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A New Proof of Poltoratskii's Theorem

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Abstract. We provide a new simple proof to the celebrated theorem of Poltoratskii concerning ratios of Borel transforms of measures. That is, we show that for any complex Borel measure μ on $\mathbb{R}$ and any $f \in L^1(\mathbb{R}, d\mu)$, $\lim_{\epsilon\to 0}(F_{f\mu}(E+i\epsilon)/F_\mu(E+i\epsilon)) = f(E)$ a.e. w.r.t. $\mu_{\rm sing}$, where $\mu_{\rm sing}$ is the part of μ which is singular with respect to Lebesgue measure and F denotes a Borel transform, namely, $F_{f\mu}(z) = \int (x-z)^{-1} f(x)\, d\mu(x)$ and $F_\mu(z) = \int (x-z)^{-1} d\mu(x)$.

1. Introduction

Let μ be a complex Borel measure on $\mathbb{R}$. Given $f \in L^1(\mathbb{R}, d\mu)$, we denote by $f\mu$ the complex measure obtained by multiplying f and μ, namely, $(f\mu)(S) = \int_S f(x)\, d\mu(x)$ for any Borel set $S \subset \mathbb{R}$. We denote by μ_{sing} the part of μ which is singular with respect to the Lebesgue measure on $\mathbb{R}$. Given any complex measure ν on $\mathbb{R}$, we define its Borel (a.k.a. Stieltjes or Borel-Stieltjes) transform, for $z \in \mathbb{C} \setminus \mathbb{R}$, by

$$F_\nu(z) = \int_{\mathbb{R}} \frac{d\nu(x)}{x - z}\,. \tag{1.1}$$

$F_\nu(z)$ is a well defined analytic function of z on $\mathbb{C} \setminus \mathbb{R}$.

The purpose of this paper is to provide a new simple proof to the following theorem of Poltoratskii (essentially, Theorem 2.7 of [4]):

Theorem 1.1. *For any complex valued Borel measure μ on $\mathbb{R}$ and for any $f \in L^1(\mathbb{R}, d\mu)$,*

$$\lim_{\epsilon \to 0} \frac{F_{f\mu}(E + i\epsilon)}{F_\mu(E + i\epsilon)} = f(E) \tag{1.2}$$

for a.e. $E \in \mathbb{R}$ with respect to μ_{sing}.

Remarks. 1. Poltoratskii actually considers measures on the unit circle (rather than on $\mathbb{R}$). The transition between his setting and ours is elementary and standard.

2. Poltoratskii considers the general *nontangential* limits of approaching E. We prefer to consider only the "radial" limits here (approaching E from above at an angle of 90 degrees, so that our ϵ's are positive) since we find them more transparent and since this is what one usually cares about in applications to spectral theory. Our proof is nevertheless fully valid for the more general nontangential limits (one essentially just needs to replace the various limits of $\epsilon \to 0$ by some appropriate notation for nontangential limits).

3. It is easy to see that in Theorem 1.1 one cannot replace μ_{sing} by μ. This is because F_μ and $F_{f\mu}$ have finite limits a.e. with respect to the absolutely continuous part of μ and finite limits of the form $\lim_{\epsilon \to 0} F_{f\mu}(E + i\epsilon)$ depend on values of f away from the point E.

4. Theorem 1.1 is obvious for the special case where μ_{sing} is a pure point measure, since one can easily see by dominated convergence that for every $E \in \mathbb{R}$, $\lim_{\epsilon \to 0} \epsilon F_{f\mu}(E + i\epsilon) = if(E)\mu(\{E\})$ and thus (1.2) clearly holds whenever E is a point mass of μ. The main point of the theorem is thus in showing that the result also extends to the singular continuous part of μ.

Poltoratskii's proof of the above Theorem 1.1 is somewhat complicated, partly since it is done in the framework of a theory that is also concerned with other questions. In light of recent applications to the spectral theory of random operators [2,3], there is natural interest in having a short self-contained proof that would be, in particular, easily accessible to spectral analysts. The proof below aims to achieve this goal.

2. Proof of Theorem 1.1

We first need to recall some well known elementary facts.

