. (6.37) can be expressed in terms of the geometric resolvent identity eq. (5.5):

$$\mathbf{1}_{\mathcal{C}}\left(\frac{1}{z-H_\ell^P} - \frac{1}{z-H}\right)\mathbf{1}_{\mathcal{C}} \ = \ \mathbf{1}_{\mathcal{C}}\frac{1}{z-H_\ell^P}\,[H,\Theta]\,\frac{1}{z-H}\mathbf{1}_{\mathcal{C}} \tag{6.39}$$

with Θ any function which is identically one in a neighborhood of $\mathcal{C}$ and supported inside $\Lambda_{\ell/2-r_o}$. We choose Θ with $\boldsymbol{\nabla}\Theta$ supported in a strip of width one near the boundary of $\Lambda_{\ell/2-r_o}$.

Using a "commutator argument" similar to that in the proof of Lemmas 4.3 and 4.4, we may show that for $z \in \mathcal{J}_\delta \times \{[-1, 0) \cup (0, 1]\}$,

$$[H,\Theta]\,\frac{1}{z-H}\mathbf{1}_{\mathcal{C}} \ = \ T\Psi\frac{1}{z-H}\mathbf{1}_{\mathcal{C}} \ , \tag{6.40}$$

where T is a trace class operator with

$$\mathrm{Tr}\,|T| \ \le \ C(E)\ \ell^{d-1} \tag{6.41}$$

and Ψ is a function which is identically one on the support of $\boldsymbol{\nabla}\Theta$ and also supported in a strip of width order one near the boundary of $\Lambda_{\ell/2-r_o}$. Therefore

$$\begin{aligned}\left|\mathrm{Tr}\,1_{\mathcal{C}}\frac{1}{z-H_\ell^P}\,[H,\Theta]\,\frac{1}{z-H}\mathbf{1}_{\mathcal{C}}\right| &\le C(E)\ell^{d-1}\ \left\|1_{\mathcal{C}}\frac{1}{z-H_\ell^P}\Psi\right\|\ \left\|\Psi\frac{1}{z-H}\mathbf{1}_{\mathcal{C}}\right\| \\ &\le C(E)\ \ell^{d-1}\ \frac{1}{y^2}\mathrm{e}^{-\mu(E)y\ell} \ ,\end{aligned} \tag{6.42}$$

where we have used the Combes-Thomas bound [18, 11] to estimate the resolvent norms.

Putting this into eq. (6.37) we find that

$$\begin{aligned}\left|\mathrm{Tr}\,\mathbf{1}_{\mathcal{C}}(f(H_\ell^P) - f(H))\mathbf{1}_{\mathcal{C}}\right| \ &\le C(E)\int_{\mathcal{J}_\delta\times[-1,1]} \frac{y^n}{\delta^{n+1}}\ell^{d-1}\ \frac{1}{y^2}\mathrm{e}^{-\mu(E)y\ell}\,\mathrm{d}x\mathrm{d}y \\ &\le C(E,n)\,\ell^{d-n}\delta^{-n} \ ,\end{aligned} \tag{6.43}$$

which completes the proof of eq. (6.35).

We now recall some standard facts from the Bloch/Floquet theory of periodic operators, c.f. [53]. For each $\mathbf{k} \in BZ$ with $BZ = [0, 2\pi]^d$ define the restriction of H_ℓ^P to $\Lambda_{\ell/2}$ with quasi-periodic boundary conditions at quasi-momentum $\mathbf{k}/\ell$ to be the self-adjoint operator $H_{\ell;\mathbf{k}}$ on $L^2(\Lambda_{\ell/2})$ that agrees with H when applied to functions compactly supported in the interior of $\Lambda_{\ell/2}$ and whose domain includes all functions on $\Lambda_{\ell/2}$ of the form $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x/\ell}\phi(x)$ with $\phi(x)$ smooth and periodic.

It is well known that the periodic density of states κ_ℓ^P may be obtained as an average of the densities for $H_{\ell;\mathbf{k}}$:

$$\kappa_\ell^P(\mathcal{J}_{\delta/2}) \;=\; \frac{1}{(2\pi)^d}\int_{BZ}\kappa_{\ell;\mathbf{k}}(\mathcal{J}_{\delta/2})\,\mathrm{d}\mathbf{k}\;, \tag{6.44}$$

where the quasi-periodic densities $\kappa_{\ell;\mathbf{k}}$ are defined by

$$\kappa_{\ell;\mathbf{k}}(A) \;:=\; \frac{1}{\ell^d}\mathbb{E}\left(\mathrm{Tr}\,P_A(H_{l;\mathbf{k}})\right)\;. \tag{6.45}$$

