Construction of periodic solutions of the nonlinear wave equation with Dirichlet boundary conditions by the Lindstedt series method

Guido Gentile† **and Vieri Mastropietro**∗

† Dipartimento di Matematica, Università di Roma Tre, Roma, I-00146

∗ Dipartimento di Matematica, Università di Roma "Tor Vergata", Roma, I-00133

Abstract. *We introduce and prove the convergence of a "renormalized" Lindstedt series for the oscillatory solutions of the nonlinear wave equation with Dirichlet boundary conditions, via Renormalization Group methods.*

1. Introduction

We consider the *nonlinear wave equation* in $d = 1$ given by

$$\begin{cases} u_{tt} - u_{xx} + \mu u = f(u), \\ u(0,t) = u(\pi,t) = 0. \end{cases} \tag{1.1}$$

where μ is a positive real parameter (*mass*) and $f(u)$ is an odd analytic function (*perturbation*):

$$f(u) = \sum_{N=1}^{\infty} f_{2N+1} u^{2N+1}, \qquad |f_s| \le F\, e^{\xi s}, \tag{1.2}$$

for some real constants F and ξ. Let us call

$$M = \inf \{ s \ge 1 \; : \; f_s \ne 0 \} . \tag{1.3}$$

Dirichlet boundary conditions allow us to use as a basis in $L^2([0,\pi])$ the set of functions $\{e_m\} = \{\sin mx, m \in \mathbb{N}\}$.

If $f = 0$ every solution of (1.1) can be written as

$$u(x,t) = \sum_{m=0}^{\infty} A_m \cos(\omega_m t + \theta_m) \sin mx, \tag{1.4}$$

where $\omega_m = \sqrt{m^2 + \mu}$ and θ_m is an arbitrary phase. In particular

$$\varepsilon \cos \omega_1 t \sin x \tag{1.5}$$

is a solution for all values of the real parameter ε.

If a nonlinear term (perturbation) is added to the wave equation ($f \neq 0$) one can ask if periodic solutions close to (1.5) exist. This is not a trivial problem because, as we shall see later, small divisors appear of the form $-\omega_1^2 n^2 + \omega_m^2$, with $m \geq 2$, and they can be arbitrarily small when n and m are large. We look for which conditions the periodic solution (1.5) can be continued for $f \neq 0$ into

$$u(x,t;\Omega,\varepsilon) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} e^{in\Omega t + imx} \hat{u}_{n,m}(\Omega,\varepsilon), \tag{1.6}$$

possibly with a different frequency Ω, with $\Omega = \omega_1 + O(\varepsilon)$. Of course by the symmetry of (1.1) if such a solution exists it verifies

$$\hat{u}_{n,m} = -\hat{u}_{n,-m} = \hat{u}_{-n,m} \tag{1.7}$$

for all $n, m \in \mathbb{Z}$ (here and henceforth we shorten $\hat{u}_{n,m}(\Omega,\varepsilon)$ with $\hat{u}_{n,m}$).

Remark 1. For finite dimensional Hamiltonian systems existence of periodic orbits follows from the Lyapunov center theorem [14], which states that, for a Hamiltonian system with an elliptic equilibrium at 0, by denoting with $\omega_1, \ldots, \omega_n$ the frequencies of the linear oscillations close to 0, if a nonresonance condition holds, namely if ω_1 is such that $\omega_1 \ell - \omega_j \neq 0$ for all $\ell \in \mathbb{N}$ and for all $j \neq 1$, then the periodic solution of the linearized system with frequency ω_1 and small amplitude can be continued into periodic solutions with frequency close to ω_1. Existence of periodic orbits in the case in which the frequencies are possibly resonant was proved by the Weinstein theorem [17]. The infinite dimensional case poses difficulties which are qualitatively different with respect to the finite dimensional case; the denominators can be arbitrarily small and the problem appears to be related to KAM problems, like the existence of lower dimensional tori in a Hamiltonian system.

We shall prove the following result.

Theorem. *There exists a set $\mathcal{M} \subset \mathbb{R}$ of full Lebesgue measure and for all $\mu \in \mathcal{M}$ there exist $\varepsilon_0 > 0$ and, for all $\varepsilon_* \in (0,\varepsilon_0)$, a set $\mathcal{E} \subset [-\varepsilon_*,\varepsilon_*]$ with complement of relative Lebesgue measure tending to 0 as $\varepsilon_* \to 0$ such that for all $\varepsilon \in \mathcal{E}$ there exists a value $\Omega = \Omega(\varepsilon)$ such that $u(x,t;\Omega(\varepsilon),\varepsilon)$ is a solution of (1.1), analytic in (x,t) and $2\pi/\Omega(\varepsilon)$-periodic in t, with*

$$\begin{aligned} &|\Omega(\varepsilon) - \omega_1| \leq C|\varepsilon|^{M-1}, \\ &|u(x,t,\Omega(\varepsilon),\varepsilon) - \varepsilon \cos\omega_1 t \sin x| \leq C|\varepsilon|^M, \end{aligned} \tag{1.8}$$

for a suitable constant C.

Remark 2. The above theorem can be immediately extended to equations of the form

$$u_{tt} - Du = f(u), \tag{1.9}$$

where D is a self-adjoint strictly positive operator with non-degenerate eigenvalues μ_k such that $\mu_k \geq ck^\gamma$ with $c, \gamma \geq 1$ positive constants and such that, if $\gamma > 1$,

$\mu_{k'} - \mu_k \geq c(k'^\gamma - k^\gamma)$ for all $k' > k \geq 1$, while, if $\gamma = 1$, $\mu_{k'} - \mu_k = c_\ell(1 + O(k^{-\xi}))$ for all $k' - k = \ell \geq 1$ and with $c_\ell, \xi > 0$.

Inserting (1.6) into (1.1) gives

$$\sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} e^{i\Omega nt + imx} \hat{u}_{n,m} \left[-\Omega^2 n^2 + \omega_m^2 \right] = f(u), \tag{1.10}$$

which, if we expand

$$f(u) = \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} e^{i\Omega nt + imx} \hat{f}_{n,m}(u), \tag{1.11}$$

can be written in Fourier space as

$$\hat{u}_{n,m} \left[-\Omega^2 n^2 + \omega_m^2 \right] = \hat{f}_{n,m}(u). \tag{1.12}$$

In order to solve such equations it is convenient to write, for any $|m| \geq 1$,

$$\omega_m^2 = (\omega_m^2 - \nu_m) + \nu_m \equiv \tilde{\omega}_m^2 + \nu_m, \qquad \nu_m = \nu_{-m}, \tag{1.13}$$

so that (1.12) can be written as

$$\hat{u}_{n,m} \left[-\Omega^2 n^2 + \tilde{\omega}_m^2 \right] = \nu_m \hat{u}_{n,m} + \hat{f}_{n,m}(u). \tag{1.14}$$

We shall call *counterterms* the quantities ν_m.

We shall look for periodic solutions with period $2\pi/\Omega$, where $\Omega \equiv \tilde{\omega}_1$. The analysis will be performed in two steps. First we consider ε and $\tilde{\omega} = \{\tilde{\omega}_m\}_{|m|\geq 1}$ as independent parameters and we assume the following Diophantine conditions:

$$\begin{aligned} &|\tilde{\omega}_1 n \pm \tilde{\omega}_m| \geq C_0 |n|^{-\tau} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m \in \mathbb{Z} \setminus \{0, \pm 1\}, \\ &|\tilde{\omega}_1 n \pm (\tilde{\omega}_{m'} \pm \tilde{\omega}_m)| \geq C_0 |n|^{-\tau} \qquad \forall n \in \mathbb{Z} \setminus \{0\} \text{ and } \forall m, m' \in \mathbb{Z} \setminus \{0, \pm 1\}, \end{aligned} \tag{1.15}$$

where C_0 and τ are two positive constants; as usual in literature, we shall call the two conditions in (1.15), respectively, as the first and the second Mel'nikov conditions.

We shall prove that there is a choice of the sequence $\nu(\tilde{\omega}, \varepsilon) = \{\nu_m(\tilde{\omega}, \varepsilon)\}_{|m|\geq 1}$, analytically depending on ε (for ε small enough), such that there exist $\hat{u}_{n,m}(\tilde{\omega}, \varepsilon)$ solving (1.14) which are analytic in ε and are the Fourier coefficients of a function $u(x, t; \tilde{\omega}, \varepsilon)$ verifying the bound (1.8).

The second step consists in proving that there are functions $\tilde{\omega}_m = \tilde{\omega}_m(\omega, \varepsilon)$, with $\omega = \{\omega_m\}_{|m|\geq 1}$, solving $\tilde{\omega}_m^2 + \nu_m(\tilde{\omega}, \varepsilon) = \omega_m^2$; they are obtained by requesting that there is a set $\mathcal{M}$ such that for all $\mu \in \mathcal{M}$ there exists a set $\mathcal{E}$ such that the functions $\tilde{\omega}_m(\omega, \varepsilon)$ are well defined and verify (1.15) for $\varepsilon \in \mathcal{E}$. Then we have to check that both $\mathcal{M}$ and $\mathcal{E}$ have nonvanishing measure. Hence $\hat{u}_{n,m}(\tilde{\omega}(\omega, \varepsilon), \varepsilon)$ is indeed the Fourier transform of a periodic solution of (1.1).

Remark 3. We write the decomposition (1.13) by explicitly requesting the parity property $\nu_m = \nu_{-m}$, and we will check at the end that indeed it is possible to find functions ν_m with such parity property so that $\hat{u}_{n,m}(\tilde{\omega}, \varepsilon)$ is analytic in ε.

The above result is not new, as it was proved by Kuksin [13], Pöschel [15] and Wayne [16] by showing that the nonlinear wave equation can be written as an infinite-dimensional Hamiltonian system and extending in a suitable way KAM techniques;

the existence of a periodic or quasi-periodic solution means the existence of a finite-dimensional invariant torus. Indeed much more general results exist about the existence of periodic or quasi-periodic solutions of nonlinear PDEs, based on the above KAM methods or on the Lyapunov-Schmidt reduction, see Craig and Wayne [3] and Bourgain [2]; in particular the second Mel'nikov conditions can be removed. Our approach is based on expanding the periodic solution as a power series of ε; such power series are the analogues of the *Lindstedt series* for KAM invariant tori in finite-dimensional Hamiltonian systems. Contrary to what happens in such a case (at least for maximal invariant tori, see [6]), such series are *not* analytic in the perturbative parameter. However, while $\hat{u}_{n,m}(\tilde{\omega}(\omega,\varepsilon),\varepsilon)$ is not analytic in ε, it turns out that the function $\hat{u}_{n,m}(\tilde{\omega},\varepsilon)$ (that is the function obtained by keeping fixed the parameters $\tilde{\omega}$, without expliciting the dependence of $\tilde{\omega}$ on ε) is analytic in ε, provided that $\tilde{\omega}$ satisfies the conditions (1.15) and ε is small enough. By inverting (1.13) one writes $\tilde{\omega}$ as a function of ω and $\varepsilon \in \mathcal{E}$. The main advantage of such an approach is that it is rather constructive, in the sense that we obtain an explicit representation of the periodic solution on the nonlinear wave equation. As a new result, it turns out the analyticity of $\hat{u}_{n,m}(\tilde{\omega},\varepsilon)$ as a function of ε, at fixed $\tilde{\omega}$. Our analysis of the Lindstedt series for $\hat{u}_{n,m}(\tilde{\omega},\varepsilon)$ is based on Renormalization Group methods similar to the ones applied to the problem of the convergence of the series of classical KAM tori by Gallavotti [5] (see also Eliasson [4]). In the latter problem one has to exploit a number of partial cancellations in the perturbative expansion which at the end ensure the analiticity of the series, while in the case of Lindstedt series for PDE such cancellations are absent. On the contrary here one has to perform a suitable resummation of the formal perturbative expansion, which, at the end, implies non-analyticity of the solution; for similar results along the same direction see also [6] and [7]. Our method is based on techniques which were developed for the proof of the convergence of perturbative series for quantities of interest in quantum field theory or statistical mechanics.

Finally one could try to study by the same methods the case of quasi-periodic solutions with Dirichlet boundary conditions, or the case of both periodic and quasi-periodic solutions with periodic boundary conditions (in which the eigenvalues are degenerate), and we hope to come back on that in the near future.

2. Recursive relations

By (1.7), we can rewrite (1.14) as

$$\hat{u}_{n,m}\left[-\tilde{\omega}_1^2 n^2 + \tilde{\omega}_m^2\right] = \nu_m^{(a)} \hat{u}_{n,m} + \nu_m^{(b)} \hat{u}_{n,-m} + \hat{f}_{n,m}(u), \tag{2.1}$$

with $\nu_m^{(a)} - \nu_m^{(b)} = \nu_m$.

We shall look for $\hat{u}_{n,m}$ in the form of a power series expansion in ε and in the parameters $\nu_m^{(c)}$, with $c = a, b$ and $|m| \geq 1$,

$$\hat{u}_{n,m} = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\underline{k}=(k_1^{(a)},k_1^{(b)},k_2^{(a)},k_2^{(b)},\ldots)} \hat{u}_{n,m}^{(k,\underline{k})} \prod_{m'=2}^{\infty} \prod_{c=a,b} (\nu_{m'}^{(c)})^{k_{m'}^{(c)}}, \tag{2.2}$$

with $k \geq 1$ and $k_{m'}^{(c)} \geq 0$ for all $m' \geq 2$ and $c = a, b$. We set also $|\underline{k}| = k_1^{(a)} + k_1^{(b)} + k_2^{(a)} + k_2^{(b)} + \ldots$; of course we are using the symmetry property in (1.13) to restrict the dependence only on the positive labels m'. We shall refer to (2.3) as the *Lindstedt series* of the periodic solution, for the manifest analogy with the case of finite-dimensional Hamiltonian systems.

Remark 4. We shall see that for fixed k the vector $\underline{k}$ can have only a finite number ($\leq k$) of components different from zero, which gives sense to the sum appearing in (2.2).

For $|m| > 1$ one has $\hat{u}_{n,m}^{(1,\underline{0})} = 0$, while for $m = \pm 1$ one has

$$\hat{u}_{n,m}^{(1,\underline{0})} = \delta_{n,\pm 1} \frac{m}{4i}; \tag{2.3}$$

this simply follows from the fact that we are looking for a solution which, to first order in ε, reduces to (1.5).

Moreover, for $m = \pm 1$ and $n = \pm 1$, we shall choose

$$\hat{u}_{n,m}^{(k,\underline{k})} = 0 \qquad \forall (k, \underline{k}) \neq (1, \underline{0}). \tag{2.4}$$

By inserting (2.2) into (1.14) we obtain for $(k, \underline{k}) \neq (1, \underline{0})$ the recursive equations

$$\begin{aligned} \hat{u}_{n,m}^{(k,\underline{k})} = & \frac{1}{-\tilde{\omega}_1^2 n^2 + \tilde{\omega}_m^2} \Big[\Big(1 - \delta_{k_m^{(a)},0}\Big) \hat{u}_{n,m}^{(k,\underline{k}-\underline{\delta}_m^{(a)})} + \Big(1 - \delta_{k_m^{(b)},0}\Big) \hat{u}_{n,-m}^{(k,\underline{k}-\underline{\delta}_m^{(b)})} \Big) \\ & + \sum_{N=1}^{\infty} f_{2N+1} \sum_{\substack{k_1+k_2+\ldots+k_{2N+1}=k \\ \underline{k}_1+\underline{k}_2+\ldots+\underline{k}_{2N+1}=\underline{k}}} \sum_{\substack{n_1+n_2+\ldots n_{2N+1}=n \\ m_1+m_2+\ldots m_{2N+1}=m}} \prod_{j=1}^{2N+1} \hat{u}_{n_j,m_j}^{(k_j,\underline{k}_j)} \Big], \end{aligned} \tag{2.5}$$

which hold both for $|m| > 1$ and for $|m| = 1$ with $|n| \neq 1$. Here $\underline{\delta}_m^{(c)}$ is the vector $\underline{k}$ with components $k_{m'}^{(c')} = 0$ for all $(m', c') \neq (m, c)$ and with $k_m^{(c)} = 1$.

In the case $m = 1$ and $|n| = 1$ we obtain from (1.14), (1.10) and (2.3)

$$\begin{aligned} 0 = & \left(-\tilde{\omega}_1^2 + \tilde{\omega}_1^2\right) \frac{1}{4i} = \\ & = \frac{\nu_1}{4i} + \sum_{N=1}^{\infty} f_{2N+1} \sum_{\substack{n_1+n_2+\ldots n_{2N+1}=n \\ m_1+m_2+\ldots m_{2N+1}=1}} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2N+1},m_{2N+1}}, \end{aligned} \tag{2.6}$$

which will be regarded as an equation for fixing ν_1, as a suitable function of ε:

$$\nu_1 = -4i \sum_{N=1}^{\infty} f_{2N+1} \sum_{\substack{n_1+n_2+\ldots n_{2N+1}=n \\ m_1+m_2+\ldots m_{2N+1}=1}} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2N+1},m_{2N+1}}. \tag{2.7}$$

It is important to realize that, thanks to the parity property (1.7), even if in principle (2.7) represent two equations (one for $n = 1$ and one for $n = -1$), so that compatibility problems could arise, in fact the right hand side of (2.7) depends only on the modulus of n: this means that the value of ν_1 computed if $n=-1$ is the same as the one computed if $n=1$, hence (2.7) represents only one equation and it fixes ν_1.

Remark 5. If one computes ν_{-1} (corresponding to $m=-1$), one finds $\nu_{-1}=\nu_1$ (simply because for $m=-1$ both terms in the right hand side of (2.6) change sign), so that the parity property $\nu_m=\nu_{-m}$ is immediately seen to be satisfied for $m=1$.

Remark 6. Note also that the quantity $\tilde\omega_1 n\pm\tilde\omega_m$ has modulus bounded by a constant for all $(n,m)\neq(\pm1,\pm1)$ such that either $|n|=1$ or $|m|=1$.

3. Tree representation of the Lindstedt series

Note that (2.5) expresses $\hat u^{(k,\underline{k})}_{n,m}$ in terms of coefficients $\hat u^{(k',\underline{k}')}_{n',m'}$ with $k'+|\underline{k}'|<k+|\underline{k}|$, so that, by iterating (2.5), at the end one can express $\hat u^{(k,\underline{k})}_{n,m}$, for all $(k,\underline{k})\neq(1,\underline{0})$, in terms of the only coefficients $\hat u^{(1,\underline{0})}_{\pm1,\pm1}$, as given by (2.3).

