Glauber dynamics of continuous particle systems

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Abstract

This paper is devoted to the construction and study of an equilibrium Glauber-type dynamics of infinite continuous particle systems. This dynamics is a special case of a spatial birth and death process. On the space Γ of all locally finite subsets (configurations) in $\mathbb{R}^d$, we fix a Gibbs measure μ corresponding to a general pair potential ϕ and activity $z>0$. We consider a Dirichlet form $\mathcal{E}$ on $L^2(\Gamma,\mu)$ which corresponds to the generator H of the Glauber dynamics. We prove the existence of a Markov process $\mathbf{M}$ on Γ that is properly associated with $\mathcal{E}$. In the case of a positive potential ϕ which satisfies $\delta:=\int_{\mathbb{R}^d}(1-e^{-\phi(x)})\,z\,dx<1$, we also prove that the generator H has a spectral gap $\geq 1-\delta$. Furthermore, for any pure Gibbs state μ, we derive a Poincaré inequality. The results about the spectral gap and the Poincaré inequality are a generalization and a refinement of a recent result from [6].

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1 Introduction

This paper is devoted to the construction and study of an equilibrium Glauber-type dynamics (GD) of infinite continuous particle systems. This dynamics is a special case of a spatial birth and death process on $\mathbb{R}^d$. For a system of particles in a bounded volume, such processes were introduced and studied by C. Preston in [19]. In the latter case, the total number of particles is finite at any moment of time.

In the recent paper by Bertini *et al.*, [6], the generator of the GD in a finite volume was studied. This generator corresponds to a special case of birth and death coefficients in Preston's dynamics. A positive, finite range, pair potential ϕ and an activity $z>0$ were fixed which satisfy the condition of the low activity-high temperature regime. Then, with any finite volume $\Lambda\subset\mathbb{R}^d$ and a boundary condition η outside Λ, one may associate a finite volume Gibbs measure $\mu_{\Lambda,\eta}$. A non-local Dirichlet form $\mathcal{E}_{\Lambda,\eta}$ on $L^2(\mu_{\Lambda,\eta})$ was considered which corresponds to the generator of the GD on Λ. It was shown that the generator $H_{\Lambda,\eta}$ of

$\mathcal{E}_{\Lambda,\eta}$ has a spectral gap which is uniformly positive with respect to all finite volumes Λ and boundary conditions η.

In this paper, we discuss the GD in the infinite volume. The problem of construction of a spatial birth and death process in the infinite volume was initiated in paper [8], where it was solved in a very special case of nearest neighbor birth and death processes on the real line.

So, we consider the space Γ of all locally finite subsets (configurations) in $\mathbb{R}^d$, and a grand canonical Gibbs measure μ on Γ which corresponds to a pair potential ϕ and activity $z > 0$. The measure μ is supposed to be either of the Ruelle type or corresponding to a positive potential ϕ satisfying the integrability condition. In Section 2, we shortly recall some facts about Gibbs measures which we use later on.

In Section 3, we consider the following bilinear form on $L^2(\Gamma, \mu)$ which is defined on a proper set of cylinder functions:

$$\mathcal{E}(F, G) = \int_\Gamma \sum_{x\in\gamma} (F(\gamma \setminus x) - F(\gamma))\,(G(\gamma \setminus x) - G(\gamma))\,\mu(d\gamma) \tag{1.1}$$

(here and below, for simplicity of notations we will just write x instead of $\{x\}$ for any $x \in \mathbb{R}^d$). We prove that this form is closable and its closure is a Dirichlet form. By using the general theory of Dirichlet forms (cf. [14]), we prove that there exists a Hunt process $\mathbf{M}$ on Γ properly associated with $\mathcal{E}$. In particular, $\mathbf{M}$ is a conservative Markov process on Γ with *cadlag* paths. By construction, $\mathbf{M}$ is an equilibrium GD on Γ with the stationary measure μ. Let us mention that the birth and death coefficients were supposed to be bounded in [8], which is not the case for the GD, provided the potential ϕ has a negative part.

In the case where the interaction between the particles is absent (i.e., $\phi = 0$ and, therefore, μ is the Poisson measure π_z with intensity z), the Markov process corresponding to the Dirichlet form (1.1) was explicitly constructed and studied by D. Surgailis [24, 25].

In Sections 4 and 5, we only consider the case of a positive potential ϕ and study the problem of the spectral gap for the generator H of the Dirichlet form $\mathcal{E}$.

Let us recall that the Poisson measure π_z possesses the chaos decomposition property, and hence the space $L^2(\Gamma, \pi_z)$ is unitarily isomorphic to the symmetric Fock space over $L^2(\mathbb{R}^d, z\,dx)$, see e.g. [24]. Under this isomorphism, the operator H goes over into the number operator N in the Fock space, see [1, Theorem 5.1]. Evidently, N (and thus H) has spectral gap 1. Therefore, one may expect that, at least in the case of a “small perturbation” of the Poisson measure, the operator H still has a spectral gap.

