Rotating Singular Perturbations of the Laplacian

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Abstract

We study a system of a quantum particle interacting with a singular time-dependent uniformly rotating potential in 2 and 3 dimensions: in particular we consider an interaction with support on a point (rotating point interaction) and on a set of codimension 1 (rotating blade). We prove the existence of the Hamiltonians of such systems as suitable self-adjoint operators and we give an explicit expression for their unitary semigroups. Moreover we analyze the asymptotic limit of large angular velocity and we prove strong convergence of the time-dependent propagator to some one-parameter unitary group as $\omega \to \infty$.

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1 Introduction

In this paper we shall study systems defined by formal time-dependent Schrödinger operators on $L^2(\mathbb{R}^n)$, $n = 2, 3$

$$H(t) = H_0 + V_t = -\Delta + V_t \tag{1.1}$$

with uniformly rotating potentials

$$V_t(\vec{x}) = V(\mathcal{R}^{-1}(t)\,\vec{x}) \tag{1.2}$$

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where V is a singular potential (e.g. $V(\vec{x}) = \delta(\vec{x} - \vec{y}_0)$) and $\mathcal{R}(t)$ a rotation on the $x, y-$plane with period $2\pi/\omega$:

$$\mathcal{R}(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & & 1 \end{pmatrix}$$

Regular rotating potentials were studied by Enss et al. [6] in order to extract information about the scattering of a quantum particle: indeed they considered a class of potentials such that the kinetic energy of the system remains bounded on the range of wave operators and they proved existence and completeness of the wave operators.

Our purpose is to define in a rigorous way the time-dependent Hamiltonians (1.1) when the potential has a more singular behavior: we shall study rotating point perturbations[1] of the Laplacian in 2 and 3 dimensions and rotating blades, namely rotating singular potentials supported over a set of codimension 1 (a segment in 2 dimensions and an half-disk in 3 dimensions respectively).

As pointed out by Enss et al., the uniformly rotating Hamiltonians can be studied in a simpler way than general time-dependent operators, indeed, considering the time evolution $U_{\mathrm{rot}}(t, s)$ of the system in a uniformly rotating frame around the $z-$axis, it is easy to see that the following relation with the time evolution in the inertial frame $U_{\mathrm{inert}}(t, s)$ holds

$$U_{\mathrm{inert}}(t, s) = R(t)\, U_{\mathrm{rot}}(t - s)\, R^{\dagger}(s) \tag{1.3}$$

where $R(t)\Psi(\vec{x}) = \Psi(\mathcal{R}(t)^{-1}\, \vec{x})$ and $U_{\mathrm{rot}}(t, s) = U_{\mathrm{rot}}(t - s)$ is the one-parameter unitary group

$$U_{\mathrm{rot}}(t - s) = e^{-iK(t-s)} \tag{1.4}$$

with a time-independent generator K, formally defined in the following way

$$K = H_0 - \omega J + V \tag{1.5}$$

Here J stands for the third component of the angular momentum and V is the time-independent potential (1.2).

Using this trick we shall define the previous time-dependent Hamiltonians considering the corresponding formal time-independent generators in the rotating frame and studying their self-adjoint extensions.

The last goal of this work will be the analysis of the asymptotic limit of the systems when the angular velocity $\omega \to \infty$: by means of the explicit expression of resolvents of singular perturbations of the Laplacian, we shall prove convergence in strong sense of $U_{\mathrm{inert}}(t, s)$ to some one-parameter unitary group $U_{\mathrm{asympt}}(t-s)$ with time-independent generator H_{asympt}. Moreover we shall see that, for point interactions, H_{asympt} is the Laplacian with singular perturbation on a circle, while the asymptotic limit of the rotating blade is simply a regular potential supported on a compact set. The same study was performed by Enss et al. [7] for regular rotating potentials.

[1] Point interactions were introduces for the first time in a rigorous way by Berezin and Faddeev in 1961 [3]. For general references about fixed and time-dependent point interactions see [2, 4, 5, 8].

2 The Rotating Point Interaction in 3D

2.1 The Hamiltonian

The system we shall study is defined by the formal time-dependent Hamiltonian

$$H(t) = H_0 + a\, \delta^{(3)}(\vec{x} - \vec{y}(t)) \tag{2.1}$$

where $\vec{y}(t) = \mathcal{R}(t)\vec{y}_0$.
According to the previous scheme, the formal generator of time evolution in the uniformly rotating frame (with angular velocity ω) is given by

$$K = H_0 - \omega J + a\, \delta^{(3)}(\vec{x} - \vec{y}_0)$$

Therefore the Hamiltonian of the system is a self-adjoint extension of the operator

$$K_{y_0} = H_\omega$$
$$\mathcal{D}(K_{y_0}) = C_0^\infty(\mathbb{R}^3 - \{\vec{y}_0\})$$

The operator K_{y_0} is symmetric and then closable; let $\dot{K}_{y_0}$ be its closure, with domain $\mathcal{D}(\dot{K}_{y_0})$.
The function

$$\mathcal{G}_z(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{l=0}^\infty \sum_{m=-l}^{l} \frac{1}{k^2 - m\omega - z}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x}) \tag{2.2}$$

for $\vec{x} \in \mathbb{R}^3 - \{\vec{y}_0\}$ and $z \in \mathbb{C} - \mathbb{R}$, is the unique solution of

$$\dot{K}^*_{y_0} \Psi_z(\vec{x}) = z \Psi_z(\vec{x})$$

with $\Psi \in \mathcal{D}(\dot{K}^*_{y_0})$ (see Proposition A.1).
The operator $\dot{K}_{y_0}$ has then deficiency indexes $(1,1)$ and its self-adjoint extensions are given by the one-parameter family of operators K_{α,y_0}, $\alpha \in [0, 2\pi)$:

$$\mathcal{D}(K_{\alpha,y_0}) = \{f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_- \mid g \in \mathcal{D}(\dot{K}_{y_0}), c \in \mathbb{C}\} \tag{2.3}$$

$$K_{\alpha,y_0}(f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_-) = \dot{K}_{y_0} g + ic\mathcal{G}_+ - ice^{i\alpha}\mathcal{G}_- \tag{2.4}$$

where

$$\mathcal{G}_\pm(\vec{x}) = \mathcal{G}_{\pm i}(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{l=0}^\infty \sum_{m=-l}^{l} \frac{1}{k^2 - m\omega \mp i}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

for $\vec{x} \in \mathbb{R}^3 - \{\vec{y}_0\}$.
Moreover the self-adjoint extension K_{π,y_0} corresponds to the "free" Hamiltonian $\dot{H}_\omega$: indeed, if $\Psi \in \mathcal{D}(K_{\pi,y_0})$,

$$\Psi = f + c(\mathcal{G}_+ - \mathcal{G}_-)$$

and the difference $\mathcal{G}_+ - \mathcal{G}_-$ is a continuous function at $\vec{x} = \vec{y}_0$, which belongs to the domain of H_ω, so that K_{π,y_0} becomes exactly the operator $\dot{H}_\omega$.
Using this result and applying the Krein's theory of self-adjoint extensions, it is easy to obtain the following

Theorem 2.1 *The resolvent of K_{α,y_0} has integral kernel given by*

$$(K_{\alpha,y_0}-z)^{-1}(\vec{x},\vec{x}')=\mathcal{G}_z(\vec{x},\vec{x}')+\lambda(z,\alpha)\mathcal{G}^*_{\bar{z}}(\vec{x}',\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0) \tag{2.5}$$

with $z\in\varrho(K_{\alpha,y_0})$, $\vec{x},\vec{x}'\in\mathbb{R}^3$, $\vec{x}\neq\vec{x}'$, $\vec{x},\vec{x}'\neq\vec{y}_0$ and

$$\frac{1}{\lambda(z,\alpha)}=\frac{1}{\lambda(-i,\alpha)}-(z+i)\big(\mathcal{G}_{\bar{z}}(\vec{x}),\mathcal{G}_-(\vec{x})\big) \tag{2.6}$$

$$\lambda(-i,\alpha)=\frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2} \tag{2.7}$$

Proof: Since $\dot{K}_{y_0}$ is a densely defined, closed, symmetric operator with deficiency indexes $(1,1)$, we can apply the Krein's theory (cfr. [2, 11]) to classify all its self-adjoint extensions: from the Krein's formula we immediately obtain

$$(K_{\alpha,y_0}-z)^{-1}-(K_{\pi,y_0}-z)^{-1}=\lambda(z,\alpha)\big(\mathcal{G}_{\bar{z}}(\vec{x}),\cdot\big)\mathcal{G}_z(\vec{x})$$

for $z\in\varrho(K_{\alpha,y_0})\cap\varrho(H_\omega)$. It follows that $(K_{\alpha,y_0}-z)^{-1}$ has integral kernel given by

$$(K_{\alpha,y_0}-z)^{-1}(\vec{x},\vec{x}')=(\dot{H}_\omega-z)^{-1}(\vec{x},\vec{x}')+\lambda(z,\alpha)\mathcal{G}^*_{\bar{z}}(\vec{x}',\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0)$$

Moreover $\lambda(z,\alpha)$ satisfies the following equation

$$\frac{1}{\lambda(z,\alpha)}=\frac{1}{\lambda(z',\alpha)}-(z-z')\big(\mathcal{G}_{\bar{z}}(\vec{x}),\mathcal{G}_{z'}(\vec{x})\big)$$

The explicit expression of the factor $\lambda(-i,\alpha)$ is proved in the following Theorem.

□

Theorem 2.2 *The domain $\mathcal{D}(K_{\alpha,y_0})$, $\alpha\in[0,2\pi)$, consists of all elements $\Psi\in\mathbb{R}^3$ which can be decomposed in the following way*

$$\Psi(\vec{x})=\Phi_z(\vec{x})+\lambda(z,\alpha)\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0)$$

for $\vec{x}\neq\vec{y}_0$, $\Phi_z\in\mathcal{D}(\dot{H}_\omega)$ and $z\in\varrho(K_{\alpha,y_0})$. The previous decomposition is unique and on every Ψ of this form

$$(K_{\alpha,y_0}-z)\Psi=(H_\omega-z)\Phi_z$$

Proof: First of all we observe that functions belonging to $\mathcal{D}(\dot{H}_\omega)$ are Hölder continuous functions with exponent smaller than $1/2$ in every compact subset of $\mathbb{R}^3$. Indeed the domain of self-adjointness of $\dot{H}_\omega$ contains functions in $H^2_{\text{loc}}(\mathbb{R}^3)$: on every compact set $S\subset\mathbb{R}^3$, the domain[2] of H_0^S is strictly contained on the domain of J^S, since J^S is a bounded operator on $\mathcal{D}(\dot{H}_0^S)=H^2(S)$, therefore $\mathcal{D}(\dot{H}^S_\omega)=\mathcal{D}(\dot{H}^S_0)=H^2(S)$. Hence it makes sense to write $\Phi(\vec{y}_0)$ for every $\Phi\in\mathcal{D}(\dot{H}_\omega)$ and $\vec{y}_0\in\mathbb{R}^3$.
Moreover

$$\mathcal{D}(K_{\alpha,y_0})=(K_{\alpha,y_0}-z)^{-1}(\dot{H}_\omega-z)\mathcal{D}(\dot{H}_\omega)$$

[2] The notation A^S denotes the restriction of the operator A to the Hilbert space $L^2(S)$.

and the claim follows from the expression of the resolvent given in the previous Theorem 2.2.
To prove the uniqueness of the decomposition let $\Psi = 0$, so that

$$\Phi_z(\vec{x}) = -\frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2}\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x})$$

but $\Phi_z(\vec{x})$ must be continuous at $\vec{x} = \vec{y}_0$: it follows that $\Phi_z(\vec{y}_0) = 0$ and then $\Phi_z = 0$.
Finally the last equality of the Theorem easily follows from

$$(K_{\alpha,y_0} - z)^{-1}(\dot{H}_\omega - z)\Phi_z = \Phi_z + \lambda(z,\alpha)\big(\mathcal{G}_{\bar{z}}(\vec{x}), (\dot{H}_\omega - z)\Phi_z(\vec{x})\big)\mathcal{G}_z = \Psi$$

To prove the explicit expression of $\lambda(-i,\alpha)$ it is sufficient to study the behavior of functions in $\mathcal{D}(K_{\alpha,y_0})$ at $\vec{y}_0$. Let $\Psi(\vec{x}) \in \mathcal{D}(K_{\alpha,y_0})$,

$$\Psi(\vec{x}) = f(\vec{x}) + c\mathcal{G}_+(\vec{x}) + ce^{i\alpha}\mathcal{G}_-(\vec{x})$$

with $f \in \mathcal{D}(\dot{H}_{y_0})$ and $c \in \mathbb{C}$.
Since

$$\mathcal{G}_+(\vec{x}) = \int_0^\infty dk \sum_{l=0}^\infty \sum_{m=-l}^l \left[\frac{1}{k^2 - m\omega + i} + \frac{2i}{|k^2 - m\omega - i|^2}\right] \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x}) =$$

$$= \mathcal{G}_-(\vec{x}) + 2ig(\vec{x},\vec{y}_0)$$

where

$$g(\vec{x},\vec{y}_0) = \int_0^\infty dk \sum_{l=0}^\infty \sum_{m=-l}^l \frac{1}{|k^2 - m\omega - i|^2}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

belongs to $\mathcal{D}(\dot{H}_\omega)$, $\forall \vec{y}_0 \in \mathbb{R}^3$, we obtain

$$\Psi(\vec{x}) = f(\vec{x}) + 2ic\, g(\vec{x},\vec{y}_0) + c(1+e^{i\alpha})\mathcal{G}_-(\vec{x})$$

and

$$\lim_{\vec{x}\to\vec{y}_0} \Big[\Psi(\vec{x}) - c(1+e^{i\alpha})\mathcal{G}_-(\vec{x})\Big] = 2ic\|\mathcal{G}_-(\vec{x})\|^2_{L^2}$$