It is immediate to verify that, by such an iteration procedure, $u^{(k,\underline{k})}_{n,m}$ can be written for all $(k,\underline{k})\neq(0,\underline{0})$ as sum over (rooted) trees in the following way

$$\hat u^{(k,\underline{k})}_{n,m}=\sum_{\theta\in\Theta^{*(k,\underline{k})}_{n,m}}\mathrm{Val}(\theta), \tag{3.1}$$

where, if, given a tree θ, we denote with $L(\theta)$, $V(\theta)$ and $E(\theta)$ the set of lines, nodes and end-points, respectively, of θ one has

$$\mathrm{Val}(\theta)=\Big(\prod_{\ell\in L(\theta)}g_\ell\Big)\Big(\prod_{v\in V(\theta)}\eta_v\Big)\Big(\prod_{v\in E(\theta)}\frac{\varepsilon m_{n_v}}{4i}\Big). \tag{3.2}$$

and $\Theta^{*(k,\underline{k})}_{n,m}$ is the set of all the possible trees defined according to the following rules (see Figure 3.1).

(1) To each end-point $v\in E(\theta)$ one associates the mode label (n_v,m_v), with $m_v=\pm1$ and $n_v=\pm1$, such that

$$\sum_{v\in E(\theta)}n_v=n\qquad\text{and}\qquad\sum_{v\in E(\theta)}m_v=m, \tag{3.3}$$

and the factor $\varepsilon m_v/4i$.

(2) To each line $\ell\in L(\theta)$ one associates the *propagator*

$$g_\ell\equiv g(\tilde\omega_1 n_\ell,m_\ell)=\begin{cases}\dfrac{1}{-\tilde\omega_1^2n_\ell^2+\tilde\omega_{m_\ell}^2}, & (n_\ell,m_\ell)\neq(\pm1,\pm1),\\ 1, & (n_\ell,m_\ell)=(\pm1,\pm1),\end{cases} \tag{3.4}$$

with *momentum* (n_ℓ,m_ℓ), where one has

$$n_\ell=\sum_{w\in E_\ell}n_w\qquad\text{and}\qquad m_\ell=\sum_{w\in E_\ell}m_w, \tag{3.5}$$

if we denote by E_ℓ the set of end-points of the subtree with root line ℓ. Only the lines coming out from the end-points can have momentum $(n_\ell, m_\ell) = (\pm 1, \pm 1)$.
(3) For each node $v \in V(\theta)$ one has s_v entering lines. If $s_v = 1$ and the momentum of the exiting line ℓ is (n_ℓ, m_ℓ), the momentum of the entering line can be only $(n_\ell, \pm m_\ell)$. One associates to such a node v a factor $\eta_v = \nu^{(c)}_{m_\ell}$, where $c = a$ if the momentum of the entering line is (n_ℓ, m_ℓ) and $c = b$ if the momentum of the entering line is $(n_\ell, -m_\ell)$; we call ν-vertex a node of this kind. If $s_v > 1$ one associates to the node a factor $\eta_v = f_{s_v}$; in particular s_v is odd and if $s_v > 1$ then $s_v \geq M$.
(4) One has the constraints

$$\begin{aligned} &|E(\theta)| = \sum_{v \in E(\theta)} |n_v| = \sum_{v \in E(\theta)} |m_v| = k, \\ &\sum_{m=1}^{\infty} \sum_{c=a,b} k^{(c)}_m = |\underline{k}|, \end{aligned} \tag{3.6}$$

and the number of vertices v with $s_v = 1$ such that a factor $\nu^{(c)}_m$ is associated to them is $k^{(c)}_m$.

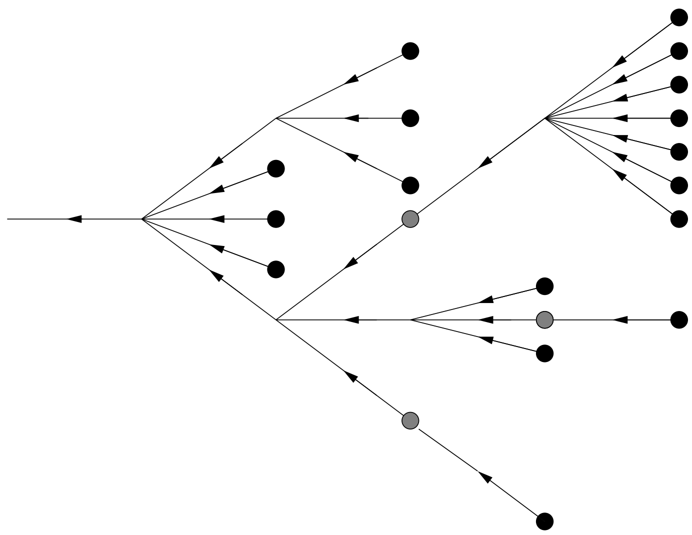

FIGURE 3.1. A tree θ belonging to the set $\Theta^{(k,\underline{k})*}_{n,m}$ for the model (1.1) with $M = 3$. The end-points are represented by black bullets, while the ν-vertices are represented as grey bullets.

Remark 7. It was showed in [1] that by assuming the first Diophantine condition in (1.15) with $\tau = 1$ then the proof of the existence of periodic solution becomes quite simpler. This can be understood in the formalism described here by noting that if $\tau = 1$ each propagator g_ℓ is bounded by a constant and hence each tree is bounded by C^k: for an explicit discussion see [9]. Of course assuming $\tau = 1$ one can prove the existence of periodic solutions only for a *zero measure set* of values of ε.

In the case $\tau > 1$ it is not true that the propagators are bounded by a constant; there is instead a small divisor problem and a multiscale analysis is necessary: to such a case the rest of the paper is devoted.

4. Multiscale decomposition: clusters and self-energy graphs

We assume the Diophantine conditions (1.15). We introduce a multiscale decomposition of the propagator. Let $\chi(x)$ be a C^∞ non-increasing function such that $\chi(x) = 0$ if $|x| \geq C_0$ and $\chi(x) = 1$ if $|x| \leq 2C_0$ (C_0 is the same contant appearing in (1.15)), and let $\chi_h(x) = \chi(2^h x) - \chi(2^{h+1}x)$ for $h \geq 0$, and $\chi_{-1}(x) = 1 - \chi(x)$; such functions realize a smooth partition of the unity as

$$1 = \chi_{-1}(x) + \sum_{h=0}^{\infty} \chi_h(x) = \sum_{h=-1}^{\infty} \chi_h(x). \tag{4.1}$$

Note that if $\chi_h(x) \neq 0$ for $h \geq 0$ one has $2^{-h-1}C_0 \leq |x| \leq 2^{-h+1}C_0$, while if $\chi_{-1}(x) \neq 0$ one has $|x| \geq C_0$.

We write the propagator as sum of propagators on single scales in the following way:

$$g(\tilde{\omega}_1 n, m) = \sum_{h=-1}^{\infty} \frac{\chi_h(|\tilde{\omega}_1 n| - \tilde{\omega}_m)}{-\tilde{\omega}_1^2 n^2 + \tilde{\omega}_m^2} = \sum_{h=-1}^{\infty} g^{(h)}(\tilde{\omega}_1 n, m). \tag{4.2}$$

Note that we can bound $|g^{(h)}(\tilde{\omega}_1 n, m)| \leq 2^{-h+1}C_0$.

This means that we can attach a scale label $h \geq -1$ to each line $\ell \in L(\theta)$, which is the scale of the propagator which is associated to ℓ. We can denote with $\Theta_{n,m}^{(k,\underline{k})}$ the set of trees which differ from the previous ones simply because the lines carry also the scale labels. The set $\Theta_{n,m}^{(k,\underline{k})}$ is defined according to the rules (1)÷(4) of Section 3, by changing item (2) into

(2') To each line $\ell \in L(\theta)$ one associates a scale label

$$h_\ell = \begin{cases} 0, 1, 2, 3, \ldots, & \text{if } (n_\ell, m_\ell) \neq (\pm 1, \pm 1), \\ -1, & \text{if } (n_\ell, m_\ell) = (\pm 1, \pm 1), \end{cases} \tag{4.3}$$

and the *propagator*

$$g_\ell^{(h_\ell)} \equiv g^{(n_\ell)}(\tilde{\omega}_1 n_\ell, m_\ell) = \begin{cases} \dfrac{\chi_{h_\ell}(|\tilde{\omega}_1 n_\ell| - \tilde{\omega}_{m_\ell})}{-\tilde{\omega}_1^2 n_\ell^2 + \tilde{\omega}_{m_\ell}^2}, & \text{if } (n_\ell, m_\ell) \neq (\pm 1, \pm 1), \\ 1, & \text{if } (n_\ell, m_\ell) = (\pm 1, \pm 1), \end{cases} \tag{4.4}$$

with *momentum* (n_ℓ, m_ℓ).

Looking at the scale labels we identify the connected cluster T of nodes which are linked by a continuous path of lines with the same scale label h_T or a lower one and which are maximal; we shall say that the cluster has scale h_T. We shall denote with $V(T)$ and $E(T)$ the set of and the set of end-points, respectively, which are contained inside the cluster T, and with $L(T)$ the set of lines connecting them.

Therefore an inclusion relation is established between clusters, in such a way that the innermost clusters are the clusters with lowest scale, and so on. The value of a tree can be written then as

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{v \in V(\theta)} \eta_v \Big) \Big(\prod_{v \in E(\theta)} \frac{\varepsilon m_v}{4i} \Big), \tag{4.5}$$

so that (3.1) is replaced with

$$\hat{u}_{n,m}^{(k,\underline{k})} = \sum_{\theta \in \Theta_{n,m}^{(k,\underline{k})}} \mathrm{Val}(\theta), \tag{4.6}$$

with the new definition

$$\mathrm{Val}(\theta) = \Big(\prod_{\ell \in L(\theta)} g_\ell^{(h_\ell)} \Big) \Big(\prod_{v \in V(\theta)} \eta_v \Big) \Big(\prod_{v \in E(\theta)} \frac{\varepsilon m_{n_v}}{4i} \Big). \tag{4.7}$$

for the tree value $\mathrm{Val}(\theta)$.

Each cluster T has an arbitrary number of lines entering it (incoming lines), but only one or zero line coming from it (outcoming line); we shall denote the latter (when it exists) with ℓ_T^1. We shall call *external lines* of the cluster T the lines which either enter or come out from T, and we shall denote by $h_T^{(e)}$ the maximum among the scales of the external lines of T. Define $K(T)$ as the number of end-points contained inside T, so that $K(T) = |E(T)|$.

The clusters with only one incoming line ℓ_T^2 such that one has

$$n_{\ell_T^1} = n_{\ell_T^2} \text{ and } m_{\ell_T^1} = \pm m_{\ell_T^2} \tag{4.8}$$

will be called *self-energy graphs* or *resonances* (the first nome is usual in quantum field theory, the second one was introduced by Eliasson in his basic paper [4]). In such a case the line ℓ_T^1 will be called a *resonant line.* If (n, m) is the momentum of the line ℓ_T^2 we shall call (n, m) the momentum of the self-energy graph; then for any line $\ell \in L(T)$ one can write $(n_\ell, m_\ell) = (n_\ell^0, m_\ell^0) + \eta_\ell(n, m)$, where $\eta_\ell = 1$ if the line ℓ is along the path connecting the external lines of T and $\eta_\ell = 0$ otherwise.

The *value* of the self-energy graph T is defined as

$$\mathcal{V}_T^h(\tilde{\omega}_1 n, m) = \Big(\prod_{\ell \in T} g_\ell^{(h_\ell)} \Big) \Big(\prod_{v \in V(T)} \eta_v \Big) \Big(\prod_{v \in E(T)} \frac{\varepsilon m_v}{4i} \Big). \tag{4.9}$$

where $h = h_T^{(e)}$ is the maximum between the scales of the two external lines of T (they can differ at most by a unit), and one has

$$n(T) \equiv \sum_{v \in E(T)} n_v = 0, \qquad m(T) \equiv \sum_{v \in E(T)} m_v \in \{0, 2m\}, \tag{4.10}$$

by definition of self-energy graph; one has $c = a$ when $m(T) = 0$ and $c = b$ when $m(T) = 2m$.

Remark 7. Note that, if $|m| = 1$ and $|n| \neq 1$ the scale of the external lines is such that $h \leq h_0$, where h_0 is a suitable constant (see Remark 6): one can take $h_0 = 0$ for

C_0 small enough. In particular this implies that no self-energy graph with $m = 1$ is possible; we shall repeatedly use such a property in the following.

Given a tree θ, we shall denote by $N_h(\theta)$ the number of lines with scale h, and by $C_h(\theta)$ the number of clusters with scale h. We will get immediately the following bound, for $\theta \in \Theta^{(k,\underline{k})}_{n,m}$ and assuming $|\nu^{(c)}_m| \leq C|\varepsilon|$,

$$|\mathrm{Val}(\theta)| \leq |\varepsilon|^{k+|\underline{k}|} D^{k+|\underline{k}|} \prod_{h=0}^{\infty} 2^{hN_h(\theta)}, \tag{4.11}$$

if D is a suitable constant. Note that one has $|n_\ell|, |m_\ell| \leq k$ for all trees $\theta \in \Theta^{(k,\underline{k})}_{n,m}$ and for all lines $\ell \in L(\theta)$. The following result is proved in Appendix A1.

Lemma 1. *For any tree $\theta \in \Theta^{(k,\underline{k})}_{n,m}$ and for all $h \geq 0$ one has*

$$N_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M^{\nu}_h(\theta), \tag{4.12}$$

where $K(\theta) = \sum_{v\in E(\theta)} |n_v|$, while $S_h(\theta)$ is the number of self-energy graphs T in θ with $h^{(e)}_T = h$ and $M^{\nu}_h(\theta)$ is the number of ν-vertices in θ such that the maximum scale of the two external lines is h.

Let us consider a tree with no self-energy graphs and with no ν-vertices. Then by (4.10) and (4.12) we get

$$|\mathrm{Val}(\theta)| \leq |\varepsilon|^k D^k \prod_{h=0}^{\infty} 2^{4hk2^{-(h-2)/\tau}} \leq \tilde{D}^k |\varepsilon|^k, \tag{4.13}$$

for a suitable constant $\tilde{D}$, and we have used that one has $k = K(\theta)$; see (3.6). On the other hand the bound for a generic tree *with* self-energy graphs is quite bad; this is not just a technical problem, as one can easily identify trees whose value is essentially given by $C^k k!^{\alpha} \varepsilon^k$, for some positive constants α and C. We have then to define a different expansion, as it will be shown in the following section.

5. The renormalized expansion

We have seen that the expansion envisaged in the previous sections can be written as sum over trees and some of such trees (the trees containing self-energy graphs) cannot be bounded by $C^k|\varepsilon|^k$, as they are of order of $C^k|\varepsilon|^k k!^{\alpha}$. In this section we will set up a different expansion, which, by choosing in a proper way the parameters ν_m, $m > 1$, can be written as sum over trees in which all of them can be bounded by $C^k|\varepsilon|^k$.

We introduce a *localization operator* acting on the self-energy graphs in the following way, for $m > 1$,

$$\mathcal{L}\mathcal{V}^h_T(\tilde{\omega}_1 n, m) = \mathcal{V}^h_T(\tilde{\omega}_m, m), \tag{5.1}$$

and we define the *regularization operator* $\mathcal{R} = \mathbb{1} - \mathcal{L}$ as

$$\mathcal{R}\mathcal{V}^h_T(\tilde{\omega}_1 n, m) = \mathcal{V}^h_T(\tilde{\omega}_1 n, m) - \mathcal{V}^h_T(\tilde{\omega}_m, m). \tag{5.2}$$

We shall define, for $m > 1$ and $h \geq 0$, the *running coupling constants*

$$2^{-h}\nu_{h,m}^{(c)} = \nu_m^{(c)} + \sum_{T\in\mathcal{T}_{<h}^{(c)}} \mathcal{V}_T^h(\tilde{\omega}_m, m), \tag{5.3}$$

where $c = a, b$, and $\mathcal{T}_{<h}^{(a)}$ denotes the set of self-energy graphs T with $m_{\ell_T^1} = m_{\ell_T^2}$ and $h_T < h$, while $\mathcal{T}_{<h}^{(b)}$ denotes the set of self-energy graphs T with $m_{\ell_T^1} = -m_{\ell_T^2}$, and $h_T < h$. We shall set also $\nu_m^{(c)} = \nu_{-1,m}^{(c)}$.

Then one obtains

$$\hat{u}_{n,m} = \sum_{k=1}^{\infty}\sum_{\underline{k}}\sum_{\theta\in\Theta_{n,m}^{(k,\underline{k})\mathcal{R}}} \mathrm{Val}(\theta), \tag{5.4}$$

where the set $\Theta_{n,m}^{(k,\underline{k})\mathcal{R}}$ is the set of *renormalized trees*, which are defined as in the previous section (see rules (1), (2'), (3) and (4)) except that the following rules are added:

(5) to each self-energy graph (with $|m| > 1$) the $\mathcal{R}$ operation is applied;

(6) there are nodes v with $s_v = 1$ such that, if the momenta of the external lines are, respectively, (n, m) and $(n, \pm m)$, with $m > 1$, and h is the maximal scale of the external lines (they can differ at most by 1), then a factor $2^{-h}\nu_{h,m}^{(c)}$, $c = a, b$ is associated to v (where a corresponds to the sign $+$ and b to the sign $-$).

We call *regularized self-energy graphs* the self-energy graphs on which $\mathcal{R}$ applies, and we denote with $\mathcal{S}(\theta)$ the set of regularized self-energy graphs contained in θ. We still call ν-vertices the nodes v with $s_v = 1$.

In the following it will be useful to define also the *renormalized self-energy graphs*, which are defined as the self-energy graphs except that the same items (1) and (2) as for the renormalized trees apply to the self-energy graphs and nodes contained inside.

The crucial point is now that, if $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$ for some constant C, then each renormalied tree θ admit a good bound. First of all note that a bound like (4.12) holds also for a renormalized tree θ, namely one has the following result (proved in Appendix A2).

Lemma 2. *For any tree $\theta \in \Theta_{n,m}^{(k,\underline{k})\mathcal{R}}$ one has*

$$N_h(\theta) \leq 4K(\theta)2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M_h^{\nu}(\theta), \tag{5.5}$$

where the notations are as in lemma 1.

Therefore, by using lemma 2, we obtain the following result (which is proved in Appendix A3), which, under the assumption that the running coupling constants remain bounded of order ε, allows us to prove the convergence of the Linstedt series as a function of ε and the parameters $\nu_{m',h'}^{(c)}$, with $c = a, b$ and $|m'| > 1$.

Lemma 3. *Assume that for all $|m| > 1$ and all $h \geq 0$ there exist a constant C such that one has $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$, with $c = a, b$. Then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ and for all $(n, m) \neq (\pm 1, \pm 1)$ one has*

$$|\hat{u}_{n,m}| \leq D_0 D^{|n|+|m|}|\varepsilon|^{(|n|+|m|)/2}, \tag{5.6}$$

where D and D_0 are positive constants. Moreover $\hat{u}_{n,m}$ depend analytically on ε and on all the parameters $\nu^{(c)}_{m',h'}$, with $c = a, b$ and $|m'| > 1$.