One way to prove the existence of a spectral gap of a generator H_E of a Dirichlet form E is to derive a coercivity identity for H_E on a class $\mathcal{C}$ of “nice functions,” and using it, to show that, for each $F \in \mathcal{C}$, $\|H_E F\|^2 \geq G(H_E F, F)$ with $G > 0$. If one additionally knows that the operator H_E is essentially selfadjoint on $\mathcal{C}$, the latter estimate implies that H_E has a spectral gap $\geq G$. In the case of a probability measure defined on a Hilbert space, this approach was developed in [10], see also [4, Ch. 6, Sect. 4].

So, having in mind this idea, we first prove in Section 4 that the operator H is essentially selfadjoint. This is technically the most difficult part of the paper. Then, in Section 5, we prove a coercivity identity for the operator H on cylinder functions, and using it and the essential selfadjointness of H, we show that the set $(0, 1-\delta)$ does not belong to the spectrum of H, provided that $\delta := \int_{\mathbb{R}^d} (1 - e^{-\phi(x)})\, z\, dx < 1$. This statement leads us to the Poincaré inequality if we are able to show that zero is a nondegenerate eigenvalue of H. We prove the latter statement for any μ that is an extreme point in the set of all Gibbs measures corresponding to ϕ and z. In the low activity-high temperature regime, the latter set consists of exactly one point, which is therefore extreme.

Thus, compared with the result of [6], the progress achieved in the study of the spectral gap is as follows:

1. We work in the whole space $\mathbb{R}^d$, instead of taking finite volumes Λ in $\mathbb{R}^d$ and boundary conditions η;

2. We do not suppose that the potential ϕ has a finite range;

3. The essential selfadjointness of H is proven;

4. For $\delta < 1$, an explicit estimate for the value of the spectral gap of H is found, and a Poincaré inequality is proven for any pure Gibbs state.

In a forthcoming paper, we are going to discuss the existence problem for general birth and death processes on configuration spaces and study a scaling limit of these processes.

2 Gibbs measures on configuration spaces

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite:

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \mid |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where ε_x is the Dirac measure with mass at x, $\sum_{x\in\varnothing} \varepsilon_x :=$zero measure, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps $\Gamma \ni \gamma \mapsto \langle f, \gamma\rangle := \int_{\mathbb{R}^d} f(x)\, \gamma(dx) = \sum_{x\in\gamma} f(x)$, $f \in \mathcal{D}$, are continuous. Here, $\mathcal{D} := C_0(\mathbb{R}^d)$ is the space of all continuous real-valued functions on $\mathbb{R}^d$ with compact support. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ.

Now, we proceed to consider Gibbs measures on Γ. A pair potential is a Borel measurable function $\phi \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. A grand canonical Gibbs measure μ (or just Gibbs measure for short) corresponding to the pair potential ϕ and activity $z > 0$ is usually defined through the Dobrushin–Lanford–Ruelle equation, see e.g. [22]. However, it is convenient for us to give an equivalent definition through the Georgii–Nguyen–Zessin identity ([18, Theorem 2], see also [12, Theorem 2.2.4]).

For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$, we define a relative energy of interaction between a particle located at x and the configuration γ as follows:

$$E(x, \gamma) := \begin{cases} \sum_{y\in\gamma} \phi(x - y), & \text{if } \sum_{y\in\gamma} |\phi(x - y)| < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a Gibbs measure if it satisfies

$$\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, F(\gamma, x) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx\, \exp\left[-E(x,\gamma)\right] F(\gamma \cup x, x) \tag{2.1}$$

for any measurable function $F : \Gamma \times \mathbb{R}^d \to [0, +\infty]$. (Notice that any fixed set $\gamma \in \Gamma$ has zero Lebesgue measure, so that the expression $E(x,\gamma)$ on the right hand side of (2.1) is a.s. well-defined.) Let $\mathcal{G}(z,\phi)$ denote the set of all Gibbs measures corresponding to z and ϕ. In particular, if $\phi \equiv 0$, then (2.1) is the Mecke identity, which holds if and only if μ is the Poisson measure π_z with intensity measure $z\,dx$.

Let us now describe some classes of Gibbs measures which appear in classical statistical mechanics of continuous systems. For every $r = (r^1, \dots, r^d) \in \mathbb{Z}^d$, we define a cube

$$Q_r := \left\{ x \in \mathbb{R}^d \mid r^i - \frac{1}{2} \le x^i < r^i + \frac{1}{2} \right\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$, we set $\gamma_r := \gamma_{Q_r}$, $r \in \mathbb{Z}^d$. For $N \in \mathbb{N}$ let Λ_N be the cube with side length $2N-1$ centered at the origin in $\mathbb{R}^d$, Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r.