Now consider the probability space $\Omega' = \Omega\times BZ$ with associated measure $\mathrm{Prob}' = \mathrm{Prob}\,\times\,\mathrm{d}\mathbf{k}/(2\pi)^d$, and let $H_\ell = H_{\ell;\omega'}$ be a random Schrödinger operator with $\omega' = (\omega,\mathbf{k}) \in \Omega'$ distributed according to Prob$'$. As in the proof of Thms. 6.1 and 6.3, we define complementary good and bad sets, $\Omega_G := \{\omega' : \mathrm{dist}(\sigma(H_{\ell;\omega'}),E) > \delta/2\}$ and $\Omega_B = \Omega'\setminus\Omega_G$.

Using eq. (6.44) and eq. (6.35) we see that

$$\begin{aligned}\mathrm{Prob}'(\Omega_B) \;\le\; \mathbb{E}'\left(\mathrm{Tr}\,P_{\mathcal{J}_{\delta/2}}(H_{\ell;\omega'})\right) \;&=\ell^d\kappa_\ell^P(\mathcal{J}_{\delta/2})\\ &\le \ell^d\kappa(\mathcal{J}_\delta) \;+\; C_n\ell^{2d-n}\delta^{-n}\;.\end{aligned} \tag{6.46}$$

On the other hand, the improved Combes-Thomas bound [11] shows that there are $A<\infty$ and $\mu>0$ such that for $\omega'\in\Omega_G$ and $E'\in\mathcal{J}_{\delta/4}$

$$\left\|\chi_0\frac{1}{H_{\ell;\omega'}-E'}\delta\Lambda_\ell\right\| \;\le\; A\,\delta^{-1}\,\mathrm{e}^{-\mu\delta^{1/2}\ell}\;. \tag{6.47}$$

Arguing as in the proof of Theorem 6.1 this implies

$$\begin{aligned}\mathbb{E}'\left(\left\|\chi_0\frac{1}{H_{\ell;\omega'}-E'}\delta\Lambda_\ell\right\|^s\right) \;&\le\; A^s\,\delta^{-s}\,\mathrm{e}^{-s\mu\delta^{1/2}\ell}\\ &+\; C(t,\lambda,E)^{s/t}\ell^{(2+s)(d-1)}\left(\ell^d\kappa(\mathcal{J}_\delta) \;+\; C_n\ell^{2d-n}\delta^{-n}\right)^{1-s/t}\;,\end{aligned} \tag{6.48}$$

for any $s<t<1$. Thus

$$\ell^{2(d-1)}\mathbb{E}'\left(\left\|\chi_0\frac{1}{H_{\ell;\omega'}-E'}\delta\Lambda_\ell\right\|^s\right) \;\le\; C\,\max(A_1,A_2,A_3) \tag{6.49}$$

with

$$\begin{aligned}A_1 \;&= \delta^{-s}\ell^{2(d-1)}\,\mathrm{e}^{-s\mu\delta^{1/2}\ell}\;,\\ A_2 \;&= \ell^{(2+s)(d-1)+(1-s/t)d}\kappa(\mathcal{J}_\delta)^{1-s/t}\;,\\ A_3 \;&= \ell^{(2+s)(d-1)+(1-s/t)(2d-n)}\delta^{-n(1-s/t)}\;.\end{aligned} \tag{6.50}$$

By the analogue of Theorem 1.2 discussed above, in which the Dirichlet operator $H^{(\Lambda_\ell)}$ is replaced by the random quasi-periodic operator H_ℓ, we see that there is fixed quantity $B = B(s,\lambda,E)$ such that if $\max_j(A_j) < B$ then the conclusion of the present theorem holds, i.e., the interval $\mathcal{J}_{\delta/4}$ is contained in the localization regime. Note that the infinite volume operator does not depend on the quasi-momentum $\mathbf{k}$. Nonetheless, the finite volume quasi-periodic operators may be used because locally, i.e., for functions supported in the interior of the cube Λ_ℓ, they agree with the infinite volume operator.