Remark 8. One should compare the above power series expansion with the analogous Lindstedt series for invariant KAM tori in quasi-integrable Hamiltonian system (see [4], [5] and [8]). The main difference is that in the latter case the analogous of $\nu^{(c)}_{h,m}$ are exactly vanishing, as a consequence of peculiar cancellations. Hence one can prove analyticity in ε for the power series. In the present case $\nu^{(c)}_{h,m}$ are not vanishing, and we have to choose carefully the counteterms ν_m so that $|\nu^{(c)}_{h,m}| \le C|\varepsilon|$; for a similar approach , see for instance [10] or [11].

The quantities $\nu^{(c)}_{h,m}$, for $h+1 \ge 0$ and $|m| > 1$, verify the recursive relations

$$\nu^{(c)}_{h+1,m} = 2\nu^{(c)}_{h,m} + \beta^{(c)}_{h,m}(\tilde{\omega}, \varepsilon, \{\nu^{(c')}_{h',m'}\}), \tag{5.7}$$

where, by defining $\mathcal{T}^{(c)}_h$ as the set of self-energy graphs in $\mathcal{T}^{(c)}_{<h+1}$ which are on scale h, the *beta function*

$$\beta^{(c)}_{h,m} \equiv \beta^{(c)}_{h,m}(\tilde{\omega}, \varepsilon, \{\nu^{(c')}_{h',m'}\}) = 2^{h+1} \sum_{T \in \mathcal{T}^{(c)}_h} \mathcal{V}^{h+1}_T(\tilde{\omega}_m, m), \tag{5.8}$$

depends only on the scales $h' \le h$.

The validity of (5.7) can be checked from (5.3) by noting that the dependence of the value $\mathcal{V}^h_T(\tilde{\omega}_1 n, m)$ of a self-energy graph T on h is only through the constraint that the scale h_T of T has to be such that $h_T < h$. Then, subtracting (5.3) from the equivalent expression for $h+1$, one finds (5.7).

In order to obtain a bound on the beta function, hence on the running coupling constants, we need to bound $\mathcal{V}^{h+1}_T(\tilde{\omega}_m, m)$ for $T \in \mathcal{T}^{(c)}_h$. First of all we note that a bound like that of lemma 2 applies also to the renormalized self-energy graphs.

To see this we first need to introduce some auxiliary notations.

We enlarge the set of trees by allowing more general values for the mode labels of the end-points. More precisely we define $\widetilde{\Theta}^{(k,\underline{k})\mathcal{R}}_{n,m}$ as the set $\Theta^{(k,\underline{k})\mathcal{R}}_{n,m}$ introduced after (5.4), but by changing item (1) into the following one:
(1′) We divide the set $\widetilde{E}(\theta)$ of end-points into three sets $E(\theta)$, $E_1(\theta)$ and $E_0(\theta)$, where $E(\theta)$ is defined as before, $E_1(\theta)$ is formed by end-points v with mode label $(n_v, m_v) \in \mathbb{Z} \times \mathbb{Z} \setminus \{\pm, \pm\}$, and $E_0(\theta)$ is either the empty set or a single end-point v_0 which has labels $(\overline{n} + \overline{\omega}_m, \overline{m} + m)$, where $(\overline{n}, \overline{m}, m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\overline{\omega}_m = \tilde{\omega}_m / \tilde{\omega}_1$ (so that $\overline{\omega}_m$ is not an integer). The factors corresponding to the end-points in $E_1(\theta) \cup E_0(\theta)$ are just 1.

Then we have the following generalization of lemma 2 (proved in Appendix A4).

Lemma 4. *For any tree $\theta \in \widetilde{\Theta}^{(k,\underline{k})\mathcal{R}}_{n,m}$ one has*

$$N_h(\theta) \le 4\widetilde{K}(\theta) 2^{(2-h)/\tau} - C_h(\theta) + S_h(\theta) + M^{\nu}_h(\theta), \tag{5.9}$$

where

$$\widetilde{K}(\theta) = \begin{cases} \sum_{v \in E(\theta) \cup E_1(\theta)} |n_v| & \textit{if } E_0(\theta) = \emptyset, \\ \sum_{v \in E(\theta) \cup E_1(\theta)} |n_v| + |\overline{n}| & \textit{if } E_0(\theta) \neq \emptyset, \end{cases} \tag{5.10}$$

while the other notations are as in lemma 1.

By using the just proved result we can show that the bound of lemma 2 extends also to self-energy graphs, as we anticipated, so obtaining the following result (see Appendix A5 for the proof). The idea is that one can deal with the self-energy graphs by imagining them as they were trees, but to do this one has allow more generalvalues for the mode labels of the end-points: the definition of the set $\theta \in \widetilde{\Theta}^{(k,\underline{k})\mathcal{R}}_{n,m}$ takes into account exactly such a generalization.

Lemma 5. *Consider $T \in \mathcal{T}^{(c)}_h$. If $N_{h'}(T)$ is the number of lines internal to T with scale $h' \leq h$ and $C_{h'}(T)$ is the number of clusters on scale h' contained in T, then one has*

$$N_{h'}(T) \leq 4K(T) 2^{(2-h')/\tau} - C_{h'}(T) + S_{h'}(T) + M^{\nu}_{h'}(T), \tag{5.11}$$

where $K(T) = \sum_{v \in V(T)} |n_v|$, while $S_{h'}(T)$ and $M^{\nu}_{h'}(T)$ are, respectively, the number of self-energy graphs T' with $h^{(e)}_{T'} = h'$ contained in T and the number of ν-vertices contained in T.

Then in Appendix A6 we prove the following bound for $\mathcal{V}^{h+1}_T(\tilde{\omega}_m, m)$. The main point is that, thanks to lemma 5, we can eesentially reason for the self-energy graphs as for the trees.

Lemma 6. *Assume that for all $|m| > 1$ and all $h \geq 0$ there exist a constant C such that one has $|\nu^{(c)}_{h,m}| \leq C|\varepsilon|$, with $c = a, b$. Then for all $h \geq 0$ and for all $T \in \mathcal{T}^{(c)}_h$ one has*

$$|\mathcal{V}^{h+1}_T(\tilde{\omega}_m, m)| \leq B_0 B^{K(T)} |\varepsilon|^{2^{(h-1)/\tau}/2} |\varepsilon|^{K(T)/2}, \tag{5.12}$$

where $K(T) = \sum_{v \in V(T)} |n_v|$, for some positive constants B_0 and B.

Note that in each contribution to $\beta^{(c)}_{h,m}$ containing a $\nu^{(c')}_{h',m}$, there are at least $M-1$ end-points; in fact the self-energy graphs (with an incoming external line carrying momentum (n,m)) with only ν-vertices are such that all the propagators have argument given by $\tilde{\omega}_1 n - \tilde{\omega}_m$; hence when the $\mathcal{L}$ operation is applied (consisting in replacing $\tilde{\omega}_1 n$ with $\tilde{\omega}_m$) they are vanishing by the support properties of the propagators. This, together with the inequality $M \geq 3$, implies that we can bound in (A2.2)

$$\left| \beta^{(c)}_{h,m} \right| \leq C|\varepsilon|^{M-1} \leq C|\varepsilon|^2, \tag{5.13}$$

which follows from the very definition for the contributions arising from self-energy graphs containing only one node (hence on scale $h = -1$), and from the bound (A3.2) for the contribution containing at least two nodes (so that one line can be on scale $h \geq 0$).

6. The choice of the counterterms

In this section we show that it is possible to choose $\nu^{(c)} = \{\nu_m^{(c)}\}_{|m|\geq 1}$ such that, for a suitable positive constant C, one has $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$ for all $h \geq 0$ and for all $|m| > 1$.

For any sequence $a \equiv \{a_m\}_{|m|\geq 1}$ we introduce the norm

$$\|a\|_\infty = \sup_{|m|\geq 1} |a_m|, \tag{6.1}$$

and the seminorm

$$|a|_\infty = \sup_{|m|>1} |a_m|, \tag{6.2}$$

which will be useful in order to bound the counterterms.

We prove the following lemma in Appendix A7.

Lemma 7. *There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there is a family of intervals $I_{c,m}^{(\bar{h})}$, $\bar{h} \geq 0$, $|m| > 1$, $c = a, b$, such that $I_{c,m}^{(\bar{h}+1)} \subset I_{c,m}^{(\bar{h})}$, $|I_{c,m}^{\bar{h}}| \leq 2|\varepsilon|(\sqrt{2})^{-(\bar{h}+1)}$ and, if $\nu_m^{(c)} \in I_{c,m}^{(\bar{h})}$, then*

$$|\nu_h^{(c)}|_\infty \leq C|\varepsilon|, \qquad \bar{h} \geq h \geq 0, \tag{6.3}$$

for some positive constant C. Finally one has $\nu_{h,-m}^{(c)} = \nu_{h,m}^{(c)}$, $c = a, b$, for all $\bar{h} \geq h \geq 0$ and for all $m > 1$.

It will be useful to really construct the $\nu_{h,m}^{(c)}$ by a contraction method. By iterating (5.7) we find, for $m > 1$

$$\nu_{h,m}^{(c)} = 2^{h+1} \left(\nu_m^{(c)} + \sum_{k=-1}^{h-1} 2^{-k-2} \beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\}) \right). \tag{6.4}$$

where $\beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\})$ depends on $\nu_{k',m'}$ with $k' \leq k-1$. If we put $h = \bar{h}$ in (6.4) we get the following identity

$$\nu_m^{(c)} = -\sum_{k=-1}^{\bar{h}-1} 2^{-k-2} \beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\}) + 2^{-\bar{h}-1} \nu_{\bar{h},m}^{(c)} \tag{6.5}$$

and (6.4), (6.5) are equivalent, for $\bar{h} > h \geq 0$, to

$$\nu_{h,m}^{(c)} = -2^{h+1} \left(\sum_{k=h}^{\bar{h}-1} 2^{-k-2} \beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')}\}) \right) + 2^{h-\bar{h}} \nu_{\bar{h},m}^{(c)}. \tag{6.6}$$

The sequences $\{\nu_{h,m}^{(c)}\}_{|m|>1}$, $\bar{h} > h \geq 0$, parametrized by $\{\nu_{\bar{h},m}^{(c)}\}_{|m|>1}$ such that $|\nu_{\bar{h}}^{(c)}|_\infty \leq C|\varepsilon|$, can be obtained as the limit as $q \to \infty$ of the sequences $\{\nu_{h,m}^{(c)(q)}\}$, $q \geq 0$, defined recursively as

$$\begin{aligned} \nu_{h,m}^{(c)(0)} &= 0, \\ \nu_m^{(c)(q)} &= -\sum_{k=h}^{\bar{h}-1} 2^{-k-2} \beta_{k,m}^{(c)}(\tilde{\omega}, \varepsilon, \{\nu_{k',m'}^{(c')(q-1)}\}) + 2^{h-\bar{h}} \nu_{\bar{h},m}^{(c)}. \end{aligned} \tag{6.7}$$

In fact, it is easy to show inductively that, if ε is small enough, $|\nu_h^{(q)}|_\infty \le C|\varepsilon|^2$, so that (6.7) is meaningful, and

$$\max_{0\le h\le \bar h} |\nu_h^{(q)} - \nu_h^{(q-1)}|_\infty \le (C|\varepsilon|)^q. \tag{6.8}$$

For $q=1$ this is true as $\nu_h^{(c)(0)} = 0$; for $q>1$ it follows trivially by the fact that $\beta_k^{(c)}(\tilde\omega,\varepsilon,\{\nu_{k',m'}^{(c')(q-1)}\}) - \beta_k^{(c)}(\tilde\omega,\varepsilon,\{\nu_{k',m'}^{(c')(q-2)}\})$ can be written as a sum of terms in which there are at least one ν-vertex, with a difference $\nu_{h'}^{(c')(q-1)} - \nu_{h'}^{(c')(q-2)}$, with $h'\ge k$, in place of the corresponding $\nu_{h'}^{(c')}$, and one node carrying an ε. Then $\nu_h^{(q)}$ converges as $q\to\infty$, for $\bar h < h \le 1$, to a limit ν_h, satisfying the bound $|\nu_h|_\infty \le C|\varepsilon|$. Since the solution is unique, it must coincide with the one implied by lemma 7.

The above prescription fixes $\nu_{h,m}^{(c)}$ for all $|m|>1$. The values $\nu_{\pm1}^{(c)}$ are then fixed by inserting the values of $\nu_{h,m}^{(c)}$ for all $|m|>1$ into the expressions of $\hat u_{n,m}$ appearing in the right hand side of (2.7).

Finally note that one can choose $\nu_m^{(c)} = \nu_{-m}^{(c)}$ and then $\nu_{h,m}^{(c)} = \nu_{h,-m}^{(c)}$ for any $|m|>1$ and any $\bar h \ge h \ge 0$; this follows from the fact that the function $\beta_{k,m}^{(c)}$ in (6.7) is even under the exchange $m\to -m$; it depends on m through $\tilde\omega_m$ (which is an even function of m), through the end-points $v\in E(\theta)$ (which are odd under the exchange $m\to-m$; but their number must be even) and finally through $\nu_{k,m}^{(q-1)}$ which are assumed inductively to be even.

7. Construction of the perturbed frequencies

In the following it will be convenient to set $\omega = \{\omega_m\}_{|m|\ge1}$ and $\tilde\omega = \{\tilde\omega_m\}_{|m|\ge1}$. By the analysis of the previous sections we have found the counterterms $\{\nu_m(\tilde\omega,\varepsilon)\}_{|m|\ge1}$ as functions of ε and $\tilde\omega$. We have now to invert the relations

$$\tilde\omega_m^2 + \nu_m(\tilde\omega,\varepsilon) = \omega_m^2. \tag{7.1}$$

This is given by the following result.

Proposition 1. *For all $\mu_0>0$ there exists a full measure set $\mathcal M\subset[0,\mu_0]$ such that for all $\mu\in\mathcal M$ there are $\xi>0$ and a set $\mathcal E\subset[-\varepsilon_0,\varepsilon_0]$ with complement of relative Lebesgue measure of order ε_0^ξ such that for all $\varepsilon\in\mathcal E$ there exists $\tilde\omega=\tilde\omega(\omega,\varepsilon)$ solution of (7.1).*

We shall show that there exists a sequence of sets $\{\mathcal E^{(p)}\}_{p=0}^\infty$ in $[-\varepsilon_0,\varepsilon_0]$, such that $\mathcal E^{(p+1)}\subset\mathcal E^{(p)}$, and a sequence of functions $\{\tilde\omega^{(p)}(\omega,\varepsilon)\}_{p=0}^\infty$, with each $\tilde\omega^{(p)}\equiv\tilde\omega^{(p)}(\omega,\varepsilon)$ defined for $\varepsilon\in\mathcal E^{(p)}$, such that for all $\varepsilon\in\mathcal E$, with

$$\mathcal E = \bigcap_{p=0}^\infty \mathcal E^{(p)} = \lim_{p\to\infty}\mathcal E^{(p)}, \tag{7.2}$$

there exists the limit

$$\tilde\omega^{(\infty)}(\omega,\varepsilon) = \lim_{p\to\infty}\tilde\omega^{(p)}(\omega,\varepsilon), \tag{7.3}$$

and it solves (7.1).

To fulfill the program above we shall follow an iterative scheme by setting

$$\begin{aligned} &\tilde{\omega}_m^{(0)2} = \omega_m^2, \\ &\tilde{\omega}_m^{(p)2} \equiv \tilde{\omega}_m^{(p)2}(\omega,\varepsilon) = \omega_m^2 - \nu_m(\tilde{\omega}^{(p-1)},\varepsilon), \qquad p \ge 1, \end{aligned} \tag{7.4}$$

and reducing recursively the set of admissible values of ε.

We impose on $\omega \equiv \tilde{\omega}^{(0)}$ the Diophantine conditions

$$\begin{aligned} &|\omega_1 n \pm \omega_m| \ge C_0 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z}\setminus\{0\} \text{ and } \forall m \in \mathbb{Z}\setminus\{0,\pm 1\}, \\ &|\omega_1 n \pm m| \ge C_0 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z}\setminus\{0\} \text{ and } \forall m \in \mathbb{Z}\setminus\{0,\pm 1\}, \\ &|\omega_1 n \pm (\omega_{m'} \pm \omega_m)| \ge C_0 |n|^{-\tau_0} \qquad \forall n \in \mathbb{Z}\setminus\{0\} \text{ and } \forall m,m' \in \mathbb{Z}\setminus\{0,\pm 1\}, \end{aligned} \tag{7.5}$$

where C_0 and τ_0 are two positive constants. This will imply some restriction on the admissible values of μ, as the following result shows (see Appendix A8 for the proof).

Lemma 8. *For all $\mu_0 > 0$ the set of values $\mu \in [0,\mu_0]$ such that (7.5) are satisfied for some positive constant C_0 is of full measure provided that one has τ_0 large enough.*

The sets $\mathcal{E}^{(p)}$ will be defined recursively as

$$\begin{aligned} \mathcal{E}^{(0)} &= (-\varepsilon_0, \varepsilon_0), \\ \mathcal{E}^{(p)} &= \Big\{ \varepsilon \in \mathcal{E}^{(p-1)} \;:\; |\tilde{\omega}_1^{(p)} n \pm \tilde{\omega}_m^{(p)}| > C_0 |n|^{-\tau}, \\ &\qquad |\tilde{\omega}_1^{(p)} n \pm (\tilde{\omega}_m^{(p)} \pm \tilde{\omega}_{m'}^{(p)})| > C_0 |n|^{-\tau} \Big\}, \qquad p \ge 1, \end{aligned} \tag{7.6}$$

for $\tau > \tau_0$ to be fixed.

In Appendix A9 we prove the following result.

Lemma 9. *For all $p \ge 1$ one has*

$$\left\| \tilde{\omega}^{(p)}(\omega,\varepsilon) - \tilde{\omega}^{(p-1)}(\omega,\varepsilon) \right\|_\infty \le C\varepsilon_0^p \qquad \forall \varepsilon \in \mathcal{E}^{(p)}, \tag{7.7}$$

for some constant C.

This implies that there exist a sequence $\{\tilde{\omega}^{(p)}\}_{p=0}^\infty$ converging to $\tilde{\omega}^{(\infty)}$ for $\varepsilon \in \mathcal{E}$. We have now to show that the set $\mathcal{E}$ has positive (large) measure.

