Boundary maps for C^*-crossed products with $\mathbb{R}$ with an application to the quantum Hall effect

J. Kellendonk[1], H. Schulz-Baldes[2],

[1] School of Mathematics, Cardiff University, Cardiff, CF24 4YH, Wales
[2] Institut für Mathematik, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

Abstract

The boundary map in K-theory arising from the Wiener-Hopf extension of a crossed product algebra with $\mathbb{R}$ is the Connes-Thom isomorphism. In this article, the Wiener Hopf extension is combined with the Heisenberg group algebra to provide an elementary construction of a corresponding map in cyclic cohomology. It then follows directly from a non-commutative Stokes theorem that this map is dual w.r.t. Connes' pairing of cyclic cohomology with K-theory. As an application, we prove equality of quantized bulk and edge conductivities for the integer quantum Hall effect described by continuous magnetic Schrödinger operators.

1 Main result

The main object of this work is to prove equality between pairings of K-group elements and cyclic cocycles of C^*-algebras and their crossed products. Reserving details and elaborations for later, it can be stated as follows.

Theorem 1 *Consider an $\mathbb{R}$-action α on a C^*-algebra $\mathcal{B}$ together with an α invariant cyclic cocycle η on $\mathcal{B}$. Denote by τ the translation action of $\mathbb{R}$ on $\mathbb{R}\cup\{\infty\}$ (with fixed point ∞). This defines a crossed product C^*-algebra $\mathcal{B}\rtimes_\alpha\mathbb{R}$, furnished with a natural derivation $\nabla f(x) = \imath x f(x)$, and an extension of this C^*-algebra by another crossed product, $C_0(\mathbb{R}\cup\{\infty\},\mathcal{B})\rtimes_{\tau\otimes\alpha}\mathbb{R}$, the so-called Wiener-Hopf extension. They form an exact sequence*

$$0 \longrightarrow \mathcal{K}\otimes\mathcal{B} \longrightarrow C_0(\mathbb{R}\cup\{\infty\},\mathcal{B})\rtimes_{\tau\otimes\alpha}\mathbb{R} \stackrel{\mathrm{ev}_\infty}{\longrightarrow} \mathcal{B}\rtimes_\alpha\mathbb{R} \longrightarrow 0\,. \qquad (1)$$

Here $\mathcal{K}$ are the compact operators on $L^2(\mathbb{R})$. If $(\Omega,\int,d)$ is an α-invariant cycle over $\mathcal{B}$ defining the cyclic n-cocycle η, a cyclic $n+1$-cocycle on $\mathcal{B}\rtimes_\alpha\mathbb{R}$ is given by

$$\#_\alpha\eta(f_0,\ldots,f_{n+1}) \;=\; \sum_{k=1}^{n+1}(-1)^k \int \mathrm{ev}_0\left(f_0 df_1\cdots\nabla f_k\cdots df_{n+1}\right)\,.$$

Then for Connes' pairing of cyclic cohomology with K-theory, one has

$$\langle \#_\alpha \eta, x \rangle \;=\; -\frac{1}{2\pi}\,\langle\, \mathrm{Tr}\otimes\eta, \partial_i x\rangle\;, \qquad x \in K_i(\mathcal{B}\rtimes_\alpha \mathbb{R})\;,$$

where ∂_i are the boundary maps, i.e., for odd n, $i=0$ and ∂_0 is the exponential map of K-theory associated to (1) *while, for even n, $i=1$ and ∂_1 is the corresponding index map.*

If we restricted attention to smooth sub-algebras of the C^*-algebras involved, this result could be obtained by combining known facts from [C81, R82, ENN88]. The wider context used here requires to work with unbounded cocycles (n-traces) [C86, C94]. However, even in the restricted context of smooth algebras, we believe that our constructions are more explicit and simpler than those in the literature and therefore more easily accessible to the non-expert in non-commutative topology who might be mostly interested in the application to the quantum Hall effect. To address the latter reader, we provide all essential steps in the proofs and avoid abstract topological arguments, e.g. do not appeal to Bott periodicity to derive our result. This makes this article self-contained.

The abovementioned simplification was possible because we use an approach to the central results of K-theory based on continuous fields of C^*-algebras. We find that these provide a very elegant way to prove Bott periodicity and the Connes-Thom isomorphism, as was shown by Elliott, Natsume and Nest in an article [ENN93] which inspired this work.

Our main motivation is the application of Theorem 1 to the quantum Hall effect and we will develop this in some detail in Section 2. The reader only interested in the above mathematical result can skip Section 2 and start directly with Section 3. Only Examples 3, 7 and 11 as well as Section 6.4 refer to Section 2 later on. However, let us advocate at this point the explicit calculations in Section 2.2 which illuminate the heart of the topological example by constructing an explicit homotopy between the Bott projection and the Landau projection.

2 Topology of the integer quantum Hall effect

In quantum Hall samples, there are two current carrying mechanisms: edge currents flow along the boundaries due to intercepted cyclotron orbits and bulk currents result from the Lorentz drift in presence of an exterior electric field. Important in the present context is that both these currents are topologically quantized by a Fredholm index resulting from a pairing of adequate elements of K-groups and cyclic cocyles. While for bulk currents, this was known for a long time [B85, AS85, K87, B88, ASS94, BES94], quantization of edge currents was only proven more recently and under a gap condition on the bulk Hamiltonian, namely, in a tight binding context in [SKR00, KRS02, EG02], and for continuous magnetic (differential) operators in [KS03]. In the continuous case, the two pairings turn out to be over two of the algebras in the Wiener-Hopf extension (1) and Theorem 1 then implies equality of bulk and edge Hall conductivities. To explain this result and the quantities involved in more detail is the object of Section 2.1.

An analogous result on equality of bulk and edge conductivities was proven in [KRS02] within the tight-binding approximation. It was re-derived by ad-hoc methods by Elbau and Graf

[EG02] and their proof may likely transpose to the continuous case as well. However, we prefer to give a more conceptual argument exhibiting this equality as a corollary of Theorem 1 and hence showing that the quantum Hall effect can be thought of as an experimental realization of Bott periodicity (or the Connes-Thom isomorphism). Equality of bulk and edge conductivities appeared already in various other guises, e.g. the framework of scaling theory [P85] and that of classical electromagnetism as resulting from a simple conservation law [F94].

In order to apply the operator-theoretic framework of Theorem 1 to the study of the non-commutative topology of the integer quantum Hall effect, it is necessary to construct C^*-algebras which are the observable algebras of the quantum mechanical system in a plane and a half-plane, modeling respectively the bulk and the edge of a sample. These algebras contain all bounded functions of the Hamiltonians and derivations thereof which are needed to construct the current operators. Moreover, the algebras are furnished with traces allowing to calculate expectation values. Using these ingredients, formulae for the bulk and edge Hall conductivities can be written out. Strictly speaking, such expressions would not require the language of C^*-algebras and hence one could dispense with it in [ASS94, KS03]. But when it comes to their comparison, we find that the language of C^*-algebras and their invariants is the most natural one and thus follow the tradition of Bellissard's philosophy of the *non-commutative Brillouin zone* [B85].

2.1 Hall conductivities as pairings

Here we summarize the framework and the results of [KS03] and then state the main theorem for quantum Hall systems. We described a quantum Hall system with disorder and a boundary by means of covariant families of integral operators on $L^2(\mathbb{R}^2)$. For this we combined a Borel probability space[1] $(\Sigma, \mathbf{P})$ whose elements describe disorder configurations in $\mathbb{R}^2$ with the space $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$ (topologically a half open interval). Points $s \in \hat{\mathbb{R}}$ describe the position of the boundary of $\{\vec{x} \in \mathbb{R}^2 | x_2 \geq -s\}$. Since disorder configurations can be translated in $\mathbb{R}^2$, the space Σ carries an $\mathbb{R}^2$-action which is required to be continuous, to leave the probability measure $\mathbf{P}$ invariant and to be ergodic w.r.t. it. The $\mathbb{R}^2$-action is simply denoted by $\sigma \mapsto \vec{x}\cdot\sigma$ and extended to an action on $\hat{\Sigma} = \Sigma \times \hat{\mathbb{R}}$ by $(\sigma, s) \mapsto (\vec{x}\cdot\sigma, s - x_2)$. A family $A = (A_{\hat{\sigma}})_{\hat{\sigma}\in\hat{\Sigma}}$ of integral operators on $L^2(\mathbb{R}^2)$ is called covariant if

$$U(\vec{\xi})A_{\hat{\sigma}}U(\vec{\xi})^* \;=\; A_{\vec{\xi}\cdot\hat{\sigma}} \;, \qquad \vec{\xi} \in \mathbb{R}^2 \;,$$

where $U(\vec{\xi})$ are the magnetic translation operators defined in Example 3 below. The system is described by a random family $H = (H_{\hat{\sigma}})_{\hat{\sigma}\in\hat{\Sigma}}$ of Hamiltonians, each given as sum of the Landau operator and a disordered potential (depending on the random variable $\sigma \in \Sigma$) and with Dirichlet boundary conditions at s. It is shown in [KS03] that sufficiently regular bounded functions of H yield covariant families of integral operators. Moreover, if we set $s = \infty$, i.e. push the boundary to $-\infty$, then we describe a disordered system without boundary by a Hamiltonian denoted $H_\infty = (H_{(\sigma,\infty)})_{\sigma\in\Sigma}$. In this framework [BES94, ASS94], the bulk Hall conductivity of a gas of independent electrons described by the planar Hamiltonian at zero-temperature and

[1] To avoid a clash in notation, we denote Σ what has been denoted Ω in [KS03].

with chemical potential μ belonging to a mobility gap in the spectrum of the Hamiltonian is given by

$$\sigma_b^\perp(\mu) \;=\; \frac{q^2}{h}\,\mathrm{ch}(P_\mu,P_\mu,P_\mu)\;,\quad \mathrm{ch}(A,B,C) = -2\pi\imath\,\mathcal{T}(A([X_1,B][X_2,C]-[X_2,B][X_1,C]))\;. \tag{2}$$

Here $P_\mu = \chi_{(-\infty,\mu]}(H_\infty)$ is the covariant family of associated Fermi projections, X_j the j-component of the position operator on $L^2(\mathbb{R}^2)$ and $\mathcal{T}$ the trace per unit volume (precise definition given in Eq. (14) below).

The main result of [KS03] is that the edge Hall conductivity of a gas of independent electrons described by the Hamiltonian H with Dirichlet boundary conditions at the edge at zero-temperature with chemical potential μ belonging to a gap Δ in the spectrum of the *planar* Hamiltonian H_∞ is given by

$$\sigma_e^\perp(\mu) \;=\; -\frac{q^2}{h}\,\xi(\mathcal{U}^*(\Delta)-1,\mathcal{U}(\Delta)-1)\;,\qquad \xi(A,B)=\hat{\mathcal{T}}(A[X_1,B])\;. \tag{3}$$

Here $\mathcal{U}(\Delta)$ is constructed from the half-planar Hamiltonian via functional calculus

$$\mathcal{U}(\Delta) \;=\; \exp(-2\pi\imath\, G(H)) \tag{4}$$

for a monotonously decreasing smooth function $G:\mathbb{R}\to\mathbb{R}$ with $G(-\infty)=1$, $G(\infty)=0$, and $\mathrm{supp}(G')\subset\Delta\backslash G^{-1}(\frac{1}{2})$, and $\hat{\mathcal{T}}$ is the trace per unit length along the boundary combined with the operator trace perpendicular to the boundary (its precise definition is given in eq. (16) below). As will be argued in Section 6.4, the following is then a corollary of Theorem 1.

Theorem 2 *Suppose that E is in a gap of H_∞, the Hamiltonian on the plane without boundary. Then*

$$\sigma_e^\perp(E) \;=\; \sigma_b^\perp(E)\;.$$

In the Landau model and its restriction to the half-space a proof can be given by explicitly calculating both pairings and then seeing that both numbers are the same [SKR00]. This calculation makes use of the translation invariance in the direction along the boundary. Not only does it exclude any type of disorder which brakes that invariance, it is also not very satisfactory in that it does not show directly that both pairings are the same.

Our proof is based on the fundamental relation between the observable algebra $\mathcal{A}_\infty$ of the plane model and that of the model with boundary $\mathcal{A}$, respectively defined by covariant families of integral operators without or with boundary. Indeed, the pushing of the boundary to infinity defines a surjective algebra morphism

$$\mathcal{A} \stackrel{\mathrm{ev}_\infty}{\longrightarrow} \mathcal{A}_\infty. \tag{5}$$

Therefore $\mathcal{A}$ is an extension of $\mathcal{A}_\infty$ by the edge algebra $\mathcal{E}:=\ker(\mathrm{ev}_\infty)$ which can be understood as the algebra of observables which are located at the boundary. This gives an exact sequence precisely of the form (1).

2.2 Landau projection versus Bott projection

This section, essentially taken from [ENN93], serves two purposes. First it makes clear why K-theory comes naturally into play and second it provides an illustrative example for the technique we use to prove the main result. That technique can be summarized by saying that one treats non-commutativity against non-trivial topology.

Let us consider the Landau Hamiltonian which describes the quantum mechanical motion of an charged particle in the plane $\mathbb{R}^2$ which is subject to a constant magnetic field of strength γ in appropriate units

$$H_\gamma \;=\; \frac{\hbar^2}{2m}(D_\gamma^* D_\gamma + \gamma)\;, \qquad D_\gamma \;=\; D_1 + \imath D_2\;,$$

where $D_j = \imath\partial_j + A_j$ with $\partial_1 A_2 - \partial_2 A_1 = -\imath[D_1, D_2] = \gamma$. The spectrum of H_γ is discrete for $\gamma \neq 0$ and $\ker(D_\gamma)$ is the eigenspace of the lowest eigenvalue. One refers to the projection Π_γ onto this eigenspace (called the lowest Landau level) also as Landau projection. In the symmetric gauge it has an integral kernel

$$\langle \vec{x}\,|\Pi_\gamma|\vec{y}\,\rangle \;=\; c_1\gamma e^{-\gamma c_2|\vec{x}-\vec{y}|^2}\Phi_\gamma(\vec{x},\vec{y}\,)\;, \qquad \Phi_\gamma(\vec{x},\vec{y}\,) \;=\; e^{-\imath\gamma(x_1y_2-y_1x_2)}\;,$$

where $c_{1,2}$ are γ-independent constants. This holds, of course, only for $\gamma \neq 0$ where we have discrete spectrum.

Note that, pointwise, $\langle \vec{x}\,|\Pi_\gamma|\vec{y}\,\rangle \stackrel{\gamma\to 0}{\longrightarrow} 0$. But in norm $\gamma \mapsto \Pi_\gamma$ cannot be continuous at $\gamma \to 0$ since the spectrum of H_0, which is essentially the Laplace operator, is all of $\mathbb{R}^+$. Nevertheless there is a way to make sense out of a limit $\gamma \to 0$ for a quantity related to Π_γ, namely its homotopy class in some naturally assigned C^*-algebra. Two projections of a C^*-algebra are homotopic if there exists a continuous projection-valued path in the algebra between them.

The C^*-algebra to which Π_γ naturally belongs is $C^*_\gamma(D_1, D_2)$, the norm closure of the algebra generated by certain regular functions of D_1, D_2. These are the smeared out Weyl-operators $T_\gamma(f) = \int_{\mathbb{R}^2} d\vec{y} f(\vec{y})T_\gamma(\vec{y})$ with $f \in L^1(\mathbb{R}^2)$ corresponding to the Landau translations

$$(T_\gamma(\vec{\xi}\,)\psi)(\vec{x}\,) \;\;=\;\; \Phi_\gamma(\vec{\xi},\vec{x}-\vec{\xi}\,)\,\psi(\vec{x}-\vec{\xi}\,)\;,$$

for $\psi \in L^2(\mathbb{R}^2)$ (*cf.* [KS03]). One finds that

$$T_\gamma(f)T_\gamma(g) \;=\; T_\gamma(f *_\gamma g)\;, \qquad f *_\gamma g(\vec{x}) \;=\; \int d\vec{y}\; f(\vec{y}\,)g(\vec{x}-\vec{y}\,)\Phi_\gamma(\vec{y},\vec{x}-\vec{y}\,)\,.$$

We may therefore also speak of $C^*_\gamma(D_1, D_2)$ as the (C^*-) closure of $L^1(\mathbb{R}^2)$ with above product. In the symmetric gauge, we can write $\Pi_\gamma = T_\gamma(k_\gamma)$ with $k_\gamma(\vec{\xi}\,) = c_1\gamma e^{-\gamma c_2\vec{\xi}^2}$ and so $\Pi_\gamma \in C^*_\gamma(D_1, D_2)$ for $\gamma \neq 0$ (we did not define Π_0). To describe the C^*-algebra for $\gamma \neq 0$, consider the representation ρ of $C^*_\gamma(D_1, D_2)$ on $L^2(\mathbb{R})$ induced by $\rho(T_\gamma(e_1))\phi(x) = \phi(x-1)$ and $\rho(T_\gamma(e_2))\phi(x) = e^{2\pi\imath\gamma x}\phi(x)$. For $\gamma \neq 0$ the representation is faithful identifying $C^*_\gamma(D_1, D_2)$ with $\mathcal{K}(L^2(\mathbb{R}))$, the compact operators over $\mathbb{R}$. This is the Schrödinger representation, $\rho(D_1)$ and $\rho(D_2)$ being momentum and position operators and γ Planck's constant. For $\gamma = 0$, however, the representation is not faithful and $C^*_\gamma(D_1, D_2)$ is isomorphic to the algebra of continuous (complex-valued) functions over $\mathbb{R}^2$ which vanish at infinity. An isomorphism is given by Fourier transform, which converts convolution into point-wise multiplication and identifies D_j with the position operator X_j. If we add a unit to the latter algebra it becomes isomorphic to $C(S^2)$, the continuous functions over the 2-sphere.