To obtain a concrete result, we let $\ell \approx \delta^{-r}$ for some $r > 1$. Then $A_1 < B$ and $A_3 < B$ for all sufficiently small δ provided we choose n sufficiently large. Thus, there is $\delta_0 > 0$ such that for $\delta < \delta_0$ the condition $\max_j(A_j) < B$ is equivalent to requiring that

$$A_2 = \delta^{-\xi(1-s/t)}\kappa(\mathcal{J}_\delta)^{1-s/t} \; < \; B \;, \tag{6.51}$$

where $\xi = rd + r(d-1)(2+s)/(1-s/t)$. With the appropriate choice of $0 < s < t < 1$ and $r > 1$ we can obtain any $\xi > 3d - 1$. This completes the proof of Theorem 6.4. $\square$

Appendix A. Technical comments

Following are some comments of technical nature, which are intended to supplement the discussion of the assumptions and results stated in the introduction.
(1) *(The operator nature of H_ω)* Under the stated assumptions, H_ω is essentially self adjoint on C_o^∞ [44]. The random potential $V_\omega(q)$ is non-negative and uniformly bounded by λb_+. The operator H_ω is bounded below, with $\sigma(H_\omega) \subset [E_0, \infty)$.
(2) For $1 \leq p < \infty$ let $\mathcal{I}_p$ denote the Schatten class of order p, i.e., the ideal of bounded operators A on $L^2(\mathbb{R}^d)$ with $\|A\|_p := (\mathrm{Tr}\,|A|^p)^{1/p} < \infty$. Then for all z in the resolvent set of H, $\rho(H)$, and all $f \in L^p(\mathbb{R}^d)$ with $p > d/2$,

$$f(q)(H_\omega - z)^{-1} \in \mathcal{I}_p \tag{A.1}$$

Also, for any $a > -E_0$,

$$f(q)(H_\omega + a)^{-r} \in \mathcal{I}_p \tag{A.2}$$

if $f \in L^p(\mathbb{R}^d)$ and $r > d/2p$. Proofs can be found, for example, in refs. [58, 60]. Under the assumptions **A2** and **A3**, which imply bounds on the random potential $V_\omega(q)$, the Schatten class bounds hold uniformly in ω.
(3) Weak solutions of $H\varphi = z\varphi$, $z \in \mathbb{C}$, have the *unique continuation property*, i.e. if φ vanishes on a non-empty open set, then φ vanishes identically. This follows from ref. [35] which requires only local form-boundedness of V_o^2, $\boldsymbol{A}^2$, and $(\boldsymbol{\nabla} \wedge \boldsymbol{A})^2$ with respect to the Laplacian. There is quite a bit of literature on unique continuation allowing more general $\boldsymbol{A}$ and V; c.f., [13, 60, 34, 42, 68, 69].
(4) The above three properties hold also for the restrictions of H to any open set Ω with Dirichlet boundary conditions and for the restriction to any cube $\Lambda_{x,L} := x + [-L/2, L/2]^d$ with Neumann or quasi-periodic boundary conditions (see Section 6.3).
(5) *(The regularization by* i0*)* For *unbounded* regions Ω one knows that the operator norm limit

$$\chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}0)^{-1}\chi_y := \lim_{\varepsilon \downarrow 0} \chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1}\chi_y \tag{A.3}$$

exists almost surely for almost every $E \in \mathbb{R}$. This follows from Fubini's theorem and the fact that, for fixed ω, $\chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}0)^{-1}\chi_y$ exists for almost every E. To prove the latter one writes

$$\begin{aligned}\chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1}\chi_y &= \chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1}P_I(H_\omega^{(\Omega)})\chi_y \\ &\quad + \chi_x(H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1}P_{\mathbb{R}\setminus I}(H_\omega^{(\Omega)})\chi_y,\end{aligned} \tag{A.4}$$

where I is a bounded interval and we denote the spectral projection onto a measurable set M for a self-adjoint operator H by $P_M(H)$. If E is in the interior of I, then the second term in the sum trivially has a limit. For the first term, a polarization argument shows that it suffices to consider $G_A(E + \mathrm{i}\varepsilon) := \mathbf{1}_A(H_\omega^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1}P_I(H_\omega^{(\Omega)})\mathbf{1}_A$ for bounded regions A. The operator function $z \mapsto G_A(z)$ defined for z in the upper half plane $\mathbb{C}_+$ is trace class valued (see eq. (3.8)) and analytic with non-negative imaginary part. It follows from a result of de Branges [22], also used in Appendix B, that the limit exists for almost every E. To conclude one exhausts $\mathbb{R}$ with bounded intervals I.

By eq. (A.3) we can extend the bound (1.7) to $\mathbb{E}(\|\chi_x(H^{(\Omega)} - E - \mathrm{i}0)^{-1}\chi_y\|^s)$ for almost every E. However, it is convenient for us to have eq. (1.7) for *all* E, which is why we prefer to state it with the supremum over $\varepsilon > 0$ outside the expectation.