It is convenient to introduce a set of variables $\mu(\tilde{\omega},\varepsilon)$ such that

$$\tilde{\omega}_m + \mu_m(\tilde{\omega},\varepsilon) = \omega_m; \tag{7.8}$$

the variables $\mu(\tilde{\omega},\varepsilon)$ and the counterterms are trivially related by

$$\nu_m(\tilde{\omega},\varepsilon) = \mu_m^2(\tilde{\omega},\varepsilon) + 2\tilde{\omega}_m \mu_m(\tilde{\omega},\varepsilon). \tag{7.9}$$

One can write $\tilde{\omega}^{(p)} = \omega - \mu(\tilde{\omega}^{(p-1)},\varepsilon)$, according to (7.4), so that the Diophantine conditions in (1.15) can be written as

$$\begin{aligned} &|\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)},\varepsilon) n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)},\varepsilon)| > C_0 |n|^{-\tau}, \\ &|\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)},\varepsilon) n \\ &\qquad \mp ((\omega_m \pm \omega_{m'}) + (\mu_m(\tilde{\omega}^{(p-1)},\varepsilon) + \mu_{m'}(\tilde{\omega}^{(p-1)},\varepsilon)))| > C_0 |n|^{-\tau}. \end{aligned} \tag{7.10}$$

Suppose that for $\varepsilon \in \mathcal{E}^{(p-1)}$ the functions $\nu(\tilde{\omega}^{(p-1)}, \varepsilon)$ are well defined; then define $\mathcal{I}^{(p)} = \mathcal{I}_1^{(p)} \cup \mathcal{I}_2^{(p)} \cup \mathcal{I}_3^{(p)}$, where $\mathcal{I}_1^{(p)}$ is the set of values ε verifying the conditions

$$\left| \omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon) n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) \right| \le C_0 \left| n \right|^{-\tau}, \tag{7.11}$$

$\mathcal{I}_2^{(p)}$ is the set of values ε verifying the conditions

$$\begin{aligned} \Big| & \omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon) n \\ & \pm ((\omega_m - \omega_{m'}) \mp (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) - \mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon))) \Big| \le C_0 |n|^{-\tau}, \end{aligned} \tag{7.12}$$

and $\mathcal{I}_3^{(p)}$ is the set of values ε verifying the conditions

$$\begin{aligned} \Big| & \omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon) n \\ & \pm \Big((\omega_m + \omega_{m'}) \mp (\mu_m(\tilde{\omega}^{(p-1)}, \varepsilon) + \mu_{m'}(\tilde{\omega}^{(p-1)}, \varepsilon)) \Big) \Big| \le C_0 |n|^{-\tau}. \end{aligned} \tag{7.13}$$

For future convenience we shall call, for $i = 1, 2, 3$, also $\mathcal{I}_i^{(p)}(n)$ the subsets of $\mathcal{I}_i^{(p)}$ which verify the Diophantine conditions (7.11), (7.12) and (7.13), respectively, for fixed n. We want to bound the measure of the set $\mathcal{I}^{(p)}$. First we need to know a little better the dependence on ε of the counterterms: this is provided by the following results (to be proved in Appendix A10 and in Appendix A11, respectively).

Lemma 10. *For all $p \ge 1$ and for all $\varepsilon \in \mathcal{E}^{(p)}$ one has*

$$\begin{aligned} \nu_1(\tilde{\omega}, \varepsilon) &= \alpha_M \varepsilon^{M-1} + O(\varepsilon^M), \\ \nu_m(\tilde{\omega}, \varepsilon) &= \beta_M \varepsilon^{M-1} + O(\varepsilon^M), \qquad m \ge 2, \end{aligned} \tag{7.14}$$

for suitable positive constants α_M e β_M such that $|n\alpha_M - \beta_M| > 1/4^{M-1}$ for all $n \in \mathbb{Z}$.

Lemma 11. *For all $p \ge 1$ and for all $\varepsilon \in \mathcal{E}^{(p)}$ one has*

$$\begin{aligned} \left| \partial_{\tilde{\omega}_{m'}} \nu_m(\tilde{\omega}^{(p)}, \varepsilon) \right| &\le C \left| \varepsilon \right|^{2(M-1)}, \qquad m \ge 2, \\ \left| \partial_\varepsilon \tilde{\omega}_m^{(p)}(\omega, \varepsilon) \right| &\le C \left| \varepsilon \right|^{M-2} \qquad m \ge 2, \end{aligned} \tag{7.15}$$

where the derivatives are in the sense of Whitney.

Now we can bound the measure of the set we have to exclude.

We start with the estimate of the measure of the set $\mathcal{I}_1^{(p)}$. When (7.11) is satisfied one must have (by using also the first of (7.5))

$$\begin{aligned} C_0 |n|^{-\tau_0} &\le |\omega_1 n - \omega_m| \\ &\le |\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}, \varepsilon) n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\ &\qquad + |\mu_1(\tilde{\omega}^{(p-1)}, \varepsilon) n - \mu_m(\tilde{\omega}^{(p-1)}, \varepsilon)| \\ &\le C_0 |n|^{-\tau} + C\varepsilon_0 |n|, \end{aligned} \tag{7.16}$$

which implies, for $|n|>1$ and $\tau>\tau_0+1$,

$$|n| \geq \mathcal{N}_0 \equiv \left(\frac{C_0}{2C\varepsilon_0}\right)^{1/(\tau_0+1)}, \tag{7.17}$$

and also

$$C_1|m| \leq |\omega_m - \mu_m(\tilde{\omega}^{(p-1)},\varepsilon)| \leq |(\omega_1-\mu_1(\tilde{\omega}^{(p-1)},\varepsilon))n| + C_0|n|^{-\tau} \leq C_2|n|, \tag{7.18}$$

which implies

$$|m| \leq \mathcal{M}_0|n|, \qquad \mathcal{M}_0 = \frac{C_2}{C_1}. \tag{7.19}$$

Let us consider the function $\mu(\tilde{\omega}^{(p-1)},\varepsilon)$: we can define a map $t \to \varepsilon(t)$ such that

$$\begin{aligned} f(\varepsilon(t)) &\equiv \omega_1 n - \mu_1(\tilde{\omega}^{(p-1)},\varepsilon(t))n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)},\varepsilon(t)) \\ &= t\frac{C_0}{|n|^\tau}, \qquad t\in[-1,1], \end{aligned} \tag{7.20}$$

describes the interval defined by (7.11); then one has

$$\int_{\mathcal{I}_1^{(p)}} \mathrm{d}\varepsilon = \sum_{|n|\geq\mathcal{N}_0} \sum_{|m|\leq\mathcal{M}_0|n|} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}\right|. \tag{7.21}$$

We have from (7.20)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{C_0}{|n|^\tau}, \tag{7.22}$$

hence

$$\int_{\mathcal{I}_1^{(p)}} \mathrm{d}\varepsilon = \sum_{|n|\geq\mathcal{N}_0} \sum_{|m|\leq\mathcal{M}_0|n|} \frac{C_0}{|n|^\tau} \int_{-1}^{1} \mathrm{d}t \left|\frac{\mathrm{d}f(\varepsilon(t))}{\mathrm{d}\varepsilon(t)}\right|^{-1}. \tag{7.23}$$

In order to perform the derivative in (7.22) we write

$$\frac{\mathrm{d}\mu_m}{\mathrm{d}\varepsilon} = \partial_\varepsilon\mu_m + \sum_{m'\in\mathbb{Z}\setminus\{0\}} \partial_{\omega_{m'}}\mu_m\partial_\varepsilon\tilde{\omega}_{m'}^{(p-1)}, \tag{7.24}$$

and the first term is bounded by the first of (7.15), while each term of the sum in (7.24) is bounded through the second of (7.15). Moreover one has

$$\left|\partial_{\omega_{m'}}\mu_m\right| \leq |\varepsilon|^{\max\{||m|-|m'||,2(M-1)\}} C^{\max\{||m|-|m'||,2(M-1)\}}, \tag{7.25}$$

because μ_m depends on ω if there are at least $2(M-1)$ end-points and if $|m'|>|m|$ there must be at least $|m'|-|m|$ end-points.

At the end we get that the sum in (7.24) is $\mathrm{O}(\varepsilon^{2(M-1)}|m|)$, and using (7.19) one has

$$\begin{aligned} f(\varepsilon(t)) &= \omega_1 n - \omega_m - \alpha_M\varepsilon^{M-1}(t) + \beta_M n\varepsilon^{M-1}(t) + O(n\varepsilon^M(t)) \\ &= t\frac{C_0}{|n|^\tau}, \end{aligned} \tag{7.26}$$

where α_M and β_M are defined in the statement of lemma 10, hence

$$\left| \frac{\partial f(\varepsilon(t))}{\partial \varepsilon(t)} \right| \geq c_M \, |n| \, |\varepsilon(t)|^{M-2} , \tag{7.27}$$

for a suitable constant c_M, and, for $|n| > 1$ and $\tau > \tau_0 + 1$,

$$\begin{aligned} C_1 |n| |\varepsilon^{M-1}(t)| &\geq \left| cn\varepsilon^{M-1}(t) + O(n\varepsilon^{2(M-1)}(t)) \right| \geq \left| |\omega_1 n - \omega_m| - \frac{|tC_0|}{|n|^\tau} \right| \\ &> \frac{C_0}{|n|^{\tau_0}} \left(1 - \frac{1}{|n|^{\tau-\tau_0}} \right) > \frac{C_0}{2|n|^{\tau_0}}, \end{aligned} \tag{7.28}$$

so that one has

$$|\varepsilon(t)| > \left(\frac{C_0}{2C_1} \right)^{1/(M-1)} \frac{1}{n^{(\tau_0+1)/(M-1)}}, \tag{7.29}$$

which, introduced into (7.23), gives together with (7.27),

$$\begin{aligned} \int_{\mathcal{I}_1^{(p)}} \mathrm{d}\varepsilon &\leq \text{const.} \sum_{|n| \geq \mathcal{N}_0} \sum_{|m| \leq \mathcal{M}_0 |n|} \frac{C_0}{|n|^\tau} \int_{-1}^{1} \frac{\mathrm{d}t}{|\varepsilon(t)|^{M-2} \, |n|} \\ &\leq \text{const.} \sum_{|n| \geq \mathcal{N}_0} \sum_{|m| \leq \mathcal{M}_0 |n|} \frac{C_0}{|n|^\tau} \frac{n^{(\tau_0+1)(M-2)/(M-1)}}{\sqrt{C_0} |n|} \\ &\leq \text{const.} \sqrt{C_0} \sum_{|n| \geq \mathcal{N}_0} \sum_{|m| \leq \mathcal{M}_0 |n|} \frac{1}{|n|^{\tau'}}, \end{aligned} \tag{7.30}$$

with

$$\tau' = \tau + 1 - \frac{\tau_0 + 1(M-2)}{M-1}. \tag{7.31}$$

Therefore the Lebesgue measure of the set $\mathcal{I}_1^{(p)}$ is bounded by

$$\begin{aligned} &\text{const.} \sum_{|n| \geq \mathcal{N}_0} \sum_{|m| \leq \mathcal{M}_0 |n|} \frac{C_0}{|n|^{\tau'}} \leq \text{const.} \sqrt{C_0} \sum_{n=\mathcal{N}_0}^{\infty} n^{1-\tau'} \\ &\quad \leq \text{const.} \sqrt{C_0} \mathcal{N}_0^{-\tau'+2} = \text{const.} \sqrt{C_0} \left(\frac{\varepsilon_0}{C_0} \right)^{(\tau'-2)/(1+\tau_0)} \leq \text{const.} \varepsilon_0^{1+\xi'}, \end{aligned} \tag{7.32}$$

provided that one has

$$\xi' = \frac{\tau' - \tau_0 - 1}{\tau_0 + 1} = \frac{\tau - \dfrac{1}{2}(3\tau_0 + 1)}{\tau_0 + 1} > 0, \tag{7.33}$$

which imposes $\tau > (3\tau_0 + 1)/2$.

Now we discuss how to bound the measure of the set $\mathcal{I}_2^{(p)}$. We start by noting that for all $p \geq 0$ one has

$$\tilde{\omega}_m^{(p)} = \omega_m + \kappa + \sigma_m^{(p)} \qquad \left| \sigma_m^{(p)} \right| < C \frac{|\varepsilon|}{m^{\xi'}} \tag{7.34}$$

where $\xi' = 1$ and $\kappa = O(\varepsilon^{2(M-1)})$ is a constant (in m).

In fact $\nu_m^{(p)}$ is given by a sum of values of renormalized self-energy graphs T, and, for each value, the dependence on m is due to the propagators of lines along the path connecting the external lines of the self-energy graph (the self-energy graphs with no path of lines connecting the external lines contribute to the constant term κ). The propagators of such lines have the form

$$\frac{\chi_h(|\tilde\omega_1^{(p)} n_\ell + \tilde\omega_{m_\ell}^{(p)}| - \tilde\omega_m^{(p)})}{-\left(|\tilde\omega_1^{(p)} n_\ell + \tilde\omega_{m_\ell}^{(p)}| + \tilde\omega_{m_\ell}^{(p)}\right)\left(|\tilde\omega_1^{(p)} n_\ell + \tilde\omega_{m_\ell}^{(p)}| - \tilde\omega_{m_\ell}^{(p)}\right)}, \tag{7.35}$$

and the second factor in the denominator is bounded proportionally to 2^{-n_ℓ}, while the first is bounded by a constant times m_ℓ; hence (7.34) holds with $\xi = 1$. As $m_\ell = m_\ell^0 + m$ and $k_T > m_\ell^0$ we get

$$\begin{aligned} \tilde\omega_{m+\ell}^{(p)} - \tilde\omega_m^{(p)} &= \omega_{m+\ell} - \omega_m + O(\varepsilon (m+\ell)^{-\xi'}) + O(\varepsilon m^{-\xi'}) \\ &= \ell + O(\ell m^{-\xi}) + O(\varepsilon (m+\ell)^{-\xi'}) + O(\varepsilon m^{-\xi'}) = \ell + O(\ell \mu_0 m^{-1}), \end{aligned} \tag{7.36}$$

where $1 = \min\{\xi, \xi'\} = \min\{1, 2\}$; hence we can bound

$$\left|\tilde\omega_{m+\ell}^{(p)} - \tilde\omega_m^{(p)} - \ell\right| \le \frac{2K\mu_0\ell}{m}, \tag{7.37}$$

with the same constant K as in (A8.4).

The conditions in (7.12) correspond to several possibilities. If $n > 0$, $m' > m > 0$ and $||\tilde\omega_{m'}^{(p)} - \tilde\omega_m^{(p)}| - \tilde\omega_1^{(p)} n| < 1$ the discussion proceeds as follows.

When the conditions (7.12) are satisfied, one has

$$\begin{aligned} C_0 |n|^{-\tau_0} &\le |\omega_1 n - (\omega_{m'} - \omega_m)| \\ &\le |\omega_1 n - \mu_1(\tilde\omega^{(p-1)}, \varepsilon) n - (\omega_{m'} + \mu_{m'}^{(p-1)}(\tilde\omega^{(p-1)}, \varepsilon)) \\ &\qquad - (\omega_m + \mu_m^{(p-1)}(\tilde\omega^{(p-1)}, \varepsilon))| \\ &\qquad + |\mu_1(\tilde\omega^{(p-1)}, \varepsilon) n - \mu_{m'}(\tilde\omega^{(p-1)}, \varepsilon) + \mu_m(\tilde\omega^{(p-1)}, \varepsilon)| \\ &\le C_0 |n|^{-\tau} + 3C\varepsilon_0 |n|, \end{aligned} \tag{7.38}$$

which implies

$$|n| \ge \mathcal{N}_1 \equiv \left(\frac{C_0}{6C\varepsilon_0}\right)^{1/(\tau_0+1)}. \tag{7.39}$$

Besides of (7.11) we eliminate also the values ε verifying

$$|\omega_1 n - \mu_1(\tilde\omega^{(p-1)}, \varepsilon) n - m| \le C_0 |n|^{-\tau}, \tag{7.40}$$

for $\tau > \tau_0 + 1$ and for all $(n, m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. This requires to take off from $\mathcal{E}^{(p)}$ a subset whose measure is bounded by a constant times $\varepsilon_0^{1+\xi}$, as it can easily checked by proceeding as in the proof of the first of (7.11).

We can bound the Lebesgue measure of the set $\mathcal{I}_2^{(p)}$ by distinguishing, for fixed (n,ℓ), with $\ell = m' - m > 0$, the values $m \le m_0$ and $m > m_0$, where m_0 is determined by the request that one has for $m > m_0$

$$\frac{2K\mu_0\ell}{m^\xi} \le \frac{C_0}{2|n|^\tau}, \tag{7.41}$$

which gives

$$m_0 = \left(\frac{2\mu_0 K\ell|n|^\tau}{C_0}\right)^{1/\xi}. \tag{7.42}$$

Therefore for $m > m_0$ and $\mathcal{L}_0$ defined by

$$\frac{C_1\ell}{\sqrt{1+\mu_0}} \le \left|\tilde\omega_{m+\ell}^{(p)} - \tilde\omega_m^{(p)}\right| < \tilde\omega_1^{(p)}|n| + 1 < C_2|n|, \qquad \mathcal{L}_0 = \frac{C_2}{C_1}, \tag{7.43}$$

where (A8.8) has been used, one has, from (7.39)

$$\begin{aligned} &\left|\omega_1 n - \mu(\tilde\omega^{(p-1)},\varepsilon)n - (\tilde\omega_{m'}^{(p)} - \tilde\omega_m^{(p)})\right| \\ &\qquad \ge \left|\omega_1 n - \mu(\tilde\omega^{(p-1)},\varepsilon)n - \ell\right| - \frac{2K\mu_0\ell}{m^\xi} \\ &\qquad \ge \frac{C_0}{|n|^\tau} - \frac{C_0}{2|n|^\tau} \ge \frac{C_0}{2|n|^\tau}, \end{aligned} \tag{7.44}$$

so that one has to exclude no further value from $\mathcal{E}^{(p-1)}$, provided one takes C_0 the double of the original value. For $m < m_0$ one has to exclude a set of measure bounded by

$$\begin{aligned} \sum_{|n|\ge\mathcal{N}_1}\sum_{\ell\le\mathcal{L}_0|n|}\sum_{|m|\le m_0}\frac{C_0}{|n|^{\tau'}} &\le \text{const.}C_0\sum_{n=\mathcal{N}_1}^{\infty} n^{1-\tau'+(1+\tau)/\xi} \\ &\le \text{const.}C_0\mathcal{N}_0^{-\tau'+1+(\tau+1)/\xi} = \text{const.}C_0\left(\frac{\varepsilon_0}{C_0}\right)^{(\tau'-2-\tau)/(1+\tau_0)} \\ &\le \text{const.}\varepsilon_0^{1+\xi''}, \end{aligned} \tag{7.45}$$

provided that τ' is such that

$$\xi'' = \frac{\tau' - 2 - \dfrac{1+\tau}{\xi} - \tau_0}{1+\tau_0} > 0. \tag{7.46}$$

Finally we study the measure of the set $\mathcal{I}_3^{(p)}$. If $n > 0$ and $\left||\tilde\omega_m^{(p)} + \tilde\omega_{m'}| - \tilde\omega n\right| \le 1$ then one has to sum over $|n| \le \mathcal{N}_0$, with $\mathcal{N}_0$ given by (7.17), while one has $m < \tilde\omega_m^{(p)} < \tilde\omega_{m'}^{(p)} + \tilde\omega_m^{(p)} < \tilde\omega_1^{(p)} n + 1 < (\tilde\omega_1^{(p)} + 1)|n|$, so that one has to sum only over the m's such that $|m| < (\tilde\omega_1^{(p)} n + 1)|n|$, while $|m'|$ is uniquely determined by the values of$n$ and m. Then one can proceed as in the previous case and in the end one excludes a further subset of $\mathcal{E}^{(p-1)}$ whose Lebesgue measure admits a bound like (7.26).