The algebras $C^*_\gamma(D_1, D_2)$ for different $\gamma \in [0, 1]$ can be combined into a larger C^*-algebra $\mathcal{F}$ known as a field of C^*-algebras with fibres $C^*_\gamma(D_1, D_2)$ over the base $[0, 1]$ [R89]. This larger C^*-algebra is the closure of the algebra of continuous functions $f : [0, 1] \to L^1(\mathbb{R}^2)$ with product

$$(fg)(\gamma) \;=\; f(\gamma) *_\gamma g(\gamma) \,.$$

Its elements are also called (continuous) sections. As above $\Pi_\gamma = T_\gamma(k_\gamma)$, but k_γ is not continuous for $\gamma \to 0$, because $\|\partial_\gamma k_\gamma\|_1 \sim \gamma^{-1}$. Therefore, $\gamma \mapsto \Pi_\gamma$ cannot be extended at $\gamma = 0$ to define an element in $\mathcal{F}$.

But $\gamma \mapsto [\Pi_\gamma \oplus 1]_0$ can be extended to $\gamma = 0$ where we denote by $[\Pi_\gamma \oplus 1]_0$ the homotopy class in $M_2(C^*_\gamma(D_1, D_2)^+)$, the algebra of 2×2-matrices with entries in $C^*_\gamma(D_1, D_2)^+$, the latter being $C^*_\gamma(D_1, D_2)$ with an added unit. This is done as follows: Take any projection-valued section $p \in M_2(\mathcal{F}^+)$ with $p(1) = \begin{pmatrix} \Pi_1 & 0 \\ 0 & 1 \end{pmatrix} = \Pi_1 \oplus 1$. Then $[\Pi_\gamma \oplus 1]_0 = [p(\gamma)]_0$ and we can extend the function $\gamma \mapsto [\Pi_\gamma \oplus 1]_0$ by $[p(0)]_0$ at $\gamma = 0$. We construct a particular example of such a section p. For a linear operator T on $L^2(\mathbb{R}^2)$, let P_T be the projection in $M_2(\mathcal{B}(L^2(\mathbb{R}^2)))$ onto the range of $1 \oplus T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ sending ψ to $\psi \oplus T\psi$. Now set $T = \lambda D_\gamma$, then

- for each γ fixed, $\lambda \in \mathbb{R}^+ \mapsto P_{\lambda D_\gamma}$ is a norm continuous path in $M_2(C^*_\gamma(D_1, D_2)^+)$,
- for each λ fixed, $\gamma \in [0, 1] \mapsto P_{\lambda D_\gamma}$ is a continuous section in $M_2(\mathcal{F}^+)$.

The second part is not obvious at $\gamma = 0$, but can be seen as follows: Defining $Q_\gamma = (D^*_\gamma D_\gamma + 1)^{-1} \begin{pmatrix} 1 & D^*_\gamma \\ D_\gamma & D^*_\gamma D_\gamma \end{pmatrix}$ one can verify $\|Q_\gamma - P_{D_\gamma}\| \leq |\gamma|$. Now Q_γ is a continuous section because one can express the integral kernel $(D^*_\gamma D_\gamma + 1)^{-1}$ as a (norm-convergent) Laplace transform of the well-known heat-kernel of $D^*_\gamma D_\gamma$. This leads to expressions of the entries of Q_γ which are all of the form $T_\gamma(f_\gamma)$ with functions f_γ whose derivatives in γ have bounded L^1-norm.

For $\gamma = 0$, the mentioned isomorphism onto $C_0(\mathbb{R}^2)$ intertwines $D_0 = Q_0$ to multiplication with the Fourier variable $k_1 + \imath k_2$ and P_{D_0} gets identified with

$$b \;=\; \frac{1}{k_1^2 + k_2^2 + 1} \begin{pmatrix} 1 & k_1 + \imath k_2 \\ k_1 - \imath k_2 & k_1^2 + k_2^2 \end{pmatrix} \,,$$

the so-called Bott projection. For any other $\gamma \neq 0$, Q_γ is not a projection. On the other hand, for $\gamma \neq 0$, $\lim_{\lambda \to \infty} P_{\lambda D_\gamma}$ tends to the projection onto $\ker(D_\gamma) \oplus \overline{\mathrm{im}\,(D_\gamma)}$ which, since $\mathrm{im}\,(D_\gamma)$ is dense, equals $\Pi_\gamma \oplus 1$. Hence $p(\gamma) = P_{\frac{1}{1-\gamma} D_\gamma}$ is an element of $M_2(C^*_\gamma(D_1, D_2)^+)$ joining $\Pi_\gamma \oplus 1$ with the Bott-projection $p(0) = b$.

The construction of a section in the C^*-field which links the Landau projection to the Bott-projection is the key argument in the proof of Bott-periodicity given in [ENN93]. Bott-periodicity says that the maps Θ and β described in Section 4.2 are isomorphisms and accounts for the fact that the natural definition of the K_i-group of a C^*-algebra for $i > 1$ reproduces only the K_{i-2}-group. In fact, the above section can be reinterpreted as defining a map $\mu : K_2(\mathbb{C}) := K_0(C_0(\mathbb{R}^2)) \to K_0(C^*_\gamma(D_1, D_2)) = K_0(\mathbb{C})$, the latter because $C^*_\gamma(D_1, D_2) \cong \mathcal{K}$ and the stability of K-groups.

We note here that we established our goal of finding a limit $\gamma \to 0$ only by using two 'tricks'. First, we added a unit to the algebra and, second, we considered projections not only in the algebras themselves, but also in the (two-by-two) matrix algebras over them. Both of these operations are basic ingredients in K-theory, even in the definition of the K-groups themselves.

Homotopy classes of projections in matrix algebras over a C^*-algebra carry an additive structure and cyclic cohomology provides additive maps on these classes to the complex numbers. Two such maps are particularly interesting as they appear in the physical context here, the trace and the Chern class. Let us denote by Tr the operator trace. Then $p \mapsto \mathrm{Tr}(\rho(p))$ is finite and constant on homotopy classes of projections $p \in C^*_\gamma(D_1, D_2)$, since it is the rank of $\rho(p)$. In addition to that, $\mathcal{T} : C^*_\gamma(D_1, D_2) \to \mathbb{C}$: $\mathcal{T}(f) = f(0)$ defines an unbounded trace and $\nabla_j : C^*_\gamma(D_1, D_2) \to C^*_\gamma(D_1, D_2)$: $\nabla_j(f)(\vec{x}) = \imath x_j f(\vec{x})$ two unbounded derivations which in the representation on $L^2(\mathbb{R}^2)$ look as $T_\gamma(\nabla_j(f)) = \imath[X_j, T_\gamma(f)]$. Let $\mathrm{ch} : C^*_\gamma(D_1, D_2)^3 \to \mathbb{C}$ be the unbounded trilinear form given by[2]

$$\mathrm{ch}(f_0, f_1, f_2) \;=\; -\,2\pi\imath\, \mathcal{T}(f_0 *_\gamma (\nabla_1 f_1 *_\gamma \nabla_2 f_2 - \nabla_2 f_1 *_\gamma \nabla_1 f_2)). \qquad (6)$$

For a projection p, $\mathrm{ch}(p, p, p)$ is called its Chern number. Note that ch depends on γ through the product $*_\gamma$. For $\gamma = 0$, Fourier transform identifies ch with the first geometric Chern class ch_g over the one-point compactification of $\mathbb{R}^2$ given by:

$$\mathrm{ch}_g(\hat{f}_0, \hat{f}_1, \hat{f}_2) \;=\; \frac{1}{2\pi\imath} \int_{\mathbb{R}^2} d\vec{k}\; \mathrm{Tr}\left(\hat{f}_0(\vec{k}\,)\left(\partial_{k_1}\hat{f}_1(\vec{k}\,)\partial_{k_2}\hat{f}_2(\vec{k}\,) - \partial_{k_2}\hat{f}_1(\vec{k}\,)\partial_{k_1}\hat{f}_2(\vec{k}\,)\right)\right) \,.$$

Lemma 1 *For $\gamma > 0$, and any projection $p \in C^*_\gamma(D_1, D_2)$:*

$$\text{(i)}\;\; \mathrm{ch}(\Pi_\gamma, \Pi_\gamma, \Pi_\gamma) \;=\; \mathrm{ch}_g(b, b, b)\,, \qquad \text{(ii)}\;\; \mathrm{ch}(p, p, p) \;=\; -\mathrm{Tr}(\rho(p))\,.$$

The first statement can be verified by a direct calculation showing that both sides are equal to -1. However, there is a much more illuminating proof (Example 16, an application of Proposition 8) which will work in greater generality and uses that one can express ch for two different values of γ as boundary cocycles of a chain as explained in detail in Section 6.2. The second statement can also be verified explicitely (see equation (30)). It expresses a periodicity result in cyclic cohomology presented in Section 6.3.

[2]We use the same letter as in (2) because it is the same cocycle if Σ consists only of one point, i.e. if there is no disorder.

2.3 Outline of the argument without disorder

We provide here a short description of the steps of the proof of Theorem 2 where for reasons of clarity we disregard disorder which amounts to taking Σ to be a single point. The observable algebra of the plane model is then $\mathcal{A}_\infty = C^*_\gamma(D_1, D_2)$ and algebra for the model with boundary is an extension of $C^*_\gamma(D_1, D_2)$ by the edge algebra $C^*_{\gamma,\hbar}(D_1, D_2, X_2)$. Here the 2-component of the position operator X_2 measures the distance to the boundary. $C^*_{\gamma,\hbar}(D_1, D_2, X_2)$ is constructed as $C^*_\gamma(D_1, D_2)$, but has one more Weyl generator satisfying $[D_2, X_2] = \imath\hbar$. We may describe it as the algebra of continuous functions $F : \mathbb{R} \to C^*_\gamma(D_1, D_2)$ which vanish at infinity with product

$$(F *_{\gamma,\hbar} G(s))(\vec{x}\,) \;=\; (F(s) *_\gamma G(s - \hbar x_2))(\vec{x}\,) \; .$$

The extension gives rise to a map $\exp : [C^*_\gamma(D_1, D_2)]_0 \to [C^*_{\gamma,\hbar}(D_1, D_2, X_2)]_1$ sending homotopy classes of projections of $C^*_\gamma(D_1, D_2)$ to homotopy classes of unitaries of $C^*_{\gamma,\hbar}(D_1, D_2, X_2)$, which in the situation of Section 2.1 yields

$$\exp[P_\mu]_0 \;=\; [\mathcal{U}(\Delta)]_1 \; .$$

Dual to the exponential map is a map on the level of cyclic cohomology which sends ch to the functional $\tilde{\xi} : C^*_{\gamma,\hbar}(D_1, D_2, X_2)^4 \to \mathbb{C}$ given by

$$\tilde{\xi}(F_0, F_1, F_2, F_3) \;=\; \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma) \int_{\mathbb{R}} ds\; \mathcal{T}(F_0 *_{\gamma,\hbar} \delta_{\sigma(1)} F_1 *_{\gamma,\hbar} \delta_{\sigma(2)} F_2 *_{\gamma,\hbar} \delta_{\sigma(3)} F_3)(s) \; ,$$

where $\delta_3(F)(s) = \imath\partial_s F(s)$ and S_3 is the permutation group of 3 elements. Duality means here that for some constant c,

$$\operatorname{ch}(p, p, p) \;=\; c\,\tilde{\xi}(u^* - 1, u - 1, u^* - 1, u - 1) \; , \qquad [u]_1 = \exp[p]_0 \; .$$

The above duality is proven by tuning $\hbar \to 0$ and applying the techniques outlined in the last section. In this case, $\lim_{\hbar\to 0}[\mathcal{U}(\Delta)]_1 = [e^{2\pi\imath\chi} P_\mu + P_\mu^\perp]_1$ where $\chi = \chi(s)$ is a switch function and $\tilde{\xi}$ changes only through the product. We prove an analogue of Lemma 1.i, namely that $\tilde{\xi}(\mathcal{U}(\Delta)^* - 1, \mathcal{U}(\Delta) - 1, \mathcal{U}(\Delta)^* - 1, \mathcal{U}(\Delta) - 1) = \xi_{\hbar=0}((e^{-2\pi\imath\chi} - 1)P_\mu, (e^{2\pi\imath\chi} - 1)P_\mu, (e^{-2\pi\imath\chi} - 1)P_\mu, (e^{2\pi\imath\chi} - 1)P_\mu)$. One then has only to show that the r.h.s. is equal to $c^{-1}\operatorname{ch}(P_\mu, P_\mu, P_\mu)$ which is a simple calculation, because χ and P_μ commute. Note that we do not turn $\gamma \to 0$ as in Section 2.2, but rather $\hbar \to 0$ which is formally the same. It is not necessary but interesting to note that, if we also turn down the magnetic field $\gamma \to 0$ then the unitalization of $C^*_{\gamma,\hbar}(D_1, D_2, X_2)$ becomes $C(S^3)$ and above arguments simply relate the non-trivial topology of S^3 with that of S^2.

Finally, an analogue of Lemma 1(ii) will show $\xi(u^* - 1, u - 1) = c\,\tilde{\xi}(u^* - 1, u - 1, u^* - 1, u - 1)$.

3 Some C^*-algebraic constructions

Recall that a complex $*$-Banach algebra is a normed algebra over $\mathbb{C}$ which is complete and equipped with an anti-linear involution $A \mapsto A^*$ satisfying $\|A^*\| = \|A\|$. Without further mentioning all our Banach algebras are complex $*$-Banach algebras. A C^*-algebra is a Banach algebra satisfying in addition $\|A^*A\| = \|A\|^2$. We discuss in the following two particular constructions which allow one to construct new C^*-algebras from given ones: the crossed product construction and continuous fields of C^*-algebras, which will be referred to as C^*-fields. Finally some extensions are presented.

3.1 Crossed products by $\mathbb{R}$

Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})$ be an action of $\mathbb{R}$ on a C^*-algebra $\mathcal{B}$. It is required to be continuous in the sense that for all $A \in \mathcal{B}$, the function $x \in \mathbb{R} \mapsto \alpha_x(A)$ is continuous. The crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{R}$ of $\mathcal{B}$ with respect to the action α of $\mathbb{R}$ is defined as follows [P79].

The linear space $C_c(\mathbb{R}, \mathcal{B})$ of compactly supported continuous functions with values in $\mathcal{B}$ is endowed with the $*$-algebra structure

$$(fg)(x) \;=\; \int_{\mathbb{R}} dy\, f(y)\, \alpha_y(g(x-y))\;, \qquad f^*(x) \;=\; \alpha_x(f(-x))^*\;. \tag{7}$$

The L^1-completion of $C_c(\mathbb{R}, \mathcal{B})$, i.e. completion w.r.t. the norm $\|f\|_1 := \int_{\mathbb{R}} dx\, \|f(x)\|_{\mathcal{B}}$, is a Banach algebra, the L^1-crossed product denoted $L^1(\mathbb{R}, \mathcal{B}, \alpha)$. The crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{R}$ is the completion of $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ w.r.t. the C^*-norm $\|f\| := \sup_\rho \|\rho(f)\|$ where the supremum is taken over all bounded $*$-representations. It is not necessary to perform the middle step via the L^1-crossed product, but it is sometimes convenient to work with it when verifying that a given operator belongs to $\mathcal{B} \rtimes_\alpha \mathbb{R}$. In this spirit, we can benefit from our results in [KS03]. By a continuity argument, one can simply work with functions $f : \mathbb{R} \to \mathcal{B}$ when performing calculations with elements of $\mathcal{B} \rtimes_\alpha \mathbb{R}$

Let $(\rho, \mathcal{H})$ be a representation of $\mathcal{B}$ which is extended to a representation $(\tilde{\rho}, L^2(\mathbb{R}, \mathcal{H}))$ by $(\tilde{\rho}(A)\psi)(x) = \rho(\alpha_{-x}(A))\psi(x)$, $A \in \mathcal{B}$. Let further $(\lambda, L^2(\mathbb{R}))$ be a unitary representation of $\mathbb{R}$. It is extended trivially to a representation on $L^2(\mathbb{R}) \otimes \mathcal{H} \cong L^2(\mathbb{R}, \mathcal{H})$. Then we get a representation $(\pi, L^2(\mathbb{R}, \mathcal{H}))$ of $\mathcal{B} \rtimes_\alpha \mathbb{R}$ by

$$\pi(f) = \int_{\mathbb{R}} dx\, \tilde{\rho}(f(x))\, \lambda(x).$$

We are interested in the case in which λ is the left regular representation so that this becomes

$$(\pi(f)\psi)(x) \;=\; \int_{\mathbb{R}} dy\, \rho(\alpha_{-x}(f(x-y)))\psi(y)\;. \tag{8}$$

Example 1 (Suspension of an algebra) Consider the trivial action id of $\mathbb{R}$ on a C^*-algebra $\mathcal{B}$. Fourier transformation on the $\mathbb{R}$-variable defines an algebra morphism from $L^1(\mathbb{R}, \mathcal{B}, \mathrm{id})$ into $C_0(\tilde{\mathbb{R}}, \mathcal{B})$, the continuous $\mathcal{B}$-valued functions over $\tilde{\mathbb{R}} \cong \mathbb{R}$, the dual group of $\mathbb{R}$, which vanish at infinity with pointwise multiplication. Its extension to the C^*-completion yields an isomorphism $\mathcal{B} \rtimes_{\mathrm{id}} \mathbb{R} \cong C_0(\mathbb{R}, \mathcal{B})$. $C_0(\mathbb{R}, \mathcal{B})$ is called the suspension of $\mathcal{B}$ and also denoted by $S\mathcal{B}$.