(6) *(The removal of* i0 *for bounded domains)* If we fix a *bounded* region Ω, then any fixed $E \in \mathbb{R}$ is almost surely not in $\sigma(H^{(\Omega)})$, as follows from the assumptions on the distribution of the random couplings stated in Section 1.2 (via analytic perturbation theory and unique continuation of eigenfunctions). Thus, $\left\| \chi_x (H^{(\Omega)} - E)^{-1} \chi_y \right\|$ is an almost surely finite random variable, and Fatou's lemma shows that the fractional moment bound (1.7) on $\mathbb{E}(\| \chi_x (H^{(\Omega)} - E - \mathrm{i}\varepsilon)^{-1} \chi_y \|^s)$ implies also such a bound for $\varepsilon = 0$.
(7) *(Results for large disorder regime)* Readers familiar with fractional-moment methods for discrete random operators will likely note the bound we obtain in Lemma 1.1 is a bit weaker than the *a-priori* bound derived in that context, which falls off like λ^{-s} for large λ. We show below – in the proof of Theorem 6.2 – that the s-moments of the resolvent of $H_\omega^{(\Omega)}$ for a bounded region Ω are $\mathcal{O}(|\Omega|^s / \lambda^s)$. Coupled with Theorem 1.2 this will allow us to conclude localization at "large disorder." However, we do not show here that the fractional-moments of the infinite volume resolvent tend to zero for large λ (although this may still be true).
(8) *(Possible coexistence of bulk localization with extended boundary states)* An operator may exhibits localization in the bulk (in terms of transition amplitides) along with extended *boundary states* occurring in certain geometries. Such situations have been studied and are of particular interest for the Quantum Hall Effect, with H_o the Landau operator ([21, 24]). Our use of the domain-adapted metric, dist_Ω - in which exponential decay is compatible with the above picture, allows the analysis of localization to proceed even in such cases. However, it is also possible to formulate other finite volume criteria which rule out extended surface states. The input conditions need to be more restrictive and involve propagators between boundary regions in arbitrary geometries. For this purpose one may present a modified version of Theorem 1.1, changed in a manner similar to what was done for discrete models in ref. [6, Theorem 1.1]. A key point is that the domain-adapted metrics can be replaced in (1.10) by the usual distance, in which case the conclusions (eq. (1.11) and pure point spectrum) hold in any sufficiently regular region and in particular, under the stronger assumptions, rule out also extended boundary states.
(9) *(Energy dependence of the bounds)* We note that Theorem 1.1 only requires exponential decay of the energy-averaged Green function. However, typically this will be established, for example by Theorem 1.2, through a bound which is uniform in energy.
(10) *(The use of trace norm)* While operator norm bounds might be considered sufficient for applications, we note that eq. (1.11) holds as a trace norm bound for the quantity on the left. This extension naturally follows from our proof (see Section 3), which uses term by term bounds on sums of rank one operators, and the latter have equal operator and trace norms.
(11) *(The assumption* **A3'***)* The condition of blow-up regularity of a random variable X requires its probability density to be absolutely continuous, with a bounded density $\rho(\cdot)$, since eq. (1.22) implies (for arbitrary n)

$$\rho(x) \;=\; n \int \rho_n(nx - y|y) \mu_n(\mathrm{d}y) \,, \tag{A.5}$$

where μ_n is the probability distribution of $Y^{(n)}$. The converse is not true: there exist bounded densities such that the associated probability measure has infinite blow up norm. However, if $\ln\rho$ is Lipschitz-continuous then $D_\rho < \infty$. In this case, a particularly simple decomposition of the random variable X with distribution $\rho(x)\mathrm{d}x$ is obtained with $X^{(n)}$ the fractional part of nX, so that $Y^{(n)}$ is integer valued and

$$\rho_n(x|j) \ := \ \frac{\rho(j/n + x/n)}{n\int_{j/n}^{(j+1)/n} \rho(x)\mathrm{d}x} \ . \tag{A.6}$$

The Lipschitz condition guarantees the uniform boundedness of ρ_n defined in this way. A particularly simple example, for which the results are already of interest, is provided by the uniform distribution in $[0, 1]$.

Appendix B. A "weak L^1" bound for resolvents of dissipative operators

The goal of this appendix is to provide a proof of Lemma 4.1. All the main arguments are taken from [49].

We will use here that a maximally dissipative operator A in $\mathcal{H}$ has a selfadjoint dilation L in a Hilbert space $\tilde{\mathcal{H}}$ which contains $\mathcal{H}$ as a subspace, i.e.

$$(A-\xi)^{-1} = P(L-\xi)^{-1}P^* \tag{B.1}$$

for every $\xi \in \mathbb{C}$ with $\operatorname{Im}\xi < 0$. Here P is the orthogonal projection onto $\mathcal{H}$ in $\tilde{\mathcal{H}}$. For this and much more on the general theory of dissipative operators see the survey [52] or the book [65] (where the equivalent theory of contractions and their unitary dilations is presented).