By summing together the bounds for $\mathcal{I}_1^{(p)}$, for $\mathcal{I}_2^{(p)}$ and for $\mathcal{I}_3^{(p)}$, then the bound

$$\mathrm{meas}(\mathcal{I}^{(p)}) \le b\varepsilon_0^{\xi'+1} \tag{7.47}$$

follows for all $p \ge 1$.

We can conclude the proof of the theorem through the following result (proved in Appendix A11), which shows that the bound (7.47) essentially extends to the union of all $\mathcal{I}^{(p)}$ (at the cost of taking a larger constant B instead of b).

Lemma 12. *Define $I^{(p)}$ as the set of values in $\mathcal{E}^{(p)}$ verifying (A9.7). Then one has*

$$\mathrm{meas}\left(\cup_{p=0}^{\infty}\mathcal{I}^{(p)}\right) \le B\varepsilon_0^{\xi'+1}, \tag{7.48}$$

for two suitable positive constants B and ξ'.

The conclusion is that one has

$$\mathrm{meas}(\mathcal{E}) = 2\varepsilon_0\left(1 - b\varepsilon_0^{\xi}\right), \tag{7.49}$$

so that the proof of the theorem is complete.

Appendix A1. Proof of lemma 1

We prove inductively the bound

$$N_h^*(\theta) \le \max\{0, 2K(\theta)2^{(2-h)/\tau} - 1\}, \tag{A1.1}$$

where $N_h^*(\theta)$ is the number of non-resonant lines in $L(\theta)$ on scale $h' \ge h$.

First of all note that for a tree θ to have a line on scale h the condition $K(\theta) > 2^{(h-1)/\tau}$ is necessary, by the first Diophantine conditions in (1.15). This means that one can have $N_h^*(\theta) \ge 1$ only if $K = K(\theta)$ is such that $K > k_0 \equiv 2^{(h-1)/\tau}$: therefore for values $K \le k_0$ the bound (A1.1) is satisfied.

If $K = K(\theta) > k_0$, we assume that the bound holds for all trees θ' with $K(\theta') < K$. Define $E_h = 2^{-1}(2^{(2-h)/\tau})^{-1}$: so we have to prove that $N_h^*(\theta) \le \max\{0, K(\theta)E_h^{-1} - 1\}$.

Call ℓ the root line of θ and $\ell_1, \ldots, \ell_m$ the $m \ge 0$ lines on scale $\ge h$ which are the closest to ℓ (i.e. such that no other line along the paths connecting the lines $\ell_1, \ldots, \ell_m$ to the root line is on scale $\ge h$).

If the root line ℓ of θ is either on scale $< h$ or on scale $\ge h$ and resonant, then

$$N_h^*(\theta) = \sum_{i=1}^{m} N_h^*(\theta_i), \tag{A1.2}$$

where θ_i is the subtree with ℓ_i as root line, hence the bound follows by the inductive hypothesis.

If the root line ℓ has scale $\ge h$ and is non-resonant, then $\ell_1, \ldots, \ell_m$ are the entering line of a cluster T.

By denoting again with θ_i the subtree having ℓ_i as root line, one has

$$N_h^*(\theta) = 1 + \sum_{i=1}^{m} N_h^*(\theta_i), \tag{A1.3}$$

so that the bound becomes trivial if either $m = 0$ or $m \geq 2$.

If $m = 1$ then one has a cluster T with two external lines ℓ and ℓ_1, which are both with scales $\geq h$; then

$$\left||\tilde{\omega}_1 n_\ell| - \tilde{\omega}_{m_\ell}\right| \leq 2^{-h+1} C_0, \qquad \left||\tilde{\omega}_1 n_{\ell_1}| - \tilde{\omega}_{m_{\ell_1}}\right| \leq 2^{-h+1} C_0, \tag{A1.4}$$

and $n_\ell \neq n_{\ell_1}$ as T is not a self-energy graph. Then, by (A1.4), one has, for suitable $\eta_\ell, \eta_{\ell_1} \in \{+,-\}$,

$$2^{-h+2} C_0 \geq \left|\tilde{\omega}_1 (n_\ell - n_{\ell_1}) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}}\right| \geq C_0 |n_\ell - n_{\ell_1}|^{-\tau}, \tag{A1.5}$$

where the second Diophantine conditions in (1.15) have been used. Hence $K(\theta) - K(\theta_1) > E_h$, which, inserted into (A1.3) with $m = 1$, gives, by using the inductive hypothesis,

$$\begin{aligned} N_h^*(\theta) &= 1 + N_h^*(\theta_1) \leq 1 + K(\theta_1) E_h^{-1} - 1 \\ &\leq 1 + \Big(K(\theta) - E_h\Big) E_h^{-1} - 1 \leq K(\theta) E_h^{-1} - 1, \end{aligned} \tag{A1.6}$$

hence the bound is proved also if the root line is on scale $\geq h$.

In the same way one proves that, if we denote with $C_h(\theta)$ the number of clusters on scale h, one has

$$C_h(\theta) \leq \max\{0, 2K(\theta) 2^{(2-h)/\tau} - 1\}. \tag{A1.7}$$

For a tree to contain a cluster on scale h it has to contain *a fortiori* a line on that scale, so that again the bound (A1.7) is trivially satisfied for $K(\theta) \leq k_0 = 2^{(h-1)/\tau}$.

For $K(\theta) = K > k_0$ we can proceed inductively as before. If the node v_0, which the root line ℓ of θ comes out from, is not in a cluster on scale h the one has

$$C_h(\theta) = \sum_{i=1}^{m} C_h(\theta_i), \tag{A1.8}$$

where $\theta_1, \ldots, \theta_m$ are the subtrees with root in v_0; in such a case the bound follows from the inductive hypothesis.

If v_0 is inside a cluster T on scale h then call $\theta_1, \ldots, \theta_m$ the entering lines of T; then one has

$$C_h(\theta) = 1 + \sum_{i=1}^{m} C_h(\theta_i), \tag{A1.9}$$

and the bound (A1.7) follows again from the inductive hypothesis for either $m = 0$ or $m \geq 2$. If $m = 1$ then one has

$$\left||\tilde{\omega}_1 n_\ell| - \tilde{\omega}_{m_\ell}\right| \leq 2^{-h+1} C_0, \qquad \left||\tilde{\omega}_1 n_{\ell_1}| - \tilde{\omega}_{m_{\ell_1}}\right| \leq 2^{-h+1} C_0, \tag{A1.10}$$

where $\ell \in L(T)$ is on scale h, while ℓ_1 is on a scale $h_{\ell_1} > h$ (by definition of cluster). Therefore, if $n_\ell = n_\ell^0$, one has, by the first Diophantine condition in (1.15),

$$2^{-h+1} C_0 \ge \left| |\tilde\omega_1 n_\ell| - \tilde\omega_{m_\ell} \right| \ge C_0 |n_\ell^0|^{-\tau}, \tag{A1.11}$$

while, if $n_\ell = n_\ell^0 + n_{\ell_1}$, one has for suitable $\eta_\ell, \eta_{\ell_1} \in \{+,-\}$,

$$\begin{aligned} 2^{-h+2} C_0 &\ge \left| \tilde\omega_1 (n_\ell - n_{\ell_1}) + \eta_\ell \tilde\omega_{m_\ell} + \eta_{\ell_1} \tilde\omega_{m_{\ell_1}} \right| \\ &\ge C_0 |n_\ell - n_{\ell_1}|^{-\tau} \ge C_0 |n_\ell^0|^{-\tau}, \end{aligned} \tag{A1.12}$$

by the second Diophantine conditions in (1.15). So in both cases one has

$$\sum_{v \in V(T)} |n_v| \ge 2^{(h-2)/\tau} > E_h, \tag{A1.13}$$

so that (A1.9) implies the bound (A1.7) also for $m = 1$. Of course one can bound the number of non-resonant lines on scale h of any tree θ by $N_h^*(\theta) + C_h(\theta) - C_h(\theta)$, with $N_h^*(\theta) + C_h(\theta) \le 4K(\theta) 2^{(2-h)/\tau}$: therefore (4.12) follows.

Appendix A2. Proof of lemma 2

One has to show inductively that, by defining $N_h^*(\theta)$ as the number of non-resonant lines on scale $h' \ge h$ in the renormalized tree θ, one has

$$N_h^*(\theta) \le \max\{0, 2K(\theta) 2^{(2-h)/\tau} - 1\}, \tag{A2.1}$$

and that an analogous bound holds for the number $C_h(\theta)$ of clusters on scale h.

Let us consider a self-energy graph T in θ which is *maximal*, *i.e.* which is not contained in any other resonance. The value of such a self-energy graph is given by the product of the values of the inner maximal self-energy graphs (on which the $\mathcal{R}$ operation is still applied) times the product of propagators corresponding to the remaining lines; there is a chain of propagators and self-energy graphs connecting the external lines such that the momentum (n_ℓ, m_ℓ) of any of such propagator or self-energy graph has the form

$$(n_\ell^0 + n, m_\ell^0 + m), \tag{A2.2}$$

where (n, m) is the momentum of the external lines, and (n_ℓ^0, m_ℓ^0) are implicitly defined; all the other propagators and self-energy graphs are independent of (n, m). Let us consider the action of $\mathcal{R}$ on the maximal self-energy graphs. The action of $\mathcal{R}$ consists in writing the self-energy graph as a sum of terms, in which one propagator $g^{(h_\ell)}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m)$ is replaced with $g^{(h_\ell)}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m) - g^{(h_\ell)}(\tilde\omega_1 n_\ell^0 + \tilde\omega_m, m_\ell^0 + m)$ or the value of one regularized inner self-energy graph $\mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m)$ is replaced with $\mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m) - \mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_m, m_\ell^0 + m)$. We can write

$$\begin{aligned} & g^{(h_\ell)}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m) - g^{(h_\ell)}(\tilde\omega n_\ell^0 + \tilde\omega_m, m_\ell^0 + m) \\ & \qquad = (\tilde\omega_1 n - \tilde\omega_m) \int_0^1 \mathrm{d}t \, \partial \frac{\chi_{h_\ell}(|\tilde\omega_1 n_\ell^0 + t(\tilde\omega_1 n - \tilde\omega_m)| - \tilde\omega_m)}{-(\tilde\omega_1 n_\ell^0 + t(\tilde\omega_1 n - \tilde\omega_m))^2 + \tilde\omega_m^2}, \end{aligned} \tag{A2.3}$$

where ∂ denotes the derivative with respect to the argument $\tilde\omega_1 n_\ell^0 + t(\tilde\omega_1 n - \tilde\omega_m)$.

On the other hand $\mathcal{L}\mathcal{V}_T^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m)$ is independent of $\omega_1 n$ (see (5.1)), so that

$$\begin{aligned}\mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_1 n, m_\ell^0 + m) - \mathcal{R}\mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_m, m_\ell^0 + m) \\ = \mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega n, m_\ell^0 + m) - \mathcal{V}_{T'}^{h_{T'}^{(e)}}(\tilde\omega_1 n_\ell^0 + \tilde\omega_m, m_\ell^0 + m),\end{aligned} \tag{A2.4}$$

and we can reason as above, writing the inner self-energy graphs as a sum of terms.

Then the proof of lemma 2 proceeds exactly as for lemma 1, but in (A1.4) one has to replace $\tilde\omega_1 n_\ell$ with $\tilde\omega_1 n_\ell^0 + t(\tilde\omega_1 n - \tilde\omega_m)$, when ℓ is along a path of lines connecting the external lines of T. But this does not change the proof because when passing from (A1.4) to (A1.5) the terms depending on t cancel out.

Appendix A3. Proof of lemma 3

By using the expression (A2.3) and (A2.4) we see that the effect of $\mathcal{R}$ is to improve by a factor $2^{-h_T^{(e)}+h_T}$ the bound of the propagator corresponding to the line ℓ. In the same way, the difference of propagators in the inner self-energy graph gives the "gain"

$$2^{-h_T^{(e)}+h_{T'}} \le 2^{-h_T^{(e)}+h_T} 2^{-h_{T'}^{(e)}+h_{T'}}, \tag{A3.1}$$

as $h_{T'}^{(e)} \le h_T$. At the end (i) the propagators are derived at most one time; (ii) the number of terms so generated is $\le k + |\underline{k}|$; (iii) to each self-energy graph T a factor $2^{-h_T^{(e)}+h_T}$ is associated.

Assuming that $|\nu_{h,m}^{(c)}| \le C\varepsilon$, for any θ one obtains, for a suitable constants $\overline{D}$ and $\overline{D}_0$,

$$\begin{aligned}|\mathrm{Val}(\theta)| \le \overline{D}_0 |\varepsilon|^{k+|\underline{k}|} \overline{D}^{k+|\underline{k}|} \\ \Big(\prod_{h=0}^{\infty} \exp\Big[h \log 2 \Big(4k 2^{-(h-2)/\tau} - C_h(T) + S_h(\theta) + M_h^\nu(\theta) \Big) \Big] \Big) \\ \Big(\prod_{T\in\mathcal{S}(\theta)} 2^{-h_T^{(e)}+h_T} \Big) \Big(\prod_{h=0}^{\infty} 2^{-hM_h^\nu(\theta)} \Big),\end{aligned} \tag{A3.2}$$

where lemma 2 has been used in order to bound the number of lines on scale h, and

$$\prod_{T\in\mathcal{S}(\theta)} 2^{-h_T^{(e)}+h_T} \tag{A3.3}$$

is due to the nontrivial action of the $\mathcal{R}$ operator on the self-energy graphs, while the factor

$$\prod_{h=0}^{\infty} 2^{-hM_\nu^h(\theta)} \tag{A3.4}$$

takes into account the 2^{-h} factors associated to the ν-vertices contributing a factor $\nu^{(c)}_{h,m}$. Then one has

$$\begin{aligned} &\Big(\prod_{h=0}^{\infty} 2^{hS_h(\theta)} \Big) \Big(\prod_{T\in\mathcal{S}(\theta)} 2^{-h_T^{(e)}} \Big) = 1, \\ &\Big(\prod_{h=0}^{\infty} 2^{-hC_h(\theta)} \Big) \Big(\prod_{T\in\mathcal{S}(\theta)} 2^{h_T} \Big) \le 1, \end{aligned} \tag{A3.5}$$

so that one finds, for suitable constants $\tilde{D}_0$ and $\tilde{D}$,

$$\sum_{\underline{k}\,:\,|\underline{k}| \text{ fixed}} \sum_{\theta\in\Theta^{(k,\underline{k})\mathcal{R}}_{n,m}} |\mathrm{Val}(\theta)| \le \tilde{D}_0 |\varepsilon|^{k+|\underline{k}|} \tilde{D}^{k+|\underline{k}|} 2^{(k+|\underline{k}|)} 4^k 2^{|\underline{k}|} \le |\varepsilon|^{k+|\underline{k}|} C^{k+|\underline{k}|}, \tag{A3.6}$$

where $2^{k+|\underline{k}|}$ is a bound on the number of trees in some $\Theta^{(k,\underline{k})}_{n,m}$ with fixed $|\underline{k}|$, 4^k is a bound on the assignments of the labels (n_v, m_v) for all end-points v, and finally $2^{|\underline{k}|}$ is a bound on the labels $c = a, b$ for all self-energy graphs.

Hence, for fixed (n, m) one has

$$\sum_{k,\underline{k}} \sum_{\theta\in\Theta^{(k,\underline{k})}_{n,m}} |\mathrm{Val}(\theta)| \le D_0 |\varepsilon|^{(|n|+|m|)/2} D^{|n|+|m|}, \tag{A3.7}$$

as $k \ge \min\{|n|, |m|\}$ and $|\underline{k}| \ge 0$, so that (5.6) is proved.

Appendix A4. Proof of lemma 4

First of all we note that the result stated in lemma 2 still holds, with no change, if we allow any integer value for the mode labels (n_v, m_v) of the end-points. The only difference is that the induction has to be performed on $\widetilde{K}(\theta)$, as it is defined in (5.10) for $E_0(\theta) = \emptyset$. So we can assume that the bound (A2.1) holds for any tree θ with $E_0(\theta) = \emptyset$.

Then we have to show that the bound (5.9) holds for trees with $E_0(\theta) = \{v_0\} \neq \emptyset$. We mimic the proof of lemma 2 (hence of lemma 1): we prove the bound

$$N^*_h(\theta) \le \max\{0, 2\widetilde{K}(\theta) 2^{(2-h)/\tau} - 1\}, \tag{A4.1}$$

for all trees θ with $E_0(\theta) \neq \emptyset$, again by induction on $\widetilde{K}(\theta)$.

For any line $\ell \in L(\theta)$ set $\eta_\ell = 1$ if the line is along the path connecting v_0 to the root and $\eta_\ell = 0$ otherwise, and write

$$n_\ell = n^0_\ell + \eta_\ell \left(\overline{n} + \overline{\omega}_m \right), \qquad m_\ell = m^0_\ell + \eta_\ell \left(\overline{m} + m \right), \tag{A4.2}$$

which implicitly defines n^0_ℓ and m^0_ℓ.

Define $k_0 = 2^{(h-1)/\tau}$. One has $N^*_h(\theta) = 0$ for $\widetilde{K}(\theta) < k_0$, because if a line $\bar{\ell} \in L(\theta)$ is indeed on scale h then $|\tilde{\omega}_1 n_{\bar{\ell}} - \tilde{\omega}_{m_{\bar{\ell}}}| < C_0 2^{1-h}$, so that (A4.2) and the Diophantine conditions (1.15) imply

$$\widetilde{K}(\theta) \ge \left| n^0_{\bar{\ell}} + \eta_{\bar{\ell}} \overline{n} \right| > 2^{(h-1)/\tau} \equiv k_0. \tag{A4.3}$$

Then, for $K \geq k_0$, we assume that the bound (A4.1) holds for all $\widetilde{K}(\theta) = K' < K$, and we show that it follows also for $\widetilde{K}(\theta) = K$.

If the root line ℓ of θ is either on scale $< h$ or on scale $\geq h$ and resonant, the bound (A4.1) follows immediately from the bound (A2.1) and from the inductive hypothesis.

The same occurs if the root line is on scale $\geq h$ and non-resonant, and, by calling $\ell_1, \ldots, \ell_m$ the lines on scale $\geq h$ which are the closest to ℓ, one has $m \geq 2$. If $m = 0$ then $N_h^*(\theta) = 1$ and $\widetilde{K}(\theta) 2^{(2-h)/\tau} \geq 1$ because one must have $\widetilde{K}(\theta) \geq k_0$.