Thus Ψ can be uniquely decomposed in

$$\Psi(\vec{x}) = \Phi(\vec{x}) + \lambda(-i,\alpha)\Phi(\vec{y}_0)\mathcal{G}_-(\vec{x})$$

with $\Phi \in \mathcal{D}(\dot{H}_\omega)$ and boundary condition

$$\lim_{\vec{x}\to\vec{y}_0} \Big[\Psi(\vec{x}) - \lambda(-i,\alpha)\Phi(\vec{y}_0)\mathcal{G}_-(\vec{x})\Big] = \Phi(\vec{y}_0)$$

Comparing the two boundary conditions we obtain

$$\Phi(\vec{y}_0) = 2ic\|\mathcal{G}_-(\vec{x})\|^2_{L^2}$$

$$c(1+e^{i\alpha}) = \lambda(-i,\alpha)\Phi(\vec{y}_0)$$

and then

$$c = \frac{\Phi(\vec{y}_0)}{2i\|\mathcal{G}_-(\vec{x})\|^2_{L^2}}$$

$$\lambda(-i,\alpha) = \frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2_{L^2}}$$

□

Theorem 2.3 *The spectrum $\sigma(K_{\alpha,y_0})$ is purely absolutely continuous and*

$$\sigma(K_{\alpha,y_0}) = \sigma_{\rm ac}(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R} \tag{2.8}$$

Proof: Considering the explicit expression of the resolvent given in Theorem 2.1, we immediately see that $\sigma(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R}$: indeed, since $(K_{\alpha,y_0} - z)^{-1} - (H_\omega - z)^{-1}$ is of rank 1 for each $z \in \mathbb{R}$ and $\alpha \in [0, 2\pi)$, Weyl's Theorem (see for example Theorem XIII.14 in [13]) implies $\sigma_{\rm ess}(K_{\alpha,y_0}) = \sigma_{\rm ess}(H_\omega)$.
In order to prove absence of pure points and singular spectrum, we are going to apply the limiting absorption principle (see Theorem XIII.19 in [13]): to this purpose we need to prove that the following inequality is satisfied for every interval $[a, b] \subset \mathbb{R}$,

$$\sup_{0<\varepsilon<1} \int_a^b dx \, \left| \Im\Big[\Big(\Psi \, , \, \big(K_{\alpha,y_0} - x - i\varepsilon\big)^{-1} \Psi \Big) \Big] \right|^p < \infty$$

with Ψ in a dense subset of $L^2(\mathbb{R}^3)$ and $p > 1$.
Since the operator H_ω has no singular spectrum, the inequality is easily satisfied if $\alpha = \pi$. So, let $\alpha \neq \pi$, from Theorem 2.1 one has

$$\Big(\Psi \, , \, \big(K_{\alpha,y_0} - x - i\varepsilon\big)^{-1} \Psi \Big) = \Big(\Psi \, , \, \big(H_\omega - x - i\varepsilon\big)^{-1} \Psi \Big) +$$
$$+ \lambda(\alpha, x + i\varepsilon) \Big(\mathcal{G}_{x-i\varepsilon} \, , \, \Psi \Big) \Big(\Psi \, , \, \mathcal{G}_{x+i\varepsilon} \Big)$$

and again the inequality holds for the first term. It is very easy to see that the second term is a bounded function of x if $\varepsilon > 0$, so that we have only to control the limit when $\varepsilon \to 0$. Since the singular spectrum of H_ω is empty, we can choose the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $(H_\omega - x)\varphi$ where $\varphi \in \mathcal{D}(H_\omega)$:

$$\Big(\mathcal{G}_{x-i\varepsilon} \, , \, \Psi \Big) \Big(\Psi \, , \, \mathcal{G}_{x+i\varepsilon} \Big) = \Big[\big(H_\omega - x - i\varepsilon\big)^{-1} \big(H_\omega - x\big) \varphi \Big](\vec{y}_0) \cdot$$
$$\cdot \Big[\big(H_\omega - x - i\varepsilon\big)^{-1} \big(H_\omega - x\big) \varphi^* \Big](\vec{y}_0) \underset{\varepsilon \to 0}{\longrightarrow} \big|\varphi(\vec{y}_0)\big|^2 < \infty$$

since functions in $\mathcal{D}(H_\omega)$ are continuous and because

$$\Big[\big(H_\omega - x - i\varepsilon\big)^{-1} \big(H_\omega - x\big) \varphi \Big](\vec{y}_0) = \varphi(\vec{y}_0) + i\varepsilon \Big[\big(H_\omega - x - i\varepsilon\big)^{-1} \varphi \Big](\vec{y}_0)$$

and

$$\lim_{\varepsilon \to 0} \left| \varepsilon \Big[\big(H_\omega - x - i\varepsilon\big)^{-1} \varphi \Big](\vec{y}_0) \right| \leq \lim_{\varepsilon \to 0} \varepsilon \, \big\|\mathcal{G}_{x-i\varepsilon}\big\| \, \|\varphi\| = 0$$

Indeed from Proposition A.1 we can easily extract the following upper bound for $\big\|\mathcal{G}_{x-i\varepsilon}\big\|$,

$$\big\|\mathcal{G}_{x-i\varepsilon}\big\| \leq \frac{C}{\sqrt{\varepsilon}}$$

Finally from equation (2.6) it follows that

$$\big|\lambda(\alpha, x + i\varepsilon)\big| \underset{\varepsilon \to 0}{\longrightarrow} 0$$

Since the previous argument applies for each interval $[a, b] \subset \mathbb{R}$, the proof is completed.

□

2.2 Asymptotic Limit of Rapid Rotation

Let $U_{\mathrm{rot}}(t-s)$ the unitary group generated by K_{α,y_0} for some $\alpha \in [0,2\pi)$, according to [6],

$$U_{\mathrm{inert}}(t,s) = R(t)\,U_{\mathrm{rot}}(t-s)\,R^{\dagger}(s)$$

In the following, we shall prove that

$$\mathrm{s-}\lim_{\omega\to\infty} U_{\mathrm{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $H_{\gamma,C}$ is an appropriate self-adjoint extension of H_C, a singular perturbation of the Laplacian supported over a circle of radius y_0 in the $x,y-$plane: let C the curve $\vec{y}(\varphi) = (y_0, \frac{\pi}{2}, \varphi)$, $\varphi \in [0,2\pi]$, and $\dot{H}_C$ the closure of the operator

$$H_C = H_0$$

$$\mathcal{D}(H_C) = C_0^{\infty}(\mathbb{R}^3 - C)$$

we first classify all the self-adjoint extensions of $\dot{H}_C$:

Proposition 2.1 *The self-adjoint extensions of the operator $\dot{H}_C$, that are invariant under rotations around the $z-$axis, are given by the one-parameter family $H_{\gamma,C}$, $\gamma \in \mathbb{R}$, with domain*

$$\mathcal{D}(H_{\gamma,C}) = \{\Psi \in L^2(\mathbb{R}^3) \mid \exists\, \xi_\Psi \in \mathcal{D}(\Gamma_{\gamma,C}(z)), \Psi - \tilde{G}_z\xi_\Psi \in H^2(\mathbb{R}^3),$$

$$\big(\Psi - \tilde{G}_z\xi_\Psi\big)\big|_C = \Gamma_{\gamma,C}(z)\xi_\Psi\} \tag{2.9}$$

$$\big(H_{\gamma,C} - z\big)\Psi = \big(H_0 - z\big)\big(\Psi - \tilde{G}_z\xi_\Psi\big) \tag{2.10}$$

where $z \in \mathbb{C}$, $\Im(z) > 0$,

$$\mathcal{D}(\Gamma_{\gamma,C}(z)) = \{\xi \in L^2([0,2\pi]) \mid \Gamma_{\gamma,C}(z)_m\xi_m \in l^2\} \tag{2.11}$$

$$\xi_m \equiv \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\phi\,\xi(\phi)e^{-im\phi}$$

$$\big(\Gamma_{\gamma,C}(z)\xi\big)(\phi) = \gamma\xi(\phi) - \int_0^{2\pi} d\phi'\,\frac{e^{i\sqrt{z}|\vec{y}(\phi)-\vec{y}(\phi')|}}{4\pi|\vec{y}(\phi)-\vec{y}(\phi')|}\xi(\phi') \tag{2.12}$$

$$\Gamma_{\gamma,C}(z)_m = \gamma - 2\pi\int_0^{\infty} dk \sum_{l=|m|}^{\infty}\frac{1}{k^2-z}\big|\varphi_{klm}(\vec{y}_0)\big|^2 \tag{2.13}$$

and

$$\big(\tilde{G}_z\xi\big)(\vec{x}) \equiv \int_0^{2\pi} d\phi\,\frac{e^{i\sqrt{z}|\vec{x}-\vec{y}(\phi)|}}{4\pi|\vec{x}-\vec{y}(\phi)|}\xi(\phi)$$

Proof: See [14, 15]. The formula for $\Gamma_{\alpha,C}(\lambda)_m$ is obtained expressing the free resolvent in terms of spherical waves.

□

Proposition 2.2 *For every $\Psi \in L^2(\mathbb{R}^3)$, $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$ and $\vec{y}_0 = (0, y_0, 0)$,*

$$\big(H_{\gamma,C} - z\big)^{-1}\Psi(\vec{x}) = \big(H_0 - z\big)^{-1}\Psi(\vec{x}) +$$

$$+ \sum_{m=-\infty}^{+\infty} \frac{2\pi}{\Gamma_{\gamma,C}(z)_m} G_z^m(\vec{x}, \vec{y}_0) \left(G_z^{m*}(\vec{x}', \vec{y}_0)\, , \Psi(\vec{x}')\right)_{L^2(\mathbb{R}^3)}$$

where

$$G_z^m(\vec{x}, \vec{y}_0) \equiv \int_0^\infty dk \sum_{l=|m|}^{\infty} \frac{1}{k^2 - z}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

Proof: The expression for the resolvent of $H_{\gamma,C}$ for a generic curve C is given in [14, 15]:

$$\big(H_{\gamma,C} - z\big)^{-1}\Psi(\vec{x}) = \big(H_0 - z\big)^{-1}\Psi(\vec{x}) + \tilde{G}_z\left[\Gamma^{-1}_{\gamma,C}(z)\Big(\big(H_0 - z\big)^{-1}\Psi\Big)\Big|_C\right]$$

Since $\Gamma_{\gamma,C}(z)$ is diagonal in the basis $e_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$ of $L^2([0, 2\pi], d\phi)$,

$$\big(\Gamma^{-1}_{\gamma,C}(z)\xi\big)(\phi) = \sum_{m=-\infty}^{\infty} \frac{1}{\Gamma_{\gamma,C}(z)_m}\, \xi_m\, e_m(\phi)$$

and therefore

$$\Gamma^{-1}_{\gamma,C}(z)\Big(\big(H_0 - z\big)^{-1}\Psi\Big)\Big|_C = \sum_{m=-\infty}^{\infty} \frac{\Big[\Big(\big(H_0 - z\big)^{-1}\Psi\Big)\Big|_C\Big]_m}{\Gamma_{\gamma,C}(z)_m}\, e_m(\phi)$$

where

$$\Big[\Big(\big(H_0 - z\big)^{-1}\Psi\Big)\Big|_C\Big]_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\phi\, e^{-im\phi} \int_{\mathbb{R}^3} d^3\vec{x}' \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{m'=-l}^{l} \frac{1}{k^2 - z} \cdot$$

$$\cdot\, \varphi^*_{klm'}(\vec{x}')\, \varphi_{klm'}(\vec{y}(\phi))\, \Psi(\vec{x}') = \sqrt{2\pi}\, e^{im\frac{\pi}{2}} \int_{\mathbb{R}^3} d^3\vec{x}'\, G_z^m(\vec{x}', \vec{y}_0)\, \Psi(\vec{x}')$$

Finally

$$(\tilde{G}_z\, e_m)(\vec{x}) = \int_0^{2\pi} \frac{d\phi}{\sqrt{2\pi}} \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{m'=-l}^{l} \frac{1}{k^2 - z}\, \varphi^*_{klm'}(\vec{y}(\phi))\, \varphi_{klm'}(\vec{x})\, e^{im\phi} =$$

$$= \sqrt{2\pi}\, e^{-im\frac{\pi}{2}}\, G_z^m(\vec{x}, \vec{y}_0)$$

□

Corollary 2.1 *If $\Psi(\vec{x}) \in L^2(\mathbb{R}^3)$, $\Psi(\vec{x}) = \chi(r) Y_{l_0}^{m_0}(\theta, \phi)$ and $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$,*

$$\Big(\big(H_{\gamma,C} - z\big)^{-1}\Psi\Big)(\vec{x}) = \int_0^\infty dr' r'^2\, g_z^{l_0}(r, r')\chi(r')\; Y_{l_0}^{m_0}(\theta, \phi) +$$

$$+ \frac{2\pi\, Y_{l_0}^{m_0}(\pi/2, 0)}{\Gamma_{\gamma,C}(z)_{m_0}}\, G_z^{m_0}(\vec{x}, \vec{y}_0) \int_0^\infty dr' r'^2\, g_z^{l_0}(y_0, r')\chi(r')$$

where

$$g_z^{l_0}(r,r') \equiv \frac{2}{\pi}\int_0^\infty dk\, \frac{k^2}{k^2-z}\, j_{l_0}(kr) j_{l_0}(kr') = \big(H_0 - z\big)^{-1}\big|_{\mathcal{H}_{l_0}^{m_0}}(r,r')$$

and $\mathcal{H}_{l_0}^{m_0}$ *is the subspace of* $L^2(\mathbb{R}^3)$ *spanned by* $Y_{l_0}^{m_0}(\theta,\phi)$*.*