We start the proof of Lemma 4.1 with two reduction steps. First, we show that it is sufficient to deal with the case $M_1 = M = M_2^*$.

Thus, let us assume that it is proven that

$$|\{v \in \mathbb{R} : \|M^*(A-v+i0)^{-1}M\|_{HS} > t\}| \leq \frac{C}{t}\|M\|_{HS}^2 \tag{B.2}$$

for all self-adjoint A in $\mathcal{H}$ and Hilbert-Schmidt operators $M : \mathcal{H}_1 \to \mathcal{H}$.

The estimate (4.8) follows from (B.2) by a polarization and a scaling argument. For this let us temporarily write $T = (A-v+i0)^{-1}$. One checks that

$$\begin{aligned} M_2TM_1 &= \frac{1}{2}(M_2+M_1^*)T(M_2^*+M_1) - \frac{i}{2}(M_2-iM_1^*)T(M_2^*+iM_1) \\ &\quad - \frac{1-i}{2}M_2TM_2^* - \frac{1-i}{2}M_1^*TM_1. \end{aligned} \tag{B.3}$$

All four terms on the r.h.s. of (B.3) are of the type which is covered by (B.2). The set $\{v : \|M_2TM_1\|_{HS} > t\}$ is contained in the union of $\{v : \|\frac{1}{2}(M_2+M_1^*)T(M_2^*+M_1)\|_{HS} > t/4\}$ and three similar sets. Applying (B.2) to all of them gives

$$|\{v : \|M_2TM_1\|_{HS} > t\}| \leq \frac{C_1}{t}(\|M_2\|_{HS}^2 + \|M_1\|_{HS}^2) \tag{B.4}$$

with a suitable constant C_1. Scaling of (B.4) yields

$$\begin{aligned} |\{v : \|M_2TM_1\|_{HS} > t\}| &= \left|\left\{v : \left\|\frac{M_2}{\|M_2\|_{HS}}T\frac{M_1}{\|M_1\|_{HS}}\right\|_{HS} > \frac{t}{\|M_1\|_{HS}\|M_2\|_{HS}}\right\}\right| \\ &\leq \frac{2C_1}{t}\|M_1\|_{HS}\|M_2\|_{HS}, \end{aligned} \tag{B.5}$$

and thus (4.8).

It remains to show (B.2), which follows if we can show that

$$|\{v \in \mathbb{R} : \|M^*(A-v+i\delta)^{-1}M\|_{HS} > t\}| \leq \frac{C}{t}\|M\|_{HS}^2 \tag{B.6}$$

for all $\delta > 0$, with $C < \infty$ independent of δ.

To show that (B.6) implies (B.2), consider the function Φ defined on $\mathbb{C}^- = \{\operatorname{Im}\xi < 0\}$ by $\Phi(\xi) = M^*(A-\xi)^{-1}M$. The function Φ is analytic and takes values in the trace class operators on $\mathcal{H}$ with non-negative imaginary part. By a result of de Branges [22], later proven independently in [9] and [15], $\Phi(v-i0) := \lim_{\delta\downarrow 0}\Phi(v-i\delta)$ exists in Hilbert-Schmidt norm for almost every $v \in \mathbb{R}$. Together with (B.3) this implies the existence statement in part (1) of Lemma 4.1. For $\delta \geq 0$ let g_δ denote the characteristic function of the set $\{v \in \mathbb{R} : \|M^*(A-v+i\delta)^{-1}M\|_{HS} > t\}$.

One checks that $g_0(v) \leq \liminf_{\delta\downarrow 0} g_\delta(v)$ for almost every v. Therefore (B.2) is a consequence of (B.6) and Fatou's lemma.

Before we proceed with the remaining proof of (B.6), we state two classical facts on Hilbert transforms which will be used.

The Hilbert transform of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by the principle-value integral

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{x-y}\, dy, \tag{B.7}$$

whenever this limit exists. The same definition applies when f takes values in a Hilbert space, in which case the r.h.s. of (B.7) is interpreted as a Bochner integral.