So the only non-trivial case is when, in the latter case, one has $m = 1$. If this happens ℓ_1 is, by construction, the root line of a tree θ_1 such that $E_0(\theta_1) \neq \emptyset$, and $\widetilde{K}(\theta) = \widetilde{K}(T) + \widetilde{K}(\theta_1)$, where T is the cluster which has ℓ and ℓ_1 as external lines and we have defined

$$\widetilde{K}(T) \equiv \sum_{v \in E(T) \cup E_1(T)} |n_v|, \tag{A4.4}$$

which satisfies the bound $\widetilde{K}(T) \geq |n_{\ell_1} - n_\ell|$.

Moreover one has

$$\begin{aligned} \big| |\tilde{\omega}_1 n_\ell^0 + \tilde{\omega}_1 \overline{n} + \tilde{\omega}_m| - \tilde{\omega}_{m_\ell} \big| &\leq 2^{-h'+1} C_0, \\ \big| |\tilde{\omega}_1 n_{\ell_1}^0 + \tilde{\omega}_1 \overline{n} + \tilde{\omega}_m| - \tilde{\omega}_{m_{\ell_1}} \big| &\leq 2^{-h'+1} C_0, \end{aligned} \tag{A4.5}$$

so that, for suitable $\eta_\ell, \eta_{\ell_1} \in \{-, +\}$, we obtain

$$\begin{aligned} 2^{-h'+2} C_0 &\geq \big| \tilde{\omega}_1 (n_\ell^0 - n_{\ell_1}^0) + \eta_\ell \tilde{\omega}_{m_\ell} + \eta_{\ell_1} \tilde{\omega}_{m_{\ell_1}} \big| \\ &\geq C_0 |n_\ell^0 - n_{\ell_1}^0|^{-\tau} \equiv C_0 |n_\ell - n_{\ell_1}|^{-\tau}, \end{aligned} \tag{A4.6}$$

by the second Diophantine conditions in (1.15), as the quantities $\tilde{\omega}_1 \overline{n} + \tilde{\omega}_m$ appearing in (A4.4) cancel out. Therefore one obtains

$$N_h^*(\theta) \leq 1 + \widetilde{K}(\theta_1) E_h^{-1} - \widetilde{K}(\theta) E_h^{-1} - 1 \leq \widetilde{K}(\theta) E_h^{-1} - 1, \tag{A4.7}$$

by using the same notations as in (A1.6), hence the first bound in (A4.1) is proved.

Analogously one proves the bound

$$C_h(\theta) \leq \max\{0, 2\widetilde{K}(\theta) 2^{(2-h)/\tau} - 1\}, \tag{A4.8}$$

so that (5.9) follows.

Appendix A5. Proof of lemma 5

Consider a self-energy graph $T \in \mathcal{T}_h^{(c)}$. We can consider the tree θ obtained from T by adding to it the outcoming line ℓ_T^1 and replacing the entering line ℓ_T^2 with a line emerging from an end-point v_0 which carries a mode label $(\tilde{\omega}_m, m)$; by construction one has

$$\mathcal{V}_T^h(\tilde{\omega}_m, m) = \mathrm{Val}'(\theta), \tag{A5.1}$$

where $\mathrm{Val}'(\theta)$ differs from (4.5) as in the first product the line ℓ_1 coming out from T is missing, and $E_0(\theta) = \{v_0\}$.

We want to prove that one has

$$N^*_{h'}(T) \le \max\{0, 2K(T)2^{(2-h')/\tau} - 1\}, \tag{A5.2}$$

if $N^*_{h'}(T)$ denotes the number of non-resonant lines on scale $\le h'$ internal to T.

Let v be the node such that ℓ^1_T comes out from v, and call $\ell_1, \ldots, \ell_{s_v}$ the lines entering v; denote with $\theta_1, \ldots, \theta_{s_v}$ the trees which have $\ell_1, \ldots, \ell_{s_v}$ as root lines. By construction one has

$$N^*_{h'}(T) = \sum_{j=1}^{s_v} N^*_{h'}(\theta_j), \tag{A5.3}$$

and (only) one of the subtrees θ_j, say θ_1, contains the end-point v_0. Therefore in (5.7) we can bound

$$\sum_{j=2}^{s_v} N^*_{h'}(T) \le \sum_{j=2}^{s_v} \max\{0, 2K(\theta_j)2^{(2-h')/\tau} - 1\}, \tag{A5.4}$$

by using (A1.1), and we can bound

$$N_{h'}(\theta_1) \le K(\theta_1)2^{(2-h')/\tau} - 1, \tag{A5.5}$$

by using (5.9) (with $\overline{n} = 0$).

Analogously one can prove that one has

$$C_{h'}(T) \le \max\{0, 2K(\theta_j)2^{(2-h')/\tau} - 1\}, \tag{A5.6}$$

if $C_{h'}(T)$ denotes the number of clusters on scale $\le h'$ internal to T.

Then the bound (A4.7) follows.

Appendix A6. Proof of lemma 6

By using lemma 5 and the cancellations discussed in Appendix A3 we obtain for all $T \in \mathcal{T}^{(c)}_h$

$$\begin{aligned} \left|\mathcal{V}^{h+1}_T(\tilde{\omega}_m, m)\right| \le\; & C^k |\varepsilon|^{K(T)+|\underline{K}(T)|} \\ & \prod_{h'=h}^{\infty} \exp\left[4k \log 2 h' 2^{(2-h')/\tau} - C_{h'}(T) + S_{h'}(T) + M^{\nu}_{h'}(T)\right] \\ & \Big(\prod_{T' \subset T} 2^{-h^{(e)}_T + h_T}\Big)\Big(\prod_{h'=h}^{\infty} 2^{-h' M^{\nu}_{h'}(T)}\Big). \end{aligned} \tag{A6.1}$$

The main difference with respect to the previous case is that, given a self-energy graph $T \in \mathcal{T}^{(c)}_h$, there is at least a line $\ell \in L(T)$ on scale $h_\ell = h$ and with propagator

$$\frac{1}{-\tilde{\omega}_1^2 (n^0_\ell + \eta_\ell \overline{\omega}_m)^2 + \tilde{\omega}^2_{m^0_\ell + \eta_\ell m}}, \tag{A6.2}$$

where $\eta_\ell = 1$ if the line ℓ belongs to the path of lines connecting the entering line (carrying a momentum (n, m)) of T with the line coming out of T, and $\eta_\ell = 0$ otherwise. Then by (1.15) one has

$$C_0 |n_\ell^0|^{-\tau} \leq \left| \tilde{\omega}_1 n_\ell^0 + \eta_\ell \tilde{\omega}_m \pm \tilde{\omega}_{m_\ell^0 + \eta_\ell m} \right| \leq C_0 2^{-h+1}, \tag{A6.3}$$

so that $|n_\ell^0| \geq 2^{(h-1)/\tau}$. On the other hand one has $|n_\ell^0| \leq K(T)$, hence $K(T) \geq 2^{(h-1)/\tau}$; so we get the bound (A4.8).

Appendix A7. Proof of lemma 7

The proof is done by induction on $\bar{h}$. Let us define $J_{c,m}^{(h)} = [-|\varepsilon|, |\varepsilon|]$ and call $J^{(h)} = \times_{|m|>1, c=a,b} J_{c,m}^{(h)}$ and $I^{(h)} = \times_{|m|>1, c=a,b} I_{c,m}^{(h)}$.

We suppose that there exists $I^{(\bar{h})}$ such that, if ν spans $I^{(\bar{h})}$ then $\nu_{\bar{h}}$ spans $J^{(\bar{h})}$ and $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$ for $\bar{h} \geq h \geq 0$; we want to show that the same holds for $\bar{h}+1$. Let us call $\tilde{J}^{(\bar{h}+1)}$ the interval spanned by $\{\nu_{\bar{h}+1,m}^{(c)}\}_{|m|>1, c=a,b}$ when $\{\nu_m^{(c)}\}_{|m|>1, c=a,b}$ span $I^{(\bar{h})}$. For any $\{\nu_m^{(c)}\}_{|m|>1, c=a,b} \in I^{(\bar{h})}$ one has $\{\nu_{\bar{h}+1,m}\}_{|m|>1, c=a,b} \in [-2|\varepsilon| - C|\varepsilon|^2, 2|\varepsilon| + C|\varepsilon|^2]$, where the bound (5.13) has been used. This means that $J^{(\bar{h}+1)}$ is strictly contained in $\tilde{J}^{(\bar{h}+1)}$.

On the other hand it is obvious that there is a one-to-one correspondence between $\{\nu_m^{(c)}\}_{|m|>1, c=a,b}$ and the sequence $\{\nu_{h,m}^{(c)}\}_{|m|>1, c=a,b}$, $\bar{h}+1 \geq h \geq 0$. Hence there is a set $I^{(\bar{h}+1)} \subset I^{(\bar{h})}$ such that, if $\{\nu_m^{(c)}\}_{|m|>1, c=a,b}$ spans $I^{(\bar{h}+1)}$, then $\{\nu_{\bar{h}+1,m}^{(c)}\}_{|m|>1, c=a,b}$ spans the interval $J^{(\bar{h})}$ and, for ε small enough, $|\nu_h|_\infty \leq C|\varepsilon|$ for $\bar{h}+1 \geq h \geq 0$.

The previous computations also show that the inductive hypothesis is verified also for $\bar{h} = 0$ so that we have proved that there exists a decreasing sets of intervals $I^{(\bar{h})}$ such that if $\{\nu_m^{(c)}\}_{|m|>1, c=a,b} \in I^{(\bar{h})}$ then the sequence $\{\nu_{h,m}^{(c)}\}_{|m|>1, c=a,b}$ is well defined for $h \leq \bar{h}$ and it verifies $|\nu_{h,m}^{(c)}| \leq C|\varepsilon|$. In order to prove the bound on the size of $I_{c,m}^{(\bar{h})}$ let us denote by $\{\nu_{h,m}^{(c)}\}_{|m|>1, c=a,b}$ and $\{\nu_{h,m}'^{(c)}\}_{|m|>1, c=a,b}$, $0 \leq h \leq \bar{h}$, the sequences corresponding to $\{\nu_m^{(c)}\}_{|m|>1, c=a,b}$ and $\{\nu_m'^{(c)}\}_{|m|>1, c=a,b}$ in $I^{(\bar{h})}$. We have

$$\nu_{h+1,m}^{(c)} - \nu_{h+1,m}'^{(c)} = 2\left(\nu_{h,m}^{(c)} - \nu_{h,m}'^{(c)}\right) + \beta_{h,m}^{(c)} - \beta_{h,m}'^{(c)}, \tag{A7.1}$$

where $\beta_{h,m}^{(c)}$ and $\beta_{h,m}'^{(c)}$ are shorthands for the beta functions. Then, as $|\nu_k - \nu_k'|_\infty \leq |\nu_h - \nu_h'|_\infty$ for all $k \leq h$, we have

$$|\nu_h - \nu_h'|_\infty \leq \frac{1}{2} |\nu_{h+1} - \nu_{h+1}'|_\infty + C|\varepsilon|^2 |\nu_h - \nu_h'|_\infty. \tag{A7.2}$$

Hence if ε is small enough then one has

$$|\nu - \nu'|_\infty \leq (\sqrt{2})^{-(\bar{h}+1)} |\nu_{\bar{h}} - \nu_{\bar{h}}'|_\infty. \tag{A7.3}$$

Since, by definition, if ν spans $I^{(\bar{h})}$, then $\nu_{\bar{h}}$ spans the interval $J^{(\bar{h})}$, of size $2|\varepsilon|$, the size of $I^{(\bar{h})}$ is bounded by $2|\varepsilon|(\sqrt{2})^{(-\bar{h}-1)}$.

Appendix A8. Proof of lemma 8

(i) Assume μ to be Diophantine with Diophantine constants C_1, τ_1, *i.e.*

$$|\mu p + q| > C_1 |p|^{-\tau_1} \qquad \forall (p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}; \tag{A8.1}$$

such μ have full measure in $\mathbb{R}$ if $\tau_1 > 1$. From (A8.1), by recalling that $\omega_m^2 = m^2 + \mu$, we have that, for all $(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

$$\begin{aligned} |(\omega_1 n - m)(\omega_1 n + m)| &= \left|\mu n^2 + \left(n^2 - m^2\right)\right| > \frac{C_1}{|n|^{2\tau_1}}, \\ |(\omega_1 n - \omega_m)(\omega_1 n + \omega_m)| &= \left|\mu\left(n^2 - 1\right) + \left(n^2 - m^2\right)\right| > \frac{C_1}{|n|^{2\tau_1}}, \end{aligned} \tag{A8.2}$$

which imply

$$\begin{aligned} |\omega_1 n \pm m| &> \frac{C_2}{|n|^{2\tau_1+1}}, \\ |\omega_1 n \pm \omega_m| &> \frac{C_2}{|n|^{2\tau_1+1}}, \end{aligned} \tag{A8.3}$$

from which the first two of (7.5) follow provided that one has $C_2 \geq C_0$ and $\tau_0 \geq 2\tau_1 + 1 > 3$.

(ii) Then we consider the last of (7.5). Suppose without loss of generality $n > 0$ and $m' \geq m > 0$. We discuss separately the cases $|\omega_1 n - (\omega_{m'} - \omega_m)|$ and $|\omega_1 n - (\omega_{m'} + \omega_m)|$.

Let us consider first the case $|\omega_1 n - (\omega_{m'} - \omega_m)|$; if $m' = m + \ell$ we can write

$$\omega_{m+\ell} - \omega_m = \ell + c_{\ell,m}, \qquad 0 > c_{\ell,m} > -K\mu_0 \frac{\ell}{m^\xi}, \tag{A8.4}$$

for $\xi = 2$ and a suitable positive K (one can take $K = 2$), and we consider separately the case $|c_{\ell,m}| < C_2 |n|^{-2\tau_1 - 1}/2$ and $|c_{\ell,m}| \geq C_2 |n|^{-2\tau_1-1}/2$.

In the first case by the second of (A8.3) one has

$$|\omega_1 n - \omega_{m+\ell} + \omega_m| \geq |\omega_1 n - \ell| - |c_{\ell,m}| > \frac{1}{2} \frac{C_2}{|n|^{2\tau_1+1}} > \frac{C_0}{|n|^{\tau_0}}, \tag{A8.5}$$

where the last inequality requires $C_0 \leq C_2/2$.

In the second case we can assume $|\omega_1 n - \omega_{m+\ell} + \omega_m| < 1$, because otherwise (7.5) is trivially satisfied: hence we have to consider only values of n, m, ℓ such that $0 < \omega_{m+\ell} - \omega_m < \omega_1 n + 1$ (recall that we are assuming $\ell, m, \omega_1 n > 0$).

The condition $|c_{\ell,m}| \geq C_2 |n|^{-2\tau_1-1}/2$ and the last inequality in (A8.4) imply

$$|m| \leq M \equiv \left(\frac{2\mu_0 K \ell |n|^{2\tau_1+1}}{C_2}\right)^{1/\xi}, \tag{A8.6}$$

whereas the inequality

$$\begin{aligned} \omega_{m+\ell} - \omega_m &= \frac{\omega_{m+\ell}^2 - \omega_m^2}{\omega_{m+\ell} + \omega_m} = \frac{\ell}{\sqrt{1+\mu}} \frac{(\ell + m)\sqrt{1+\mu} + m\sqrt{1+\mu}}{\omega_{m+\ell} + \omega_m} \\ &\geq \frac{\ell}{\sqrt{1+\mu}} \frac{\omega_{m+\ell} + \omega_m}{\omega_{m+\ell} + \omega_m} \geq \frac{\ell}{\sqrt{1+\mu}} \geq \frac{\ell}{\sqrt{1+\mu_0}} \end{aligned} \tag{A8.7}$$

implies

$$\frac{\ell}{\sqrt{1+\mu_0}} \le \omega_{m+\ell} - \omega_m \le \omega_1 n + 1, \tag{A8.8}$$

so that one must have

$$\ell < L(|\omega_1 n| + 1), \qquad L = \sqrt{1+\mu_0}. \tag{A8.9}$$

At fixed n, m, m' we can define a map $t \to \mu(t)$ such that

$$f(\mu(t)) \equiv \omega_1(\mu(t))n - \omega_{m+\ell}(\mu(t)) + \omega_m(\mu(t)) = t\frac{C_0}{|n|^{\tau_0}}. \tag{A8.10}$$

Then one has, if $\mathcal{I}$ is the set of μ not verifying the Diophantine condition $|\omega_1 n - \omega_{m+\ell} + \omega_m| \ge C_0 |n|^{-\tau_0}$,

$$\int_{\mathcal{I}} \mathrm{d}\mu = \sum_{n=1}^{\infty} \sum_{\ell=1}^{L(|\omega n|+1)} \sum_{|m|\le M} \int_{-1}^{1} \mathrm{d}t \left| \frac{\mathrm{d}\mu(t)}{\mathrm{d}t} \right|, \tag{A8.11}$$

where $\mathrm{d}\mu(t)/\mathrm{d}t$ can be obtained by noting that deriving (A8.10) with respect to t gives

$$\frac{\mathrm{d}f(\mu(t))}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{C_0}{|n|^{\tau_0}}, \tag{A8.12}$$

Moreover

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\mu} &= \frac{1}{2}\frac{1}{\omega_1\omega_{m+\ell}\omega_m}\left(n\omega_m\omega_{m+\ell} + \omega_1\left(\omega_{m+\ell} - \omega_m\right)\right) \\ &\ge \frac{n\omega_{m+\ell}\omega_m}{2\omega_1\omega_{m+\ell}\omega_m} \ge \frac{1}{2\sqrt{1+\mu}} \ge \frac{1}{2\sqrt{1+\mu_0}} = \frac{1}{2L}, \end{aligned} \tag{A8.13}$$

so that we find

$$\int_{\mathcal{I}} \mathrm{d}\mu \le 4L \sum_{n=01}^{\infty} \sum_{\ell=1}^{L(|\omega n|+1)} \sum_{|m|\le M} \frac{C_0}{|n|^{\tau_0}}. \tag{A8.14}$$

We can bound (A8.14) with a constant times

$$(1+\mu_0)^{1+1/2\xi}\mu_0^{1/\xi} C_0 \sum_{n=1}^{\infty} n^{-(\tau_0 - 1 - (2\tau_1+2)/\xi)}, \tag{A8.15}$$

which is proportional to C_0 provided that one has

$$\tau_0 > 2 + \frac{2\tau_1 + 2}{\xi} > 4. \tag{A8.16}$$

(iii) Now we pass to the case $|\omega_1 n - \omega_{m'} - \omega_m|$. We can have $|\omega_1 n - \omega_{m'} - \omega_m| < 1$ only if $(\omega_1 + 1)n \ge 1 + \omega_1 n > \omega_{m'} + \omega_m > \omega_m > |m|$, and for fixed m there is a unique $|m'|$ such that $|\omega_{m'} + \omega_m - \omega n| \le 1$ (so that the summation over m' gives only a factor 2). Moreover for $n = 1$ one has $\omega_m + \omega_{m'} > 2\omega_1$, so that, if $C_0 \le 1 \le \omega_1$, the Diophantine condition $|\omega_1 n - \omega_{m'} - \omega_m| \geq C_0 |n|^{-\tau_0}$ is automatically satisfied for $n = 1$

At fixed n, m, m', with $n \geq 2$, by defining a map $t \to \mu(t)$ through

$$f(\mu(t)) \equiv \omega_1(\mu(t))n - \omega_{m+\ell}(\mu(t)) - \omega_m(\mu(t)) = t\frac{C_0}{|n|^{\tau_0}}, \qquad t \in [-1, 1], \tag{A8.17}$$

and using that for $n \geq 2$ one has

$$n\omega_{m'}\omega_m \geq n\frac{\omega_{m'} + \omega_m}{2}\omega_m \geq \omega_m \left(\omega_{m'} + \omega_m\right), \tag{A8.18}$$

we have

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\mu} &= \frac{1}{2}\frac{1}{\omega_1\omega_{m'}\omega_m}\left(n\omega_m\omega_{m+\ell} - \omega_1\left(\omega_{m'} + \omega_m\right)\right) \\ &\geq \frac{1}{4}\frac{\omega_m - \omega_1}{\omega_1\omega_m}\frac{\omega_{m'} + \omega_m}{\omega_{m'}} \geq \frac{1}{4}\frac{\omega_m^2 - \omega_1^2}{\omega_1\omega_m\left(\omega_m + \omega_1\right)} \\ &\geq \frac{1}{16}\frac{m^2}{\omega_m^2\omega_1} \geq \frac{1}{16}\frac{1}{(1+\mu)^{3/2}} = \frac{1}{16L^3}, \end{aligned} \tag{A8.19}$$

so that we we have to exclude further a set $\mathcal{I}'$ of values of μ with measure bounded by

$$\int_{\mathcal{I}'} \mathrm{d}\mu \leq 32L^3 \sum_{n=1}^{\infty} \sum_{|m|<(\omega+1)|n|} \frac{C_0}{|n|^{\tau_0}}, \tag{A8.20}$$

which is bounded by a constant times

$$\left(1 + \mu_0\right)^2 C_0 \sum_{n=1}^{\infty} n^{-\tau_0+1}, \tag{A8.21}$$

which is proportional to C_0 if τ_0 satisfes (A8.16).