Proof: The result follows from a straightforward calculation: indeed, if $\Psi(\vec{x}) = \chi(r) Y_{l_0}^{m_0}(\theta,\phi)$,

$$\Big(G_z^{m*}(\vec{x}', \vec{y}_0)\, , \Psi(\vec{x}')\Big) = \delta_{m,m_0}\, Y_{l_0}^{m_0}(\pi/2, 0) \int_0^\infty dr' r'^2\, g_z^{l_0}(y_0, r')\chi(r')$$

and

$$\Big(\big(H_0 - z\big)^{-1}\Psi\Big)(\vec{x}) = \int_0^\infty dr' r'^2\, g_z^{l_0}(r,r')\chi(r')\; Y_{l_0}^{m_0}(\theta,\phi)$$

□

Now we can state the main result:

Theorem 2.4 *For every* $t, s \in \mathbb{R}$*,*

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $\gamma(\alpha, y_0) \in \mathbb{R}$ *and*

$$\gamma(\alpha, y_0) = 2\pi \int_0^\infty dk \sum_{l=0}^{\infty} \left[\frac{2i}{(1+e^{i\alpha})|k^2+i|^2} + \frac{1}{k^2+i}\right] \big|\varphi_{kl0}(\vec{y}_0)\big|^2$$

Proof: First we observe that (see Lemma 2.1)

$$\text{s}-\lim_{\omega\to\infty}\int_{-\infty}^0 dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_{\gamma,C} - z\big)^{-1} = \int_{-\infty}^0 dt\, e^{-izt}\, e^{iH_{\gamma,C}t}$$

and, since the previous equality holds for every $z \in \mathbb{C}$, $\Im(z) > 0$, we obtain

$$\text{s}-\lim_{\omega\to\infty} U^*_{\text{inert}}(t,0) = e^{iH_{\gamma,C}t}$$

and therefore

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,0) = e^{-iH_{\gamma,C}t}$$

The result then follows from the property of the 2-parameters unitary group $U_{\text{inert}}(t,s)$:

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = \text{s}-\lim_{\omega\to\infty}\Big[U_{\text{inert}}(t,0)\, U^*_{\text{inert}}(s,0)\Big] = e^{-iH_{\gamma,C}(t-s)}$$

The explicit expression of the parameter $\gamma(\alpha, y_0)$ is proved in the following Lemma 2.1.

□

Lemma 2.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\mathrm{s-}\lim_{\omega\to\infty} \int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\mathrm{inert}}(t,0) = -i\big(H_{\gamma,C} - z\big)^{-1}$$

Proof: We shall verify the equality on the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $\Psi(\vec{x}) = \chi(r) Y^{m_0}_{l_0}(\theta,\phi)$, with $l_0 = 0,\ldots\infty$ and $m_0 = -l_0,\ldots,l_0$,

$$U^*_{\mathrm{inert}}(t,0)\Psi(\vec{x}) = e^{iK_{\alpha,y_0}t} R^*(t)\Psi(\vec{x}) = e^{i(K_{\alpha,y_0}+m_0\omega)t}\Psi(\vec{x})$$

Therefore

$$\int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\mathrm{inert}}(t,0)\Psi(\vec{x}) = \int_{-\infty}^{0} dt\, e^{-izt}\, e^{i(K_{\alpha,y_0}+m_0\omega)t}\Psi(\vec{x}) =$$

$$\int_{-\infty}^{0} dt\, e^{-i(z-m_0\omega)t}\, e^{iK_{\alpha,y_0}t}\Psi(\vec{x}) = -i\big(K_{\alpha,y_0} + m_0\omega - z\big)^{-1}\Psi(\vec{x})$$

Hence we have now to prove that

$$\lim_{\omega\to\infty} \big(K_{\alpha,y_0} + m_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_{\gamma,C} - z\big)^{-1}\Psi(\vec{x})$$

First of all we observe that, for each $z \in \mathbb{C}$, $\Im(z) > 0$, $m_0 \in \mathbb{Z}$ and $\vec{y}_0 = (0, y_0, 0)$,

$$\lim_{\omega\to\infty} \mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0) = G^{m_0}_z(\vec{x},\vec{y}_0)$$

in the norm topology of $L^2(\mathbb{R}^3)$: since

$$\mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0) = G^{m_0}_z(\vec{x},\vec{y}_0) + R^{m_0}_z(\vec{x},\vec{y}_0)$$

with

$$R^{m_0}_z(\vec{x},\vec{y}_0) = \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{k^2 - (m-m_0)\omega - z}\, \varphi^*_{klm}(\vec{y}_0)\, \varphi_{klm}(\vec{x})$$

it is sufficient to prove that

$$\lim_{\omega\to\infty} \big\| R^{m_0}_z(\vec{x},\vec{y}_0)\big\|_{L^2(\mathbb{R}^3)} = 0$$

but

$$\big\| R^{m_0}_z(\vec{x},\vec{y}_0)\big\|^2_{L^2(\mathbb{R}^3)} = \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{|k^2 - (m-m_0)\omega - z|^2} |\varphi_{klm}(\vec{y}_0)|^2$$

and the right hand side is bounded for each $\omega \in \mathbb{R}$ (see Proposition A.1), so that we can exchange the limit with the integration

$$\lim_{\omega\to\infty} \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{1}{|k^2 - (m-m_0)\omega - z|^2} |\varphi_{klm}(\vec{y}_0)|^2 =$$

$$= \int_0^\infty dk \sum_{l=0}^{\infty} \sum_{\substack{m=-l \\ m\neq m_0}}^{l} |\varphi_{klm}(\vec{y}_0)|^2 \lim_{\omega\to\infty} \frac{1}{|k^2-(m-m_0)\omega - z|^2} = 0$$

Now, since (see Theorem 2.1)

$$\Big[\big(K_{\alpha,y_0}+m_0\omega - z\big)^{-1}\Psi\Big](\vec{x}) = \Big(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} +$$

$$+\lambda(z-m_0\omega,\alpha)\Big(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}',\vec{y}_0),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)}\mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0)$$

and

$$\lim_{\omega\to\infty}\Big(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} = e^{im_0\frac{\pi}{2}}\Big(G_z^{m_0\,*}(\vec{x},\vec{x}'),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} =$$

$$= \int_0^\infty dr' r'^2 g_z^{l_0}(y_0,r')\,\chi(r')\ \ Y_{l_0}^{m_0}(\pi/2,\pi/2)$$

$$\lim_{\omega\to\infty}\Big(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}',\vec{y}_0),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} = e^{-im_0\frac{\pi}{2}}\Big(G_{\bar{z}}^{m_0}(\vec{x}',\vec{y}_0),\Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)}$$

$$\lim_{\omega\to\infty}\mathcal{G}_{z-m_0\omega}(\vec{x},\vec{y}_0) = e^{im_0\frac{\pi}{2}}G_z^{m_0}(\vec{x},\vec{y}_0)$$

we obtain

$$\lim_{\omega\to\infty}\big(K_{\alpha,y_0}+m_0\omega-z\big)^{-1}\Psi(\vec{x}) = \int_0^\infty dr' r'^2\ g_z^{l_0}(r,r')\chi(r')\ \ Y_{l_0}^{m_0}(\theta,\phi) +$$

$$+\,\beta(z,\alpha)\,G_z^{m_0}(\vec{x},\vec{y}_0)\int_0^\infty dr' r'^2\ g_z^{l_0}(y_0,r')\chi(r') = \big(H_{\gamma,C}-z\big)^{-1}\Psi(\vec{x})$$

with[3]

$$\beta(z,\alpha) = \lim_{\omega\to\infty}\lambda(z-m_0\omega,\alpha)$$

and

$$\frac{\Gamma_{\gamma,C}(z)_{m_0}}{2\pi} = \frac{1}{\beta(z,\alpha)}$$

It remains to prove the explicit expression of $\gamma(\alpha,y_0)$: using the relation (see Theorem 2.1)

$$\frac{1}{\lambda(z-m_0\omega,\alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z-m_0\omega+i)\Big(\mathcal{G}_{-m_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_-(\vec{x})\Big)$$

we obtain

$$\frac{1}{\beta(z,\alpha)} = \lim_{\omega\to\infty}\left[\frac{1}{\lambda(-i,\alpha)} - (z-m_0\omega+i)\Big(\mathcal{G}_{-m_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_-(\vec{x})\Big)\right] =$$

$$= \frac{2i}{1+e^{i\alpha}}\int_0^\infty dk\sum_{l=0}^{\infty}\frac{1}{|k^2+i|^2}\big|\varphi_{kl0}(\vec{y}_0)\big|^2 + \int_0^\infty dk\sum_{l=0}^{\infty}\frac{1}{k^2+i}\big|\varphi_{kl0}(\vec{y}_0)\big|^2 +$$

[3] Actually λ is a function of $z-m_0\omega$ and ω, since the Green's function $\mathcal{G}_-(\vec{x})$ depends on ω.

$$-\int_0^\infty dk \sum_{l=|m_0|}^{\infty} \frac{1}{k^2-z} \left|\varphi_{klm_0}(\vec{y}_0)\right|^2$$

and hence the result. We want to stress that, as it was expected, $\gamma \in \mathbb{R}$:

$$\Im\left\{\frac{2i}{(1+e^{i\alpha})|k^2+i|^2} + \frac{1}{k^2+i}\right\} = \frac{1}{|k^2+i|^2}\left\{\Im\left[\frac{2i}{1+e^{i\alpha}}\right] - 1\right\} =$$

$$= \frac{1}{|k^2+i|^2}\left\{\frac{\Im\left[2i+2ie^{-i\alpha}\right]}{2+2\cos\alpha} - 1\right\} = 0$$

□

3 The Rotating Point Interaction in 2D

3.1 The Hamiltonian

The system we shall study is defined by the formal time-dependent Hamiltonian

$$H(t) = H_0 + a\,\delta^{(2)}(\vec{x} - \vec{y}(t)) \tag{3.1}$$

where $\vec{y}(t) = \mathcal{R}(t)\vec{y}_0$.
The formal generator of time evolution in the uniformly rotating frame (with angular velocity ω) is given by

$$K = H_0 - \omega J + a\,\delta^{(2)}(\vec{x} - \vec{y}_0)$$

Therefore the Hamiltonian of the system is a self-adjoint extension of the operator

$$K_{y_0} = H_\omega$$

$$\mathcal{D}(K_{y_0}) = C_0^\infty(\mathbb{R}^2 - \{\vec{y}_0\})$$

According to the discussion of Section 2, the Hamiltonian is given by the self-adjoint operator

$$\mathcal{D}(K_{\alpha,y_0}) = \{f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_- | g \in \mathcal{D}(\dot{K}_{y_0}), c \in \mathbb{C}\} \tag{3.2}$$

$$K_{\alpha,y_0}(f + c\mathcal{G}_+ + ce^{i\alpha}\mathcal{G}_-) = \dot{K}_{y_0} g + ic\mathcal{G}_+ - ice^{i\alpha}\mathcal{G}_- \tag{3.3}$$

with $\alpha \in [0, 2\pi)$ and where

$$\mathcal{G}_\pm(\vec{x}) = \mathcal{G}_{\pm i}(\vec{x}, \vec{y}_0)$$

$$\mathcal{G}_z(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{n=-\infty}^{\infty} \frac{1}{k^2 - n\omega - z}\,\varphi^*_{kn}(\vec{y}_0)\,\varphi_{kn}(\vec{x}) \tag{3.4}$$

for $\vec{x} \in \mathbb{R}^2 - \{\vec{y}_0\}$.
Like in the 3D case, the self-adjoint extension K_{π,y_0} corresponds to the "free" Hamiltonian $\dot{H}_\omega$ and

Theorem 3.1 *The resolvent of K_{α,y_0} has integral kernel given by*

$$(K_{\alpha,y_0}-z)^{-1}(\vec{x},\vec{x}') = \mathcal{G}_z(\vec{x},\vec{x}') + \lambda(z,\alpha)\mathcal{G}^*_{\bar{z}}(\vec{x}',\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0) \tag{3.5}$$

with $z \in \varrho(K_{\alpha,y_0})$, $\vec{x},\vec{x}' \in \mathbb{R}^2$, $\vec{x} \neq \vec{x}'$, $\vec{x},\vec{x}' \neq \vec{y}_0$ and

$$\frac{1}{\lambda(z,\alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z+i)\big(\mathcal{G}_{\bar{z}}(\vec{x}),\mathcal{G}_-(\vec{x})\big) \tag{3.6}$$

$$\lambda(-i,\alpha) = \frac{1+e^{i\alpha}}{2i\|\mathcal{G}_-(\vec{x})\|^2} \tag{3.7}$$

Proof: See the Proof of Theorem 2.1 and Proposition A.2.

□

Theorem 3.2 *The domain $\mathcal{D}(K_{\alpha,y_0})$, $\alpha \in [0,2\pi)$, consists of all elements $\Psi \in \mathbb{R}^3$ which can be decomposed in the following way*

$$\Psi(\vec{x}) = \Phi_z(\vec{x}) + \lambda(z,\alpha)\Phi_z(\vec{y}_0)\mathcal{G}_z(\vec{x},\vec{y}_0)$$

for $\vec{x} \neq \vec{y}_0$, $\Phi_z \in \mathcal{D}(\dot{H}_\omega)$ and $z \in \varrho(K_{\alpha,y_0})$. The previous decomposition is unique and on every Ψ of this form we obtain

$$(K_{\alpha,y_0}-z)\Psi = (H_\omega - z)\Phi_z$$

Proof: See the Proof of Theorem 2.2.

□

Theorem 3.3 *The spectrum $\sigma(K_{\alpha,y_0})$ is purely absolutely continuous and*

$$\sigma(K_{\alpha,y_0}) = \sigma_{\mathrm{ac}}(K_{\alpha,y_0}) = \sigma(H_\omega) = \mathbb{R} \tag{3.8}$$

Proof: See the Proof of Theorem 2.3, Theorem 2.1 and Proposition A.2.