Proposition 2.1. *For any positive Borel measure μ on $\mathbb{R}$, μ_{sing} is supported on $\{E : \lim_{\epsilon\to 0} |F_\mu(E+i\epsilon)| = \infty\}$.*

Proof. We have

$$|F_\mu(E+i\epsilon)| \geq |\operatorname{Im} F_\mu(E+i\epsilon)| = \int_{\mathbb{R}} \frac{\epsilon\, d\mu(x)}{(x-E)^2+\epsilon^2} \geq \frac{\mu(E-\epsilon, E+\epsilon)}{2\epsilon}, \tag{2.1}$$

so the result follows from the theorem of de la Vallée Poussin [7] (also see Theorem 7.15 of [6]). $\square$

Proposition 2.2. *For any finite positive Borel measure μ on $\mathbb{R}$ and for any real valued $f \in L^1(\mathbb{R}, d\mu)$,*

$$\lim_{\epsilon\to 0} \frac{\operatorname{Im} F_{f\mu}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} = f(E) \tag{2.2}$$

for a.e. $E \in \mathbb{R}$ with respect to μ. Moreover, for any two finite positive Borel measures μ,ν on $\mathbb{R}$,

$$\lim_{\epsilon\to 0} \frac{\operatorname{Im} F_\nu(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} = 0 \tag{2.3}$$

for a.e. $E \in \mathbb{R}$ with respect to the part of μ that is singular with respect to ν.

Proof. Since μ is positive and f is real valued, we have

$$\operatorname{Im} F_{f\mu}(E+i\epsilon) = \int_{\mathbb{R}} \frac{\epsilon\, f(x)\, d\mu(x)}{(x-E)^2+\epsilon^2}$$

(and similarly for F_μ and F_ν), namely, the imaginary parts of the Borel transforms coincide with Poisson integrals of the corresponding measures. It is thus obvious that (2.2) is equivalent to

$$\limsup_{\epsilon\to 0} \frac{1}{\operatorname{Im} F_\mu(E+i\epsilon)} \int_{\mathbb{R}} \frac{\epsilon\,(f(x)-f(E))\,d\mu(x)}{(x-E)^2+\epsilon^2} = 0\,, \tag{2.4}$$

and that it holds a.e. w.r.t. μ if f happens to be continuous. Since the continuous functions are dense in $L^1(\mathbb{R}, d\mu)$, we can always find a continuous g such that $\int |f-g|\,d\mu \equiv \|f-g\|_1$ is arbitrarily small. Let $h = f - g$, then by writing $f = g + h$, we see that a.e. w.r.t. μ, the upper limit in (2.4) is bounded by

$$\limsup_{\epsilon\to 0} \frac{\operatorname{Im} F_{|h|\mu}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} + |h|(E)\,.$$

Defining the maximal function

$$M_h(E) \equiv \sup_{\epsilon>0} \frac{(|h|\mu)(E-\epsilon, E+\epsilon)}{\mu(E-\epsilon, E+\epsilon)}$$

and noting that $\epsilon^{-1} \operatorname{Im} F_\mu(E+i\epsilon) = \int_0^{1/\epsilon^2} dt\, \mu\left(E-\sqrt{t^{-1}-\epsilon^2}, E+\sqrt{t^{-1}-\epsilon^2}\right)$ (and similarly for $|h|\mu$), we see that $M_h(E) \geq \operatorname{Im} F_{|h|\mu}(E+i\epsilon)/\operatorname{Im} F_\mu(E+i\epsilon)$ for any $\epsilon > 0$, at any E. Thus, (2.4) would follow a.e. w.r.t. μ if we can show that by making $\|h\|_1$ small we can make $M_h(E)$ arbitrarily small outside sets of arbitrarily small measure μ. To see this, note that for any $t > 0$, each point of the set $\{E : M_h(E) > t\}$ is the center of a closed interval I for which $(|h|\mu)(I) > t\,\mu(I)$. Therefore, for every $-\infty < a < b < \infty$, we can apply the Besicovitch covering theorem (see, e.g., section 1.5.2 of [1]) to extract a countable covering of $[a,b] \cap \{E : M_h(E) > t\}$ by such intervals that has the form $\bigcup_{k=1}^N \bigcup_{j=1}^\infty I_{jk}$, where for each $k \in \{1, ..., N\}$, the intervals I_{jk}, $j = 1, 2, ...$, are disjoint, and where N is some universal integer. Thus,

$$\mu([a,b] \cap \{E : M_h(E) > t\}) \leq \sum_{k=1}^N \sum_{j=1}^\infty \mu(I_{jk}) < \sum_{k=1}^N \sum_{j=1}^\infty \frac{(|h|\mu)(I_{jk})}{t} \leq N \frac{\|h\|_1}{t}\,,$$