Example 2 ($C^*_\gamma(D_1, D_2)$ revisited) In the previous example, let $\mathcal{B} = \mathbb{C}$ and define for $\gamma \in \mathbb{R}$ the $\mathbb{R}$-action $\hat{\tau}^\gamma$ on $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R}$ by

$$\hat{\tau}^\gamma_t(f)(x) \;=\; e^{\imath\gamma x t} f(x)\;.$$

For $\gamma = 0$, this action is trivial and while for $\gamma \neq 0$ it is a scaled version of the dual action of $\mathbb{R}$. Fourier transform $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \cong S\mathbb{C}$ yields $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\hat{\tau}^\gamma} \mathbb{R} \cong S\mathbb{C} \rtimes_{\tau^\gamma} \mathbb{R}$ with $\tau^\gamma_y(f)(x) = f(x - \gamma y)$ for $f \in S\mathbb{C}$. We claim that $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\hat{\tau}^\gamma} \mathbb{R}$ is also isomorphic $C^*_\gamma(D_1, D_2)$. For this purpose, we choose the Landau gauge $\Phi_\gamma(\vec{\xi}, \vec{x}\,) = e^{-\imath\gamma\xi_1 x_2}$ in which one can quickly show that $L^1(\mathbb{R}^2)$ with product $*_\gamma$ is isomorphic to $L^1(\mathbb{R}, L^1(\mathbb{R}, \mathbb{C}, \mathrm{id}), \hat{\tau}^\gamma)$. But another (useful) way of seeing this is to realize that, starting out with the identity representation $\rho : \mathbb{C} \to \mathbb{C}$ and applying twice the procedure (8), we obtain, first, a faithful representation of $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R}$ on $L^2(\mathbb{R})$ and then the representation π of $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\hat{\tau}^\gamma} \mathbb{R}$ on $L^2(\mathbb{R}^2)$ assigning to $F : \mathbb{R} \to (\mathbb{R} \to \mathbb{C})$ the integral operator on $L^2(\mathbb{R}^2)$ with kernel

$$\langle \vec{x}\,|\pi(F)|\vec{y}\,\rangle \;=\; e^{-\imath\gamma(x_1-y_1)x_2} F(x_2 - y_2)(x_1 - y_1)\,.$$

Setting $f(\vec{x}) = e^{-\imath\gamma x_1 x_2} F(x_2)(x_1)$ we thus find $\pi(F) = T_\gamma(f)$. If $\gamma \neq 0$, then π is faithful so that this also shows our claim.

Example 3 (Observable algebra with disorder and boundary) Consider the set Σ (of disorder configurations) from [KS03] with its $\mathbb{R}^2$ action denoted $\vec{x} \cdot \sigma$, $\vec{x} \in \mathbb{R}^2$, $\sigma \in \Sigma$. The first component of this action corresponds to translating the disorder configuration along the boundary and yields an $\mathbb{R}$-action on $C_0(\Sigma)$ given by

$$\beta^{\|}_{x_1}(f)(\sigma) = f((x_1, 0) \cdot \sigma)\,.$$

Let $\gamma \in \mathbb{R}$ and define an $\mathbb{R}$-action on the crossed product $C_0(\Sigma) \rtimes_{\beta^{\|}} \mathbb{R}$ by $(f : \mathbb{R} \to C_0(\Sigma))$

$$\beta^{\perp}_{x_2}(f)(x_1)(\sigma) \;=\; e^{\imath\gamma x_1 x_2} f(x_1)((0, x_2) \cdot \sigma)\,.$$

We denote $\mathcal{A}_\infty = C_0(\Sigma) \rtimes_{\beta^{\|}} \mathbb{R} \rtimes_{\beta^{\perp}} \mathbb{R}$ and call $\mathcal{A}_\infty$ the observable algebra for the planar model. To fit this example with the general set-up of this work, set $\mathcal{B} = C_0(\Sigma) \rtimes_{\beta^{\|}} \mathbb{R}$, $\alpha = \beta^\perp$ so that $\mathcal{B} \rtimes_\alpha \mathbb{R} = \mathcal{A}_\infty$.

Following the philosophy in [KS03], we can view $\mathcal{A}_\infty$ as being obtained from a larger algebra which is based on the space $\hat{\Sigma} = \Sigma \times (\mathbb{R} \cup \infty)$ with its $\mathbb{R}^2$ action $\vec{x} \cdot (\sigma, s) = (\vec{x} \cdot \sigma, s + x_2)$. This algebra is $\mathcal{A} = C_0(\hat{\Sigma}) \rtimes_{\beta^{\|}} \mathbb{R} \rtimes_{\tau\otimes\beta^{\perp}} \mathbb{R}$ where we extend $\beta^{\|}$ trivially to the second factor of $\hat{\Sigma}$ which should be thought of as the (approximate) position of the boundary (possibly pushed to infinity) and

$$(\tau \otimes \beta^{\perp})_{x_2}(f)(x_1)(\hat{\sigma}) = \beta^{\perp}_{x_2} f(x_1)((0, x_2) \cdot \hat{\sigma})\,.$$

$\mathcal{A}$ is the observable algebra for the half-space. Its relation with the covariant operator families of [KS03] is as follows.

A point $\hat{\sigma} \in \hat{\Sigma}$ defines a one-dimensional representation of $C_0(\hat{\Sigma})$, $\rho_{\hat{\sigma}} : C_0(\hat{\Sigma}) \to \mathbb{C}$: $\rho_{\hat{\sigma}}(f) = f(\hat{\sigma})$. Applying (8) twice, we get a representation $\pi_{\hat{\sigma}}$ of $\mathcal{A}$ on $L^2(\mathbb{R}^2)$. If $F : \mathbb{R} \to (\mathbb{R} \to C_0(\hat{\Sigma}))$ then the integral kernel of $\pi_{\hat{\sigma}}(F)$ is

$$\begin{aligned}\langle \vec{x}\,|\pi_{\hat{\sigma}}(F)|\vec{y}\,\rangle \;&=\; \rho_{\hat{\sigma}} \beta^{\|}_{-x_1} \left((\tau \otimes \beta^{\perp})_{-x_2} \left(F(x_2 - y_2) \right) (x_1 - y_1) \right) \\ &=\; e^{-\imath\gamma(x_1-y_1)x_2} F(x_2 - y_2)(x_1 - y_1)(-\vec{x} \cdot \hat{\sigma})\,.\end{aligned}$$

It follows that $\langle \vec{x}-\vec{\xi}\,|\pi_{\hat\sigma}(F)|\vec{y}-\vec{\xi}\,\rangle = e^{\imath\gamma(x_1-y_1)\xi_2}\langle \vec{x}\,|\pi_{\vec{\xi}\cdot\hat\sigma}(F)|\vec{y}\,\rangle$ and hence

$$U(\vec{\xi}\,)\pi_{\hat\sigma}(F)U(\vec{\xi}\,)^* \;=\; \pi_{\vec{\xi}\cdot\hat\sigma}(F)\;,$$

where the magnetic translation operators $U(\vec{\xi}\,)$ are defined by

$$(U(\vec{\xi}\,)\psi)(\vec{x}\,) \;=\; \hat\Phi(\vec{\xi},\vec{x}-\vec{\xi}\,)\,\psi(\vec{x}-\vec{\xi}\,)\;, \qquad \hat\Phi(\vec{\xi},\vec{x}\,) \;=\; e^{-\imath\gamma\xi_2 x_1}\;.$$

The collection $\pi(F) = (\pi_{\hat\sigma}(F))_{\hat\sigma\in\hat\Sigma}$ forms therefore a covariant family of bounded integral operators. By construction the operators are weakly continuous in $\hat\sigma$ and the norm $\|\pi(F)\|_\infty$ which we defined in [KS03] to be the essential supremum over $\|\pi_{\hat\sigma}(F)\|$ is bounded by the C^*-norm $\|F\|$. If we apply the above construction to the direct sum representation $\rho = \bigoplus_{\hat\sigma\in\hat\Sigma}\rho_{\hat\sigma}$ we obtain a representation π which also decomposes into a direct sum representation, namely on $\bigoplus_{\hat\sigma\in\hat\Sigma} L^2(\mathbb{R}^2)$, and we can interprete the covariant family $(\pi_{\hat\sigma}(F))_{\hat\sigma\in\hat\Sigma}$ as the representative $\pi(F)$ of F. Since the direct sum representation ρ is faithful, also π is faithful and so, first, identifies $\mathcal{A}$ with a sub-algebra of the completion of the algebra of weakly continuous covariant families of bounded integral operators denoted $\mathcal{A}$ in [KS03], and second, implies $\|\pi(F)\|_\infty = \|F\|$. The estimates established in [KS03] now show that for potentials $V_\sigma(\vec{x}) := V(-\vec{x}\cdot\sigma)$ with $V\in C(\Sigma)$ and differentiable along the flow of the $\mathbb{R}^2$ action and $F\in C^k_c(\mathbb{R})$ with $k>6$, the covariant families $F(H)$ and $D_jF(H)$ can be viewed as elements of $L^1(\mathbb{R}, L^1(\mathbb{R}, C_0(\hat\Sigma),\beta^{\|}),\tau\otimes\beta^{\perp})$ and hence of $\mathcal{A}$.

3.2 C^*-fields

We follow the exposition of [L98] in defining a continuous field of C^*-algebras (or simply a C^*-field) $(\mathcal{C},\{\mathcal{C}^\hbar,\varphi_\hbar\}_{\hbar\in I})$ over a locally compact Hausdorff space I. This consists of a C^*-algebra $\mathcal{C}$ (also called the total algebra of the field), a collection of C^*-algebras $\{\mathcal{C}^\hbar\}_{\hbar\in I}$, one for each point of the space I, with surjective algebra homomorphisms $\varphi_\hbar:\mathcal{C}\to\mathcal{C}^\hbar$ such that,

1. for $a\in\mathcal{C}$, $\|a\| = \sup_{\hbar\in I}\|\varphi_\hbar(a)\|$,

2. for all $a\in\mathcal{C}$, $\hbar\mapsto\|\varphi_\hbar(a)\|$ is a function in $C_0(I)$,

3. $\mathcal{C}$ is a left $C_0(I)$ module and, for $f\in C_0(I), a\in\mathcal{C}$ we have $\varphi_\hbar(fa) = f(\hbar)\varphi_\hbar(a)$.

The construction is reminiscent of a fibre bundle, except there is no typical fibre, the algebras $\mathcal{C}^\hbar$ need not to be isomorphic even if I is connected, and so one cannot define how the $\mathcal{C}^\hbar$ are topologically glued together using local trivializations. This information is contained in the algebra $\mathcal{C}$, the total C^*-algebra of the field. In fact, continuous sections of the field are collections $\{a_\hbar\}_{\hbar\in I}$ for which exist $a\in\mathcal{C}$ such that $\varphi_\hbar(a) = a_\hbar$. $\mathcal{C}$ can therefore be seen as the algebra of continuous sections with pointwise (in $\hbar$) multiplication. A C^*-field is called trivial if $\mathcal{C} = C_0(I,\mathcal{B})$ for some C^*-algebra $\mathcal{B}$, $\mathcal{C}^\hbar = \mathcal{B}$ and $\varphi_\hbar$ the evaluation at $\hbar$.

All we are interested in here concerns the more special set up in which $I\subset\mathbb{R}$ and we have a collection of continuous $\mathbb{R}$-actions $\{\alpha^\hbar\}_{\hbar\in I}$ on a single C^*-algebra $\mathcal{B}$, $\alpha^\hbar:\mathbb{R}\to\mathrm{Aut}(\mathcal{B})$. Collecting these together we get an $\mathbb{R}$ action $\tilde\alpha:\mathbb{R}\to\mathrm{Aut}(C_0(I,\mathcal{B}))$ by

$$\tilde\alpha_t(f)(\hbar) = \alpha^\hbar_t(f(\hbar))\;,$$

which is continuous provided the above expression is continuous in $\hbar$ for all t and f which we hereby assume. Then $(C_0(I,\mathcal{B})\rtimes_{\tilde{\alpha}}\mathbb{R},\{\mathcal{B}\rtimes_{\alpha^\hbar}\mathbb{R},\mathrm{ev}_\hbar\}_{\hbar\in I})$ is a continuous field of C^*-algebras [R89].

Example 4 (Heisenberg group algebra) The (polarized) Heisenberg group $\mathbb{H}_3$ is $\mathbb{R}^3$ as topological space, but with (non-abelian) multiplication

$$(a_1,a_2,a_3)(b_1,b_2,b_3)=(a_1+b_1,a_2+b_2,a_3+b_3+a_1b_2)\ .$$

It contains the subgroup $\mathbb{R}^2=\{(a_1,a_2,a_3)\in\mathbb{H}_3|a_1=0\}$ so that $\mathbb{H}_3$ can be identified with the semi-direct product $\mathbb{R}^2\rtimes_{\tilde{\tau}}\mathbb{R}$ where $\tilde{\tau}_{a_1}(a_2,a_3)=(a_2,a_3+a_1a_2)$. The Heisenberg group algebra (i.e. the crossed product $\mathbb{C}\rtimes_{\mathrm{id}}\mathbb{H}_3$ defined in a similar way as for $\mathbb{R}$) can therefore be identified with the C^*-algebra $C_0(\mathbb{R}^2)\rtimes_{\tilde{\tau}}\mathbb{R}$ with $\tilde{\tau}_{a_1}(f)(a_2,a_3)=f(a_2,a_3-a_1a_2)$. Let $\varphi_{a_2}:C_0(\mathbb{R}^2)\rtimes_{\tilde{\tau}}\mathbb{R}\to C_0(\mathbb{R})\rtimes_{\tau^{a_2}}\mathbb{R}$ be evaluation of the 2-component at a_2, i.e. $\varphi_{a_2}(f)(a_1)(a_3)=f(a_1)(a_2,a_3)$. Then $\mathrm{im}\,(\varphi_{a_2})\cong C_0(\mathbb{R})\rtimes_{\tau^{a_2}}\mathbb{R}$ where $\tau^{a_2}_{a_1}(g)(a_3)=g(a_3-a_2a_1)$ for $g:\mathbb{R}\to C_0(\mathbb{R})$. Furthermore $(C_0(\mathbb{R}^2)\rtimes_{\tilde{\tau}}\mathbb{R},\{C_0(\mathbb{R})\rtimes_{\tau^{a_2}}\mathbb{R},\varphi_{a_2}\}_{a_2\in\mathbb{R}})$ is a C^*-field. Therefore a_2 plays the role of $\hbar$. Note also that $C_0(\mathbb{R})\rtimes_{\tau^{a_2}}\mathbb{R}\cong C^*_\gamma(D_1,D_2)$ if $\gamma=a_2$ (cf. Example 2). In this situation, a_1,a_3 correspond to the space directions of D_1,D_2 and the C^*-field $\mathcal{F}$ of Section 2.2 is the restriction of the above C^*-field to $a_2=\gamma\in[0,1]$.

Example 5 If we have an $\mathbb{R}$-action α on a C^*-algebra $\mathcal{B}$ we can extend the above field of the Heisenberg group algebra in the following way: With the above $\mathbb{R}$-action $\tilde{\tau}$ on $C_0(\mathbb{R}^2)$ define $\tilde{\tau}\otimes\alpha:\mathbb{R}\to\mathrm{Aut}\,C_0(\mathbb{R}^2,\mathcal{B})$ by $(\tilde{\tau}\otimes\alpha)_{a_1}(f)(a_2,a_3)=\alpha_{a_1}(f(a_2,a_3-a_2a_1))$. Setting $a_2=\hbar$ as above, this then yields a C^*-field $(C_0(\mathbb{R}^2,\mathcal{B})\rtimes_{\tilde{\tau}\otimes\alpha}\mathbb{R},\{C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^\hbar\otimes\alpha}\mathbb{R},\varphi_\hbar\}_{\hbar\in\mathbb{R}})$ which will be of crucial importance later on. This C^*-field is trivial away from $\hbar=0$, i.e., for $\hbar\neq0$, $C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^1\otimes\alpha}\mathbb{R}=S\mathcal{B}\rtimes_{\tau^1\otimes\alpha}\mathbb{R}\cong C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^\hbar\otimes\alpha}\mathbb{R}$ and $\ker(\varphi_0)\cong C_0(\mathbb{R}\backslash\{0\},C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^1\otimes\alpha}\mathbb{R})$. Moreover, $C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^0\otimes\alpha}\mathbb{R}\cong S(\mathcal{B}\rtimes_\alpha\mathbb{R})$ which is not isomorphic to $C_0(\mathbb{R},\mathcal{B})\rtimes_{\tau^1\otimes\alpha}\mathbb{R}$.

3.3 Extensions

Suppose that we have a surjective morphism between C^*-algebras $q:\mathcal{C}\to\mathcal{B}$. One then says that $\mathcal{C}$ is an extension of $\mathcal{B}$ by the ideal $\mathcal{J}:=\ker(q)$.[3]

Example 6 (Cone of an algebra) We constructed the suspension $S\mathcal{B}=C_0(\mathbb{R},\mathcal{B})$ of an algebra $\mathcal{B}$ in Example 1. Its cone is given by $C\mathcal{B}=C_0(\hat{\mathbb{R}},\mathcal{B})$ where $\hat{\mathbb{R}}=\mathbb{R}\cup\infty$. The cone is an extension of $\mathcal{B}$ by the ideal $S\mathcal{B}$, the morphism being $q=\mathrm{ev}_\infty$, the evaluation at ∞.

Example 7 (Pushing the boundary to infinity) The second component in $(\sigma,s)\in\hat{\Sigma}$ describes the position of the boundary and is allowed to take the value ∞. The evaluation map $\mathrm{ev}_\infty:\mathcal{A}\to\mathcal{A}_\infty$, $\mathrm{ev}_\infty(F)(x_2)(x_1)(\sigma)=F(x_2)(x_1)(\sigma,\infty)$ has the effect of pushing the boundary to $-\infty$. The algebra $\mathcal{A}_\infty=C(\Sigma)\rtimes_{\beta^\|}\mathbb{R}\rtimes_{\beta^\perp}\mathbb{R}$ is therefore the observable algebra of the model with disorder, but without a boundary (planar model). If Σ consists of one point only so that there is no disorder, then $\mathcal{A}_\infty=C^*_\gamma(D_1,D_2)$ and the ideal is $\ker(\mathrm{ev}_\infty)=C^*_{\gamma,\hbar}(D_1,D_2,X_2)$.

[3]Some authors call $\mathcal{C}$ an extension of $\mathcal{J}$ by $\mathcal{B}$.