□

3.2 Asymptotic Limit of Rapid Rotation

As in the 3D case, we shall prove that

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $H_{\gamma,C}$ is an appropriate self adjoint extension of H_C, a singular perturbation of the Laplacian supported over a circle of radius y_0: let C the curve $\vec{y}(\theta) = (y_0,\theta)$, $\theta \in [0,2\pi]$, and $\dot{H}_C$ the closure of the operator

$$H_C = H_0$$

$$\mathcal{D}(H_C) = C_0^\infty(\mathbb{R}^2 - C)$$

Proposition 3.1 *The self-adjoint extensions of the operator $\dot{H}_C$, that are invariant under rotations around the $z-$axis, are given by the one-parameter family of operators $H_{\gamma,C}$, $\gamma \in \mathbb{R}$, with domain*

$$\mathcal{D}(H_{\gamma,C}) = \{\Psi \in L^2(\mathbb{R}^2) \mid \exists\, \xi_\Psi \in \mathcal{D}(\Gamma_{\gamma,C}(z)), \Psi - \tilde{G}_z\xi_\Psi \in H^2(\mathbb{R}^2),$$
$$\big(\Psi - \tilde{G}_z\xi_\Psi\big)\big|_C = \Gamma_{\gamma,C}(z)\xi_\Psi\} \tag{3.9}$$
$$\big(H_{\gamma,C} - z\big)\Psi = \big(H_0 - z\big)\big(\Psi - \tilde{G}_z\xi_\Psi\big) \tag{3.10}$$

where $z \in \mathbb{C}$, $\Im(z) > 0$,

$$\mathcal{D}(\Gamma_{\gamma,C}(z)) = \{\xi \in L^2([0,2\pi]) \mid \Gamma_{\gamma,C}(z)_n\xi_n \in l^2\} \tag{3.11}$$
$$\xi_n \equiv \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} d\theta\, \xi(\theta)e^{-in\theta} = \Big(e_n\ ,\ \xi_\Psi\Big)_{L^2([0,2\pi],d\theta)}$$
$$\big(\Gamma_{\gamma,C}(z)\xi\big)(\theta) \equiv \frac{\xi(\theta)}{\gamma} - \int_0^{2\pi} d\theta' \frac{e^{i\sqrt{z}|\vec{y}(\theta)-\vec{y}(\theta')|}}{4\pi|\vec{y}(\theta)-\vec{y}(\theta')|}\xi(\theta') \tag{3.12}$$
$$\Gamma_{\gamma,C}(z)_n = \frac{1}{\gamma} - 2\pi\int_0^\infty dk \frac{1}{k^2-z}\big|\varphi_{kn}(\vec{y}_0)\big|^2 \tag{3.13}$$

and

$$\big(\tilde{G}_z\xi\big)(\vec{x}) \equiv \int_0^{2\pi} d\theta \frac{e^{i\sqrt{z}|\vec{x}-\vec{y}(\theta)|}}{4\pi|\vec{x}-\vec{y}(\theta)|}\xi(\theta)$$

Proof: Singular perturbations of the Laplacian supported on a curve in $\mathbb{R}^2$ are analogous to singular perturbations supported on a surface in $\mathbb{R}^3$: indeed the quadratic form

$$\mathcal{F}(\Psi,\Psi) \equiv \int_{\mathbb{R}^2} d^2\vec{x}\, \big|\nabla\Psi\big|^2 - \int_C d\theta\, \gamma(\theta)\big|\Psi(\vec{y}(\theta))\big|^2$$

is easily seen to be a closed semibounded quadratic form (see for example [14, 15] and the discussion of Section 5) on

$$\mathcal{D}(F) = \big\{\Psi \in L^2(\mathbb{R}^2) \mid \exists\, \xi_\Psi \in L^2(C), \Psi - \tilde{G}_z\xi_\Psi \in H^1(\mathbb{R}^2)\big\}$$

and it can be proved that it is associated to the self-adjoint operator $H_{\gamma,C}$.

□

Proposition 3.2 *If $\Psi(\vec{x}) \in L^2(\mathbb{R}^2)$, $\Psi(\vec{x}) = \chi(r)e_{n_0}(\theta)$ and $z \in \varrho(H_{\gamma,C})$, $\Im(z) > 0$,*

$$\Big(\big(H_{\gamma,C} - z\big)^{-1}\Psi\Big)(\vec{x}) = \int_0^\infty dr' r'\, g_z^{n_0}(r,r')\chi(r') +$$
$$+ \frac{2\pi}{\Gamma_{\gamma,C}(z)_{n_0}}\, G_z^{n_0}(\vec{x},\vec{y}_0)\int_0^\infty dr' r'\, g_z^{n_0}(y_0,r')\chi(r')$$

where

$$g_z^{n_0}(r,r') \equiv \int_0^\infty dk\, \frac{k}{k^2-z}\, J_{|n_0|}(kr)J_{|n_0|}(kr') = \big(H_0 - z\big)^{-1}\big|_{\mathcal{H}_{n_0}}(r,r')$$

and

$$G_z^n(\vec{x},\vec{y}_0) \equiv \int_0^\infty dk \frac{1}{k^2-z}\, \varphi^*_{kn}(\vec{y}_0)\, \varphi_{kn}(\vec{x})$$

Proof: See the Proof of Proposition 2.2 and Corollary 2.1.

□

Theorem 3.4 *For every $t, s \in \mathbb{R}$,*

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_{\gamma,C}(t-s)}$$

where $\gamma(\alpha, y_0) \in \mathbb{R}$ and

$$\gamma(\alpha, y_0) = \int_0^\infty dk\, k \left[\frac{2i}{(1+e^{i\alpha})|k^2+i|^2} + \frac{1}{k^2+i}\right] J_0^2(ky_0)$$

Proof: See the Proof of Theorem 2.4 and the following Lemma 3.1.

□

Lemma 3.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s}-\lim_{\omega\to\infty} \int_{-\infty}^0 dt\, e^{-izt}\, U^*_{\text{inert}}(t,0) = -i\big(H_{\gamma,C} - z\big)^{-1}$$

Proof: The first part of the proof is analogous to the Proof of Lemma 2.1 (the only difference is the dense subset of $L^2(\mathbb{R}^2)$ given by functions of the form $\Psi(\vec{x}) = \chi(r) e_{n_0}(\theta)$, with $n_0 \in \mathbb{Z}$).
Hence it remains to prove that

$$\lim_{\omega\to\infty} \big(K_{\alpha,y_0} + n_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_{\gamma,C} - z\big)^{-1}\Psi(\vec{x})$$

Now, for each $z \in \mathbb{C}$, $\Im(z) > 0$, $n_0 \in \mathbb{Z}$ and $\vec{y}_0 = (0, y_0)$,

$$\lim_{\omega\to\infty} \mathcal{G}_{z-n_0\omega}(\vec{x}, \vec{y}_0) = G_z^{n_0}(\vec{x}, \vec{y}_0)$$

in the norm topology of $L^2(\mathbb{R}^2)$: since

$$\mathcal{G}_{z-n_0\omega}(\vec{x}, \vec{y}_0) = G_z^{n_0}(\vec{x}, \vec{y}_0) + R_z^{n_0}(\vec{x}, \vec{y}_0)$$

with

$$R_z^{n_0}(\vec{x}, \vec{y}_0) = \int_0^\infty dk \sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} \frac{1}{k^2 - (n-n_0)\omega - z}\, \varphi^*_{kn}(\vec{y}_0)\, \varphi_{kn}(\vec{x})$$

it is sufficient to prove that

$$\lim_{\omega\to\infty} \big\|R_z^{n_0}(\vec{x}, \vec{y}_0)\big\|_{L^2(\mathbb{R}^2)} = 0$$

but

$$\big\|R_z^{n_0}(\vec{x}, \vec{y}_0)\big\|^2_{L^2(\mathbb{R}^2)} = \int_0^\infty dk \sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} \frac{1}{|k^2 - (n-n_0)\omega - z|^2} |\varphi_{kn}(\vec{y}_0)|^2$$

and the right hand side is bounded (see Proposition A.2) for each $\omega \in \mathbb{R}$, so that exchanging the limit with the integration, we obtain the result.
Now, substituting in the expression of the resolvent (see Theorem 3.1),

$$\Big[\big(K_{\alpha,y_0} + m_0\omega - z\big)^{-1}\Psi\Big](\vec{x}) = \Big(\mathcal{G}^*_{z-m_0\omega}(\vec{x},\vec{x}'), \Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} +$$

$$+\lambda(z - m_0\omega, \alpha)\Big(\mathcal{G}_{\bar{z}-m_0\omega}(\vec{x}', \vec{y}_0), \Psi(\vec{x}')\Big)_{L^2(\mathbb{R}^3)} \mathcal{G}_{z-m_0\omega}(\vec{x}, \vec{y}_0)$$

the result follows from a straightforward calculation. Moreover we obtain the same relation between γ and α:

$$\frac{\Gamma_{\gamma,C}(z)_{n_0}}{2\pi} = \frac{1}{\beta(z,\alpha)}$$

where

$$\beta(z,\alpha) = \lim_{\omega\to\infty} \lambda(z - n_0\omega, \alpha)$$

but

$$\frac{1}{\lambda(z - n_0\omega, \alpha)} = \frac{1}{\lambda(-i,\alpha)} - (z - n_0\omega + i)\Big(\mathcal{G}_{-n_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_{-}(\vec{x})\Big)$$

and then

$$\frac{1}{\beta(z,\alpha)} = \lim_{\omega\to\infty}\left[\frac{1}{\lambda(-i,\alpha)} - (z - n_0\omega + i)\Big(\mathcal{G}_{-n_0\omega+\bar{z}}(\vec{x}),\ \mathcal{G}_{-}(\vec{x})\Big)\right] =$$

$$= \frac{2i}{1+e^{i\alpha}}\int_0^\infty dk \frac{1}{|k^2+i|^2}\left|\varphi_{k0}(\vec{y}_0)\right|^2 + \int_0^\infty dk \frac{1}{k^2+i}\left|\varphi_{k0}(\vec{y}_0)\right|^2 +$$

$$-\int_0^\infty dk \frac{1}{k^2 - z}\left|\varphi_{kn_0}(\vec{y}_0)\right|^2$$

□

4 The Rotating Blade in 3D

4.1 The Hamiltonian

The formal time-dependent Hamiltonian of the system is given by

$$H(t) = H_0 + a\,R(t)\,\Theta_D(x,z)\,\delta(y) \tag{4.1}$$

where $R(t)\Psi(\vec{x}) = \Psi(\mathcal{R}(t)^{-1}\,\vec{x})$ and $\Theta_D(x,z)$ is the characteristic function of the half-disk $0 \le r \le A$, $\phi = 0$. Therefore in the rotating frame the formal generator of time evolution is

$$K = H_0 - \omega J + a\,\Theta_D(x,z)\,\delta(y)$$

or more rigorously a self-adjoint extension of the symmetric operator

$$K_D = H_\omega$$

$$\mathcal{D}(K_D) = C_0^\infty(\mathbb{R}^3 - D)$$

where D is the half-disk $D \equiv \{(r, \theta, \phi) \in \mathbb{R}^3 \mid 0 \le r \le A,\ 0 \le \theta \le \pi,\ \phi = 0\}$.
The Hamiltonian cannot be easily defined with the method of quadratic form, because of its unboundedness from below. Hence we shall pursue a different strategy: we shall define a sequence of cut-off Hamiltonians which converge to the operator H_ω in the strong resolvent sense and that are self-adjoint and bounded from below; then we shall add the singular perturbation and prove that the so obtained operators are self-adjoint. Finally we shall prove that the limit (in the strong resolvent sense) of the sequence of cut-off perturbed Hamiltonians is a self-adjoint operator that we shall identify with the Hamiltonian of the system.
So let

$$H_\omega^L = H_\omega \, \Pi_L \tag{4.2}$$

where Π_L is the projector on the subspace of $L^2(\mathbb{R}^3)$ generated by functions of the form $\chi(r) Y_l^m(\theta, \phi)$, with $l \le L$. It is very easy to prove that the operator H_ω^L is self-adjoint on the domain $H^2(\mathbb{R}^3)$: the operator J is bounded on the domain of the projector Π_L and therefore it is an infinitesimally bounded perturbation of H_0, so that we can apply the Kato Theorem [9]. Moreover for each $z \in \varrho(H_\omega^L)$ the resolvent $(H_\omega^L - z)^{-1}$ is given by an integral operator with kernel

$$\mathcal{G}_z^L(\vec{x}, \vec{x}') = \int_0^\infty dk \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{\varphi_{klm}^*(\vec{x}')\, \varphi_{klm}(\vec{x})}{k^2 - m\omega - z} \tag{4.3}$$

Proposition 4.1 *The sequence of cut-off Hamiltonians converge as $L \to \infty$ in the strong resolvent sense to the self-adjoint operator H_ω.*

Proof: For each $L \in \mathbb{N}$ and $z \in \mathbb{C} - \mathbb{R}$, the function $\mathcal{G}_z^L(\vec{x}, \vec{x}')$ belongs to $L^2(\mathbb{R}^3, d^3\vec{x})$:

$$\left\| \mathcal{G}_z^L(\vec{x}, \vec{x}') \right\|^2 \le \left\| \mathcal{G}_z(\vec{x}, \vec{x}') \right\|^2 < \infty$$

and then the result is a straightforward consequence of Proposition A.1. The operator H_ω was studied in [6, 16].

□

Now we can defined the perturbed cut-off Hamiltonians with the method of quadratic form: let[4]

$$\mathcal{F}_{\alpha,L}(\Psi, \Psi) = F_{\omega,L}(\Psi, \Psi) - \int_D d\mu_D(\vec{r})\, \alpha(\vec{r}) \left| \Psi\big|_D(\vec{r}) \right|^2 \tag{4.4}$$

where $F_{\omega,L}$ is the closed[5] semibounded quadratic form associated to H_ω^L. The form $\mathcal{F}_{\alpha,L}$ is well defined if $\Psi \in \mathcal{D}(F_{\omega,L})$ and α is a smooth real function on D bounded away from 0.