and we obtain the desired result. Finally, given two finite positive Borel measures μ,ν on $\mathbb{R}$, we note that $\nu = f(\mu+\nu)$ for some $f \in L^1(\mathbb{R}, d(\mu+\nu))$ obeying $f(E) = 0$ a.e. with respect to the part of μ that is singular with respect to ν. Since

$$\frac{\operatorname{Im} F_\nu(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} = \frac{\operatorname{Im} F_{f(\mu+\nu)}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} \leq \frac{\operatorname{Im} F_{f(\mu+\nu)}(E+i\epsilon)}{\operatorname{Im} F_{\mu+\nu}(E+i\epsilon)}\,,$$

we thus see that (2.3) holds for the appropriate set of E's. $\square$

Remarks. 1. While Proposition 2.2 is very well known, we are not familiar with a reference that really has its proof, which is why we included a full proof here. We note, however, that this proof is just a variant of a proof of the more commonly encountered fact that $\lim_{\epsilon\to 0}((f\mu)(E-\epsilon, E+\epsilon)/\mu(E-\epsilon, E+\epsilon)) = f(E)$ a.e. w.r.t. μ.

2. To prove Proposition 2.2 for the more general case of nontangential limits, see the proof of Lemma 1.2 in [4]. Alternatively, one can also use chapter 11 of [6], who proves a variant of Proposition 2.2 for nontangential limits and the special case where μ is Lebesgue measure on the circle (but the case of a general μ is similar).

We can now prove the main theorem:

Proof of Theorem 1.1. We first consider the case of a positive (finite) measure μ and a real valued $f \in L^2(\mathbb{R}, d\mu)$. Without loss, we assume that μ is compactly supported (this is just a convenience to avoid unbounded operators below). Let A be the operator of multiplication by the parameter on $L^2(\mathbb{R}, d\mu)$, namely, $(Ag)(x) = xg(x)$ for any $g \in L^2(\mathbb{R}, d\mu)$, let $\mathbf{1}$ denote the constant function $\mathbf{1}(x) = 1 \;\; \forall x \in \mathbb{R}$, and let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the scalar product in $L^2(\mathbb{R}, d\mu)$. We note that $\mathbf{1}$ is a cyclic vector for A and that μ is the spectral measure for $\mathbf{1}$ and A, namely, the unique Borel measure on $\mathbb{R}$ obeying $\langle\mathbf{1}, g(A)\mathbf{1}\rangle = \int g(x)\, d\mu(x)$ for any continuous function g on the spectrum of A.

We define the operator A_1 on $L^2(\mathbb{R}, d\mu)$ by

$$A_1 = A + \langle\mathbf{1},\,\cdot\,\rangle\mathbf{1}\,, \tag{2.5}$$

and denote by μ_1 the spectral measure for $\mathbf{1}$ and A_1. Since $\mathbf{1}$ is easily seen to also be a cyclic vector for A_1, it follows from the spectral theorem [5] that there exists a unitary operator $U : L^2(\mathbb{R}, d\mu) \longrightarrow L^2(\mathbb{R}, d\mu_1)$, so that UA_1U^{-1} is the operator of multiplication by the parameter on $L^2(\mathbb{R}, d\mu_1)$ and $U\mathbf{1} = \mathbf{1}$. Moreover, since $Ux^n = UA^n\mathbf{1} = UAU^{-1}Ux^{n-1} = xUx^{n-1} - \langle\mathbf{1}, x^{n-1}\rangle\mathbf{1}$, one easily sees (by induction on the degree) that U takes polynomials with real coefficients to polynomials with real coefficients and thus it takes real valued functions to real valued functions. In particular, Uf is a real valued element of $L^2(\mathbb{R}, d\mu_1)$.