Example 8 (Wiener-Hopf extension) Let α be an $\mathbb{R}$-action on $\mathcal{B}$. We extend the $\mathbb{R}$-action $\tau^1 \otimes \alpha$ on the suspension $S\mathcal{B} = C_0(\mathbb{R}, \mathcal{B})$ of Example 5 to the cone $C\mathcal{B} \cong C_0(\mathbb{R} \cup \infty) \otimes \mathcal{B}$ by setting $\tau_t f(\infty) = f(\infty)$. Hence ∞ is a fixed point and therefore

$$\mathrm{ev}_\infty : C\mathcal{B} \rtimes_{\tau^1 \otimes \alpha} \mathbb{R} \longrightarrow \mathcal{B} \rtimes_\alpha \mathbb{R} \tag{9}$$

is a surjective algebra-homomorphism. The corresponding extension is called the Wiener-Hopf extension for an $\mathbb{R}$-action α on $\mathcal{B}$ [R89]. The ideal is $\ker(\mathrm{ev}_\infty) = S\mathcal{B} \rtimes_{\tau^1 \otimes \alpha} \mathbb{R}$ which appeared in Example 5, it will be shown to be isomorphic to $\mathcal{K} \otimes \mathcal{B}$ in Section 6.3. These from the ingredients of the exact sequence (1).

Example 9 One obtains the C^*-field $(C_0(\mathbb{R}, C\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R}, \{C\mathcal{B} \rtimes_{\tau^\hbar \otimes \alpha} \mathbb{R}, \varphi_\hbar\}_{\hbar \in \mathbb{R}})$ by repeating the constructions of Example 5. Evaluation at ∞ gives a surjection

$$C_0(\mathbb{R}, C\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R} \stackrel{\mathrm{ev}_\infty}{\longrightarrow} C_0(\mathbb{R}, \mathcal{B}) \rtimes_\alpha \mathbb{R} \ , \tag{10}$$

whose kernel is $C_0(\mathbb{R}, S\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R}$. Each algebra is the total algebra of a C^*-field so that one actually has a field of extensions, the fibre at $\hbar = 1$ being the Wiener-Hopf extension.

4 K-theory: homotopy classes of projections and unitaries

Consider projections in a Banach algebra $\mathcal{B}$. Two projections are homotopic if they can be joined by a projection valued continuous path in $\mathcal{B}$. We denote by $[\mathcal{B}]_0$ the homotopy classes of projections of $\mathcal{B}$. If $\mathcal{B}$ has no unit, like in the cases we are interested in, there are typically too few projections and so one adds a unit obtaining the algebra $\mathcal{B}^+ = \mathcal{B} \times \mathbb{C}$ with $(A, \lambda)(A', \lambda') = (AA' + \lambda A' + A\lambda', \lambda\lambda')$. Furthermore one includes $M_n(\mathcal{B}^+)$, the algebra of $n \times n$ matrices with entries in $\mathcal{B}^+$, as a sub-algebra in $M_{n+1}(\mathcal{B}^+)$ by enlarging the matrix with a row and column of zeros. This allows to consider projections of $M_n(\mathcal{B}^+)$ for arbitrary n in one common algebra. This algebra $M_\infty(\mathcal{B}^+) \cong \mathcal{K} \otimes \mathcal{B}^+$ is technically the C^*-inductive limit of the $M_n(\mathcal{B}^+)$ for $n \to \infty$ [W93].

Considering projections of matrix algebras over $\mathcal{B}$ allows one to define addition of homotopy classes of projections, $[p]_0 + [q]_0 = [\mathrm{diag}(p, q)]_0$. This turns $[M_\infty(\mathcal{B}^+)]_0$ into a monoid. The K_0-group $K_0(\mathcal{B})$ of $\mathcal{B}$ is obtained from the monoid $[M_\infty(\mathcal{B}^+)]_0$ by Grothendieck's construction and then getting rid of the added unit in a clever way. For the proof of Theorem 1, we do not need to know the technicalities of these constructions but present them here for sake of completion. First define $K_{00}(\mathcal{B}) = [M_\infty(\mathcal{B}^+)]_0 \times [M_\infty(\mathcal{B}^+)]_0 / \sim$ with $([p]_0, [q]_0) \sim ([p']_0, [q']_0)$ if there exists an $[r]_0 \in [M_\infty(\mathcal{B}^+)]_0$ such that $[p]_0 + [q']_0 + [r]_0 = [p']_0 + [q]_0 + [r]_0$. Now the projection onto the identity part of $\mathcal{B}$ is an algebra homomorphism $\mathcal{B}^+ \to \mathbb{C}$ inducing a map $K_{00}(\mathcal{B}) \to \mathbf{Z}$ whose kernel is by definition $K_0(\mathcal{B})$.

In a similar spirit on considers homotopy classes of unitaries. To make that possible at all in a non-unital algebra one has again to adjoin a unit. Let $U(\mathcal{B})$ be the group of unitaries $u \in \mathcal{B}^+$ such that $u - 1 \in \mathcal{B}$ (the 1 is here the unit in $\mathcal{B}^+$) and denote by $[\mathcal{B}]_1$ its homotopy classes.

$U(M_n(\mathcal{B}))$ is a subgroup of $U(M_{n+1}(\mathcal{B}))$ if embedded by enlarging the matrix with a 1 at the lower right corner and filling in the last row and column with zeros. This allows to consider elements of $U(M_n(\mathcal{B}))$ for arbitrary n in one common group $U(M_\infty(\mathcal{B}))$ and $K_1(\mathcal{B})$ is simply $[M_\infty(\mathcal{B})]_1$. The (non-abelian) product in $U(M_n(\mathcal{B}))$ induces a product in $[M_\infty(\mathcal{B})]_1$ which is abelian and therefore denoted additively. To see this let $u, v \in U(M_n(\mathcal{B}))$. They are identified with $\mathrm{diag}(u,1)$ and $\mathrm{diag}(v,1)$ in $U(M_{2n}(\mathcal{B}))$. Moreover $U(M_{2n}(\mathcal{B}))$ contains a continuous path linking $\mathrm{diag}(v,1)$ and $\mathrm{diag}(1,v)$. Therefore one has $[u]_1 + [v]_1 := [\mathrm{diag}(u,1)\mathrm{diag}(1,v)]_1 = [\mathrm{diag}(v,1)\mathrm{diag}(1,u)]_1$.

4.1 Elliott-Natsume-Nest map

Suppose we have a continuous field of C^*-algebras $(\mathcal{C}, \{\mathcal{C}^\hbar, \varphi_\hbar\}_{\hbar\in I})$ over $I = [0,1]$ which is trivial away from $\hbar = 0$. This means that there are isomorphisms $\phi_\hbar : \mathcal{C}^1 \to \mathcal{C}^\hbar$ for $\hbar > 0$ such that $\phi : C_0((0,1], \mathcal{C}^1) \to \ker(\varphi_0)$: $\varphi_\hbar(\phi(f)) = \phi_\hbar(f(\hbar))$ is an isomorphism. The following theorem shows that in this situation one obtains maps $[\mathcal{C}^0]_i \to [\mathcal{C}^1]_i$ which induce homomorphisms $K_i(\mathcal{C}^0) \to K_i(\mathcal{C}^1)$. We call these maps ENN-maps.

Theorem 3 [ENN93] *Consider a continuous field of C^*-algebras $(\mathcal{C}, \{\mathcal{C}^\hbar, \varphi_\hbar\}_{\hbar\in I})$ over $I = [0,1]$ which is trivial away from $\hbar = 0$. For any projection $p \in \mathcal{C}^0$ there is a projection valued section $\tilde{p} \in \mathcal{C}$ such that $\varphi_0(\tilde{p}) = p$. For any $u \in U(\mathcal{C}^0)$ there is a section $\tilde{u} \in U(\mathcal{C})$ such that $\varphi_0(\tilde{u}) = u$. The maps $\mu_i : [\mathcal{C}^0]_i \to [\mathcal{C}^1]_i$: $\mu_0([p]_0) = [\varphi_1(\tilde{p})]_0$, $\mu_1([u]_1) = [\varphi_1(\tilde{u})]_1$ are well-defined and induce homomorphisms $\mu_i : K_i(\mathcal{C}^0) \to K_i(\mathcal{C}^1)$.*

Proof: (We only show how these maps are constructed, for the rest see [ENN93].) Let p be a projection in $\mathcal{C}^0$. Since φ_0 is surjective, there exists a selfadjoint section $x \in \mathcal{C}$ with $\varphi_0(x) = p$. By property 2 of C^*-fields, we find for any $\delta > 0$ an ϵ such that $\|\varphi_\hbar(x^2 - x)\| < \delta$ for $\hbar < \epsilon$. For small δ the spectrum of $\varphi_\hbar(x)$ is close to $\{0,1\}$ and we can find a continuous function $f : \mathbb{R} \to \mathbb{R}$, vanishing for $t < a$ and being 1 for $t > b$ where $0 < a < b < 1$ and (a,b) does not intersect the spectrum of $\varphi_\hbar(x)$. Then $f(x)$ is another section with $\varphi_0(f(x)) = p$ but such that $\varphi_\hbar(x)$ are projections for $\hbar < \epsilon$. Now the section can be extended by the constant section since the field is trivial away from $\hbar = 0$. The resulting section is $\tilde{p}$.

The choice of x is not canonical but it is not hard to see that the homotopy class of $\tilde{p}$ is uniquely determined since any other choice $\tilde{p}'$ needs to be close to $\tilde{p}$ at small $\hbar$. The case of unitaries works in a similar way.

It can be shown [ENN93] that under the conditions of the theorem and with canonically extended $\varphi_\hbar$, the field $(M_n(\mathcal{C}^+), \{M_n(\mathcal{C}^{\hbar+}), \varphi_\hbar\}_{\hbar\in I})$ is a continuous field of C^*-algebras which is trivial away from 0. The above construction applies therefore also to elements in $[M_n(\mathcal{C}^{0+})]_0$ and $[M_n(\mathcal{C}^0)]_1$ and it is possible to establish that it induces homomorphisms between the corresponding K-groups. □

Example 10 Let us view Section 2.2 in the light of the last theorem. As established in Example 4, the algebras $C^*_\gamma(D_1, D_2)$ for different γ constitute a C^*-field $\mathcal{F}$. When restricted to $I = [0,\gamma]$, this C^*-field is trivial away from 0. Starting with the Bott-projection b, the above theorem insures the existence of a continuous section in $\mathcal{F}$ whose evaluation at 0 is b and claims that the homotopy class of the evalutation at γ is independent of the choice of that section. We have constructed in Section 2.2 explicitely such a section and have seen that it joins the Bott-projection to the Landau projection. Thus we conclude that $\mu_0([b]_0) = [\Pi_\gamma]_0 + [1]_0$.

4.2 Exponential and index map

Suppose given an extension $\mathcal{C} \stackrel{q}{\to} \mathcal{B}$ by $\mathcal{J} := \ker(q)$. What interests us here are two maps which measure in some sense the extend to which the map induced by q on homotopy classes is not surjective. The first of these maps is

$$\exp \,:\, [\mathcal{B}]_0 \to [\mathcal{J}]_1 \,.$$

It induces a homomorphism $\exp : K_0(\mathcal{B}) \to K_1(\mathcal{J})$ which is called the exponential map. It is defined as follows: First let p be a projection in $\mathcal{B}$. Since q is surjective, there exists an $x \in \mathcal{C}$ such that $q(x) = p$. Since p is self-adjoint we can choose x self-adjoint and define

$$\exp[p]_0 \;:=\; [u]_1 \,, \qquad u \;=\; e^{2\pi\imath x} \,.$$

Now $\exp[p]_0$ is well-defined as an element of $[\mathcal{C}]_1$ because, first, any two preimages of p are connected by a continuous path of self-adjoint elements, and second, the exponential function is continuous and thus maps continuous paths onto continuous paths. But using the power series expansion of $e^{2\pi\imath x}$ and the fact that $x^n - x \in \ker(q) = \mathcal{J}$ one sees immediately that $e^{2\pi\imath x} - 1 \in \mathcal{J}$ so that $[e^{2\pi\imath x}]_1$ belongs to $[\mathcal{J}]_1$. This is actually all we need for our application, but we mention that this map extends to a map $\exp : [M_n(\mathcal{B}^+)]_0 \to [M_n(\mathcal{J})]_1$ if one extends q to a map $M_n(\mathcal{C}^+) \to M_n(\mathcal{B}^+)$ by setting $q(1) = 1$ (on added units) and applying it component wise to matrices over the algebras.

If we apply the above to the cone (Example 6) given by $C\mathcal{B} \stackrel{\mathrm{ev}_\infty}{\to} \mathcal{B}$, then $\ker(\mathrm{ev}_\infty)$ is the suspension of $\mathcal{B}$ and the exponential map is the so-called Bott map $\exp = \beta : K_0(\mathcal{B}) \to K_1(S\mathcal{B})$,

$$\beta[p]_0 = [e^{2\pi\imath\chi p}]_1$$

where $\chi : \mathbb{R} \to [0,1]$ is a continuous function with $\lim_{t\to-\infty}\chi(t) = 0$ and $\lim_{t\to\infty}\chi(t) = 1$.

The above exact sequence also gives rise to a map $\mathrm{ind} : K_1(\mathcal{B}) \to K_0(\mathcal{J})$ defined as follows. Given $V \in U(M_n(\mathcal{B}))$ defining a class in $K_1(\mathcal{B})$, let $W \in U(M_{2n}(\mathcal{B}))$ be a lift of $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$. Then

$$\mathrm{ind}([V]_1) \;=\; \left[W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^*\right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_0 \,.$$

The index map of the extension defined by $C\mathcal{B} \stackrel{\mathrm{ev}_\infty}{\to} \mathcal{B}$ is denoted by Θ. The fact [W93] that the compositions $\Theta\beta : K_0(\mathcal{B}) \to K_0(SS\mathcal{B})$ and $\beta\Theta : K_1(\mathcal{B}) \to K_1(SS\mathcal{B})$ are isomorphisms is called Bott-periodicity.

4.3 exp and ind of Wiener-Hopf extension

We want to express the exponential and the index map of the Wiener-Hopf extension $\mathrm{ev}_\infty : C\mathcal{B} \rtimes_{\tau^1\otimes\alpha} \mathbb{R} \longrightarrow \mathcal{B} \rtimes_\alpha \mathbb{R}$ (discussed in Example 8) using an ENN-map. Herefore we use the C^*-fields of Example 9 restricted to $[0,1] \in \mathbb{R}$, namely $(C_0([0,1], C\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}, \{C\mathcal{B} \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R}, \varphi_\hbar\}_{\hbar\in[0,1]})$ and

$$C([0,1], C\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R} \stackrel{\mathrm{ev}_\infty}{\longrightarrow} C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R}\ . \tag{11}$$

The C^*-field corresponding to $C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R}$ is trivial and that corresponding to the ideal $C([0,1], S\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}$ satisfies the conditions of Theorem 3 to give rise to ENN-maps

$$\mu_i : [S\mathcal{B} \rtimes_{\mathrm{id}\otimes\alpha} \mathbb{R}]_i \to [S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R}]_i\ . \tag{12}$$

Proposition 1 *Let* exp *and* ind *be exponential and index map of the Wiener-Hopf extension (9). Then $\mu_1\beta = \exp$ and $\mu_0\Theta = \mathrm{ind}$. Here we have used the identification $C\mathcal{B} \rtimes_{\mathrm{id}\otimes\alpha} \mathbb{R} \cong C(\mathcal{B} \rtimes_\alpha \mathbb{R})$.*

Proof: A projection $p \in \mathcal{B} \rtimes_\alpha \mathbb{R}$ defines a constant section in $C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R}$. If $x \in C([0,1], C\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}$ is a self-adjoint lift of the constant section under (11) then, by definition, $\mu_1[e^{2\pi i\varphi_0(x)}]_1 = [e^{2\pi\imath\varphi_1(x)}]_1$. Furthermore $\exp[p]_0 = [e^{2\pi\imath\varphi_1(x)}]_1$ since $\varphi_1(x)$ is a lift of p in (9). The claim follows since $\varphi_0(x)$ is a lift of p in the extension $C\mathcal{B} \rtimes_{\mathrm{id}\otimes\alpha} \mathbb{R} \stackrel{\varphi_\infty}{\to} \mathcal{B} \rtimes_\alpha \mathbb{R}$, and χp a lift of p in the extension $C(\mathcal{B} \rtimes_\alpha \mathbb{R}) \stackrel{\varphi_\infty}{\to} \mathcal{B} \rtimes_\alpha \mathbb{R}$. Under the identification stated in the lemma $e^{2\pi\imath\chi p}$ is therefore homotopic to $e^{2\pi\imath\varphi_0(x)}$. The argument involving the index map is similar. □

5 Cyclic cohomology

Given a complex algebra $\mathcal{B}$ let $C^n_\lambda(\mathcal{B})$ be the set of $n+1$-linear functionals on $\mathcal{B}$ which are cyclic in the sense that

$$\varphi(A_1, \cdots, A_n, A_0) \;=\; (-1)^n\, \varphi(A_0, \cdots, A_n).$$

Define the boundary operator $b : C^n_\lambda(\mathcal{B}) \to C^{n+1}_\lambda(\mathcal{B})$:

$$b\varphi(A_0, \cdots, A_{n+1}) \;=\; \sum_{j=0}^{n} (-1)^j \varphi(A_0, \cdots, A_j A_{j+1}, \cdots, A_{n+1}) + (-1)^{n+1}\varphi(A_{n+1}A_0, \cdots, A_n)\ .$$

Definition 1 *The cyclic cohomology $HC(\mathcal{B})$ of $\mathcal{B}$ is the cohomology of the complex*

$$0 \;\to\; C^0_\lambda(\mathcal{B}) \;\to\; \cdots \;\to\; C^n_\lambda(\mathcal{B}) \;\stackrel{b}{\to}\; C^{n+1}_\lambda(\mathcal{B}) \;\to\; \cdots\ .$$

An element $\varphi \in C^n_\lambda(\mathcal{B})$ satisfying $b\varphi = 0$ is called a cyclic n-cocycle.