Proposition 4.2 *Let $z \in \mathbb{C} - \mathbb{R}$, the form $\mathcal{F}_{\alpha,L}$ can be written in the following way,*

$$\mathcal{F}_{\alpha,L}(\Psi, \Psi) = \mathcal{F}_{\omega,L}^z(\Psi, \Psi) + \Phi_{\alpha,L}^z(\xi_\Psi, \xi_\Psi) - 2\Im(z)\, \Im\Big[\big(\Psi\ ,\ \tilde{\mathcal{G}}_z^L \xi_\Psi\big) \Big] \tag{4.5}$$

[4]Here $d\mu_D(\vec{r})$ stands for the restriction of the Lebesgue measure to D, namely $d\mu_D(\vec{r}) \equiv r^2\, dr\, d\cos\theta$ for $\vec{r} = (r, \theta) \in D$; $\vec{r}$ denotes the restriction of $\vec{x} \in \mathbb{R}^3$ to D, i.e. $\vec{r} \equiv (r, \theta)$.

[5]The form $F_{\omega,L}$ is closed on the domain $\mathcal{D}(F_{\omega,L}) = H^1(\mathbb{R}^3)$.

where

$$\mathcal{F}^z_{\omega,L}(\Psi,\Psi) = F_{\omega,L}(\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi, \Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi) - \Re(z)\|\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 + \Re(z)\|\Psi\|^2 \tag{4.6}$$

$$\Phi^z_{\alpha,L}(\xi_\Psi,\xi_\Psi) = \Re\Big[\big(\xi_\Psi\ ,\ \Gamma^L_\alpha(z)\,\xi_\Psi\big)_{L^2(D,d\mu_D)}\Big] \tag{4.7}$$

and

$$\Big[\Gamma^L_\alpha(z)\,\xi_\Psi\Big](\vec{r}) = \frac{\xi_\Psi(\vec{r})}{\alpha(\vec{r})} - \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}^L_z(\vec{x},\vec{x}\,')\big|_{\vec{x},\vec{x}\,'\in D}\,\xi_\Psi(\vec{r}\,') \tag{4.8}$$

$$\big(\tilde{\mathcal{G}}^L_z\xi\big)(\vec{x}) \equiv \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}^L_z(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D}\,\xi(\vec{r}\,')$$

Proof: The result follows from a simple calculation: setting

$$\xi_\Psi(\vec{r}) = \alpha(\vec{r})\ \Psi\big|_D(\vec{r}) \tag{4.9}$$

one has

$$\mathcal{F}_{\alpha,L}(\Psi,\Psi) - F_{\omega,L}(\Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi, \Psi - \tilde{\mathcal{G}}^L_z\xi_\Psi) = \big(\tilde{\mathcal{G}}^L_z\xi\ ,H^L_\omega(\Psi - \tilde{\mathcal{G}}^L_z\xi)\big) +$$

$$+\big(\Psi\ ,\ H^L_\omega\tilde{\mathcal{G}}^L_z\xi\big) - \int_D d\mu_D\,\frac{|\xi_\Psi|^2}{\alpha} =$$

$$= \int_D d\mu_D\,\frac{|\xi_\Psi|^2}{\alpha} - \big(\tilde{\mathcal{G}}^L_z\xi\ ,\ (H^L_\omega - z^*)\tilde{\mathcal{G}}^L_z\xi\big) - z^*\big\|\tilde{\mathcal{G}}^L_z\xi\big\|^2 + 2\Re\Big[z\big(\Psi\ ,\ \tilde{\mathcal{G}}^L_z\xi\big)\Big] =$$

$$= \Phi^z_{\alpha,L}(\xi_\Psi,\xi_\Psi) - \Re(z)\big\|\tilde{\mathcal{G}}^L_z\xi\big\|^2 + 2\Re\Big[z\big(\Psi\ ,\ \tilde{\mathcal{G}}^L_z\xi\big)\Big]$$

since

$$\Im(z)\ \big\|\tilde{\mathcal{G}}^L_z\xi\big\|^2 = \Im\Big[\big(\tilde{\mathcal{G}}^L_z\xi\ ,\ (H^L_\omega - z^*)\tilde{\mathcal{G}}^L_z\xi\big)\Big]$$

but

$$\big\|\tilde{\mathcal{G}}^L_z\xi\big\|^2 = \big\|\Psi - \tilde{\mathcal{G}}^L_z\xi\big\|^2 - \big\|\Psi\big\|^2 + 2\Re\Big[\big(\Psi\ ,\ \tilde{\mathcal{G}}^L_z\xi\big)\Big]$$

so that we obtain the result.

□

Of course the form $\mathcal{F}_{\alpha,L}$ is independent on z and the decomposition $\Psi = \varphi_z + \tilde{\mathcal{G}}^L_z\xi_\Psi$ is unique, since $\tilde{\mathcal{G}}^L_z\xi_\Psi \notin \mathcal{D}(F_{\omega,L})$ if $\xi_\Psi \in L^2(D,d\mu_D)$. Moreover the form $\Phi^z_{\alpha,L}(\xi,\xi)$ is bounded and one can choose $z \in \mathbb{C}$ such that the form satisfies another useful inequality:

Proposition 4.3 *The form $\Phi^z_{\alpha,L}(\xi,\xi)$ is bounded for each $\xi \in L^2(D,d\mu_D)$.*

Proof: The first term of the form is of course bounded if $\xi \in L^2(D,d\mu_D)$ and

$$\left|\int_D d\mu_D\ \xi\ \big(\tilde{\mathcal{G}}^L_z\xi\big)^*\big|_D\right| \le \|\xi\|_{L^2(D,d\mu_D)}\ \big\|\big(\tilde{\mathcal{G}}^L_z\xi\big)\big|_D\big\|_{L^2(D,d\mu_D)}$$

but we are going to prove that the function $(\tilde{\mathcal{G}}^L_z\xi)|_D(\vec{r})$ is bounded $\forall\,\vec{r}\in D$, so that

$$\big\|\big(\tilde{\mathcal{G}}^L_z\xi\big)\big|_D\big\|_{L^2(D,d\mu_D)} < C(A)\ \|\xi\|^2_{L^2(D,d\mu_D)}$$

and hence the result. Indeed

$$\left|(\tilde{\mathcal{G}}^L_z\xi)\big|_D(\vec{r})\right|^2 = \left|\Big(\mathcal{G}^L_z(\vec{x}',\vec{x})\big|_{\vec{x},\vec{x}'\in D}\,,\,\xi(\vec{r}')\Big)_{L^2(D,d\mu_D)}\right|^2 \leq$$

$$\leq \left\|\mathcal{G}^L_z(\vec{x}',\vec{x})\right\|^2_{L^2(D,d\mu_D(\vec{r}'))}\,\left\|\xi\right\|^2_{L^2(D,d\mu_D)} \leq C\,\left\|\xi\right\|^2_{L^2(D,d\mu_D)}$$

since the Green's function $\mathcal{G}^L_z(\vec{x},\vec{y}_0)$ belongs to $L^2(\mathbb{R}^3)$, for each $z \in \mathbb{C}-\mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$.

□

Proposition 4.4 *For each smooth real function α on D bounded away from 0, there exists $\zeta \in \mathbb{R}$, $\zeta < 0$ such that, for each $z \in \mathbb{C}-\mathbb{R}$, $\Re(z) < \zeta$, the following inequality holds*

$$\Phi^z_{\alpha,L}(\xi,\xi) - 2\Im(z)\,\Im\Big[\big(\Psi\,,\,\tilde{\mathcal{G}}^L_z\xi_\Psi\big)\Big] - \big(\Re(z)+\omega L\big)\,\|\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 > 0 \qquad (4.10)$$

Proof: We first point out that (see Proposition A.1)

$$\lim_{\Re(z)\to\infty}\left\|\mathcal{G}^L_z(\vec{x},\vec{y}_0)\right\| \leq C(\Im(z)) < \infty$$

Thus, since the form $\Phi^z_{\alpha,L}(\xi,\xi)$ remains bounded for each $z \in \mathbb{C}-\mathbb{R}$, $\Im(z) \neq 0$, and

$$\lim_{\Re(z)\to\infty}\Re(z)\|\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 = \infty$$

$$\left|\Im(z)\,\Im\Big[\big(\Psi\,,\,\tilde{\mathcal{G}}^L_z\xi_\Psi\big)\Big]\right| \leq C(\Im(z))\,\left\|\xi\right\|^2$$

we can always found a ζ satisfying the requirement.

□

Now we can prove that the complete form $\mathcal{F}_{\alpha,L}$ is closed and bounded from below:

Theorem 4.1 *The form $\mathcal{F}_{\alpha,L}$ is bounded from below and closed on the domain*

$$\mathcal{D}(\mathcal{F}_{\alpha,L}) = \left\{\Psi \in L^2(\mathbb{R}^3)\mid \exists\xi_\Psi \in L^2(D,d\mu_D), \Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi \in H^1(\mathbb{R}^3)\right\} \qquad (4.11)$$

where $z \in \mathbb{C}-\mathbb{R}$.

Proof: Semiboundedness is trivial thanks to Proposition 4.4: since the form $\mathcal{F}_{\alpha,L}$ does not depend on z, we can choose $z \in \mathbb{C}-\mathbb{R}$, $\Re(z) < \zeta$, so that the inequality (4.10) applies and

$$\mathcal{F}_{\alpha,L}(\Psi,\Psi) \geq F_{\omega,L}(\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi,\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi) + \omega L\,\|\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi\|^2 + \Re(z)\,\|\Psi\|^2 \geq$$

$$\geq F_0(\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi,\Psi-\tilde{\mathcal{G}}^L_z\xi_\Psi) + \Re(z)\,\|\Psi\|^2 \geq \Re(z)\,\|\Psi\|^2$$

So it remains to prove closure: if $\Psi_n = \varphi_n + \tilde{\mathcal{G}}_z\xi_n$ a sequence in $\mathcal{D}(\mathcal{F}_{\alpha,L})$ converging to Ψ in the norm topology of $L^2(\mathbb{R}^3)$, such that[6]

$$\lim_{n,m\to\infty}\big(\mathcal{F}_{\alpha,L}-\Re(z)\big)(\Psi_n-\Psi_m) = 0$$

[6] F_0 is simply the form associated to the free Hamiltonian, i.e. $F_0(\Psi,\Psi) = \int|\nabla\Psi|^2$.

$$\lim_{n,m\to\infty} \big(\mathcal{F}_{\alpha,L} - \Re(z)\big)(\Psi_n - \Psi_m) \geq \lim_{n,m\to\infty} F_0(\varphi_n - \varphi_m) \geq 0$$

so that

$$\lim_{n,m\to\infty} F_0(\varphi_n - \varphi_m) = 0$$

and

$$\lim_{n,m\to\infty} \Phi^z_{\alpha,L}(\xi_n - \xi_m) = 0$$

The result easily follows, because F_0 and $\Phi^z_{\alpha,L}$ are closed forms (see Proposition 4.3).

□

Thus the form $\mathcal{F}_{\alpha,L}$ defines a semibounded self-adjoint operator:

Proposition 4.5 *The operators K^L_α defined below are self-adjoint:*

$$\mathcal{D}(K^L_\alpha) = \big\{\Psi \in L^2(\mathbb{R}^3) \mid \exists \xi_\Psi \in L^2(D, d\mu_D), \Psi - \tilde{\mathcal{G}}^L_z \xi_\Psi \in \mathcal{D}(H^L_\omega),$$

$$\big(\Psi - \tilde{\mathcal{G}}^L_z \xi_\Psi\big)\big|_D = \Gamma^L_\alpha(z)\xi_\Psi\big\} \tag{4.12}$$

$$\big(K^L_\alpha - z\big)\Psi = \big(H^L_\omega - z\big)\big(\Psi - \tilde{\mathcal{G}}^L_z \xi_\Psi\big) \tag{4.13}$$

where $\alpha \in \mathrm{C}(D)$, $\alpha(\vec{r}) \neq 0$, for each $\vec{r} \in D$.
Moreover

$$\Big[\big(K^L_\alpha - z\big)^{-1}\Psi\Big](\vec{x}) = \Big[\big(H^L_\omega - z\big)^{-1}\Psi\Big](\vec{x}) +$$

$$+ \int_D d^2\vec{r}\,' \ \big[\Gamma^L_\alpha(z)\big]^{-1}\Big[\big(H^L_\omega - z\big)^{-1}\Psi\Big]\Big|_D(\vec{r}\,')\, \mathcal{G}^L_z(\vec{x}, \vec{x}\,')\big|_{\vec{x}\,'\in D} \tag{4.14}$$

for each $z \in \varrho(K_\alpha)$.

Proof: The result easily follows from Theorem 4.1. The explicit expression of the resolvent is a direct consequence of the equation (4.13). We want only to remark that the operator $\Gamma^L_\alpha(z)$ is invertible if $\Im(z) \neq 0$: the form $\Phi^z_{\alpha,L}$ can be written in the following way

$$\Phi^z_{\alpha,L}(\xi, \xi) \equiv \int_D d\mu_D \ \frac{|\xi|^2}{\alpha} - \Re(z)\big\|\tilde{\mathcal{G}}^L_z \xi\big\|^2$$

Since $\big\|\tilde{\mathcal{G}}^L_z \xi\big\|^2$ is bounded by $C(\Im(z))\,\|\xi\|^2$, if $\Im(z) \neq 0$, we can always choose the real part of z is such a way that the form is positive.