Note that we have the following equalities between Borel transforms and resolvent matrix elements in $L^2(\mathbb{R}, d\mu)$: $F_\mu(z) = \langle\mathbf{1}, (A-z)^{-1}\mathbf{1}\rangle$, $F_{f\mu}(z) = \langle\mathbf{1}, (A-z)^{-1}f\rangle$, and $F_{(Uf)\mu_1}(z) = \langle\mathbf{1}, (A_1-z)^{-1}f\rangle$. By the resolvent formula (namely, by the basic operator formula $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$), we have for any two elements $h, g \in L^2(\mathbb{R}, d\mu)$,

$$\langle g, (A_1-z)^{-1}h\rangle = \langle g, (A-z)^{-1}h\rangle - \langle g, (A-z)^{-1}\mathbf{1}\rangle\langle\mathbf{1}, (A_1-z)^{-1}h\rangle\,. \tag{2.6}$$

By setting $g = h = \mathbf{1}$ in (2.6), we have $F_{\mu_1}(z) = F_\mu(z) - F_\mu(z)F_{\mu_1}(z)$ from which we get

$$F_{\mu_1}(z) = \frac{F_\mu(z)}{1 + F_\mu(z)} . \tag{2.7}$$

Equation (2.7) is the central starting formula for developing the theory of rank one perturbations [8]. In particular, we see from (2.7) and Proposition 2.1 that μ_{sing} and $\mu_{1,\text{sing}}$ are mutually singular and thus that μ_{sing} is singular w.r.t. μ_1. From (2.7) we also get

$$\operatorname{Im} F_{\mu_1}(z) = \frac{\operatorname{Im} F_\mu(z)}{|1 + F_\mu(z)|^2} . \tag{2.8}$$

Going back to (2.6) and setting $g = \mathbf{1}$, $h = f$, we get

$$F_{f\mu}(z) = (1 + F_\mu(z))F_{(Uf)\mu_1}(z) . \tag{2.9}$$

By taking imaginary parts of the two sides of (2.9) and dividing by $\operatorname{Im} F_\mu(z)$, we have

$$\frac{\operatorname{Im} F_{f\mu}(z)}{\operatorname{Im} F_\mu(z)} = \operatorname{Re} F_{(Uf)\mu_1}(z) + L(z) \tag{2.10}$$

with

$$L(z) = \frac{\operatorname{Re}(1 + F_\mu(z))}{\operatorname{Im} F_\mu(z)} \operatorname{Im} F_{(Uf)\mu_1}(z) . \tag{2.11}$$

Applying the Cauchy-Schwartz inequality we get

$$\begin{aligned} |\operatorname{Im} F_{(Uf)\mu_1}(E + i\epsilon)| &= \left| \int_{\mathbb{R}} \frac{\epsilon\,(Uf)(x)\, d\mu_1(x)}{(x - E)^2 + \epsilon^2} \right| \\ &\leq \sqrt{\left(\int_{\mathbb{R}} \frac{\epsilon\, d\mu_1(x)}{(x - E)^2 + \epsilon^2} \right) \left(\int_{\mathbb{R}} \frac{\epsilon\,(Uf)^2(x)\, d\mu_1(x)}{(x - E)^2 + \epsilon^2} \right)} \\ &= \sqrt{\operatorname{Im} F_{\mu_1}(E + i\epsilon)\ \operatorname{Im} F_{(Uf)^2\mu_1}(E + i\epsilon)} \\ &= \sqrt{\operatorname{Im} F_{\mu_1}(E + i\epsilon)\ \operatorname{Im} F_\mu(E + i\epsilon)}\ \sqrt{\frac{\operatorname{Im} F_{(Uf)^2\mu_1}(E + i\epsilon)}{\operatorname{Im} F_\mu(E + i\epsilon)}} , \end{aligned} \tag{2.12}$$

where the last equality is just multiplying and dividing by $\operatorname{Im} F_\mu(E+i\epsilon)$. Since $\mu_{\rm sing}$ is singular w.r.t. μ_1, it is also singular w.r.t. $(Uf)^2\mu_1$ and we thus have by Proposition 2.2,

$$\lim_{\epsilon\to 0} \frac{\operatorname{Im} F_{(Uf)^2\mu_1}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} = 0 \tag{2.13}$$

a.e. w.r.t. $\mu_{\rm sing}$. Also, we see from (2.8) that

$$\operatorname{Im} F_{\mu_1}(z)\ \operatorname{Im} F_\mu(z) = \frac{(\operatorname{Im} F_\mu(z))^2}{|1+F_\mu(z)|^2} \le 1\,, \tag{2.14}$$

and thus (2.12) implies

$$\lim_{\epsilon\to 0} |\operatorname{Im} F_{(Uf)\mu_1}(E+i\epsilon)| = 0 \tag{2.15}$$

a.e. w.r.t. $\mu_{\rm sing}$. By using (2.12) and the equality in (2.14) to estimate $|\operatorname{Im} F_{(Uf)\mu_1}(z)|$ in (2.11), we get