5.1 Cycles

A very convenient way of looking at cyclic cocycles is in terms of characters of graded differential algebras with graded closed traces $(\Omega, d, \int)$ over $\mathcal{B}$. Here $\Omega = \bigoplus_{n\in\mathbb{N}_0} \Omega^n$ is a graded algebra (we denote by $\deg(a)$ the degree of a homogeneous element a) and d a graded differential on Ω of degree 1. A graded trace on the subspace Ω^n is a linear functional $\int : \Omega^n \to \mathbb{C}$ which is cyclic in the sense that $\int w_1 w_2 = (-1)^{\deg(w_1)\deg(w_2)} \int w_2 w_1$. It is closed if it vanishes on $d(\Omega^{n-1})$. In the situation below there is a largest number n for which Ω^n is non-trivial. This n is called the top degree of Ω. The graded trace will be a graded trace on the sub-space of top degree.

Definition 2 *An n-dimensional cycle is a graded differential algebra (Ω, d) of top degree n together with a closed graded trace $\int$ on Ω^n. A cycle $(\Omega, d, \int)$ is called a cycle over $\mathcal{B}$ if there is an algebra homomorphism $\mathcal{B} \to \Omega^0$.*

We will assume here that the homomorphism $\mathcal{B} \to \Omega^0$ is injective and hence identify $\mathcal{B}$ with a sub-algebra of Ω^0. The connection with cyclic cocycles is given by the following proposition [C94].

Proposition 2 *Any cycle of dimension n over $\mathcal{B}$ defines a cyclic n-cocycle through what is called its character:*

$$\eta(A_0, \ldots, A_n) \;=\; \int A_0 dA_1 \cdots dA_n \;.$$

Conversely, any cyclic n-cocycle arises as the character of an n-cycle.

A (bounded) trace over $\mathcal{B}$ is an example of a cyclic 0-cocycle. Taking $\Omega = \mathcal{B}$, $d = 0$, $\int$ to be that trace, we have a realization of the trace as character of a 0-cycle.

In many cases, however, the cyclic cohomology of C^*-algebras contains only the (bounded) traces and is too small for many purposes. Therefore one considers also characters of n-cycles over dense sub-algebras $\mathcal{B}'$ of $\mathcal{B}$ satisfying a continuity condition. These are called higher traces [C94]. It will be useful to relax the requirement of $\mathcal{B}$ being a C^*-algebra and consider more generally Banach algebras.

Definition 3 *An n-trace on a Banach algebra $\mathcal{B}$ is the character of an n-cycle $(\Omega', d, \int)$ over a dense sub-algebra $\mathcal{B}'$ of $\mathcal{B}$ such that for all $A_1, \ldots, A_n \in \mathcal{B}'$ exists a constant $C = C(A_1, \ldots, A_n)$ such that*

$$\left| \int (X_1 dA_1) \cdots (X_n dA_n) \right| \;\leq\; C \|X_1\| \cdots \|X_n\| \;, \tag{13}$$

for all $X_j \in \mathcal{B}'^{+}$.

Condition (13) may be rephrased by saying that for all $A_1, \ldots, A_n \in \mathcal{B}'$ the a priori densely defined multi-linear functional

$$\mathcal{B}^{\times n} \to \mathbb{C} : \quad (X_1, \ldots, X_n) \;\mapsto\; \int (X_1 dA_1) \cdots (X_n dA_n)$$

extends to a bounded multi-linear functional. Denoting by $p(A_1,\dots,A_n)$ the norm of that functional, i.e. the best possible constant C in (13), we have a family of maps $\mathcal{B}^{\times n} \to \mathbb{R}$ satisfying

$$p(A_1,\dots,\lambda A_j+\lambda' A_j',\dots,A_n) \;\le\; |\lambda|\, p(A_1,\dots,A_j,\dots,A_n)+|\lambda'|\, p(A_1,\dots,A_j',\dots,A_n)\;.$$

But since d is a derivation, it also satisfies

$$p(A_1,\dots,A_jA_j',\dots,A_n) \;\le\; \|A_j'\|\, p(A_1,\dots,A_j,\dots,A_n)+\|A_j\|\, p(A_1,\dots,A_j',\dots,A_n)\;.$$

For simplicity, rather than considering cycles $(\Omega',d,\int)$ over a dense sub-algebra $\mathcal{B}'$, we shall consider triples $(\Omega,d,\int)$ as in Definition 2 with $\mathcal{B}\subset\Omega^0$, but allowing for the possibility that d and $\int$ are only densely defined. If their character is densely defined and satisfies (13), we call the triple $(\Omega,d,\int)$ an unbounded n-cycle. The role of (13) is to insure the existence of a third algebra $\mathcal{B}''$, $\mathcal{B}'\subset\mathcal{B}''\subset\mathcal{B}$, to which the character can be extended (by continuity) and such that the inclusion $i:\mathcal{B}''\hookrightarrow\mathcal{B}$ induces an isomorphism between $K(\mathcal{B}'')$ and $K(\mathcal{B})$ [C86]. We comment on that in Section 6.1.

An example of a cycle for the commutative algebra $\mathcal{B}=C(M)$ of continuous functions over a compact manifold without boundary is given by the algebra of exterior forms with its usual differential $(\Omega(M),d)$ and graded trace equal to integration of n-forms, $n=\dim(M)$. This is an unbounded cycle. One may take $\mathcal{B}'=C^\infty(M)$ and $p(A_1,\dots,A_n)=\int|dA_1\cdots dA_n|$ where (locally) $|dA_1\cdots dA_n|=|f|d\,\mathrm{vol}$ if $dA_1\cdots dA_n=f\,d\,\mathrm{vol}$. Note that $p(A_1,\dots,A_n)$ is not continuous in A_j w.r.t. the supremum norm.

A 0-trace is a (possibly densely defined unbounded) linear functional tr which is cyclic and satisfies (13). A positive trace is a positive linear functional tr which is cyclic. It might be unbounded (with dense domain), but it always satisfies $|\mathrm{tr}(AX)|\le\mathrm{tr}(|A|)\|X\|$ if A is trace class and hence (13) holds with $\mathcal{B}'$ being the ideal of trace class elements.

Here we need to construct higher traces on a Banach algebra $\mathcal{B}$ on which is given a differentiable action of $\mathbb{R}^n$ leaving a (possibly unbounded) trace invariant. This is essentially Ex. 12, p. 254 of [C94].

Proposition 3 *Let $\mathcal{B}$ be a Banach algebra with a differentiable action of $\mathbb{R}^n$ and $\mathcal{T}$ be a (possibly unbounded) invariant positive trace on $\mathcal{B}$. Denote by ∇_j, $j=1,\cdots,n$, commuting closed derivations defined by the action and suppose that*

$$\mathcal{B}' \;=\; \{A\in\bigcap_{j=1}^{n}\mathrm{dom}(\nabla_j)|\;\exists\, j \text{ s.t. } \nabla_jA \text{ traceclass}\}$$

is dense in $\mathcal{B}$. Then $(\Omega,d,\int)$ is an unbounded n-cycle over $\mathcal{B}$ where

$$\Omega \;:=\; \mathcal{B}\otimes\Lambda\mathbb{C}^n\;,$$

the tensor product of $\mathcal{B}$ with the Grassmann algebra $\Lambda\mathbb{C}^n$ with generators e_j, $j=1,\dots,n$,

$$d(A\otimes v) \;=\; \sum_{j=1}^{n}\nabla_jA\otimes e_jv\;,$$

and $\int = \mathcal{T} \otimes \imath$ *with* $\imath(e_1 \cdots e_n) = 1$*, explicitly*

$$\int A_0 dA_1 \cdots dA_n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \mathcal{T}(A_0 \nabla_{\sigma(1)} A_1 \cdots \nabla_{\sigma(n)} A_n)\ .$$

Proof: The algebraic aspects of this proposition are straightforward to show, see e.g. [KRS02]. Since trace class operators form an ideal, $\mathcal{B}'$ is a sub-algebra. Then (13) follows from

$$|\mathcal{T}((X_1\nabla_1 A_1)\cdots(X_n\nabla_n A_n))| \ \le\ \|X_1\|\cdots\|X_n\|\,\|\nabla_1 A_1\|\cdots\|\nabla_{n-1}A_{n-1}\|\ \mathcal{T}(|\nabla_n A_n|)\ ,$$

and the cyclicity of $\mathcal{T}$. □

Example 11 The Chern classes of (2) and (6) and the non-commutative winding number of (3) are examples of this construction with $n = 2$ and $n = 1$ respectively. In the first case the algebra is that of Example 7, namely $\mathcal{A}_\infty = C(\Sigma) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\beta^\perp} \mathbb{R}$, with trace $\mathcal{T} : \mathcal{A}_\infty \to \mathbb{C}$:

$$\mathcal{T}(F) \ =\ \int_\Sigma d\mathbf{P}(\sigma) F(0)(0)(\sigma) \tag{14}$$

and derivations

$$\nabla_j(F)(x_2)(x_1) \ =\ \imath x_j F(x_2)(x_1)\,. \tag{15}$$

Then ch is then $-2\pi\imath$ times the character of the 2-cycle constructed from these data using Proposition 3. Its domain includes the dense sub-algebra $C_c(\mathbb{R}, C_c(\mathbb{R}, C(\Sigma)))$. This case specializes to (6) if one has no disorder so that the algebra reduces to $C^*_\gamma(D_1, D_2)$. Finally, in the case of ξ, the algebra is the ideal $\mathcal{E} = \ker(\mathrm{ev}_\infty) \subset \mathcal{A} = C_0(\hat{\Sigma}) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\tau\otimes\beta^\perp} \mathbb{R}$ with trace $\hat{\mathcal{T}} : \mathcal{A} \to \mathbb{C}$:

$$\hat{\mathcal{T}}(F) \ =\ \int_{\hat{\Sigma}} d\mathbf{P}(\sigma) ds F(0)(0)(\sigma, s) \tag{16}$$

and derivation

$$\nabla_1(F)(x_2)(x_1) \ =\ \imath x_1 F(x_2)(x_1). \tag{17}$$

The character of the 1-cycle constructed from these data using Proposition 3 is ξ and its domain contains $C_c(\mathbb{R}, C_c(\mathbb{R}, C(\hat{\Sigma}))) \cap \mathcal{E}$.

The following is an extension of the above construction.

Proposition 4 *Let* $(\Omega, d, \int)$ *be a (possibly unbounded) k-cycle over the Banach algebra* $\mathcal{B}$ *which is invariant under a differentiable action of* $\mathbb{R}^n$ *in the sense that this action commutes with d and leaves* $\int$ *invariant. Denote by* ∇_j*,* $j = 1, \cdots, n$*, commuting closed derivations defined by the action and suppose that* $\bigcap_{j=1}^n \mathrm{dom}(\nabla_j) \cap \mathcal{B}'$ *is a dense sub-algebra of* $\mathcal{B}$ *such that on* $\mathcal{B}' \subset \mathcal{B}$ *the character of* $(\Omega, d, \int)$ *is fully defined. Taking* $\Omega' = \Omega \hat{\otimes} \Lambda\mathbb{C}^n$*, the graded tensor product,* $d' = d\hat{\otimes}1 + \delta$ *with* $\delta(w\hat{\otimes}v) = (-1)^{\partial w} \sum_j \nabla_j w \hat{\otimes} e_j v$ *and* $\int' = \int \hat{\otimes}\, \imath$*, one obtains a* $k+n$*-cycle* $(\Omega', d', \int')$ *over* $\mathcal{B}$*.*

Proof: The algebraic aspects are straightforward and again given in [KRS02]. The only point to settle is condition (13). It follows iteratively from the case $n = 1$. For $n = 1$, using cyclicity,

$$\begin{aligned}\left|\int' (X_1 d'A_1)\dots(X_{k+1}d'A_{k+1})\right| &\leq \sum_{j=1}^{k+1}\left|\int (X_j\nabla_1 A_j)(X_{j+1}\delta A_{j+1})\cdots(X_{j-1}\delta A_{j-1})\right| \\ &\leq \sum_{j=1}^{k+1}\|X_1\|\cdots\|X_{k+1}\|\,\|\nabla_1 A_j\|\,C_j,\end{aligned}$$

where C_j depends only on $A_1,\dots,A_{j-1},A_{j+1},\dots,A_{k+1}$. This inequality shows also that the character of the cycle is defined on $\bigcap_{j=1}^n \mathrm{dom}(\nabla_j)\cap\mathcal{B}'$. □

5.2 Cyclic cocycles for crossed products with $\mathbb{R}$

An action of $\mathbb{R}$ on a graded differential algebra (Ω, d) is a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\Omega)$ such that $\forall\, t \in \mathbb{R}$, α_t has degree 0 and commutes with d. If Ω is a Banach algebra or even a C^*-algebra we require in addition that for all $A \in \mathcal{B}$, $t \mapsto \alpha_t(A)$ is continuous and $\|\alpha_t\| = 1$. Therefore we can form the crossed product $L^1(\Omega,\mathbb{R},\alpha)$ or even $\Omega\rtimes_\alpha\mathbb{R}$.

Definition 4 *A n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$ is called invariant under an action α of $\mathbb{R}$ on Ω if the graded trace $\int$ is invariant under it. If $(\Omega, d, \int)$ is unbounded, we require in addition that the norms $p(A_1,\dots,A_n)$* cf. Definition 3) *satisfy that*

$$Q(A_1,\dots,A_n) := \sup_{t_i\in\mathbb{R}} p(\alpha_{t_1}(A_1),\dots,\alpha_{t_n}(A_n)) \tag{18}$$

is finite for all $A_j \in \mathcal{B}' \subset \mathcal{B}$ where $\mathcal{B}'$ is a dense sub-algebra on which the character of the n-cycle is fully defined. A cyclic cocycle of $\mathcal{B}$ is invariant under an action α of $\mathbb{R}$ if it is the character of an α-invariant cycle $(\Omega, d, \int)$.

We note that, by the cyclicity of the graded trace, the above additional condition is equivalent to demanding that $\sup_{t_1\in\mathbb{R}} p(\alpha_{t_1}(A_1), A_2,\dots,A_n)$ exists for all $A_j \in \mathcal{B}'$. Furthermore, Q inherits the properties of p, i.e.

$$Q(A_1,\dots,\lambda A_j+\lambda' A_j',\dots,A_n) \leq |\lambda|\,Q(A_1,\dots,A_j,\dots,A_n)+|\lambda'|\,Q(A_1,\dots,A_j',\dots,A_n)\,, \tag{19}$$

$$Q(A_1,\dots,A_jA_j',\dots,A_n) \leq \|A_j'\|\,Q(A_1,\dots,A_j,\dots,A_n)+\|A_j\|\,Q(A_1,\dots,A_j',\dots,A_n)\,. \tag{20}$$

Proposition 5 *Let $(\Omega, d, \int)$ be an α-invariant (possibly unbounded) n-cycle over the Banach algebra $\mathcal{B}$ and $\nabla : L^1(\mathbb{R},\mathcal{B},\alpha) \to L^1(\mathbb{R},\mathcal{B},\alpha)$ be the derivation $\nabla f(x) = \imath x f(x)$. Then $(\Omega_\alpha, d_\alpha, \int_\alpha)$ is an unbounded $n+1$-cycle over $L^1(\mathbb{R},\mathcal{B},\alpha)$ where*

$$\Omega_\alpha = L^1(\mathbb{R},\Omega,\alpha)\,\hat{\otimes}\,\Lambda\mathbb{C},$$

$$d_\alpha(\omega\hat{\otimes}v) = d'\omega\hat{\otimes}v+(-1)^{\deg(\omega)}\,\nabla\omega\hat{\otimes}\,e_1 v\,, \qquad d'\omega(x) = d(\omega(x))\,, \quad \nabla\omega(x) = \imath x\omega(x)\,,$$

and $\int_\alpha = \int \mathrm{ev}_0\hat{\otimes}\,\imath$, i.e.

$$\int_\alpha f_0 d_\alpha f_1\cdots d_\alpha f_{n+1} = \sum_{j=1}^{n+1}(-1)^{n+1-j}\int (f_0 df_1\cdots df_{j-1}(\nabla f_j)df_{j+1}\cdots df_{n+1})\,(0).$$

Proof: We first show that the triple $(L^1(\mathbb{R},\Omega,\alpha),d',\int \mathrm{ev}_0)$ defines an unbounded n-cycle over $L^1(\mathbb{R},\mathcal{B},\alpha)$. The required algebraic properties are straightforwardly checked (*cf.* [KRS02]) and we focus here on the continuity aspects (13). Let $\mathcal{B}'\subset\mathcal{B}$ be a dense sub-algebra on which the character of $(\Omega,d,\int)$ is fully defined and

$$\mathcal{V}^{\mathrm{fin}} \;:=\; \bigcup_{V\subset\mathcal{B}',\,\dim V<\infty} C_c(\mathbb{R},\overline{V})$$

where $\overline{V}:=\bigcup_{t\in\mathbb{R}}\alpha_t(V)$ is the orbit of V under the action. The space $\mathcal{V}^{\mathrm{fin}}$ is linear and since $Q(A_1,\ldots,A_n)$ is finite for all $A_j\in\mathcal{B}'$, we obtain from (19) that

$$\overline{Q}(f_1,\ldots,f_n) \;:=\; \sup_{t_j\in\mathbb{R}} Q(f_1(t_1),\ldots,f_n(t_n))$$

is finite for all $f_j\in\mathcal{V}^{\mathrm{fin}}$. Furthermore, $\mathcal{V}^{\mathrm{fin}}$ is clearly dense in $C_c(\mathbb{R},\mathcal{B}')$ in the L^1-norm. Now let $\mathcal{A}^{fin}$ be the sub-algebra of $L^1(\mathbb{R},\mathcal{B},\alpha)$ generated algebraically by $\mathcal{V}^{\mathrm{fin}}$, i.e. it consists of finite twisted convolution products of elements of $\mathcal{V}^{\mathrm{fin}}$. Since, by (20),

$$Q(f_1g_1(t),f_2(t)\ldots) \;\leq\; \int ds\,[\|f_1(s)\|\,Q(g_1(t-s),f_2(t)\ldots) \;+\; \|g_1(t-s)\|Q(f_1(s),f_2(t)\ldots)]\;,$$

we obtain

$$\overline{Q}(f_1g_1,f_2\ldots,f_n) \;\leq\; \|f_1\|_{L^1}\,\overline{Q}(g_1,f_2\ldots)+\|g_1\|_{L^1}\,\overline{Q}(f_1,f_2\ldots)\,.$$

Therefore $\overline{Q}(f_1,\ldots,f_n)$ is finite for all $f_j\in\mathcal{A}^{\mathrm{fin}}$. The character of $(L^1(\mathbb{R},\Omega,\alpha),d',\int \mathrm{ev}_0)$ is now restricted to the dense sub-algebra $\mathcal{A}^{\mathrm{fin}}$. Then we obtain

$$\left|\int \mathrm{ev}_0(X_1d'f_1)\ldots(X_nd'f_n)\right| \;\leq\; \|X_1\|_{L^1}\cdots\|X_n\|_{L^1}|\mathrm{supp}(f_1)|\cdots|\mathrm{supp}(f_n)|\;\overline{Q}(f_1,\ldots,f_n)\,. \tag{21}$$

Here $|\mathrm{supp}(f)|$ is the (finite) length of the support of f. This shows that there exists a dense sub-algebra of the Banach algebra $L^1(\mathbb{R},\mathcal{B},\alpha)$ on which the character of $(L^1(\mathbb{R},\Omega,\alpha),d',\int \mathrm{ev}_0)$ satisfies (13).