Proposition B.1. *Suppose that $\Phi \in H^2(\mathbb{C}^-)$, i.e. $\Phi : \mathbb{C}^- \to \mathbb{C}$ is analytic and*

$$\sup_{y>0} \int_{\mathbb{R}} |\Phi(x-iy)|^2\, dx < \infty. \tag{B.8}$$

Then the boundary value $\Phi(x) = \lim_{y\downarrow 0} \Phi(x-iy)$ exists for almost every $x \in \mathbb{R}$, $\Phi \in L^2(\mathbb{R})$, and its real and imaginary parts are conjugate, i.e.

$$\operatorname{Re}\Phi(x) = H(\operatorname{Im}\Phi)(x) \quad \textit{for a.e. } x \in \mathbb{R}. \tag{B.9}$$

Proposition B.2. *Let $\mathcal{H}$ be a separable Hilbert space and $f \in L^1(\mathbb{R},\mathcal{H})$ in the sense of Bochner integration. Then the Hilbert transform $Hf(y) \in \mathcal{H}$ exists for a.e. $y \in \mathbb{R}$, and there exists a constant $C < \infty$, independent of f, such that for all $t > 0$*

$$|\{y \in \mathbb{R} : \|Hf(y)\|_{\mathcal{H}} > t\}| \leq \frac{C}{t} \int_{\mathbb{R}} \|f(x)\|_{\mathcal{H}}\, dx. \tag{B.10}$$

A modern proof of Proposition B.1 can be found in [27]. Proposition B.2, i.e. the weak-L^1-property of the Hilbert transform, is well known for the case $\mathcal{H} = \mathbb{C}$. A proof in the context of more general Calderon-Zygmund inequalities can be found in [63, Chapter 2], where it is remarked that the result extends to the $\mathcal{H}$-valued case. Detailed proofs of such results for vector-valued functions, which contain Proposition B.2 as a special case, can be found in [55].

We now apply these facts to $T_\delta(v) := M^*(A - v + i\delta)^{-1}M$. The trace class norm will be denoted $\|\cdot\|_1$. The real and imaginary parts of bounded operators are defined as usual by $\operatorname{Re} C = \frac{1}{2}(C + C^*)$ and $\operatorname{Im} C = \frac{1}{2i}(C - C^*)$.

Lemma B.3. *For every $\delta > 0$ it holds that*

$$\int_{\mathbb{R}} \|\operatorname{Im} T_\delta(v)\|_{HS}\, dv \leq \int_{\mathbb{R}} \|\operatorname{Im} T_\delta(v)\|_1\, dv = \pi \|M\|_{HS}^2. \tag{B.11}$$

Proof. The first part of (B.11) follows from $\|\cdot\|_{HS} \leq \|\cdot\|_1$. Let $E(t)$ be the spectral resolution of the selfadjoint dilation L of A and $\phi \in \mathcal{H}_1$. Then by (B.1), the spectral theorem, and Fubini

$$\begin{aligned} \int \langle \operatorname{Im} T_\delta(v)\phi, \phi\rangle_{\mathcal{H}_1}\, dv &= \int\int \frac{\delta}{(x-v)^2 + \delta^2}\, d\|E(x)P^*M\phi\|_{\tilde{\mathcal{H}}}^2\, dv \\ &= \pi \|M\phi\|_{\mathcal{H}_1}^2. \end{aligned} \tag{B.12}$$

Let (ϕ_n) be an orthonormal basis in $\mathcal{H}_1$. We have $\operatorname{Im} T_\delta(v) \geq 0$ and thus by (B.12)

$$\begin{aligned} \int \|\operatorname{Im} T_\delta(v)\|_1 \, dv &= \int \operatorname{Tr}(\operatorname{Im} T_\delta(v)) \, dv \\ &= \int \sum_n \langle \operatorname{Im} T_\delta(v)\phi_n, \phi_n \rangle_{\mathcal{H}_1} \, dv \\ &= \pi \|M\|_{HS}^2. \end{aligned} \tag{B.13}$$

□

Lemma B.4. *Let $\mathcal{H}_{HS}$ denote the separable Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}_1$.*

(a) If $\phi \in \mathcal{H}_1$, then $\langle T_\delta(\cdot)\phi, \phi\rangle \in H^2(\mathbb{C}^-)$ for each fixed $\delta > 0$.

(b) For almost every $v \in \mathbb{R}$ one has

$$\operatorname{Re} T_\delta(v) = H(\operatorname{Im} T_\delta)(v) \tag{B.14}$$

in the sense of Hilbert transforms of $\mathcal{H}_{HS}$-valued functions.