$$\begin{aligned} |L(E+i\epsilon)| &\le \frac{|\operatorname{Re}(1+F_\mu(E+i\epsilon))|}{|\operatorname{Im} F_\mu(E+i\epsilon)|}\ \frac{|\operatorname{Im} F_\mu(E+i\epsilon)|}{|1+F_\mu(E+i\epsilon)|}\ \sqrt{\frac{\operatorname{Im} F_{(Uf)^2\mu_1}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)}} \\ &\le \sqrt{\frac{\operatorname{Im} F_{(Uf)^2\mu_1}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)}}\,, \end{aligned} \tag{2.16}$$

and we thus see from (2.13) that

$$\lim_{\epsilon\to 0} |L(E+i\epsilon)| = 0 \tag{2.17}$$

a.e. w.r.t. $\mu_{\rm sing}$. Combining (2.10), (2.15), and (2.17) with Proposition 2.2, now yields

$$\lim_{\epsilon\to 0} F_{(Uf)\mu_1}(E+i\epsilon) = \lim_{\epsilon\to 0} \frac{\operatorname{Im} F_{f\mu}(E+i\epsilon)}{\operatorname{Im} F_\mu(E+i\epsilon)} = f(E) \tag{2.18}$$

a.e. w.r.t. $\mu_{\rm sing}$. Thus, by going back to (2.9), dividing the two sides by $F_\mu(z)$, and applying Proposition 2.1, we get

$$\lim_{\epsilon\to 0} \frac{F_{f\mu}(E+i\epsilon)}{F_\mu(E+i\epsilon)} = \lim_{\epsilon\to 0} \left(\frac{1}{F_\mu(E+i\epsilon)} + 1\right) F_{(Uf)\mu_1}(E+i\epsilon) = f(E) \tag{2.19}$$

a.e. w.r.t. $\mu_{\rm sing}$.

This completes the proof for the special case of a positive μ and a real valued $f \in L^2(\mathbb{R}, d\mu)$. To complete the proof of the general case, we essentially just follow Poltoratskii [4]: Given a positive $f \in L^1(\mathbb{R}, d\mu)$ (μ is still positive here), let $g = 1/(1+f)$ and $\nu = (1+f)\mu$. Then $g \in L^2(\mathbb{R}, d\nu)$ (in fact, g is bounded) and μ is absolutely continuous with respect to ν. Thus, by (2.19),

$$\lim_{\epsilon\to 0} \frac{F_\mu(E+i\epsilon)}{F_{(1+f)\mu}(E+i\epsilon)} = \lim_{\epsilon\to 0} \frac{F_{g\nu}(E+i\epsilon)}{F_\nu(E+i\epsilon)} = g(E) = \frac{1}{1+f(E)} \tag{2.20}$$

a.e. w.r.t. $\mu_{\rm sing}$, and so we have by the linearity of the Borel transform,

$$\lim_{\epsilon\to 0} \frac{F_{f\mu}(E+i\epsilon)}{F_\mu(E+i\epsilon)} = \lim_{\epsilon\to 0} \frac{F_{(1+f)\mu}(E+i\epsilon)}{F_\mu(E+i\epsilon)} - 1 = f(E) \tag{2.21}$$

a.e. w.r.t. $\mu_{\rm sing}$. Since every complex valued $f \in L^1(\mathbb{R}, d\mu)$ is a linear combination of four positive functions in $L^1(\mathbb{R}, d\mu)$, the linearity of the Borel transform immediately implies the result for any such $f \in L^1(\mathbb{R}, d\mu)$. Finally, if μ is complex valued, then we have $\mu = g|\mu|$ where $|\mu|$ is positive and $g \in L^1(\mathbb{R}, d|\mu|)$ with $|g| = 1$, so

$$\lim_{\epsilon\to 0} \frac{F_{f\mu}(E+i\epsilon)}{F_\mu(E+i\epsilon)} = \lim_{\epsilon\to 0} \frac{F_{fg|\mu|}(E+i\epsilon)}{F_{g|\mu|}(E+i\epsilon)} = \lim_{\epsilon\to 0} \frac{F_{fg|\mu|}(E+i\epsilon)}{F_{|\mu|}(E+i\epsilon)} \frac{F_{|\mu|}(E+i\epsilon)}{F_{g|\mu|}(E+i\epsilon)} = f(E) \tag{2.22}$$

a.e. w.r.t. $\mu_{\rm sing}$. This completes the proof. $\square$

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