Appendix A9. Proof of lemma 9

Proof. (i) We shall prove inductively on p the bounds (7.7). From (7.4) we have

$$|\tilde{\omega}_m^{(p)}(\omega,\varepsilon) - \tilde{\omega}_m^{(p-1)}(\omega,\varepsilon)| \leq C|\nu_m(\tilde{\omega}^{(p-1)}(\omega,\varepsilon),\varepsilon) - \nu_m(\tilde{\omega}^{(p-2)}(\omega,\varepsilon),\varepsilon)|, \tag{A9.1}$$

as we can bound $|\tilde{\omega}_m^{(p)}(\omega,\varepsilon) + \tilde{\omega}_m^{(p-1)}(\omega,\varepsilon)| \geq 1$ for $\varepsilon \in \mathcal{E}^{(p)}$.

We set, for $|m| > 1$

$$\Delta\nu_{h,m} \equiv \nu_{h,m}(\tilde{\omega}^{(p-1)}(\omega,\varepsilon),\varepsilon) - \nu_{h,m}(\tilde{\omega}^{(p-2)}(\omega,\varepsilon),\varepsilon) = \lim_{q\to\infty} \Delta\nu_{h,m}^{(q)}, \tag{A9.2}$$

where we have used the notations (6.7) to define

$$\Delta\nu_{h,m}^{(q)} = \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-1)}(\omega,\varepsilon),\varepsilon) - \nu_{h,m}^{(q)}(\tilde{\omega}^{(p-2)}(\omega,\varepsilon),\varepsilon). \tag{A9.3}$$

We want to prove inductively on q the bound

$$\left|\Delta\nu_{h,m}^{(q)}\right| \le C\varepsilon^2 \|\tilde\omega^{(p-1)}(\omega,\varepsilon)-\tilde\omega^{(p-2)}(\omega,\varepsilon)\|_\infty, \tag{A9.4}$$

for some constant C, uniformly in q, h and m.
(ii) For $q=0$ the bound (A9.4) is trivially satisfied. Then assume that (A9.4) hold for all $q'<q$.

For simplicity we set $\tilde\omega=\tilde\omega^{(p-1)}(\omega,\varepsilon),\varepsilon)$ and $\tilde\omega'=\tilde\omega^{(p-2)}(\omega,\varepsilon)$. We can write, from (6.7) and (6.6), for $|m|>1$

$$\Delta\nu_{h,m}^{(q)} = -\sum_{k=h}^{\bar h-1} 2^{-k-2}\Big(\beta_{k,m}^{(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde\omega',\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega',\varepsilon)\})\Big), \tag{A9.5}$$

where we recall that $\beta_{k,m}^{(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\})$ depend only on $\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)$ with $k'\le k-1$, and we can set

$$\begin{aligned}&\beta_{k,m}^{(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\})\\ &\qquad \beta_{k,m}^{(a)(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(b)(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}),\end{aligned} \tag{A9.6}$$

according to the settings in (2.1). Then we can split the differences in (A9.5) into

$$\begin{aligned}&\beta_{k,m}^{(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde\omega',\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega',\varepsilon)\})\\ &\quad = \Big(\beta_{k,m}^{(q)}(\tilde\omega,\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde\omega',\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\})\Big)\\ &\qquad + \Big(\beta_{k,m}^{(q)}(\tilde\omega',\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde\omega',\varepsilon,\{\nu_{k'}^{(q-1)}(\tilde\omega',\varepsilon)\})\Big),\end{aligned} \tag{A9.7}$$

and we bound separately the two terms.
(iii) The second term can be expressed as sum of trees θ which differ from the previously considered ones as, among the nodes v with only one entering line, vthere are some with $\nu_{k_v}^{(c_v)(q-1)}(\omega,\varepsilon)$, some with $\nu_{k_v}^{(c_v)(q-1)}(\tilde\omega',\varepsilon)$ and one with $\nu_{k_v}^{(c_v)(q-1)}(\tilde\omega,\varepsilon) - \nu_{k_v}^{(c_v)(q-1)}(\tilde\omega',\varepsilon)$. Then, by the inductive hypothesis, we can bound

$$\begin{aligned}&\Big|\beta_{k,m}^{(q)}(\tilde\omega',\{\nu_{k'}^{(q-1)}(\tilde\omega,\varepsilon)\}) - \beta_{k,m}^{(q)}(\tilde\omega',\{\nu_{k'}^{(q-1)}(\tilde\omega',\varepsilon)\})\Big|\\ &\qquad \le D_1\varepsilon^2 \sup_{h'\ge0}\sup_{|m'|\ge1} \Delta\nu_{h',m'}^{(q-1)} \le D_1 C\varepsilon^4\,\|\tilde\omega'-\tilde\omega\|_\infty,\end{aligned} \tag{A9.8}$$

where we have used that for all q' the trees contributing to $\beta_{k,m}^{(q')}$ contain at least two end-points, so that the first nonvanishing contribution to the left hand side of (A9.8) is to second order.
(iv) We are left with the first term in (A9.7). We write $\beta_{k,m}^{(c)(q)}$ as in (5.8), with k replacing h; for each $T\in\mathcal{T}_k^{(c)}$ we can order the lines in $L(T)$ and construct a set of k_T subsets $L_1(T),\dots,L_{k_T}(T)$ of $L(T)$, with $|L_j(T)|=j$, in the following way. Set $L_1(T)=\emptyset$, $L_2(T)=\ell_1$, if ℓ_1 is a line connected to the outcoming line of T, and,

inductively for $k_T \geq 3$ and $2 \leq j \leq k_T - 1$, $L_{j+1}(T) = L_j(T) \cup \ell_j$, where the line $\ell_j \in L(T) \setminus L_j(T)$ is connected to $L_j(T)$; of course $L_{k_T}(T) = L(T)$. Then

$$\begin{aligned} \mathcal{V}_T^{k+1}(\tilde{\omega}_m, m) = \varepsilon^{k_T} \Big(\prod_{v \in V(T)} \eta_v \Big) & \qquad (A9.9) \\ \sum_{j=1}^{k_T} \Big[\Big(\prod_{\ell \in L_j(T)} g_\ell^{(h_\ell)}(\tilde{\omega}') \Big) \Big(g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) \Big) \Big(\prod_{\ell \in L(T) \setminus L_j(T)} g_\ell^{(h_\ell)}(\tilde{\omega}) \Big) \Big], \end{aligned}$$

where we have use the shortened notation $g_\ell^{(h)}(\tilde{\omega}) = g^{(h)}(\tilde{\omega} n_\ell, m_\ell)$, and, by construction, the sets $L_j(T)$ are connected (while the sets $L(T) \setminus L_j(T)$ in general are not).

In (A9.9) one has, by proceeding as in (A2.3)

$$\left| g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) \right| \leq C \left| n_{\ell_j} \right| 2^{-2h_{\ell_j}} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty , \qquad (A9.10)$$

Note that the above formula holds for any value of $\tilde{\omega}, \tilde{\omega}'$, thanks to the compact support properties of the functions χ_h in the definition of $g_{\ell_j}^{(h_{\ell_j})}$.

The main difference in bounding the other factors appearing in (A9.9), with respect to the analysis performed in Section 5, is that we cannot apply directly lemma 5, because in (A9.9) there are some propagators with $\tilde{\omega}$ and some other propagators with $\tilde{\omega}'$. However this is not a problem as we can divide the tree in the union of (properly modified, see lemma 4) subtrees, each one having all propagtors with the same kind of frequency (that is either $\tilde{\omega}$ or $\tilde{\omega}'$), and we can bound separately each of such trees.

Consider a self-energy graph T. For fixed j (see (A9.9)) call θ_0 the tree obtained in the following way. Call L_0 the set of lines containing the line ℓ_T^1 coming out from T, all the lines in $L_j(T)$ and all the $s \geq 0$ lines in L_0', where L_0' is the set of lines in $L(T) \setminus L_j(T)$ connected to $L_j(T)$. Call V_0 the set of nodes connected by lines in $L_j(T)$. We add s end-points attaching them at the beginning of the s lines in L_0'; we call E'' the set of such end-points and E' the set of end-points connected to lines in $L_j(T)$, and we set $\tilde{E} = E' \cup E''$. Then we call θ_0 the tree such that $L(\theta_0) = L_0$, $V(\theta_0) = V_0$ and $\tilde{E}(\theta_0) = \tilde{E}$; note that, by construction, such a tree has also end-points with mode labels different from $(\pm 1, \pm 1)$, and $|E_0(\theta_0)| = 1$ and $|E_1(\theta_0)| \geq 0$ (we use the notations introduced before lemma 4).

Consider also the subtrees which have as root lines the s lines in L_0, respectively. One of such trees, say θ_1', contains the node v such that the entering line of T enters v; call θ_1 the tree obtained from θ_1' by replacing the line ℓ_T^2 with a line emerging from an end-point v_0 carrying a mode label $(\overline{\omega}_m, m)$. Then we denote with $\theta_1, \ldots, \theta_s$ the s subtrees obtained in this way. For all $i = 2, \ldots, s$ one has $\tilde{E}(\theta_i) = E(\theta_i)$, while for $s = 1$ one has $\tilde{E}(\theta_i) = E(\theta_i) + E_0(\theta_i)$.

Finally associate to each end-point v_i in $E_1(\theta)$ a mode label (n_i, m_i) if (n_i, m_i) is the momentum flowing through the root line of the subtree θ_i. Moreover if v_1 is the end-point in $E_0(\theta_0)$ the corresponding mode labels are $(\overline{n} + \overline{\omega}_m, \overline{m} + m)$, with $\overline{n} = \sum_{v \in E(\theta_1)} n_v = n_1$ and $\overline{m} = \sum_{v \in E(\theta_1)} m_v = m_1$.

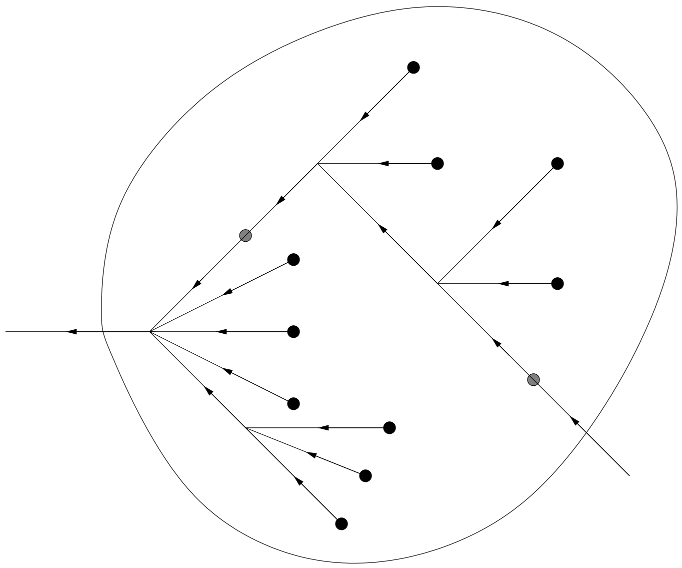

FIGURE A9.1. A self-energy graph T and the costruction of the trees θ_0, θ_1, ..., θ_s. Call $\ell^{(1)},\ldots,\ell^{(5)}$ the five lines connected to ℓ_T^1, ordered up to down. Consider the summand $j=4$ in (A9.9), and suppose that one has $\ell_j=\ell_4=\ell^{(1)}$. The set L_0 is formed by the line ℓ_T^1 coming out from T and by the lines $\ell^{(2)},\ell^{(3)}$ and $\ell^{(4)}$ in $L_4(T)$, while L_0' is formed by the lines $\ell^{(1)}$ and $\ell^{(5)}$. The trees θ_1' and θ_2 have ℓ_1 and ℓ_5 as root lines, respectively. In particular θ_1 contains the node v such that the line ℓ_T^2 enters v, and the root line of θ_1 is the line ℓ_4.

One among the trees $\theta_1,\ldots,\theta_s$ has as root line the line ℓ_j; suppose for simplicity that θ_1 is such a tree (if not the discussion proceeds in a similar way). For an example and an explicit construction see Figures A9.1 and A9.2.
(v) We can distinguish two cases. If there are no self-energy graphs T' internal to T with outcoming line in L_0 and entering line outside L_0, then one can write

$$\mathcal{V}_T^{k+1}(\tilde\omega_m,m)=\mathrm{Val}'(\theta_0)\left(g_{\ell_j}^{(h_{\ell_j})}(\tilde\omega')-g_{\ell_j}^{(h_{\ell_j})}(\tilde\omega)\right)\mathrm{Val}'(\theta_1)\Big(\prod_{i=2}^{s}\mathrm{Val}(\theta_i)\Big), \qquad (A9.11)$$

where $\mathrm{Val}'(\theta)$ is defined after (A4.8).

Note that $\mathrm{Val}(\theta_0)$, $\mathrm{Val}'(\theta_1)$ and $\mathrm{Val}(\theta_i)$, for $i=2,\ldots,s$, admit the bounds of lemma 2 and lemma 4. Therefore one has

$$\begin{aligned}&\prod_{i=2}^{s}|\mathrm{Val}(\theta_i)|\le D_0^s\prod_{i=1}^{s}D^{K(\theta_i)}|\varepsilon|^{K(\theta_i)},\\ &\big|\mathrm{Val}'(\theta_1)\big|\le D_0D^{K(\theta_1)}|\varepsilon|^{K(\theta_1)},\end{aligned} \qquad (A9.12)$$

where, for all $i=1,\ldots,s$, the mode labels n_i and m_i are given by the sum of the labels n_v and m_v, respectively, of the end-points of θ_i. The first bound in (A9.12)

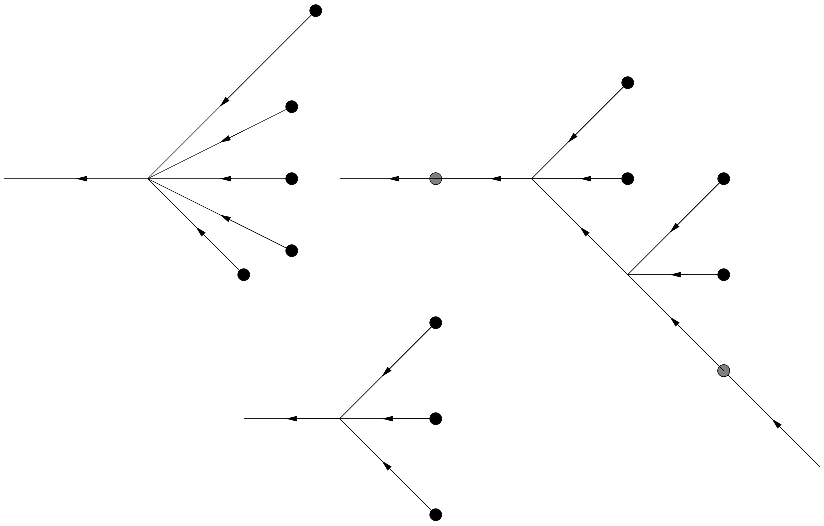

FIGURE A9.2. The trees θ_0, θ_1 and θ_2 obtained from T according to the prescription given in the text: the tree θ_1 is obtained from θ_1' by replacing the line ℓ_T^2 with a line emerging from an end-point v_0.

follows from lemma 2 (as for $i = 2, \ldots, s$ the trees θ_i have $\tilde{E}(\theta_i) = E(\theta_i)$), while the second one follows from lemma 4 and the fact that one has $\widetilde{E}(\theta_1) = E(\theta_1) + E_0(\theta_1)$ and $(\overline{n}, \overline{m}) = (0, 0)$ for the node $v_0 \in E_0(\theta_1)$, so that $\widetilde{K}(\theta_1) = K(\theta_1)$.

Analogously the bound of $\mathrm{Val}'(\theta_0)$ can be obtained from lemma 4, which gives

$$\left|\mathrm{Val}'(\theta_0)\right| \le D_0 D^{\widetilde{K}(\theta_0)} |\varepsilon|^{K(\theta_0)}, \tag{A9.13}$$

where $\widetilde{K}(\theta_0) = K(\theta_0) + |n_1| + \ldots + |n_s| \le K(\theta_0) + K(\theta_1) + \ldots + K(\theta_s)$.

We can take into account the bound (A9.10) simply by noting that the factor $2^{-2h_{\ell_j}}$ can be associated to the tree θ_1, so that in (A9.11) we can replace $\mathrm{Val}'(\theta_1)$ with a quantity which can be bounded by replacing the propagators of $\mathrm{Val}'(\theta_1)$ by their squares (of course we are using that the scales are all positive).