Now on $L^1(\mathbb{R},\mathcal{B},\alpha)$ and $L^1(\mathbb{R},\Omega,\alpha)$ we have the dual action of $\tilde{\mathbb{R}}$ and its corresponding derivation is ∇. It is densely defined and satisfies the conditions of Proposition 4. Applying that proposition, one obtains $(L^1(\mathbb{R},\Omega,\alpha),d_\alpha,\int_\alpha)$, which is an unbounded $n+1$-cycle. □

If ξ is the character of $(\Omega,d,\int)$, we denote by $\#_\alpha\xi$ the character of $(\Omega_\alpha,d_\alpha,\int_\alpha)$. Restricted to the context of smooth $\mathbb{R}$-actions on smooth sub-algebras of C^*-algebras Proposition 5 can be viewed as a special case of a result stated in the article [ENN88]. In that article, an $n+1$-cycle over a smooth sub-algebra of $\mathcal{B}\rtimes_\alpha\mathbb{R}$ is constructed from an n-cycle $(\Omega,d,\int)$ over $\mathcal{B}$ even if $\int$ is not invariant under α.

Whenever we have several commuting $\mathbb{R}$ actions leaving a cycle invariant, we can iterate this construction, since we can extend a second action β in Ω to an action on Ω_α commuting with the differential d_α and leaving $\int_\alpha$ invariant by evaluating it pointwise on functions $f:\mathbb{R}\to\Omega$ and keeping the new Grassmann generator of Ω_α fixed.

Example 12 (Suspension of cocycles) Recall from Example 1 that $\mathcal{B} \rtimes_{\mathrm{id}} \mathbb{R}$ is via Fourier transform isomorphic to the suspension $S\mathcal{B}$. For a given n-trace ξ over $\mathcal{B}$ the above construction yields an $n+1$-trace $\#_{\mathrm{id}}\xi$ over $L^1(\mathbb{R},\mathcal{B},\mathrm{id})$. When intertwined with the Fourier transform one obtains a $n+1$-trace which we denote by ξ^s over a dense Banach sub-algebra of $S\mathcal{B}$. However, since the conditions of Definition 4 are trivially satisfied and the linear space $\mathcal{V}^{\mathrm{fin}}$ used in the proof of Proposition 5 is a dense sub-algebra of $C_0(\mathbb{R},\mathcal{B})$ under pointwise multiplication we can simplify the arguments of Proposition 5 thereby improving (21), namely, for $f_i \in \mathcal{V}^{\mathrm{fin}}$,

$$\left|\int_{\mathbb{R}} ds \int (X_1 d' f_1)(s)\dots(X_n d' f_n)(s)\right| \;\leq\; \|X_1\|\cdots\|X_n\|\,\tilde{Q}(f_1,\dots,f_n) \tag{22}$$

where $\tilde{Q}(f_1,\dots,f_n) = |\mathrm{supp}(f_1)|\cdots|\mathrm{supp}(f_n)|\,\sup_{t_j\in\mathbb{R}} p(f_1(t_1),\dots,f_n(t_n))$ and the norm is here the C^*-norm on $S\mathcal{B}$.

As a result, if $(\Omega, d, \int)$ is an n-cycle over a Banach algebra $\mathcal{B}$ whose character is ξ and $\partial_s : S\mathcal{B} \to S\mathcal{B}$ the derivative w.r.t. the suspension variable, then ξ^s is the character of the unbounded $n+1$-cycle $(\Omega^s, d^s, \int^s)$ over $S\mathcal{B}$ where $\Omega^s := S\Omega\hat{\otimes}\Lambda\mathbb{C}$ and $(\omega \in S\Omega)$

$$d^s(\omega\hat{\otimes}v) \;=\; d'\omega\hat{\otimes}\, v \;+\; (-1)^{\deg(\omega)}\,\partial_s\omega\hat{\otimes}\, e_1 v\;,$$

where $d'\omega(s) = d(\omega(s))$ and $\int^s = \int_{\mathbb{R}} ds \int \hat{\otimes}\, \imath$.

Example 11 revisited We can view the Chern classes of Example 11 as being constructed from 0-traces using Proposition 5. In fact, $\mathrm{ch} = -2\pi\imath\, \#_{\beta^\perp}\#_{\beta^\parallel} \int_\Sigma d\mathbf{P}$ as can be directly verified. Without disorder this specializes to $\mathrm{ch} = -2\pi\imath\, \#_{\hat{\tau}^\gamma}\#_{\mathrm{id}}\mathrm{Tr}$ where $\mathrm{Tr} : \mathbb{C} \to \mathbb{C}$ defined by $\mathrm{Tr}(1) = 1$. For later purposes, we note that under Fourier transform of the first variable ch intertwines to $-\imath\, \#_{\tau^\gamma}\mathrm{Tr}^s$ (without disorder). If, moreover, $\gamma = 0$, then one obtains the geometric Chern character ch_g.

Example 13 (Canonical cocycle of the Heisenberg group algebra) We first construct the canonical cocycle of the group algebra of $\mathbb{R}^2$ which is equal to $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R} \cong SS\mathbb{C}$. On $\mathbb{C}$ we consider the trace Tr, a 0-cocycle which is the character of the 0-cycle $(\mathbb{C}, 0, \mathrm{id})$. Then we apply the construction of Example 12 twice to obtain the double suspension of Tr for the algebra $SS\mathbb{C} \cong \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R}$, namely $(\mathbb{C}^{ss}, d^{ss}, \mathrm{Tr}^{ss})$ where $\mathbb{C}^{ss} = SS\mathbb{C}\hat{\otimes}\Lambda\mathbb{C}^2$, $d^{ss} = \partial_{a_3}\hat{\otimes}e_1 + \partial_{a_2}\hat{\otimes}e_2$ and $\mathrm{Tr}^{ss} = \int_{\mathbb{R}^2} da_2 da_3 \mathrm{Tr} \otimes \imath$ (we use the notation of Example 4). This 2-trace is referred to as the canonical cocycle of the group algebra of $\mathbb{R}^2$. Its character is precisely ch_g.

Recall that the Heisenberg group algebra is isomorphic to $SS\mathbb{C}\#_{\tilde{\tau}}\mathbb{R}$ and so we seek to apply Proposition 5 w.r.t. the action $\tilde{\tau}$ to the above double suspension. Since this action does not commute with ∂_{a_3}, we cannot extend it trivially to the Grassmann generators. Instead we set

$$\begin{aligned}
(\tilde{\tau}_{a_1}(f\hat{\otimes}1))(a_2)(a_3) &= f(a_2)(a_3 - a_2 a_1)\hat{\otimes}1\;,\\
\tilde{\tau}_{a_1}(1\hat{\otimes}e_1) &= 1\hat{\otimes}e_1\;,\\
\tilde{\tau}_{a_1}(1\hat{\otimes}e_2) &= 1\hat{\otimes}e_2 - a_1(1\hat{\otimes}e_1)\;.
\end{aligned}$$

We claim that $\tilde{\tau}_{a_1}$ commutes with d^{ss}: it suffices to check this for elements of degree 0 in e_1 and e_2 where we get

$$\begin{aligned}
\tilde{\tau}_{a_1} d^{ss}(f\hat{\otimes}1)(a_2)(a_3) &= \partial_1 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}e_1 + \partial_2 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}(e_2 - a_1 e_1) \,, \\
d^{ss}\tilde{\tau}_{a_1}(f\hat{\otimes}1)(a_2)(a_3) &= \left(\partial_1 f(a_2)(a_3 - a_2 a_1) - a_1 \partial_2 f(a_2)(a_3 - a_2 a_1)\right)\hat{\otimes}\, e_1 \\
&\quad + \partial_2 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}e_2 \,.
\end{aligned}$$

Furthermore, $\tilde{\tau}$ leaves the graded trace Tr^{ss} invariant, because of $\tilde{\tau}_{a_1}(1\hat{\otimes}e_1e_2) = 1\hat{\otimes}e_1e_2$ and the translation invariance of the Lebesgue measure on $\mathbb{R}^2$. In order to apply Proposition 5, we show that the 2-cycle $(\mathbb{C}^{ss}, d^{ss}, \mathrm{Tr}^{ss})$ satisfies the uniform bound (18) w.r.t. the dense sub-algebra $C^1_c(\mathbb{R}^2, \mathbb{C}) \subset SS\mathbb{C}$ given by continuously differentiable functions with compact support. One finds for the norms $p(f_1, f_2)$ (*cf.* Definition 3), using $\partial_{a_2}\tilde{\tau}_t(f_1) = \tilde{\tau}_t(\partial_1 f_1) - t\,\tilde{\tau}_t(\partial_2 f_1)$ and $\partial_{a_3}\tilde{\tau}_t(f_1) = \tilde{\tau}_t(\partial_2 f_1)$,

$$\begin{aligned}
p(\tilde{\tau}_t(f_1), f_2) \;\leq\; & \left(\|\tilde{\tau}_t(\partial_1 f_1)\|\,\|\partial_2 f_2\| \;+\; \|\tilde{\tau}_t(\partial_2 f_1)\|\,\|\partial_1 f_2\|\right)|\mathrm{supp}(f_2)| \\
& + \|\tilde{\tau}_t(\partial_2 f_1)\|\,\|\partial_2 f_2\|\;|t|\;|\mathrm{supp}(\tilde{\tau}_t f_1)\cap \mathrm{supp}(f_2)| \,.
\end{aligned}$$

Here $\|.\|$ is the supremum norm on $SS\mathbb{C}$. Since $|t|\;|\mathrm{supp}(\tilde{\tau}_t f_1)\cap\mathrm{supp}(f_2)|$ is bounded in t for any two compactly supported functions f_1, f_2, we obtain the desired result, namely that $p(\tilde{\tau}_t(f_1), f_2)$ is bounded in t.

Hence we are in a position to apply Proposition 5 from which we then obtain a 3-cycle over $L^1(\mathbb{R}, SS\mathbb{C}, \tilde{\tau})$ given by

$$\left(L^1(\mathbb{R}, SS\mathbb{C}\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau})\hat{\otimes}\Lambda\mathbb{C}, d^{ss}_{\tilde{\tau}}, \int_{\mathbb{R}^2} da_2 da_3\, \mathrm{Tr}\,\mathrm{ev}_0 \otimes \imath\right) \,.$$

Its character is the canonical cocycle of the Heisenberg group algebra. Proposition 5 insures that this character is a 3-trace on the Banach sub-algebra $L^1(\mathbb{R}, SS\mathbb{C}, \tilde{\tau})$ of the Heisenberg group algebra, but this is sufficient for our purposes.

Example 14 If $(\Omega, d, \int)$ is an α-invariant n-cycle over $\mathcal{B}$, then the above construction straightforwardly generalizes to the C^*-field $SS\mathcal{B}\rtimes_{\tilde{\tau}\otimes\alpha}\mathbb{R}$ from Example 5 and we obtain the $n+3$-cycle

$$\left(L^1(\mathbb{R}, SS\Omega\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau}\otimes\alpha)\hat{\otimes}\Lambda\mathbb{C}, d^{ss}_{\tilde{\tau}\otimes\alpha}, \int_{\mathbb{R}^2} da_2 da_3 \int \mathrm{ev}_0 \otimes \imath\right) \,.$$

whose character is an $n+3$-trace on the Banach sub-algebra $L^1(\mathbb{R}, SS\mathcal{B}, \tilde{\tau}\otimes\alpha)$. The details of the proof that the intermediate $n+2$-cycle is $\tilde{\tau}\otimes\alpha$-invariant are worked out as above and we only indicate how to show the bound (18) for the $n+2$-cycle

$$\left(SS\Omega\hat{\otimes}\Lambda\mathbb{C}^2, d^{ss}, \int_{\mathbb{R}^2} da_2 da_3 \int \otimes\, \imath\right) \,.$$

For that consider the dense sub-algebra $\mathcal{V}^{\mathrm{fin}} = \bigcup_{V\cup\mathcal{B}', \dim V<\infty} C^1_c(\mathbb{R}^2, V)$ of $SS\mathcal{B}$. Then

$$\tilde{Q}(f_1, \ldots, f_n) := |\mathrm{supp}(f_1)| \ldots |\mathrm{supp}(f_n)| \sup_{s_j, t_j\in\mathbb{R}^2} p(\alpha_{s_1}(f_1(t_1)), \ldots, \alpha_{s_n}(f_n(t_n)))$$

is finite for all $f_j \in \mathcal{V}^{\text{fin}}$, and we get the desired bound (18) from

$$\begin{aligned} p(\tilde{\tau} \otimes \alpha_t(f_1), f_2, \ldots, f_{n+2}) \;\leq\; & \sum_{k \neq j} \|\partial_1 f_k\| \, \|\partial_2 f_j\| \tilde{Q}(\ldots \text{no } f_k \,\&\, f_j \ldots) \\ & + \sum_{j \neq 1} \|\partial_2 f_1\| \, \|\partial_2 f_j\| \tilde{Q}(\ldots \text{no } f_1 \,\&\, f_j \ldots) |t| |\text{supp}(\tilde{\tau}_t f_1) \cap \text{supp}(f_j)| \,. \end{aligned}$$

5.3 Chains and boundaries

The interest in the following definition is that the graded traces are not supposed to be closed.

Definition 5 *An n-dimensional chain $(\Omega, d, \int, \partial\Omega)$ is a graded differential algebra (Ω, d) of top degree n together with a graded trace $\int$ on Ω^n and a surjective homomorphism of graded algebras of degree zero $r : \Omega \to \partial\Omega$ onto a graded algebra of top degree $n-1$ such that $d\ker(r) \subset \ker(r)$ and $\int d\omega = 0$ if $r(\omega) = 0$. The chain is called a chain over $\mathcal{B}$ if there exists an algebra homomorphism $\mathcal{B} \to \Omega^0$.*

As for cycles, we consider here only the case that $\mathcal{B} \subset \Omega^0$.

In the framework of chains over Banach algebras we need to allow their characters (defined as for cycles using $\int$) to be unbounded and only densely defined. We speak about an unbounded chain (over $\mathcal{B}$) if their character satisfies the continuity condition (13). Connes analysis of [C86] in which he shows that characters satisfying (13) can be extended by continuity to a dense sub-algebra $\mathcal{B}''$ which is closed under functional holomorphic calculus does not require $\int$ to be closed and therefore extends to the case of chains.

An example of an unbounded chain for a commutative algebra $C(M)$ of functions over a compact manifold is obtained when one looks at the algebra of exterior forms with its usual differential and integration structure, but M has a boundary ∂M. The map r is then simply the restriction to the boundary and $\partial\Omega = \Omega(\partial M)$. In this context Stokes' Theorem relates integration of exact $\dim(M)$-forms over M to the integration of a form over the boundary ∂M. The following definition is motivated so that such a theorem holds automatically in the non-commutative setting.

Definition 6 *The boundary of an n-dimensional chain $(\Omega, d, \int, \partial\Omega, r)$ is the $n-1$-dimensional cycle $(\partial\Omega, d', \int')$ where*

$$d'\omega' \;=\; rd\omega \;, \qquad \int' \omega' \;=\; \int d\omega \;,$$

for some $\omega \in r^{-1}(\omega')$.

In the framework of chains over Banach algebras we require that r is continuous (which is automatic for C^*-algebras) and therefore maps sub-algebras which are closed under holomorphic functional calculus to algebras which are closed under holomorphic functional calculus. If the character of the chain extends to a dense sub-algebra $\mathcal{B}'' \subset \mathcal{B}$ then the character of the boundary of the chain (defined using $\int'$) extends to $r(\mathcal{B}'')$ and thus (13) guarantees that the latter extends to a dense sub-algebra of $r(\mathcal{B})$ which is closed under holomorphic functional calculus.

The following example plays the analog in cyclic cohomology of the ENN-map in (12).