□

At last we can remove the cut-off in the angular momentum and define the Hamiltonian of the system:

Theorem 4.2 *For each $\alpha \in \mathrm{C}(D)$, $\alpha(\vec{r}) \neq 0$, $\forall\, \vec{r} \in D$, the sequence of semibounded self-adjoint operators K^L_α converge as $L \to \infty$ in the strong resolvent sense to the self-adjoint (unbounded from below) operator K_α:*

$$\mathcal{D}(K_\alpha) = \big\{\Psi \in L^2(\mathbb{R}^3) \mid \exists \xi_\Psi \in L^2(D, d\mu_D), \Psi - \tilde{\mathcal{G}}_z \xi_\Psi \in \mathcal{D}(H_\omega),$$

$$\big(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi\big)\big|_D = \Gamma_\alpha(z)\xi_\Psi\big\} \tag{4.15}$$

$$\left(K_\alpha - z\right)\Psi = \left(H_\omega - z\right)\left(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi\right) \tag{4.16}$$

where

$$\left[\Gamma_\alpha(z)\,\xi_\Psi\right](\vec{r}) = \frac{\xi_\Psi(\vec{r})}{\alpha(\vec{r})} - \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}_z(\vec{x},\vec{x}\,')\big|_{\vec{x},\vec{x}\,'\in D}\,\xi_\Psi(\vec{r}\,') \tag{4.17}$$

$$\left(\tilde{\mathcal{G}}_z\xi\right)(\vec{x}) \equiv \int_D d\mu_D(\vec{r}\,')\ \mathcal{G}_z(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D}\,\xi(\vec{r}\,')$$

Moreover the resolvent of K_α *is*

$$\left[\left(K_\alpha - z\right)^{-1}\Psi\right](\vec{x}) = \left[\left(H_\omega - z\right)^{-1}\Psi\right](\vec{x}) +$$

$$+\int_D d^2\vec{r}\,'\ \Gamma_\alpha^{-1}(z)\left[\left(H_\omega - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}\,')\,\mathcal{G}_z(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D} \tag{4.18}$$

for each $z \in \varrho(K_\alpha)$.

Proof: The key point of the proof is the application of the Trotter-Kato Theorem (see Theorem VIII.22 in [12]) to the sequence of self-adjoint operators K_α^L: we shall prove that $(K_\alpha^L - z)^{-1}$ converge in the strong sense for all $z \in \mathbb{C} - \mathbb{R}$ to the operator $(K_\alpha - z)^{-1}$, then the Trotter-Kato Theorem guarantees that there exists a self-adjoint operator T such that K_α^L converges in the strong resolvent sense to T. The identification of T with K_α is then trivial.
So we shall start with the analysis of the sequence of bounded operators $(K_\alpha - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, defined in (4.14): thanks to Proposition 4.1, the first part of the resolvent converges in the strong sense to $(H_\omega - z)^{-1}$, so that, in order to prove convergence of the whole operator, we need to consider the second part,

$$\int_D d^2\vec{r}\,'\ \left[\Gamma_\alpha^L(z)\right]^{-1}\left[\left(H_\omega^L - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}\,')\,\mathcal{G}_z^L(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D}$$

but, for the same reason,

$$\lim_{L\to\infty} \mathcal{G}_z^L(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D} = \mathcal{G}_z(\vec{x},\vec{x}\,')\big|_{\vec{x}\,'\in D}$$

in $L^2(\mathbb{R}^3)$ and

$$\lim_{L\to\infty}\left[\left(H_\omega^L - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}\,') = \left[\left(H_\omega - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}\,')$$

in $L^2(D, d\mu_D)$, for all $\Psi \in L^2(\mathbb{R}^3)$. Hence, to complete the first part of the proof, it is sufficient to show that

$$\lim_{L\to\infty}\left[\Gamma_\alpha^L(z)\right]^{-1} = \Gamma_\alpha^{-1}(z)$$

in the norm topology of $L^2(D, d\mu_D)$, but this is again a consequence of Proposition 4.1: for each L the operator $\Gamma_\alpha^L(z)$ is invertible (see the Proof of Proposition 4.5) and, in the same way, we can prove that $\Gamma_\alpha^{-1}(z)$ is bounded and well defined, if $\Im(z) \neq 0$; moreover it is easy to see that

$$\lim_{L\to\infty}\Gamma_\alpha^L(z) = \Gamma_\alpha(z)$$

We have then proved that, for each $z \in \mathbb{C} - \mathbb{R}$,

$$\mathrm{s-}\lim_{L \to \infty} \left(K_\alpha^L - z \right)^{-1} = \left(K_\alpha - z \right)^{-1}$$

and the operator $(K_\alpha - z)^{-1}$ has of course a dense range. Thus the Trotter-Kato Theorem applies and the limiting self-adjoint operator T is immediately identified with K_α: the domain of K_α is given by functions of the form $(K_\alpha - z)^{-1}\Psi$, $\Psi \in L^2(\mathbb{R}^3)$, and the action of the operator on its domain follows from (4.18).

□

Theorem 4.3 *The spectrum of K_α is purely absolutely continuous and*

$$\sigma(K_\alpha) = \sigma_{\mathrm{ac}}(K_\alpha) = \sigma(H_\omega) = \mathbb{R}$$

Proof: First of all we shall prove that the operator

$$\mathcal{R}_\alpha^z \equiv \left(K_\alpha - z \right)^{-1} - \left(H_\omega - z \right)^{-1}$$

is a compact operator $\forall\, z \in \mathbb{C} - \mathbb{R}$. Let Ψ_n a weakly convergent sequence in $L^2(\mathbb{R}^3)$, namely $(\varphi\,, \Psi_n - \Psi_m) \to 0$ when $n, m \to \infty$ for each $\varphi \in L^2(\mathbb{R}^3)$,

$$\mathcal{R}_\alpha^z\, (\Psi_n - \Psi_m) = \int_D d^2\vec{r}'\ \Gamma_\alpha^{-1}(z) \Big[\left(H_\omega - z \right)^{-1} (\Psi_n - \Psi_m) \Big] \Big|_D\, \mathcal{G}_z(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D}$$

and

$$\begin{aligned} \left\| \mathcal{R}_\alpha^z\, (\Psi_n - \Psi_m) \right\| &\le \left\| \mathcal{G}_z \right\| \left\| \Gamma_\alpha^{-1}(z) \right\| \left| \Big(\mathcal{G}_{z^*}^*\ ,\ \Psi_n - \Psi_m \Big) \right| \le \\ &\le C \left| \Big(\mathcal{G}_{z^*}\ ,\ \Psi_n - \Psi_m \Big) \right| \underset{n,m \to \infty}{\longrightarrow} 0 \end{aligned}$$

since the operator $\Gamma_\alpha^{-1}(z)$ is bounded (see the Proof of Theorem 4.2). Therefore we can apply the Weyl theorem and thus

$$\sigma_{\mathrm{ess}}(K_\alpha) = \sigma_{\mathrm{ess}}(H_\omega) = \mathbb{R}$$

It remains to prove that the singular and pure points spectrum of K_α are empty, but it can be seen that the limiting absorption principle works. To show that the condition of the principle is satisfied, we have to consider the scalar product (where $z = x + i\varepsilon$)

$$\begin{aligned} \left| \Big(\Psi\,, \mathcal{R}_\alpha^z \Psi \Big) \right| &= \left| \int_D d^2\vec{r}'\ \Gamma_\alpha^{-1}(z) \Big[\left(H_\omega - z \right)^{-1} \Psi \Big] \Big|_D \Big(\Psi\ ,\ \mathcal{G}_z(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D} \Big) \right| \le \\ &\le \left\| \Gamma_\alpha^{-1}(z) \right\| \left| \int_D d^2\vec{r}'\ \Big(\mathcal{G}_{z^*}(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D}\ ,\ \Psi \Big) \Big(\Psi\ ,\ \mathcal{G}_z(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D} \Big) \right| \end{aligned}$$

The operator $\Gamma_\alpha^{-1}(z)$ remains bounded when $\varepsilon \to 0$ and, applying the same trick used in the Proof of Theorem 2.3, one has

$$\lim_{\varepsilon \to 0} \Big(\mathcal{G}_{x - i\varepsilon}(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D}\ ,\ \Psi \Big) \Big(\Psi\ ,\ \mathcal{G}_{x + i\varepsilon}(\vec{x}, \vec{x}') \big|_{\vec{x}' \in D} \Big) = \left| \varphi(\vec{r}') \right|^2 < \infty$$

where $\Psi = (H_\omega - x)\varphi$ and $\varphi \in \mathcal{D}(H_\omega)$, so that

$$\sup_{0 < \varepsilon < 1} \int_a^b dx\ \left| \Big(\Psi\ ,\ \mathcal{R}_\alpha^{x + i\varepsilon} \Psi \Big) \right|^p < \infty$$

for some $p > 1$ and for each interval $[a, b] \subset \mathbb{R}$.

□

4.2 Asymptotic Limit of Rapid Rotation

In this Section we shall study the asymptotic limit of rapid rotation of the unitary group

$$U_{\text{inert}}(t,s) = R(t)\, U_{\text{rot}}(t-s)\, R^{\dagger}(s)$$

which represents the time evolution in the inertial frame associated to the formal time-dependent Hamiltonian defined in (4.1), while $U_{\text{rot}}(t-s)$ is the unitary group associated to the self-adjoint generator K_α: our main goal will be the proof of the following result,

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where H_α is the self-adjoint generator[7]

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r}) \tag{4.19}$$

and $\Theta_S(\vec{r})$ is the characteristic function of a sphere S of radius A centered at the origin.

Theorem 4.4 *For every $t,s \in \mathbb{R}$,*

$$\text{s}-\lim_{\omega\to\infty} U_{\text{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r})$$

Proof: See the Proof of Theorem 2.4 and the following Lemma 4.1.

□

Lemma 4.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\text{s}-\lim_{\omega\to\infty} \int_{-\infty}^{0} dt\, e^{-izt}\, U^{*}_{\text{inert}}(t,0) = -i\big(H_\alpha - z\big)^{-1}$$

Proof: Like in the Proof of Lemma 2.1, we shall prove the result on the dense subset of $L^2(\mathbb{R}^3)$ given by functions of the form $\Psi(\vec{x}) = \chi(r) Y_{l_0}^{m_0}(\theta,\phi)$, with $l_0 = 0,\ldots\infty$ and $m_0 = -l_0,\ldots,l_0$. The first part of the Proof of Lemma 2.1 still applies, so that it is sufficient to prove that

$$\lim_{\omega\to\infty} \big(K_\alpha + m_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_\alpha - z\big)^{-1}\Psi(\vec{x})$$

First of all we observe that

$$\big(K_\alpha + m_0\omega - z\big)^{-1}\Psi = \big(H_\omega + m_0\omega - z\big)^{-1}\Psi +$$

$$+ \Big(\Gamma_\alpha^{-1}(z^* - m_0\omega)\Big[\big(H_\omega + m_0\omega - z^*\big)^{-1}\Psi\Big]\Big|_D\,,\ \mathcal{G}_{z-m_0\omega}(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\Big)_{L^2(D,d\mu_D)}$$

[7] The operator H_α is easily defined with the method of quadratic form (see for example [12]): since the potential $\alpha(r)$ is bounded, it is associated to a form infinitesimally bounded w.r.t. the free Hamiltonian H_0. Hence the operator $H_0 + \alpha(r)\,\Theta_D(\vec{r})$ is self-adjoint on the domain of H_0.

and

$$\lim_{\omega\to\infty} \big(H_\omega + m_0\omega - z\big)^{-1}\Psi = \big(H_0 - z\big)^{-1}\Psi$$

as we have proved in Lemma 2.1.
Therefore we need only to study the second part of the resolvent: it is easy to see that

$$\lim_{\omega\to\infty} \Big[\big(H_\omega + m_0\omega - z\big)^{-1}\Psi\Big]\Big|_D = \Big[\big(H_0 - z\big)^{-1}\Psi\Big]\Big|_D$$

in $L^2(D, d\mu_D)$. Moreover, since $\big[\big(H_0 - z\big)^{-1}\Psi\big]\big|_D(\vec{r})$ is a function of the form $\chi(r)Y_{l_0}^{m_0}(\theta, 0)$, we can apply the result found in the following Lemma 4.2:

$$\lim_{\omega\to\infty} \Gamma_\alpha^{-1}(z - m_0\omega)\Big[\big(H_\omega + m_0\omega - z\big)^{-1}\Psi\Big]\Big|_D =$$

$$= \Xi_\alpha(z)\Big[\big(H_0 - z\big)^{-1}\Psi\Big]\Big|_D =$$

$$= \alpha(\vec{r})\Theta_D(\vec{r})\big(H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r}) - z\big)^{-1}\Psi$$

In conclusion we obtain

$$\lim_{\omega\to\infty} \big(K_\alpha + m_0\omega - z\big)^{-1}\Psi = \big(H_0 - z\big)^{-1}\Big[1 + \alpha\Theta_D\big(H_0 - \alpha\,\Theta_S - z\big)^{-1}\Big]\Psi =$$

$$= \big(H_0 - \alpha\,\Theta_S - z\big)^{-1}\Psi$$

$\square$

Lemma 4.2 *Let $\Gamma_\alpha(z)$ the operator defined in (4.17) and $\Psi(\vec{x}) \in L^2(\mathbb{R}^3)$ of the form $\Psi(\vec{x}) = \chi(r)Y_{l_0}^{m_0}(\theta, \phi)$,*

$$\lim_{\omega\to\infty} \Gamma_\alpha^{-1}(z - m_0\omega)\,\Psi|_D = \Xi_\alpha(z)\,\Psi|_D$$

in $L^2(D, d\mu_D)$, where

$$\big(\Xi_\alpha(z)\Psi|_D\big)(\vec{r}) \equiv \Big[\alpha(\vec{r})\big(H_0 - \alpha(\vec{r})\,\Theta_S(\vec{r}) - z\big)^{-1}\big(H_0 - z\big)\,\Psi|_D\Big](\vec{r}) \qquad (4.20)$$