At the end we obtain the bound

$$\begin{aligned} &\left| \mathrm{Val}'(\theta_0) \left(g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) \right) \mathrm{Val}'(\theta_1) \Big(\prod_{i=2}^{s} \mathrm{Val}(\theta_i) \Big) \right| \\ &\qquad \le C^{K(T)} D^{2K(T)} |\varepsilon|^{K(T)} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty \\ &\le C^{K(T)} D^{2K(T)} |\varepsilon|^{K(T)/2} |\varepsilon|^{2^{(k-1)/\tau}/2} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty , \end{aligned} \tag{A9.14}$$

where $K(T)$ is defined in the statement of lemma 5, and the last inequality is obtained as explained at the end of the proof of lemma 6.

(vi) Suppose now that there are self-energy graphs with outcoming line in L_0 and entering line outside L_0; in such a case we split the renormalized part of the value of the maximal T' among such self-energy graphs (note that there cannot be two such maximal self-energy graphs) as $\mathcal{R}\mathcal{V}_{T'} = \mathcal{V}_{T'} - \mathcal{L}\mathcal{V}_{T'}$, so obtaining two contributions. Moreover if in the contribution in which $\mathcal{V}_{T'}$ was selected there is still other self-energy graphs contained in T' with outcoming line in L_0 and entering line outside L_0 again we split the maximal T'' among them as $\mathcal{R}\mathcal{V}_{T''} = \mathcal{V}_{T''} - \mathcal{L}\mathcal{V}_{T''}$ and so on. At the end we obtain a sum of contributions such that no renormalized self-energy graph can have the outcoming line in L_0 and the incoming line outside L_0. If we are left with a localized self-energy graph $\mathcal{L}\mathcal{V}_{\tilde{T}}$, with external lines one in L_0 and the other outside L_0, then it can be factorized away and the remaining part can be bounded as in item (v) above. In order to bound $\mathcal{L}V_{T'}$ we note that it has exactly the form (A9.9), *i.e.* it contains a product of propagators $g_\ell^{(h_\ell)}(\tilde{\omega}')$, a product of propagators $g_\ell^{(h_\ell)}(\tilde{\omega})$ and a difference of propagators $g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})$, so that it can bounded exactly as above.

At the end, for a fixed choice of the scales, the total number of terms so obtained is bounded by $\tilde{C}^k$, for some constant $\tilde{C}$, so that a bound $\tilde{C}^k$ times the right hand side of (A9.14) follows again. In the course of the above procedure we are loosing the factor $2^{-h_T^{(e)}+h_T}$, see (A3.1), for all the self energy graphs $T', T'', ...$ with an external line in L_0 and another outside L_0; but by definition such clusters are a sequence of clusters contained one in the other, hence we have at most an extra factor 2^h for each cluster with scale h, hence the final bound is still given by (A3.2) with $\log 4$ replacing $\log 2$.
(vii) A similar, but simpler, analysis holds in the case $m = 1$; in that case ν_1 is given by (2.7).
(viii) Now we can draw the conclusion from the construction above. We can bound $\left\| \tilde{\omega}^{(p)}(\omega, \varepsilon) - \tilde{\omega}^{(p-1)}(\omega, \varepsilon) \right\|_\infty$ with a constant times $|\varepsilon|^2$ times the same expression with p replaced with $p-1$, *i.e.* $\left\| \tilde{\omega}^{(p-1)}(\omega, \varepsilon) - \tilde{\omega}^{(p-2)}(\omega, \varepsilon) \right\|_\infty$, so that, by the inductive hypothesis, the bound (7.7) follows.

Appendix A10. Proof of lemma 10

As $\nu_m = \nu_{-m}$ we can consider only $m \geq 1$. For $m \geq 2$ one has

$$\nu_m = \nu_m^{(a)} - \nu_m^{(b)}, \qquad \nu_m^{(a)} = O(\varepsilon^2), \qquad \nu_m^{(b)} = O(\varepsilon^{\min\{2m, M-1\}}), \tag{A10.1}$$

and, if $2m > M$, a trivial computation gives

$$\nu_m^{(a)} = (-1)^{(M-1)/2} M \begin{pmatrix} M-1 \\ (M-1)/2 \end{pmatrix}^2 \left(\frac{1}{4i} \right)^{M-1} \varepsilon^{M-1} + O(\varepsilon^M), \tag{A10.2}$$

where the factor M is due to the possible choices of the line entering the self-energy graph contributing to ν_m, while the square of the binomial factor is due to the possible choices of the labels $(n_1, m_1), \ldots, (n_{M-1}, m_{M-1})$ of the two nodes internal to the self-energy graph such that $n_1 + \ldots + n_{M-1} = 0$ and $m_1 + \ldots + m_{M-1} = 0$.

For $m = 1$ one has

$$\nu_1 = (-1)^{(M-1)/2} \begin{pmatrix} M \\ (M-1)/2 \end{pmatrix}^2 \cdot \left(\frac{1}{4i} \right)^{M-1} \varepsilon^{M-1} + O(\varepsilon^M), \tag{A10.3}$$

where the square of the combinatorial factor is due to the possible choices of the labels $(n_1, m_1), \ldots, (n_M, m_M)$ of the M nodes internal to the tree such that $n_1 + \ldots + n_M = 1$ and $m_1 + \ldots + m_M = 1$.

Therefore, for $2m > M$, the lower bound on $|n\alpha_M - \beta_M|$, $n \in \mathbb{Z}$, is an easy check from the explicit expressions in (A10.2) and (A10.3), which yield

$$|n\alpha_M - \beta_M| \geq \frac{M}{4^{M-1}} \begin{pmatrix} M-1 \\ (M-1)/2 \end{pmatrix}^2 \left| n - \frac{M}{\left(\dfrac{M+1}{2}\right)^2} \right| |\varepsilon|^{M-1} , \tag{A10.4}$$

where the term in the last parantheses is greater than $1/4$ (and it equalt to $1/4$ for $M = 3$ and $n = 1$).

If $2m \leq M$ becomes a little more involved, as one has to take into account also the terms $\nu_m^{(b)}$, which has a dominant contribution given by the same expression as for $\nu_m^{(a)}$ times the factor

$$\frac{(-1)^{-m}}{\left(\dfrac{M-2}{2}+1\right)\ldots\left(\dfrac{M-2}{2}+m\right)\left(\dfrac{M-2}{2}-1\right)\ldots\left(\dfrac{M-2}{2}-m\right)}, \tag{A10.5}$$

but it is easy to realize that the same conlusions still hold (with $1/4$ replaced with $1/9$, which is the value of the difference $|n\alpha_M - \beta_M|$ computed at $M = 5$ and $n = 1$ for $m = 1$).

Appendix A11. Proof of lemma 11

In order to prove the bounds (7.15) note that

$$\left| g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - \partial_{\tilde{\omega}} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})(\tilde{\omega} - \tilde{\omega}') \right| \leq C\, n_{\ell_j}^2\, 2^{-3h_{\ell_j}} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty^2 , \tag{A11.1}$$

from the compact support properties of the propagator.

Let us consider the quantity $\nu(\tilde{\omega}', \varepsilon) - \nu(\tilde{\omega}, \varepsilon) - \partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon)\,(\omega - \omega')$, where $\partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon)$ denotes the derivative in the sense of Whitney, and note that it can be expressed as a sum over trees each one containing a line with propagator $g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}') - g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega}) - \partial_{\tilde{\omega}} g_{\ell_j}^{(h_{\ell_j})}(\tilde{\omega})\,(\tilde{\omega} - \tilde{\omega}')$, by proceeding as in the proof of lemma 9. Then we find

$$\left\| \nu(\tilde{\omega}', \varepsilon) - \nu(\tilde{\omega}, \varepsilon) - \partial_{\tilde{\omega}} \nu(\tilde{\omega}, \varepsilon)\,(\tilde{\omega} - \tilde{\omega}') \right\|_\infty \leq C_1 \varepsilon^{2(M-1)} \left\| \tilde{\omega}' - \tilde{\omega} \right\|_\infty^2 , \tag{A11.2}$$

and the first bound in (7.15).

In order to prove the first bound in (7.15) we proceed by induction by assuming that it holds for $\tilde{\omega}^{(p-1)}$; then from (7.4) we have

$$\begin{aligned} 2\tilde{\omega}^{(p)} \partial_\varepsilon \tilde{\omega}_m^{(p)}(\varepsilon, \omega) &= -\partial_\varepsilon \nu_m(\tilde{\omega}_m^{(p-1)}(\varepsilon, \omega), \varepsilon) \\ &= -\partial_{\tilde{\omega}} \nu_m(\tilde{\omega}_m^{(p-1)}(\varepsilon, \omega), \varepsilon)\, \partial_\varepsilon \tilde{\omega}_m^{(p-1)}(\varepsilon, \omega) - \partial_\varepsilon \nu_m(\tilde{\omega}_m^{(p-1)}(\varepsilon, \omega), \varepsilon), \end{aligned} \tag{A11.3}$$

hence from the inductive hypothesis and the proved bound in (7.15), we obtain

$$\left\|\tilde{\omega}^{(p)}(\omega,\varepsilon') - \tilde{\omega}^{(p)}(\omega,\varepsilon) - \partial_\varepsilon\tilde{\omega}^{(p)}(\omega,\varepsilon)\,(\varepsilon-\varepsilon')\right\|_\infty \le C|\varepsilon|^{M-3}\left|\varepsilon'-\varepsilon\right|^2, \qquad (A11.4)$$

so that also the second bound in (7.15) follows.

Appendix A12. Proof of lemma 12

First of all we check that, if we call $\varepsilon_j^{(p)}(n)$ the centers of the intervals $\mathcal{I}_j^{(p)}(n)$, with $j=1,2,3$, then one has

$$\left|\varepsilon_j^{(p+1)}(n) - \varepsilon_j^{(p)}(n)\right| \le D\varepsilon_0^p, \qquad (A12.1)$$

for a suitable constant D.

The center $\varepsilon_1^{(p)}(n)$ is defined by the condition

$$\omega_1 n - \mu_1(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)),\varepsilon_1^{(p)}(n))n - \omega_m + \mu_m(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)),\varepsilon_1^{(p)}(n)) = 0, \quad (A12.2)$$

where Whitney extensions are considered outside $\mathcal{E}^{(p-1)}$; then, by subtracting (A12.2) from the equivalent expression for $p+1$, we have

$$\begin{aligned}&\Big(\mu_1(\tilde{\omega}^{(p)}(\varepsilon_1^{(p+1)}(n)),\varepsilon_1^{(p+1)}(n)) - \mu_1(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)),\varepsilon_1^{(p)}(n))\Big)\,n \\ &\qquad - \Big(\mu_m(\tilde{\omega}^{(p)}(\varepsilon_1^{(p+1)}(n)),\varepsilon_1^{(p+1)}(n)) - \mu_m(\tilde{\omega}^{(p-1)}(\varepsilon_1^{(p)}(n)),\varepsilon_1^{(p)}(n))\Big) = 0.\end{aligned} \qquad (A12.3)$$

In (7.28) we can write, by setting for simplicity $\varepsilon=\varepsilon_1^{(p)}(n)$, $\varepsilon'=\varepsilon_1^{(p+1)}(n)$, $\tilde{\omega}'=\tilde{\omega}^{(p)}$ and $\tilde{\omega}=\tilde{\omega}^{(p-1)}$,

$$\begin{aligned}&(\mu_1(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon),\varepsilon))\,n \\ &\quad = (\mu_1(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon'),\varepsilon'))\,n + (\mu_1(\tilde{\omega}(\varepsilon'),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon),\varepsilon'))\,n \\ &\qquad + (\mu_1(\tilde{\omega}(\varepsilon),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon),\varepsilon))\,n,\end{aligned} \qquad (A12.4)$$

where we can write, from lemma 11,

$$\begin{aligned}&|(\mu_1(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon'),\varepsilon'))\,n| \le C|\varepsilon|^{2(M-1)}|n|\,\|\omega'-\omega\|_\infty\,, \\ &|(\mu_1(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon),\varepsilon'))\,n| \le C|\varepsilon|^{2(M-1)}|n|\,|\varepsilon'-\varepsilon|\,,\end{aligned} \qquad (A12.5)$$

and by lemma 10 one has

$$(\mu_1(\tilde{\omega}(\varepsilon),\varepsilon') - \mu_1(\tilde{\omega}(\varepsilon),\varepsilon))\,n = \left(n\alpha_M\,(M-1)\,\varepsilon^{m-2}\left(1+O(\varepsilon^2)\right)\right)(\varepsilon'-\varepsilon)\,. \quad (A12.6)$$

Moreover

$$\begin{aligned}&\mu_m(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_m(\tilde{\omega}(\varepsilon),\varepsilon) = \mu_m(\tilde{\omega}'(\varepsilon'),\varepsilon') - \mu_m(\tilde{\omega}(\varepsilon'),\varepsilon') \\ &\qquad + \mu_m(\tilde{\omega}(\varepsilon'),\varepsilon') - \mu_m(\tilde{\omega}(\varepsilon),\varepsilon') + (\mu_m(\tilde{\omega}(\varepsilon),\varepsilon') - \mu_m(\tilde{\omega}(\varepsilon),\varepsilon)),\end{aligned} \qquad (A12.7)$$

and, from lemma 11,

$$\begin{aligned}
&|\mu_m(\tilde\omega'(\varepsilon'),\varepsilon') - \mu_m(\tilde\omega(\varepsilon'),\varepsilon')| \le C|\varepsilon|^{2(M-1)}\,\|\omega'-\omega\|_\infty\,,\\
&|\mu_m(\tilde\omega(\varepsilon'),\varepsilon') - \mu_m(\tilde\omega(\varepsilon),\varepsilon')| \le C|\varepsilon|^{2(M-1)}|n|\,|\varepsilon'-\varepsilon|\,,
\end{aligned} \tag{A12.8}$$

and by lemma 10 one has

$$\mu_m(\tilde\omega(\varepsilon),\varepsilon') - \mu_m(\tilde\omega(\varepsilon),\varepsilon) = \left(2\beta_M\varepsilon + O(\varepsilon^2)\right)(\varepsilon-\varepsilon'). \tag{A12.9}$$

Then we get, by lemma 9,

$$\left|\varepsilon_1^{(p+1)}(n) - \varepsilon_1^{(p)}(n)\right| \le C\varepsilon_0^{p+M}, \tag{A12.10}$$

for a suitable positive constant C. This proves the bound (A12.1).

Define $p_0 \equiv p_0(n,j)$ such that

$$|\varepsilon_j^{(p_0+1)}(n) - \varepsilon_j^{(p_0)}(n)| \le C_0|n|^{-\tau}. \tag{A12.11}$$

By (A12.10) one can choose

$$p_0 = p_0(n,j) \le \text{const.}\log|n|. \tag{A12.12}$$

For all $p \le p_0$ define $\mathcal{J}_j^{(p)}(n)$ as the set of values ε such that (7.11), (7.12) and (7.13) are satisfied with C_0 replaced with $2C_0$. By the definition (A12.11) all the intervals $\mathcal{I}_j^{(p)}(n)$ fall inside the union of the intervals $\mathcal{J}_j^{(0)}(n),\ldots,\mathcal{J}_j^{(p_0)}(n)$ as soon as $p > p_0$. This means that, by calling

$$\mathcal{I} = \bigcup_{p=0}^{\infty}\bigcup_{n\in\mathbb{Z}^d}\bigcup_{j=1,2,3}\mathcal{I}_j^{(p)}(n), \tag{A12.13}$$

so that $\mathcal{E} = \mathcal{E}^{(0)}\setminus\mathcal{I}$, we can bound meas$(\mathcal{I})$ with

$$\begin{aligned}
\text{meas}(\mathcal{I}) &\le \sum_{|n|\ge\mathcal{N}_0}\sum_{j=1,2}\text{meas}(\mathcal{I}^{(p)}(n)) \le \text{const.}\sum_{|n|\ge\mathcal{N}_0}\sum_{j=1,2}\sum_{p=0}^{p_0(n,j)}\frac{2C_0}{|n|^{\tau'}}\\
&\le \sum_{n=\mathcal{N}_0} n^{-\tau'}\log n \le b\varepsilon_0^{1+\xi},
\end{aligned} \tag{A12.14}$$

with τ' verifying (7.33) and (7.46), and a constant ξ smaller than ξ' in order to take into account the logarithmic corrections due to (A12.12). Then the bound (7.48) is proved.

References

[1] D. Bambusi, *Lyapunov center theorem for some nonlinear PDE's: a simple proof*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 823–837.

[2] J. Bourgain, *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. **5** (1995), 629–639.

[3] W. Craig and C.E. Wayne, *Newton's method and periodic solutions of nonlinear wave equations*, Comm. Pure Appl. Math. **46** (1993), 1409–1501.

[4] L.H. Eliasson, *Absolutely convergent series expansions for quasi-periodic motions*, Math. Phys. Electron. J. **2** (1996), Paper 4.

[5] G. Gallavotti, *Twistless KAM tori*, Comm. Math. Phys. **164** (1994), 145–154.

[6] G. Gallavotti and G. Gentile, *Hyperbolic low-dimensional invariant tori and summation of divergent series*, Comm. Math. Phys. **227** (2002), no. 3, 421–460.

[7] G. Gentile, *Quasi-periodic solutions for two-level systems*, Preprint, 2002.

[8] G. Gentile and V. Mastropietro, *Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications*, Rev. Math. Phys. **8** (1996), 393–444.

[9] G. Gentile and V. Mastropietro, *Construction of periodic solutions of the nonlinear wave equation under strong irrationality conditions by the Linsdtedt series method*, Preprint, 2002.

[10] G. Gentile and V. Mastropietro, *Anderson localization for the Holstein model*, Comm. Math. Phys. **215** (2000), no. 1, 69–103.

[11] G. Gentile and V. Mastropietro, *Renormalization Group for one-dimensional fermions A review on mathematical results*, Phys. Rep. **352** (2001), no. 4–6, 273–437.

[12] F. Harary and E.M. Palmer, *Graphical enumeration*, Academic Press, New York-London, 1973. xiv+271 pp.

[13] S.B. Kuksin, *Nearly integrable infinite-dimensional Hamiltonian systems*, Lecture Notes in Mathematics 1556, Springer, Berlin, 1994.

[14] A. M. Lyapunov, *Problème général de la stabilité du mouvement*, Ann. Sc. Fac. Toulouse **2** (1907), 203–474.

[15] J. Pöschel, *Quasi-periodic solutions for a nonlinear wave equation*, Comment. Math. Helv. **71** (1996), no. 2, 269–296.

[16] C.E. Wayne, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys. **127** (1990), no. 3, 479–528.

[17] A. Weinstein, *Normal modes for nonlinear Hamiltonian systems*, Invent. Math. **20** (1973), 47–57.

[18] H. Whitney, *Analytic extensions of differential functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89.