Example 15 Consider the continuous C^*-field $SS\mathcal{B} \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}$ from Example 5 together with an α invariant n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$. In Example 14 we have constructed an $n+3$-cycle for a dense Banach sub-algebra of the field. Now restrict this C^*-field to the interval $[\hbar_0, \hbar_1] \subset \mathbb{R}$, i.e. consider $C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \mathcal{B}) \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}$. If we repeat the construction of the $n+3$-cycle, we end up with a graded differential algebra with graded trace

$$\left(L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C}\,, d^{ss}_{\tilde{\tau}\otimes\alpha}, \int_{[\hbar_0,\hbar_1]\times\mathbb{R}} d\hbar\, ds \int \mathrm{ev}_0 \otimes \imath \right)$$

which is not closed. This is a chain if we set

$$\partial\left(L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C}\right) \;=\; \bigoplus_{j=0,1} L^1(\mathbb{R}, C_0(\mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}, \tau^{\hbar_j} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C},$$

the first Grassmann part $\Lambda\mathbb{C}$ being the sub-algebra of $\Lambda\mathbb{C}^2$ of elements not containing e_1, and

$$r(\omega) = \begin{cases} \omega|_{\hbar=\hbar_0} \oplus \omega|_{\hbar=\hbar_1} & \text{if } \omega e_1 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

To verify this let us split $d^{ss}_{\tilde{\tau}\otimes\alpha} = \delta_1 + \delta_2$ where $\delta_1(F\hat{\otimes}1) = \partial_\lambda F\hat{\otimes}e_1$. Then

$$d^{ss}_{\tilde{\tau}\otimes\alpha}F_1 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3} \;=\; \delta_1(F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}) + \delta_2(F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3})\,. \tag{23}$$

The graded trace $\int_{[\hbar_0,\hbar_1]\times\mathbb{R}} \int \mathrm{ev}_0$ applied to the second term vanishes, because $\int_{\mathbb{R}} \int \mathrm{ev}_0$ is closed w.r.t. δ_2. The graded trace applied to the first term yields, when first integrated over $[\hbar_0, \hbar_1]$ the boundary term,

$$\int_{[\hbar_0,\hbar_1]\times\mathbb{R}} \int \mathrm{ev}_0 \left(d^{ss}_{\tilde{\tau}\otimes\alpha}F_1 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}\right) \;=\; \int_{\mathbb{R}} \int \mathrm{ev}_0 \left(F_1\delta_2 F_2 \cdots \delta_2 F_{n+3}\right)\Big|_{\hbar_0}^{\hbar_1}.$$

The integrand vanishes if $F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}$ lies in the kernel of r. This proves that the above is indeed a chain, namely it shows that Stokes' Theorem holds. Since this chain is essentially the restriction of the cycle of Example 14 to a closed interval its character satisfies (13) w.r.t. to the Banach sub-algebra $L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \mathcal{B}), \tilde{\tau} \otimes \alpha)$.

6 Pairings between K-theory and cyclic cohomology

A cyclic n-cocycle φ over $\mathcal{B}$ extends to one over $\mathcal{B}^+ = \mathcal{B} \times \mathbb{C}$ via $\varphi((A_0, \lambda_0), \ldots, (A_n, \lambda_n)) = \varphi(A_0, \ldots, A_n)$. Moreover, $\mathrm{Tr} \otimes \varphi$ is a cyclic cocycle over $M_m(\mathcal{B}) \cong M_m(\mathbb{C}) \otimes \mathcal{B}$ where Tr is the standard matrix trace. Define, for a projection $p \in M_m(\mathcal{B}^+)$ or a $u \in U(M_m(\mathcal{B}))$, respectively,

$$\langle \varphi, p \rangle \;=\; c_n\, \mathrm{Tr} \otimes \varphi(p, \ldots, p)\,, \qquad \langle \varphi, u \rangle \;=\; c_n\, \mathrm{Tr} \otimes \varphi(u^* - 1, u - 1, \ldots)\,,$$

the last formula with alternating entries. The normalization constants are (chosen as in [P83], but not as in [C94]):

$$c_{2k} \;=\; \frac{1}{(2\pi\imath)^k}\,\frac{1}{k!}\,, \qquad c_{2k+1} \;=\; \frac{1}{(2\pi\imath)^{k+1}}\,\frac{1}{2^{2k+1}}\,\frac{1}{(k+\frac{1}{2})(k-\frac{1}{2})\cdots\frac{1}{2}}\,.$$

By cyclicity of the cocycles $\langle \varphi, p \rangle$ vanishes if n is odd and similarly one can show using cyclicity that $\langle \varphi, u \rangle = 0$ if n is even.

6.1 General properties of pairings

Proposition 6 *Let $\mathcal{B}$ be a C^*-algebra, φ a cyclic n-cocycle and x a projection in $M_m(\mathcal{B})$ if n is even, or a unitary of $U(M_m(\mathcal{B}))$ if n is odd. The value of $\langle\varphi, x\rangle$ depends only on the homotopy class of x.*

Proof: Let $(\Omega, d, \int)$ be a $2k$-cycle for φ and consider a differentiable projection valued path $p_t \in \mathcal{B}$. We claim that $\langle\varphi, p_t\rangle$ is independent of t. By cyclicity

$$\frac{1}{c_{2k}}\frac{d}{dt}\langle\varphi, p_t\rangle \;=\; (2k+1)\varphi(\dot{p}_t, p_t, \dots, p_t) \;=\; (2k+1)\int \dot{p}_t(dp_t)^{2k} \;.$$

Since $\dot{p}_t(dp_t)^{2k} = (1-p_t)\dot{p}_t(dp_t)^{2k}p_t + p_t\dot{p}_t(dp_t)^{2k}(1-p_t)$, cyclicity implies that the above vanishes. Since a continuous path can be approximated by differentiable paths, the first statement follows for projections in $\mathcal{B}$. Projections in $M_n(\mathcal{B}^+)$ are treated using the extension of the cycle.

Now let $u_t \in U(\mathcal{B})$ be a differentiable path and $(\Omega, d, \int)$ be a $2k+1$-cycle for φ. We claim that $\langle\varphi, u_t\rangle$ is independent of t. Using cyclicity

$$\begin{aligned}\frac{1}{c_{2k+1}}\frac{d}{dt}\langle\varphi, u_t\rangle \;&=\; (k+1)\left(\varphi(\dot{u}_t^*, u_t-1, \dots, u_t-1) - \varphi(\dot{u}_t, u_t^*-1, \dots, u_t^*-1)\right)\\ &=\; (k+1)\left(\int \dot{u}_t^* du_t(u_t^* du_t)^{2k} - \int \dot{u}_t du_t^*(u_t du_t^*)^{2k}\right) \;.\end{aligned}$$

Inserting $\dot{u}_t^* = -u_t^*\dot{u}_t u_t^*$ and $du_t^* = -u_t^*(du_t)u_t^*$, one sees that this expression vanishes. □

The proposition implies that an even cyclic cocycle defines a map $[\mathcal{B}]_0 \to \mathbb{C}$ and an odd cyclic cocycle a map $[\mathcal{B}]_1 \to \mathbb{C}$. These maps can be extended to maps $K_0(\mathcal{B}) \to \mathbb{C}$ and $K_1(\mathcal{B}) \to \mathbb{C}$, respectively. In the Section 5.1 we mentioned that there are not enough (bounded) cyclic cocycles on a C^*-algebra and therefore discussed n-traces on Banach algebras. If $\mathcal{B}$ is a Banach algebra which is densely included in the C^*-algebra $\mathcal{C}$, then any n-trace on $\mathcal{B}$ defines by extension of the above formulas a functional on $K_n(\mathcal{C})$. The reason for this is that the algebra $\mathcal{B}''$ mentionned in Section 5.1 to which the n-trace can be extended by continuity is closed under functional holomorphic calculus and therefore $K_n(\mathcal{B}) \cong K_n(\mathcal{C})$ with isomorphism induced by $\mathcal{B} \subset \mathcal{C}$ [4]. In particular, the $n+1$-trace $\#_\alpha\xi$ constructed in Proposition 5 yields a well defined functional on the K-groups of the C^*-crossed products.

The above proposition forms part of a theorem [C86, C94] stating that $\langle\cdot,\cdot\rangle$ induces maps $K_i(\mathcal{B}) \times \bigoplus_{n\geq 0} HC^{2n+i}(\mathcal{B}) \to \mathbb{C}$ which are additive in the first and the second variable and this is the reason way one speaks about a pairing between K-groups and cyclic cohomology groups. In particular, $\langle\varphi, x\rangle$ vanishes if φ is a cyclic coboundary.

Proposition 7 *Let $(\Omega, d, \int, \partial\Omega, r)$ be a (possibly unbounded) chain over Ω^0 and consider $x' \in \partial\Omega^0$, a projection if the degree of $\partial\Omega$ is even or a unitary if the degree is odd. Suppose x'*

[4] Using holomorphic functional calculus, one can show (see e.g. [RLL00]) that any generalized idempotent in such an algebra (i.e. any element whose spectrum is contained in the union of open balls $B_{\frac{1}{3}}(0) \cup B_{\frac{1}{3}}(1)$) is homotopic to a projection and that projections which are joined by a path of generalized idempotents are homotopic. Since generalized idempotents form an open set, any projection in $\mathcal{B}$ must be joined by a path of generalized idempotents to a projection in $\mathcal{B}''$ and hence homotopic to it.

has a preimage under r which is a projection if the degree of $\partial\Omega$ is even or a unitary if the degree is odd. Then x' pairs trivially with the character φ of the boundary of this chain, namely $\langle\varphi, x\rangle = 0$.

Proof: Consider first the case in which the degree of φ is $2k$ and let p is a projection valued preimage of x' under r, or, if it is not in the domain of the character of the chain, a homotopic projection in that domain. Then $p' = r(p)$ is equal or homotopic to x' and

$$\frac{1}{c_{2k}}\langle\varphi, x'\rangle \;=\; \int' p'(d'p')^{2k} \;=\; \int (dp)^{2k+1}\,.$$

As $\int$ is a graded trace, this vanishes because

$$(dp)^{2k+1} \;=\; p(dp)^{2k+1}(1-p) + (1-p)(dp)^{2k+1}p\,.$$

If the degree is $2k+1$ we let u be a unitary in the domain of the character of the chain which is homotopic to a preimage of x' under r. Then $u' = r(u)$ is homotopic to x' and

$$\frac{1}{c_{2k+1}}\langle\varphi, x'\rangle \;=\; \int'(u'^* - 1)(d'u'du'^*)^k d'u' \;=\; \int (du^*du)^{k+1} \;=\; -\int (dudu^*)^{k+1}\,.$$

The last equality follows because $\int$ is a graded trace and the degree of du and $(du^*du)^k du^*$ are both odd. On the other hand, recall $du^* = -u^*duu^*$ so that, using cyclicity again

$$\int (du^*du)^{k+1} \;=\; \int(-u^*duu^*du)^{k+1} \;=\; \int(-duu^*duu^*)^{k+1} \;=\; \int(dudu^*)^{k+1}\,,$$

which shows that $\langle\varphi, x'\rangle$ has to vanish. □

6.2 Dual to ENN-maps, exp and ind

Proposition 8 *Let η be an α-invariant cyclic n-cocycle on $\mathcal{B}$ and $SS\mathcal{B} \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}$ the C^*-field from* Example 5 *with fibre map $\varphi_\hbar$. Furthermore, $x \in M_m((SS\mathcal{B} \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R})^+)$ is a projection if n is even, or $x \in U(M_m(SS\mathcal{B} \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R})$ if η is odd. Denote $x_\hbar = \varphi_\hbar(x)$. Then the pairings*

$$\langle \#_{\tau^\hbar\otimes\alpha}\eta^s, x_\hbar\rangle$$

(for a given $\hbar$ over the algebra $S\mathcal{B} \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R}$) are independent of $\hbar$. In other words, the map $\#_{\tau^1\otimes\alpha}\eta^s \mapsto \#_{\tau^0\otimes\alpha}\eta^s$ is the dual to the ENN*-map* (12)*.*

Proof: We apply Proposition 7 to the chain constructed in Example 15. The boundary of this chain is a cycle for $S\mathcal{B}\rtimes_{\tau^{\hbar_0}\otimes\alpha}\mathbb{R}\oplus S\mathcal{B}\rtimes_{\tau^{\hbar_1}\otimes\alpha}\mathbb{R}$ and its character is given by $\#_{\tau^{\hbar_1}\otimes\alpha}\eta^s\oplus-\#_{\tau^{\hbar_0}\otimes\alpha}\eta^s$, *c.f.* equation (23). Furthermore $r(x) = (x_{\hbar_0}, x_{\hbar_1})$ is a projection (unitary) in that algebra. By Proposition 7 $(x_{\hbar_0}, x_{\hbar_1})$ therefore pairs trivially with the character of the boundary of the chain, i.e. $0 = \langle \#_{\tau^{\hbar_1}\otimes\alpha}\eta^s, x_{\hbar_1}\rangle - \langle \#_{\tau^{\hbar_0}\otimes\alpha}\eta^s, x_{\hbar_0}\rangle$. □

Example 16 Lemma 1(i) is an application of this proposition to $\mathcal{B} = \mathbb{C}$, $\alpha = \text{id}$ and $\eta = \text{Tr}$. Then $S\mathcal{B} \rtimes_{\tau^\gamma \otimes \text{id}} \mathbb{R} \cong C^*_\gamma(D_1, D_2)$ (see Example 2) and $\text{ch} = -\imath \#_{\tau^\gamma \otimes \text{id}} \text{Tr}^s$. Example 4 establishes that the $C^*_\gamma(D_1, D_2)$ for different γ constitute a C^*-field and, applying the above proposition to the section $P_{\gamma D_\gamma}$ yields $\langle \text{ch}, \Pi_\gamma \rangle = \langle \text{ch}_g, b \rangle$ as $[\Pi_\gamma]_0 = \mu_0 [b]_0$ (*cf.* Example 10).

Theorem 4 *Let x be a representative for an element in $K_i(\mathcal{B} \rtimes_\alpha \mathbb{R})$ and η be an α-invariant cyclic cocycle over $\mathcal{B}$. Then*

$$\langle \#_\alpha \eta, x \rangle \; = \; -\langle \#_{\tau \otimes \alpha} \eta^s, \partial_i(x) \rangle \;, \tag{24}$$

where $\partial_0 = \exp : K_0(\mathcal{B} \rtimes_\alpha \mathbb{R}) \to K_1(S\mathcal{B} \rtimes_{\tau \otimes \alpha} \mathbb{R})$ and $\partial_1 = \text{ind} : K_1(\mathcal{B} \rtimes_\alpha \mathbb{R}) \to K_0(S\mathcal{B} \rtimes_{\tau \otimes \alpha} \mathbb{R})$.

Proof: Applying Proposition 8 to the points $\hbar = 0$ and $\hbar = 1$ and the Proposition 1, it suffices to show that $\langle \#_\alpha \eta, x \rangle = -\langle \#_{\text{id} \otimes \alpha} \eta^s, \partial_i(x) \rangle$ where now $\partial_0 = \beta$ and $\partial_1 = \Theta$.

Let us look here at the case $i = 0$ in which x is represented by a projection p. Then $\beta[p]$ is represented by the unitary $Gp + 1$ where $G(s) = e^{2\pi \imath \chi(s)} - 1$. Since G commutes with p, the calculation is simple. If η is the character of the $2k+1$ cycle $(\Omega, d, \int)$, then $\#_{\text{id} \otimes \alpha} \eta^s$ is the character of $(\Omega^s_\alpha, d^s_\alpha, \int_{\mathbb{R}} \int \text{ev}_0)$ and we write again $d^s_\alpha = \delta_1 + d_\alpha$ where $\delta_1(f \hat{\otimes} 1) = \partial_s f \hat{\otimes} e_2$. Then $d^s_\alpha(Gp) = (\partial_s G) p \hat{\otimes} e_2 + G d_\alpha p$ so that, using $p(d_\alpha p)^m p = p(d_\alpha p)^m$ for even m and $p(d_\alpha p)^m p = 0$ for odd m, one gets

$$\begin{aligned}
\langle \#_{\text{id} \otimes \alpha} \eta^s, Gp + 1 \rangle \;&=\; c_{2k+3} \int_{\mathbb{R}} \int \text{ev}_0 \left(\bar{G} p \, d^s_\alpha(Gp) \left(d^s_\alpha(\bar{G}p) d^s_\alpha(Gp) \right)^{k+1} \right) \\
&=\; -\, (k+2) c_{2k+3} \int_{\mathbb{R}} G' G^{k+1} \bar{G}^{k+2} \int \text{ev}_0 \left(p (d_\alpha p)^{2k+2} \right) \\
&=\; -\, \langle \#_\alpha \eta, p \rangle .
\end{aligned}$$

since $\int_{\mathbb{R}} G' G^{k+1} \bar{G}^{k+2} = 2\pi \imath \, \frac{(2k+3)!}{(k+1)!(k+2)!}$.

The case $i = 1$ is more involved and since we do not need it for our application to the quantum Hall effect, we refer the reader to [P83] for this result. □

6.3 Periodicity in cyclic cohomology

Under pairing with K-theory, cyclic cohomology behaves like a periodic cohomology theory. This means, in particular, that there exists a map (denoted S in [C94]) that assigns to each cyclic n-cocycle a cyclic $n+2$-cocycle which pairs in the same way with K-theory. This is a well-known result, but for the convenience of the reader we present a detailed proof.

Recall from Definition 1 that we extend a cocycle η from $\mathcal{B}$ to matrix algebras over $\mathcal{B}$ or to $\mathcal{K} \otimes \mathcal{B}$ by the operator trace on the left factor, $\text{Tr} \otimes \eta$. As we shall see below, there is an isomorphism $S\mathcal{B} \rtimes_{\tau \otimes \alpha} \mathbb{R} \cong \mathcal{K} \otimes \mathcal{B}$ for any $\mathbb{R}$-action α on $\mathcal{B}$ and under this isomorphism $\text{Tr} \otimes \eta$ is identified (Lemma 2) with the character η^e of the n-cycle $(S\Omega \rtimes_{\tau \otimes \alpha} \mathbb{R}, d^e, \int^e)$,

$$(d^e f)(x)(s) \;=\; d(f(x)(s)) \;, \qquad \int^e \omega \;=\; \int ds \int \omega(0)(s) \;.$$

For this construction, we assumed that $(\Omega, d, \int)$ is an α-invariant n-cycle for η.