Proof. Consider arbitrary ϕ and ψ in $\mathcal{H}_1$ and use (B.1) and the spectral theorem to estimate

$$\begin{aligned} \int |\langle T_\delta(v - iy)\phi, \psi\rangle|^2 \, dv &\leq \int \|(L - v - i(y+\delta))^{-1} P^* M\phi\|^2 \, dv \, \|M\psi\|^2 \\ &= \int\int \frac{1}{(x-v)^2 + (y+\delta)^2} \, dv \, d\|E(x)P^*M\phi\|^2 \|M\psi\|^2 \\ &= \frac{\pi}{y+\delta} \|M\phi\|^2 \|M\psi\|^2 \leq \frac{\pi}{\delta} \|M\phi\|^2 \|M\psi\|^2. \end{aligned} \tag{B.15}$$

Summing this over an orthonormal basis of vectors ψ and then over an orthonormal basis of vectors ϕ leads to

$$\int \|T_\delta(v - iy)\|_{HS}^2 \, dy \leq \frac{\pi}{\delta} \|M\|_{HS}^4. \tag{B.16}$$

This implies (a). In fact, we have proven the stronger result that $T_\delta(\cdot)$ is a Hilbert-Schmidt-valued H^2-function in the lower half plane.

By Lemma B.3 and Proposition B.2, $H(\operatorname{Im} T_\delta)$ exists almost everywhere as a Hilbert-Schmidt operator. Since the strong topology is weaker than the Hilbert-Schmidt topology, this implies the existence of $H\langle \operatorname{Im} T_\delta(\cdot)\phi, \phi\rangle = \langle H(\operatorname{Im} T_\delta)(\cdot)\phi, \phi\rangle$ for every $\phi \in \mathcal{H}_1$. By (a) and Proposition B.1 the latter is equal to $\langle \operatorname{Re} T_\delta(\cdot)\phi, \phi\rangle$. We conclude (B.14) since bounded operators are determined by their quadratic form. □

We are now prepared to prove (B.6) and thereby complete the proof of Lemma 4.1. We have, using Lemma B.4(b),

$$\begin{aligned} |\{v : \|T_\delta(v)\|_{HS} > t\}| &\leq |\{v : \|\operatorname{Re} T_\delta(v)\|_{HS} > \frac{t}{2}\}| + |\{v : \|\operatorname{Im} T_\delta(v)\|_{HS} > \frac{t}{2}\}| \\ &= |\{v : \|H(\operatorname{Im} T_\delta)(v)\|_{HS} > \frac{t}{2}\}| + |\{v : \|\operatorname{Im} T_\delta(v)\|_{HS} > \frac{t}{2}\}| \\ &\leq \frac{2(C+1)}{t} \int \|\operatorname{Im} T_\delta(v)\|_{HS} \, dv, \end{aligned} \tag{B.17}$$

where in the end Proposition B.2 and Chebychev's inequality were used. Thus (B.6) is a consequence of Lemma B.3.

Appendix C. A disorder-averaged spectral shift bound

Here we present a new result, which amounts to the boundedness -under the average over local disorder (but at fixed energy !)- of the fractional moment of the spectral shift associated with the addition, to a Schrödinger operator, of a local "potential bump".

As was explained in the introduction, some of the difficulties which have in the past impeded the extention of fractional moment methods to the continuum can be traced to the lack of uniform bounds on such spectral shifts. The following results is enabled by the methods of Section 4. While the analysis presented above does not proceede through this bound, the issues involved are closely related, and the result may provide a useful tool.

Theorem C.1. *Let $H_t = \widehat{H} + tV$ where $\widehat{H}$ satisfies* **A1** *and V is a non-negative bounded function with compact support. Let U be a non-negative bounded function such that V is strictly positive throughout the set $Q = \{q : \mathrm{dist}(q, \mathrm{supp}(U)) < \delta\}$ with some $\delta > 0$ and set $v_- = \inf_{x\in Q} V(x)$. Then, for any $0 < s < \min(2/d, 1/2)$ there is $C_{s,\delta} < \infty$ such that the spectral shift function, defined as*

$$\xi(t,E) \;=\; \mathrm{Tr}\left[P\left(H_t < E\right) - P\left(H_t + U < E\right)\right] \;, \tag{C.1}$$

satisfies, for any $E \geq \inf \sigma(\widehat{H})$:

$$\int_0^1 \left|\xi(t,E)\right|^s \mathrm{d}t \;\leq\; C_{s,\delta}\, \|U\|_\infty \left(1 + |E - E_o| + \|V\|_\infty\right)^{s(2d+2)} \;, \tag{C.2}$$

with $E_o = \inf \sigma(\widehat{H})$.