Proof: First of all we are going to prove that

$$\|\ \| - \lim_{\omega\to\infty} \Gamma_\alpha(z - m_0\omega) = \Lambda_\alpha(z)$$

where

$$\big(\Lambda_\alpha(z)\,\xi\big) = \frac{\xi}{\alpha} - \int_D d\mu_D(\vec{r}\,')\ G_z^{m_0}(\vec{x}, \vec{x}\,')\big|_{\vec{x},\vec{x}\,'\in D}\xi(\vec{r}\,')$$

for the definition of $G_z^{m_0}$ see Proposition 2.2.
Indeed

$$\Gamma_\alpha(z - m_0\omega) = \Lambda_\alpha(z) + R_z^{m_0}$$

where $R_z^{m_0}$ is a bounded integral operator on $L^2(D, d\mu_D)$ with kernel

$$R_z^{m_0}(\vec{r}, \vec{r}\,') \equiv \int_0^\infty \sum_{l=0}^\infty \sum_{\substack{m=-l \\ m\neq m_0}}^{l} \frac{\varphi_{klm}(\vec{r})\,\varphi_{klm}(\vec{r}\,')}{k^2 - (m - m_0)\omega - z}$$

that goes to 0 when $\omega \to \infty$ (see the Proof of Lemma 2.1).
Moreover $\forall\, \omega \in \mathbb{R}^+$ the operator $\Gamma_\alpha(z)$ is invertible if $\Im(z) \neq 0$ (see the Proof of Theorem 4.2) and, for each $l_0 \in \mathbb{N}$, $m_0 = -l_0, \ldots, l_0$, $z \in \mathbb{C} - \mathbb{R}$ it can be seen that the operator Λ_α is also invertible: indeed, let Ψ is the dense subset of $L^2(D, d\mu_D)$ given by functions of the form $\chi(r) Y_{l_0}^{m_0}(\theta, 0)$,

$$\big(\Lambda_\alpha(z)\, \Psi|_D\big)(r,\theta) = \frac{\Psi|_D}{\alpha}(r,\theta) - \frac{Y_{l_0}^{m_0}(\theta,0)}{2\pi} \int_0^A dr'\, {r'}^2\;\, g_z^{l_0}(\vec{r}, \vec{r}')\, \chi(\vec{r}')$$

and

$$\Big[\big(H_0 - z\big)\Lambda_\alpha(z)\, \Psi|_D\Big](\vec{r}) = \Big[\big(H_0 - z\big)\frac{\Psi|_D}{\alpha}\Big](\vec{r}) - \Theta_D(\vec{r})\, \Psi|_D(\vec{r})$$

so that $\Lambda_\alpha^{-1}(z)\Psi|_D = \Xi_\alpha(z)\Psi|_D$.

□

5 The Rotating Blade in 2D

5.1 The Hamiltonian

The formal time-dependent Hamiltonian of the system is given by the operator

$$H(t) = H_0 + a\, R(t)\, \Theta_A(x)\, \delta(y) \tag{5.1}$$

where $\Theta_A(x)$ is the characteristic function of the segment $0 \leq x \leq A$. In the rotating frame the generator of time evolution is a self-adjoint extension of the symmetric operator

$$K_S = H_\omega$$

$$\mathcal{D}(K_S) = C_0^\infty(\mathbb{R}^2 - S)$$

where S is the segment $S \equiv \{(x,0) \in \mathbb{R}^2 \mid 0 \leq x \leq A\}$.
In order to rigorously define the self-adjoint extensions of the operator K_S, we shall proceed like in the 3D case, namely we shall introduce a sequence of cut-off perturbed Hamiltonians and then we shall identify their limit with the Hamiltonian of the system.
So let

$$H_\omega^N = H_\omega\, \Pi_N \tag{5.2}$$

where Π_N is the projector on the subspace of $L^2(\mathbb{R}^2)$ generated by functions of the form $\chi(r) e_n(\theta)$, with $|n| \leq N$. The operator H_ω^N is self-adjoint on the domain $H^2(\mathbb{R}^2)$ (see the discussion at the beginning of Section 4) and, for each $z \in \varrho(H_\omega^N)$, the resolvent $(H_\omega^N - z)^{-1}$ is given by an integral operator with kernel

$$\mathcal{G}_z^N(\vec{x}, \vec{x}') = \int_0^\infty dk \sum_{n=-N}^{N} \frac{\varphi_{kn}^*(\vec{x}')\, \varphi_{kn}(\vec{x})}{k^2 - \omega n - z} \tag{5.3}$$

Proposition 5.1 *The sequence of cut-off Hamiltonians converge as $N \to \infty$ in the strong resolvent sense to the self-adjoint operator H_ω.*

Proof: See the Proof of Proposition 4.1 and Proposition A.2.

□

The perturbed cut-off Hamiltonian is associated to the form

$$\mathcal{F}_{\alpha,N}(\Psi,\Psi) = F_{\omega,N}(\Psi,\Psi) - \int_S d\mu_S\, \alpha(r)\, \left|\Psi\big|_S(r)\right|^2 \tag{5.4}$$

which is well defined[8] if $\Psi \in \mathcal{D}(F_{\omega,N})$, $F_{\omega,N}$ being the closed semibounded form associated to the self-adjoint operator H^N_ω, and $\alpha \in C(S)$, $\alpha(r) \neq 0$, $\forall r \in S$.

Proposition 5.2 *Let $z \in \mathbb{C} - \mathbb{R}$, the form $\mathcal{F}_{\alpha,N}$ can be written in the following way,*

$$\mathcal{F}_{\alpha,N}(\Psi,\Psi) = \mathcal{F}^z_{\omega,N}(\Psi,\Psi) + \Phi^z_{\alpha,N}(\xi_\Psi,\xi_\Psi) - 2\Im(z)\,\Im\Big[\big(\Psi\;,\;\tilde{\mathcal{G}}^N_z\xi_\Psi\big)\Big] \tag{5.5}$$

where

$$\mathcal{F}^z_{\omega,N}(\Psi,\Psi) = F_{\omega,N}(\Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi, \Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi) - \Re(z)\|\Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi\|^2 + \Re(z)\|\Psi\|^2 \tag{5.6}$$

$$\Phi^z_{\alpha,N}(\xi_\Psi,\xi_\Psi) = \Re\Big[\big(\xi_\Psi\;,\;\Gamma^N_\alpha(z)\,\xi_\Psi\big)_{L^2(S,d\mu_S)}\Big] \tag{5.7}$$

and

$$\Big[\Gamma^N_\alpha(z)\,\xi_\Psi\Big](r) = \frac{\xi_\Psi(r)}{\alpha(r)} - \int_S d\mu_S(r')\;\mathcal{G}^N_z(\vec{x},\vec{x}')\big|_{\vec{x},\vec{x}'\in S}\,\xi_\Psi(r') \tag{5.8}$$

$$\big(\tilde{\mathcal{G}}^N_z\xi\big)(\vec{x}) \equiv \int_S d\mu_S(r')\;\mathcal{G}^N_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in S}\,\xi(r')$$

Proof: See the Proof of Proposition 4.2.

□

Now we shall prove that the properties of the form $\Phi^z_{\alpha,N}$ still hold:

Proposition 5.3 *The form $\Phi^z_{\alpha,N}(\xi,\xi)$ is bounded for each $\xi \in L^2(S,d\mu_S)$.*

Proof: Using the result proved in Proposition A.2, we can follow the Proof of Proposition 4.3.

□

Proposition 5.4 *For each smooth real function α on S bounded away from 0, there exists $\zeta \in \mathbb{R}$, $\zeta < 0$ such that, for each $z \in \mathbb{C} - \mathbb{R}$, $\Re(z) < \zeta$, the following inequality holds*

$$\Phi^z_{\alpha,N}(\xi,\xi) - 2\Im(z)\,\Im\Big[\big(\Psi\;,\;\tilde{\mathcal{G}}^N_z\xi_\Psi\big)\Big] - \big(\Re(z) + \omega N\big)\,\|\Psi - \tilde{\mathcal{G}}^N_z\xi_\Psi\|^2 > 0$$

Proof: See the Proof of Proposition 4.4 and Proposition A.2.

□

So that we can state the following Theorem,

[8] In the 2D case, the measure $d\mu_S$ is given by $r\,dr$.

Theorem 5.1 *The form $\mathcal{F}_{\alpha,N}$ is bounded from below and closed on the domain*

$$\mathcal{D}(\mathcal{F}_{\alpha,N}) = \left\{\Psi \in L^2(\mathbb{R}^2) \mid \exists \xi_\Psi \in L^2(S, rdr), \Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi \in H^1(\mathbb{R}^2)\right\} \tag{5.9}$$

Proof: See the Proof of Theorem 4.1.

□

Proposition 5.5 *The operators K_α^N defined below are self-adjoint:*

$$\mathcal{D}(K_\alpha^N) = \left\{\Psi \in L^2(\mathbb{R}^2) \mid \exists \xi_\Psi \in L^2(S, d\mu_S), \Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi \in \mathcal{D}(H_\omega^N),\right.$$

$$\left.\left(\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi\right)\big|_D = \Gamma_\alpha^N(z)\xi_\Psi\right\} \tag{5.10}$$

$$\left(K_\alpha^N - z\right)\Psi = \left(H_\omega^N - z\right)\left(\Psi - \tilde{\mathcal{G}}_z^N \xi_\Psi\right) \tag{5.11}$$

where $\alpha \in \mathrm{C}(D)$, $\alpha(\vec{r}) \neq 0$, for each $\vec{r} \in D$.
Moreover

$$\left[\left(K_\alpha^N - z\right)^{-1}\Psi\right](\vec{x}) = \left[\left(H_\omega^N - z\right)^{-1}\Psi\right](\vec{x}) +$$

$$+ \int_D d^2\vec{r}' \ \left[\Gamma_\alpha^N(z)\right]^{-1}\left[\left(H_\omega^N - z\right)^{-1}\Psi\right]\Big|_D(\vec{r}')\, \mathcal{G}_z^N(\vec{x},\vec{x}')\big|_{\vec{x}'\in D} \tag{5.12}$$

for each $z \in \varrho(K_\alpha)$.

Proof: The result follows from Theorem 5.1. Like in the 3D case it is possible to prove that the operator $\Gamma_\alpha^N(z)$ is invertible if $\Im(z) \neq 0$.

□

Theorem 5.2 *For each $\alpha \in \mathrm{C}(S)$, $\alpha(r) \neq 0$, $\forall\, r \in S$, the sequence of semi-bounded self-adjoint operators K_α^N converge as $N \to \infty$ in the strong resolvent sense to the self-adjoint (unbounded from below) operator K_α:*

$$\mathcal{D}(K_\alpha) = \left\{\Psi \in L^2(\mathbb{R}^2) \mid \exists \xi_\Psi \in L^2(S, d\mu_S), \Psi - \tilde{\mathcal{G}}_z \xi_\Psi \in \mathcal{D}(H_\omega),\right.$$

$$\left.\left(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi\right)\big|_S = \Gamma_\alpha(z)\xi_\Psi\right\} \tag{5.13}$$

$$\left(K_\alpha - z\right)\Psi = \left(H_\omega - z\right)\left(\Psi - \tilde{\mathcal{G}}_z \xi_\Psi\right) \tag{5.14}$$

where

$$\left[\Gamma_\alpha(z)\,\xi_\Psi\right](r) = \frac{\xi_\Psi(r)}{\alpha(r)} - \int_S d\mu_S(r')\ \mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x},\vec{x}'\in S}\,\xi_\Psi(r') \tag{5.15}$$

$$\left(\tilde{\mathcal{G}}_z \xi\right)(\vec{x}) \equiv \int_S d\mu_S(r')\ \mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in D}\,\xi(r')$$

Moreover the resolvent of K_α is

$$\left[\left(K_\alpha - z\right)^{-1}\Psi\right](\vec{x}) = \left[\left(H_\omega - z\right)^{-1}\Psi\right](\vec{x}) +$$

$$+ \int_S dr'\, r'\, \Gamma_\alpha^{-1}(z)\left[\left(H_\omega - z\right)^{-1}\Psi\right]\Big|_S(r')\, \mathcal{G}_z(\vec{x},\vec{x}')\big|_{\vec{x}'\in S} \tag{5.16}$$

for each $z \in \varrho(K_\alpha)$.

Proof: See the Proof of Theorem 4.2.

□

Theorem 5.3 *The spectrum of K_α is purely absolutely continuous and*

$$\sigma(K_\alpha) = \sigma_{\mathrm{ac}}(K_\alpha) = \sigma(H_\omega) = \mathbb{R}$$

Proof: See the Proof of Theorem 4.3, Theorem 5.2 and Proposition A.2.

□

5.2 Asymptotic Limit of Rapid Rotation

In this Section, we shall prove that

$$\mathrm{s-}\lim_{\omega\to\infty} U_{\mathrm{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where H_α is the self-adjoint generator

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_C(\vec{r}) \tag{5.17}$$

and $\Theta_C(r)$ is the characteristic function of a circle C of radius A centered at the origin.

Theorem 5.4 *For every $t, s \in \mathbb{R}$,*

$$\mathrm{s-}\lim_{\omega\to\infty} U_{\mathrm{inert}}(t,s) = e^{-iH_\alpha(t-s)}$$

where

$$H_\alpha = H_0 - \alpha(\vec{r})\,\Theta_C(\vec{r})$$

Proof: See the Proof of Theorem 2.4 and the following Lemma 5.1.