Proposition 9 *Let η be an α-invariant cyclic cocycle on $\mathcal{B}$ and x be a representative for an element in $K_i(S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R})$. Then*

$$\langle \#_{\tau\otimes\alpha}\eta^s, x\rangle \;=\; -\,\frac{1}{2\pi}\,\langle \eta^e, x\rangle \;. \tag{25}$$

The following two isomorphisms [R82] allow to reduce the proof of Proposition 9 to a calculation for compact operators:

$$\Psi : S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R} \to S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}\;, \qquad \Psi(f)(x)(s) \;=\; \alpha_s(f(x)(s))\;, \tag{26}$$

and (identifying $S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$ with $S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \mathcal{B}$)

$$\Phi : S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R} \to \mathcal{K}(L^2(\mathbb{R})) \otimes \mathcal{B}\;, \qquad \Phi \;=\; \rho \otimes \mathrm{id}\;, \tag{27}$$

where ρ is the representation of $S\mathbb{C} \rtimes_\tau \mathbb{R}$ on $L^2(\mathbb{R})$ given by

$$(\rho(f)\psi)(x) \;=\; \int dy\, f(y)(x)\,\psi(x-y) \;=\; \int dy\, f(x-y)(x)\,\psi(y)\;.$$

Hence the integral kernel of $\rho(f)$ is $\langle x|\rho(f)|y\rangle = f(x-y)(x)$ so that $\mathrm{Tr}(\rho(f)) = \int_{\mathbb{R}} dx f(0)(x)$.

Lemma 2 *Let η be a α-invariant cyclic cocycle over $\mathcal{B}$. Then*

$$\#_{\tau\otimes\alpha}\,\eta^s \;=\; \Psi^*\#_{\tau\otimes\mathrm{id}}\,\eta^s\;, \tag{28}$$

$$\eta^e \;=\; \Psi^*\Phi^*\mathrm{Tr}\otimes\eta\;. \tag{29}$$

Proof: Let η be the character of $(\Omega, d, \int)$. One has

$$\#_{\tau\otimes\alpha}\,\eta^s(f_0,\ldots,f_n) \;=\; \int_{\mathbb{R}} ds \int \left(f_0 d^s_{\tau\otimes\alpha} f_1 \cdots d^s_{\tau\otimes\alpha} f_n\right)(0)(s)\;,$$

where the product is that in $S\Omega \rtimes_{\tau\otimes\alpha} \mathbb{R}$. On the other hand

$$(\Psi^*\#_{\tau\otimes\mathrm{id}}\,\eta^s)(f_0,\ldots,f_n) = \int_{\mathbb{R}} ds \int \left(\Psi(f_0) d^s_{\tau\otimes\mathrm{id}}\Psi(f_1) \cdots d^s_{\tau\otimes\mathrm{id}}\Psi(f_n)\right)(0)(s),$$

with product in $S\Omega \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$. Now (28) follows from $d^s_{\tau\otimes\mathrm{id}}\Psi(f) = \Psi(d^s_{\tau\otimes\alpha} f)$ and

$$\left(\Psi(f_0) d^s_{\tau\otimes\mathrm{id}}\Psi(f_1) \cdots d^s_{\tau\otimes\mathrm{id}}\Psi(f_n)\right)(0)(s) \;=\; \alpha_s\left(f_0 d^s_{\tau\otimes\alpha} f_1 \cdots d^s_{\tau\otimes\alpha} f_n\right)(0)(s)$$

and the α-invariance of η.

For (29), let $f_j = \Psi^{-1}(g_j \otimes b_j)$ where $g_j \otimes b_j \in S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \mathcal{B} \cong S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$. Then

$$\begin{aligned}
\Psi^*\Phi^*\,\mathrm{Tr}\otimes\eta(f_0,\ldots,f_n) &= \mathrm{Tr}\otimes\eta(\rho(g_0)\otimes b_0,\ldots,\rho(g_n)\otimes b_n) \\
&= \mathrm{Tr}(\rho(g_0\cdots g_n)) \int b_0 db_1 \cdots db_n\;, \\
\eta^e(f_0,\ldots,f_n) &= \int_{\mathbb{R}} \int \mathrm{ev}_0\; \Psi^{-1}(g_0\otimes b_0) d\Psi^{-1}(g_1\otimes b_1) \cdots d\Psi^{-1}(g_n\otimes b_n) \\
&= \int_{\mathbb{R}} \mathrm{ev}_0(g_0\cdots g_n) \int b_0 db_1 \cdots b_n\;,
\end{aligned}$$

and the lemma follows from the fact that $\mathrm{Tr}(\rho(g)) = \int_{\mathbb{R}} ds\, g(0)(s)$ for $g \in S\mathbb{C} \rtimes_\tau \mathbb{R}$. □

Lemma 3 *Let η be an α-invariant cyclic cocycle over $\mathcal{B}$. Then for a projection $X \in \mathcal{K} \otimes \mathcal{B}$ or a unitary $X \in U(\mathcal{K} \otimes \mathcal{B})$ one has*

$$\langle \mathrm{Tr} \otimes \eta, X \rangle \;=\; -2\pi \, \langle \#_{\tau\otimes\mathrm{id}} \eta^s, \Phi^{-1}(X) \rangle \;.$$

Proof: We start with the case $\mathcal{B} = \mathbb{C}$ and $\eta = \mathrm{Tr}$ which is the character of $(\mathbb{C}, 0, \mathrm{Tr})$. Then $\mathrm{Tr} \otimes \eta = \mathrm{Tr}$ and $\#_{\tau\otimes\mathrm{id}}\eta^s = \#_\tau \mathrm{Tr}^s$ is the character of $(S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \Lambda\mathbb{C}^2, \delta, \int_\tau^s)$ where, for $f : \mathbb{R} \to S\mathbb{C}$, $\delta = \delta_1 + \delta_2$ with $\delta_1(f) = \partial_s f \otimes e_1$, $\delta_2(f) = \nabla_x f \otimes e_2$, $\nabla_x f(x) = \imath x f(x)$, and $\int_\tau^s f \otimes e_1 e_2 = \int_\mathbb{R} ds f(0)(s) = \mathrm{Tr}(\rho(f))$. Let M and D be operators given by $M\psi(x) = \imath x \psi(x)$ and $D\psi(x) = \psi'(x)$ with usual common domain $C^1_c(\mathbb{R})$. They satisfy $[D, M] = \imath$. Then, for differentiable f,

$$\langle x|[M, \rho(f)]|y\rangle \;=\; \imath(x-y) f(x-y)(x) \;=\; \langle x|\rho(\nabla_x f)|y\rangle \;,$$

$$\langle x|[D, \rho(f)]|y\rangle \;=\; (\partial_x + \partial_y) f(x-y)(x) \;=\; \langle x|\rho(\partial_s f)|y\rangle \;,$$

and therefore, if $p \in S\mathbb{C} \rtimes_\tau \mathbb{R}$ is a differentiable projection,

$$\langle \#_\tau \mathrm{Tr}^s, p\rangle \;=\; \frac{1}{2\pi\imath} \, \mathrm{Tr}\left(\rho(p)[[D, \rho(p)], [M, \rho(p)]]\right) \;=\; -\frac{1}{2\pi} \, \mathrm{Tr}(\rho(p)) \;. \tag{30}$$

In the last equation, we used $P[[D, P], [M, P]]P = -P[D, M]P + [PDP, PMP]$, $P = \rho(p) = P^2$. This proves the statement for $\mathcal{B} = \mathbb{C}$ and $\eta = \mathrm{Tr}$. Note that this calculation is the usual one to obtain the Chern number of the Landau projection ($\gamma = 1$) in the Schrödinger representation .

In the general case, η is the character of some n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$. Then $d^s_{\tau\otimes\mathrm{id}} = d' + \delta$ where δ is as above and $d'(f)(x)(s) = d(f(x)(s))$.

We apply this first to a projection of the form $X = \rho(p) \otimes x \in \mathcal{K} \otimes \mathcal{B}^+$ where x and p are projections. Then $d^s_{\tau\otimes\mathrm{id}}(\Phi^{-1}(X)) = p \otimes dx + \delta p \otimes x$. Let $n = 2k$, $k \geq 0$. Using $p(\delta p)p^j = 0$ if $j > 0$, $x(dx)^{l-1}x = 0$ for $l > 1$ and $\int_\mathbb{R} ds \, \mathrm{ev}_0(p(\delta p)^2) = 2\pi\imath \, \langle \#_\tau \mathrm{Tr}^s, p\rangle = -\imath \, \mathrm{Tr}(\rho(p))$, we get

$$\begin{aligned}
\langle \#_{\tau\otimes\mathrm{id}} \eta^s, \Phi^{-1}(X)\rangle \;&=\; c_{2k+2} \int^s_{\tau\otimes\mathrm{id}} \Phi^{-1}(X(d^s_{\tau\otimes\mathrm{id}}X)^{2k+2}) \\
&=\; c_{2k+2} \sum_{0<l<j\leq 2k+2} \int_\mathbb{R} ds \, \mathrm{ev}_0 \left(p(\delta p)p^{j-l-1}(\delta p)\right) \int x(dx)^{l-1}x(dx)^{j-i-1}x(dx)^{2k+2-j} \\
&=\; -\imath \, \frac{(k+1)c_{2k+2}}{c_{2k}} \, \mathrm{Tr}(\rho(p)) \, \langle \eta, x\rangle \\
&=\; -\frac{1}{2\pi} \, \langle \mathrm{Tr} \otimes \eta, X\rangle \;.
\end{aligned}$$

Next we apply this to a unitary $X = \rho(p) \otimes x + \rho(p)^\perp \otimes 1$ for $x \in U(\mathcal{B})$ and a projection p. Then $d^s_{\tau\otimes\mathrm{id}}(\Phi^{-1}(X)) = p \otimes dx + \delta p \otimes (x - 1)$. If $n = 2k + 1$, $k > 0$, one obtains, taking into account that $p(\delta p)p^k = 0$ for $k > 0$,

$$\begin{aligned}
\langle \#_{\tau\otimes\mathrm{id}} \eta^s, \Phi^{-1}(X)\rangle \;&=\; c_{2k+3} \int^s_{\tau\otimes\mathrm{id}} \Phi^{-1}((X^* - 1)d^s_{\tau\otimes\mathrm{id}}X(d^s_{\tau\otimes\mathrm{id}}X^* d^s_{\tau\otimes\mathrm{id}}X)^{k+1}) \\
&= c_{2k+3} \int_\mathbb{R} ds \, \mathrm{ev}_0 \left(p(\delta p)^2\right) \sum_{0<j<2k+3} \int (x^* - 1)dx^{(1)} \cdots dx^{(j-1)}(x^* - 1)(x - 1)dx^{(j+2)} \cdots dx^{(2k+3)},
\end{aligned}$$

where $x^{(j)} = x$ if j is odd and $x^{(j)} = x^*$ if j is even. We claim that

$$\sum_{j=1}^{2k+2} \int (x^*-1)dx^{(1)} \cdots dx^{(j-1)}(x^*-1)(x-1)dx^{(j+2)} \cdots dx^{(2k+3)} \qquad (31)$$
$$= 2(2k+3) \int (x^*-1)dx(dx^*dx)^k,$$

which then implies

$$\langle \#_{\tau\otimes\mathrm{id}}\, \eta^s, \Phi^{-1}(X)\rangle \;=\; -\imath\, \frac{2(2k+3)c_{2k+3}}{c_{2k+1}}\, \mathrm{Tr}(\rho(p))\langle \eta, x\rangle \;=\; -\frac{1}{2\pi}\, \langle \mathrm{Tr}\otimes\eta, X\rangle \;.$$

Since $(x^*-1)(x-1) = 2 - x - x^*$, equation (31) is equivalent to

$$\int \sum_{l=0}^{k} (x^*-1)(dxdx^*)^l \left((x+x^*)dx + dx(x+x^*)\right)(dx^*dx)^{k-l} = -2\int (x^*-1)(dxdx^*)^k dx \;. \quad (32)$$

Now use $dx^*x = -x^*dx$ and $dxx^* = -xdx^*$ to pull x and x^* of $(x+x^*)$ either to the right or to the left and then use cyclicity in order to obtain

$$\begin{aligned}
\text{l.h.s. of (32)} \;&=\; \sum_{l=0}^{k} \int x(x^*-1)\left((dx^*dx)^l dx(dx^*dx)^{k-l} - (dxdx^*)^l dx^*(dxdx^*)^{k-l}\right. \\
&\qquad\qquad \left. -(dx^*dx)^l dx^*(dx^*dx)^{k-l} + (dxdx^*)^l dx(dxdx^*)^{k-l}\right) \\
&=\; \sum_{l=0}^{k} \Bigg(-\eta(x, \underbrace{x^*, x, \ldots}_{2l}, x, \underbrace{x^*, x, \ldots}_{2(k-l)}) \;+\; \eta(x, \underbrace{x, x^*, \ldots}_{2l}, x^*, \underbrace{x, x^*, \ldots}_{2(k-l)}) \\
&\qquad\qquad + \eta(x, \underbrace{x^*, x, \ldots}_{2l}, x^*, \underbrace{x^*, x, \ldots}_{2(k-l)}) \;-\; \eta(x, \underbrace{x, x^*, \ldots}_{2l}, x, \underbrace{x, x^*, \ldots}_{2(k-l)}) \Bigg) \;.
\end{aligned}$$

Here the entries under-braced are alternating. For fixed l the first and the fourth term in each summand cancel by cyclic symmetry. The remaining terms are

$$\begin{aligned}
\text{l.h.s. of (32)} \;&=\; \sum_{l=0}^{k} \Bigg(-\eta(\underbrace{x, x^*, \ldots}_{2l}, \underbrace{x^*, x, \ldots}_{2(k-l+1)}) \;+\; \eta(\underbrace{x, x^*, \ldots}_{2(l+1)}, \underbrace{x^*, x, \ldots}_{2(k-l)}) \Bigg) \\
&=\; -\eta(\underbrace{x^*, x \ldots}_{2(k+1)}) \;+\; \eta(\underbrace{x, x^*, \ldots}_{2(k+1)}) \\
&=\; -\int 2(x^*-1)dx(dx^*dx)^k \;.
\end{aligned}$$

It remains to compute the pairings for unitaries X of the form $X - 1 = \sum_j a_j \otimes b_j \in \mathcal{K}\otimes\mathcal{B}$ where the sum is finite and the a_j have finite rank (for projections of the form $\sum_j a_j \otimes b_j \in \mathcal{K}\otimes\mathcal{B}^+$

the argument is similar). In that case $X-1 \in M_n \otimes \mathcal{B}$ ($M_n = M_n(\mathbb{C})$ is the associated sub algebra of $\mathcal{K}$ for some finite n). Let $e \in M_n$ be an arbitrary rank one projection, $e^\perp$ its ortho complement in M_n. Then, by the above,

$$\langle \mathrm{Tr}\otimes\eta, X\rangle \;=\; \langle \mathrm{Tr}\otimes\mathrm{Tr}\otimes\eta, e\otimes X + e^\perp\otimes 1\rangle \;=\; -2\pi \,\langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr}\otimes\eta)^s, \rho^{-1}(e)\otimes X + \rho^{-1}(e^\perp)\otimes 1\rangle\,.$$

Further let $U \in \mathcal{U}(M_n\otimes M_n)$ be a unitary such that $\mathrm{Ad}_U \in \mathrm{Aut}(M_n\otimes M_n)$ is the flip, $Ua_1\otimes a_2U^* = a_2\otimes a_1$. Since $\mathcal{U}(M_n\otimes M_n)$ is connected, a path connecting U to the identity gives rise to a homotopy in $M_n\otimes M_n\otimes\mathcal{B}^+$ between $e\otimes X+e^\perp\otimes 1$ and $\mathrm{Ad}_U\otimes\mathrm{id}(e\otimes X+e^\perp\otimes 1)$. Since $\mathrm{Ad}_U\otimes\mathrm{id}(e\otimes X) = \sum_j a_j\otimes e\otimes b_j$, we have by homotopy invariance of the pairings

$$\begin{aligned}\langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr}\otimes\eta)^s, \rho^{-1}(e)\otimes X + \rho^{-1}(e^\perp)\otimes 1\rangle \\ &= \langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr}\otimes\eta)^s, \sum_j a_j\otimes\rho^{-1}(e)\otimes b_j + \mathrm{Ad}_U\otimes\mathrm{id}(e^\perp\otimes 1)\rangle \\ &= \langle \#_{\tau\otimes\mathrm{id}}\eta^s, \sum_j a_j\otimes b_j + 1\rangle\,.\end{aligned}$$

This proves the statement. □

Proof of Proposition 9 Combine the last two lemmas with $X = \Phi\Psi(x)$. □

6.4 Proof of Theorems 1 and 2

Theorem 1 is simply obtained by combining Theorem 4 and Proposition 9. Theorem 2 is a special case of Theorem 1 but let us give some further details.

Proof of Theorem 2 Apply Theorem 4 and Proposition 9 to the algebras and actions involved in Example 3 and the cocycles from Example 11. Specifically, we take $\mathcal{B} = C(\Omega)\rtimes_{\beta^\|}\mathbb{R}$ with action $\alpha = \beta^\perp$ and 1-cocycle η on $\mathcal{B}$ given by $\eta = \#_{\beta^\|}\int_\Sigma d\mathbf{P}\,\mathrm{ev}_0$. Hence $\eta(f,g) = \int_\Sigma d\mathbf{P}(\sigma)\,(f\nabla_1 g)(0)(\sigma)$. Then ξ from (3) is given by $\xi = \imath\eta^e$ and ch from (2) by $\mathrm{ch} = -2\pi\imath\,\#_\alpha\eta$. Now let $P_\mu \in \mathcal{A}_\infty = \mathcal{B}\rtimes_\alpha\mathbb{R}$ be the element corresponding to the Fermi projection. The exponential map associated with the extension defined by $\mathcal{A}\xrightarrow{\mathrm{ev}_\infty}\mathcal{A}_\infty$ yields $[\mathcal{U}(\Delta)]_1 = \exp[P_\mu]_0$. Thus

$$\frac{1}{2\pi\imath}\,\langle \mathrm{ch}, P_\mu\rangle \;=\; \langle \#_{\tau\otimes\alpha}\eta^s, \mathcal{U}(\Delta)\rangle \;=\; -\frac{1}{2\pi\imath}\,\langle \xi, \mathcal{U}(\Delta)\rangle\,,$$

the first equality following from Theorem 4 and the second from Proposition 9. Since $c_1 = c_2$ we have

$$\mathrm{ch}(P_\mu,P_\mu,P_\mu) \;=\; -\,\xi(\mathcal{U}(\Delta)^*-1, \mathcal{U}(\Delta)-1)\,,$$

which proves Theorem 2. □

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