Proof. We claim that it suffices to prove eq. (C.2) for operators H_t restricted to bounded regions with a constant $C_{s,\delta}$ which is independent of the region. To verify this, note that strong resolvent convergence and lower semi-continuity of the trace norm imply that

$$\xi(t,E) \;\leq\; \liminf_{L\to\infty} \xi_L(t,E) \tag{C.3}$$

where $\xi_L(t,E)$ is computed with $H_t^{(\Lambda_L)_o}$ in place of H_t with $\Lambda_L = [-L, L]^d$. It is usefl to note that, because U is non-negative, the difference of projections appearing in eq. (C.1) is a positive semi-definite operator so its trace is equal to its trace norm. An application of Fatou's lemma yields eq. (C.2) for H_t provided it holds for $H_t^{(\Lambda_L)_o}$.

Throughout the rest of the proof we fix $L > 0$ and write H_t and $\widehat{H}$ for the restrictions of these operators to $[-L, L]^d$ with Dirichlet boundary conditions. We begin with the observation that

$$\xi(t,E) \;=\; \mathrm{Tr}\, P_t \;, \tag{C.4}$$

where P_t is the spectral projection to the interval $(-\infty, -1]$ of the Birman Schwinger operator

$$K_t \;=\; U^{1/2} \frac{1}{H_t - E} U^{1/2} \;. \tag{C.5}$$

This fact follows from the Birman-Schwinger representation since E becomes an eigenvalue of $H_t + \eta U$ precisely when η is equal to an eigenvalue of $-K_t^{-1}$.

Note that for any $n \geq 1$,

$$\left\| K_t^{-n} P_t (1 + K_t^{-1})^n \right\| \;\leq 1 \;. \tag{C.6}$$

Thus by the Hölder inequality for trace norms

$$\xi(t,E) \;\le\; \|K_t^n\|_{HS} \left\|(1+K_t^{-1})^{-n}\right\|_{HS} \;. \tag{C.7}$$

Noting that

$$(1+K_t^{-1})^{-1} \;=\; U^{1/2}\frac{1}{H_t+U-E}U^{1/2} \;, \tag{C.8}$$

we find

$$\int_0^1 \mathrm{d}t\,|\xi(t,E)|^s \;\le\; \left(\int_0^1 \left\|U^{1/2}\frac{1}{H_t-E}U^{1/2}\right\|_{2n}^{2ns} \mathrm{d}t\right)^{1/2} \times \left(\int_0^1 \left\|U^{1/2}\frac{1}{H_t+U-E}U^{1/2}\right\|_{2n}^{2ns} \mathrm{d}t\right)^{1/2} \;, \tag{C.9}$$

where $\|A\|_m = (\mathrm{Tr}\,|A|^m)^{1/m}$.

Lemma 4.1 and the representation eq. (4.39) used in the proof of Lemma 4.3 may be used to show that the integrals on the right hand side of eq. (C.9) are bounded if (1) $2ns < 1$ and (2) $4n > d$ (d is the dimension). We now outline the proof of this assertion.

The arguments used in Lemma 4.3 can be used to produce a representation

$$U^{1/2}\frac{1}{H_t-E}U^{1/2} \;=\; U^{1/2}T\Theta\frac{1}{H_t-E}\Theta T^\dagger U^{1/2} \;+\; B \tag{C.10}$$

with $T, T^\dagger$ Hilbert-Schmidt. In the proof of Lemma 4.3 it was noted that B is bounded. In addition, using **A1**, B can be seen to be in $\mathcal{I}_p$ for any $p > d/2$, with uniform bounds on its $\mathcal{I}_p$ norm. Since $\|\cdot\|_{2n} \le \|\cdot\|_{HS}$ we find that

$$\int_0^1 \left\|U^{1/2}\frac{1}{H_t-E}U^{1/2}\right\|_{2n}^{2ns} \mathrm{d}t \;\le\; \int_0^1 \left\|U^{1/2}T\Theta\frac{1}{H_t-E}\Theta T^\dagger U^{1/2}\right\|_{HS}^{2ns} \;+\; \mathrm{O}(1) \;, \tag{C.11}$$

with a similar expression for the other factor in eq. (C.9). The weak L^1 bound can be used to bound this final integral since

$$\Theta\frac{1}{H_t-E}\Theta \;=\; \Theta V^{-1/2}\frac{1}{t+\widehat{K}^{-1}}V^{-1/2}\Theta \;, \tag{C.12}$$

with appropriate $\widehat{K}$. Note that $\Theta V^{-1/2}$ is bounded since Θ is supported in Q. As in Sec. 4 there is slight complication due to the fact that T (and $T^\dagger$) depend on t. However, as before, the dependence is polynomial and may be handled in the same way. □

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