□

Lemma 5.1 *For every $z \in \mathbb{C}$, $\Im(z) > 0$,*

$$\mathrm{s-}\lim_{\omega\to\infty} \int_{-\infty}^{0} dt\, e^{-izt}\, U^*_{\mathrm{inert}}(t,0) = -i\big(H_\alpha - z\big)^{-1}$$

Proof: Like in the Proof of Lemma 3.1, we shall prove the result on the dense subset of $L^2(\mathbb{R}^2)$ given by functions of the form $\Psi(\vec{x}) = \chi(r)e_{n_0}(\theta)$, $n_0 \in \mathbb{Z}$. Following the Proof of Lemma 3.1, it remains to prove that

$$\lim_{\omega\to\infty} \big(K_\alpha + n_0\omega - z\big)^{-1}\Psi(\vec{x}) = \big(H_\alpha - z\big)^{-1}\Psi(\vec{x})$$

but

$$\big(K_\alpha + n_0\omega - z\big)^{-1}\Psi = \big(H_\omega + n_0\omega - z\big)^{-1}\Psi +$$

$$+\left(\Gamma_\alpha^{-1}(z^* - n_0\omega)\Big[\big(H_\omega + n_0\omega - z^*\big)^{-1}\Psi\Big]\Big|_S\,,\, \mathcal{G}_{z-n_0\omega}(\vec{x},\vec{x}')\big|_{\vec{x}'\in S}\right)_{L^2(S,d\mu_S)}$$

and

$$\lim_{\omega\to\infty} \big(H_\omega + n_0\omega - z\big)^{-1}\Psi = \big(H_0 - z\big)^{-1}\Psi$$

as we have proved in Lemma 3.1. Moreover

$$\lim_{\omega\to\infty}\Big[\big(H_\omega+n_0\omega-z\big)^{-1}\Psi\Big]\Big|_S=\Big[\big(H_0-z\big)^{-1}\Psi\Big]\Big|_S$$

in $L^2(S,d\mu_S)$ and, applying the result found in the following Lemma 5.2,

$$\lim_{\omega\to\infty}\Gamma_\alpha^{-1}(z-n_0\omega)\Big[\big(H_\omega+n_0\omega-z\big)^{-1}\Psi\Big]\Big|_S=$$

$$=\Xi_\alpha(z)\Big[\big(H_0-z\big)^{-1}\Psi\Big]\Big|_S=$$

$$=\alpha(r)\Theta_S(r)\big(H_0-\alpha(\vec r)\,\Theta_C(\vec r)-z\big)^{-1}\Psi$$

In conclusion we obtain

$$\lim_{\omega\to\infty}\big(K_\alpha+n_0\omega-z\big)^{-1}\Psi=\big(H_0-z\big)^{-1}\Big[1+\alpha\Theta_S\big(H_0-\alpha\,\Theta_C-z\big)^{-1}\Big]\Psi=$$

$$=\big(H_0-\alpha\,\Theta_C-z\big)^{-1}\Psi$$

□

Lemma 5.2 *Let $\Gamma_\alpha(z)$ the operator defined in (5.15),*

$$\lim_{\omega\to\infty}\Gamma_\alpha^{-1}(z-n_0\omega)=\Xi_\alpha(z)$$

in $L^2(S,d\mu_S)$, where

$$\big(\Xi_\alpha(z)\xi\big)(r)\equiv\Big[\alpha(r)\Big(H_0-\alpha(r)\,\Theta_C(r)-z\Big)^{-1}\big(H_0-z\big)\,\xi\Big](r)\tag{5.18}$$

Proof: First of all we are going to prove that

$$\|\ \|-\lim_{\omega\to\infty}\Gamma_\alpha(z-n_0\omega)=\Lambda_\alpha(z)$$

where

$$\Lambda_\alpha(z)\,\xi=\frac{\xi}{\alpha}-\frac{1}{2\pi}\int_S d\mu_S(r')\ \ g_z^{n_0}(r,r')\xi(r')$$

for the definition of $g_z^{n_0}$ see Proposition 3.2.
Indeed

$$\Gamma_\alpha(z-n_0\omega)=\Lambda_\alpha(z)+R_z^{n_0}$$

where $R_z^{n_0}$ is a bounded integral operator on $L^2(S,d\mu_S)$ with kernel

$$R_z^{n_0}(r,r')\equiv\int_0^\infty\sum_{\substack{n=-\infty\\ n\neq n_0}}^{\infty}\frac{\varphi_{kn}(r)\,\varphi_{kn}(r')}{k^2-(n-n_0)\omega-z}\longrightarrow 0$$

as $\omega\to\infty$ (see the Proof of Lemma 3.1).
Moreover for each $n_0\in\mathbb{Z}$ and $z\in\mathbb{C}-\mathbb{R}$ it can be seen that the operator Λ_α is invertible: indeed

$$\Big[\big(H_0-z\big)\Lambda_\alpha(z)\,\xi\Big](\vec r)=\Big[\big(H_0-z\big)\frac{\xi}{\alpha}\Big](r)-\Theta_S(r)\,\xi(r)$$

so that $\Lambda_\alpha^{-1}(z)=\Xi_\alpha(z)$.

□

6 Conclusions and Perspectives

The operators studied in Section 2 and 3 could be viewed as the Hamiltonians of quantum systems given by a particle interacting with a rotating δ-type potential. In this context the results proved about the asymptotic limit of rapid rotation have an heuristic physical meaning: if the angular velocity of the potential is huge with respect to the velocity of the particle, we expect that the particle feels a time-independent potential, which is the mean of the true potential over a period.

This result was already proved by Enss et al. [7] for regular potential, and, from this point of view, our work is only an extension of their results to singular potentials.

A future application of that study would be the analysis of the scattering of a particle by a rotating point interaction. Indeed it would be an example of time-dependent scattering that can be reduced to a stationary problem: passing to the rotating frame, we could prove in simpler way, for example, existence and completeness of the wave operators.

In Section 3 and 4 we have studied the rotating blade, namely a singular potential with codimension 1. That kind of rotating singular perturbations of the Laplacian are more interesting and could open many suggestive problems.

For example in the 3D case we could investigate the dependence of the results on the shape of the blade. While all the properties of the form and the self-adjoint extensions still hold for a blade with a general shape, because the key point is the good behavior of the Green's function on a compact subset of $\mathbb{R}^3$, the analysis of the asymptotic limit is harder.

In fact a semi-spherical shape is very useful to perform the calculation with the Green's function of H_ω expressed in terms of functions with spherical symmetry (the spherical waves), but the same goal can be reached for a blade of different form: if we take a square shaped blade and we express the resolvent of H_ω in terms of functions with cylindrical symmetry (essentially the Bessel functions), all the results still hold. On the other hand, if the blade has no symmetry, we could expect the same behavior but it is not clear at all how it can be proved.

Finally we want to mention another feature of the problem which can be investigated: the blades we have considered are finite, so it would be interesting to study an infinite blade, for example an half-line in 2D and an half-plane in 3D, but, in that case, many problems arise in the definition of the operator. In particular the form Φ_α^z should not be bounded, unless we impose some condition on the behavior at ∞ of the parameter α.

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APPENDIX

A The Green's Function of H_ω

In this Appendix we shall study the Green's function $\mathcal{G}_z(\vec{x},\vec{y}_0)$ of H_ω and we shall prove that it belongs to $L^2(\mathbb{R}^n, d^n\vec{x})$, $\forall \vec{y}_0 \in \mathbb{R}^n$ with $n = 2, 3$.
We shall start from the 3D case:

Proposition A.1 *The resolvent* $(H_\omega - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, *has the following integral representation*

$$(H_\omega - z)^{-1}\Psi(\vec{x}) = \int_{\mathbb{R}^3} d^3x' \mathcal{G}_z(\vec{x},\vec{x}')\Psi(\vec{x}')$$

with $\Psi(\vec{x}) \in L^2(\mathbb{R}^3, d^3x)$ *and*

$$\mathcal{G}_z(\vec{x},\vec{x}') = \int_0^\infty dk \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{k^2 - m\omega - z}\, \varphi^*_{klm}(\vec{x}')\, \varphi_{klm}(\vec{x}) \tag{A.1}$$

The functions $\varphi_{klm}(\vec{x})$ *are the spherical waves*[9]*:*

$$\varphi_{klm}(\vec{x}) = \sqrt{\frac{2k^2}{\pi}}\, j_l(kr) Y_l^m(\theta,\varphi)$$

Moreover, for every $\vec{y}_0 \in \mathbb{R}^3$ *and* $z \in \mathbb{C} - \mathbb{R}$, $\mathcal{G}_z(\vec{x},\vec{y}_0) \in L^2(\mathbb{R}^3, d^3\vec{x})$.

Proof: The integral representation of the Green's function of H_ω is a straightforward consequence of the eigenvectors decomposition of H_ω. Moreover in the following we shall prove that, for each $\Psi \in L^2(\mathbb{R}^3)$, $z \in \mathbb{C} - \mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$,

$$\left| \Big(\mathcal{G}_z(\vec{x},\vec{y}_0)\ ,\ \Psi(\vec{x}) \Big)_{L^2(\mathbb{R}^3, d^3\vec{x})} \right| < \infty$$

Every function $\Psi \in L^2(\mathbb{R}^3)$ can be decomposed in terms of spherical harmonics:

$$\Psi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Psi_{lm}(r)\, Y_l^m(\theta,\phi)$$

with the L^2-condition

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left\| \Psi_{lm}(r) \right\|^2_{L^2(\mathbb{R}^+, r^2 dr)} < \infty$$

Thus

$$\left| \Big(\mathcal{G}_z(\vec{x},\vec{y}_0)\ ,\ \Psi(\vec{x}) \Big) \right|^2 \le \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left| \Big(G_{z+m\omega}(\vec{x},\vec{y}_0)\ ,\ \Psi_{lm}(r) Y_l^m(\theta,\phi) \Big) \right|^2 \le$$

[9] Here $j_l(r)$ denotes the spherical Bessel function of order l (see [10, 17]) and $Y_l^m(\theta,\phi)$, with $l \in \mathbb{N}$ and $m = -l, \ldots, l$, the spherical harmonics.

$$\leq \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\| G_{z+m\omega}(\vec{x},\vec{y}_0) \right\|^2_{L^2(\mathbb{R}^3,d^3\vec{x})} \left\| \Psi_{lm}(r) Y_l^m(\theta,\phi) \right\|^2 \leq$$

$$\leq C(\Im(z)) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\| \Psi_{lm}(r) \right\|^2_{L^2(\mathbb{R}^+,r^2dr)} < \infty$$

because the Green's function of the free Hamiltonian

$$G_{z+m\omega}(\vec{x},\vec{x}) = \frac{e^{i\sqrt{z+m\omega}|\vec{x}-\vec{y}_0|}}{4\pi|\vec{x}-\vec{x}|}$$

belongs to $L^2(\mathbb{R}^3, d^3\vec{x})$ for each $z \in \mathbb{C} - \mathbb{R}$ and $\vec{y}_0 \in \mathbb{R}^3$: we have to choose the root of $z + m\omega$ with imaginary part

$$\Im\left(\sqrt{z+m\omega}\right) = \sqrt{\frac{\left[(\Re(z)+m\omega)^2 + \Im(z)^2\right]^{\frac{1}{2}} - \Re(z) - m\omega}{2}} \geq \sqrt{\frac{|\Im(z)|}{2}} > 0$$

so that $G_{z+m\omega} \in L^2$ independently on $m \in \mathbb{Z}$.

□

An analogous result can be proved in the 2D case:

Proposition A.2 *The resolvent* $(H_\omega - z)^{-1}$, $z \in \mathbb{C} - \mathbb{R}$, *has the following integral representation*

$$(H_\omega - z)^{-1}\Psi(\vec{x}) = \int_{\mathbb{R}^2} d^2x' \mathcal{G}_z(\vec{x},\vec{x}')\Psi(\vec{x}')$$

with $\Psi(\vec{x}) \in L^2(\mathbb{R}^2, d^2x)$ *and*[10]

$$\mathcal{G}_z(\vec{x},\vec{x}') \equiv \int_0^\infty dk \sum_{n=-\infty}^{\infty} \frac{1}{k^2 - \omega n - z}\, \varphi^*_{kn}(\vec{x}')\, \varphi_{kn}(\vec{x}) \qquad \text{(A.2)}$$

$$\varphi_{kn}(\vec{x}) = \sqrt{\frac{k}{2\pi}} J_{|n|}(kr)\, e^{in\theta}$$

Moreover, for every $\vec{y}_0 \in \mathbb{R}^2$ *and* $z \in \mathbb{C} - \mathbb{R}$, $\mathcal{G}_z(\vec{x},\vec{y}_0) \in L^2(\mathbb{R}^2, d^2\vec{x})$.

Proof: Following the Proof of Proposition $A.1$, we shall consider the scalar product

$$\Big(\mathcal{G}_z(\vec{x},\vec{y}_0)\,,\ \Psi(\vec{x})\Big)_{L^2(\mathbb{R}^3,d^3\vec{x})}$$

with

$$\Psi(\vec{x}) = \sum_{n=-\infty}^{\infty} \Psi_n(r)\, \frac{e^{in\theta}}{2\pi}$$

and we obtain

$$\left|\Big(\mathcal{G}_z(\vec{x},\vec{y}_0)\,,\ \Psi(\vec{x})\Big)\right|^2 \leq \sum_{n=-\infty}^{\infty} \left\| G_{z+m\omega}(\vec{x},\vec{y}_0) \right\|^2_{L^2(\mathbb{R}^3,d^3\vec{x})} \left\| \Psi_n(r) \right\|^2_{L^2(\mathbb{R}^+,r^2dr)} < \infty$$

[10] $J_n(r)$ stands for the Bessel function of order $n \in \mathbb{N}$.

since[11]

$$G_{z+n\omega}(\vec{x},\vec{y}_0) = \frac{i}{4}\, H_0^{(1)}(\sqrt{z+n\omega}\,|\vec{x}-\vec{y}_0|)$$

belongs to $L^2(\mathbb{R}^2, d^2\vec{x})$, for each $z \in \mathbb{C} - \mathbb{R}$ and $\Im(\sqrt{z+n\omega}) > 0$.

$\square$

[11] $H_0^{(1)}$ denotes the Hankel function of first kind and order zero (see [1]).

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