For $\Lambda \subset \mathbb{R}^d$, we denote $\Gamma_\Lambda := \{\gamma \in \Gamma \mid \gamma \subset \Lambda\}$. Now, we recall some standard conditions on ϕ.

(SS) (*Superstability*) There exist $A > 0$, $B \ge 0$ such that, if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$\sum_{\{x,y\} \subset \gamma} \phi(x-y) \ge \sum_{r \in \mathbb{Z}^d} \left(A|\gamma_r|^2 - B|\gamma_r| \right).$$

Notice that the superstability condition automatically implies that the potential ϕ is semi-bounded from below.

(LR) (*Lower regularity*) There exists a decreasing positive function $a\colon \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\|r\|) < \infty$$

and for any Λ', Λ'' which are finite unions of cubes Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$,

$$\sum_{x \in \gamma',\, y \in \gamma''} \phi(x-y) \ge -\sum_{r', r'' \in \mathbb{Z}^d} a(\|r' - r''\|) |\gamma'_{r'}|\, |\gamma''_{r''}|.$$

Here, $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

(I) (*Integrability*)

$$\int_{\mathbb{R}^d} |1 - \exp[-\phi(x)]|\, dx < +\infty.$$

For results related to spectral properties of the generator of the GD, we will need the following condition.

(P) (*Positivity*) $\phi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by

$$S_\infty := \bigcup_{n=1}^{\infty} S_n,$$

where

$$S_n := \left\{ \gamma \in \Gamma \mid \forall N \in \mathbb{N} \sum_{r \in \Lambda_N \cap \mathbb{Z}^d} |\gamma_r|^2 \leq n^2 |\Lambda_N \cap \mathbb{Z}^d| \right\}.$$

By $\mathcal{G}^t(z,\phi) \subset \mathcal{G}(z,\phi)$ we denote the set of all tempered grand canonical Gibbs measures (Ruelle measures for short). Due to [23] the set $\mathcal{G}^t(z,\phi)$ is non-empty for all $z > 0$ and any potential ϕ satisfying conditions (SS), (LR), and (I). Furthermore, the set $\mathcal{G}(z,\phi)$ is non-empty for all $z > 0$ and any potential ϕ satisfying (P) and (I), see [12, Proposition 2.7.15].

Let us now recall the so-called Ruelle bound (cf. [23]).

Proposition 2.1 *Suppose that either conditions* (I), (SS), (LR) *are satisfied and* $\mu \in \mathcal{G}^t(z,\phi)$, $z > 0$, *or conditions* (P), (I) *are satisfied and* $\mu \in \mathcal{G}(z,\phi)$, $z > 0$. *Then, for any* $n \in \mathbb{N}$, *there exists a non-negative measurable symmetric function* $k_\mu^{(n)}$ *on* $(\mathbb{R}^d)^n$ *such that, for any measurable symmetric function* $f^{(n)} : (\mathbb{R}^d)^n \to [0,\infty]$,

$$\int_\Gamma \sum_{\{x_1,\dots,x_n\} \subset \gamma} f^{(n)}(x_1,\dots,x_n)\, \mu(d\gamma) \\ = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} f^{(n)}(x_1,\dots,x_n) k_\mu^{(n)}(x_1,\dots,x_n)\, dx_1 \cdots dx_n, \tag{2.2}$$

and

$$\forall (x_1,\dots,x_n) \in (\mathbb{R}^d)^n : \quad k_\mu^{(n)}(x_1,\dots,x_n) \leq \xi^n, \tag{2.3}$$

where $\xi > 0$ *is independent of* n.

The functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, are called correlation functions of the measure μ, while (2.3) is called the Ruelle bound.

Notice that any measure $\mu \in \mathcal{G}(z,\phi)$ as in Proposition 2.1 satisfies

$$\int_\Gamma \langle \varphi, \gamma \rangle^n \, \mu(d\gamma) < \infty, \qquad \varphi \in \mathcal{D},\ \varphi \geq 0, n \in \mathbb{N}. \tag{2.4}$$

that is, μ has all local moments finite.

3 The Dirichlet form $\mathcal{E}$ and associated Markov process

We introduce a set $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ of all functions on Γ of the form

$$F(\gamma)=g_F(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle), \tag{3.1}$$

where $N\in\mathbb{N}$, $\varphi_1,\dots,\varphi_N\in\mathcal{D}$, and $g_F\in C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$. Here, $C_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^N$ which are bounded together with all their derivatives. For any $\gamma\in\Gamma$, we consider $T_\gamma{:=}L^2(\mathbb{R}^d,\gamma)$ as a "tangent" space to Γ at the point γ, and for any $F\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ we define the "gradient" of F at γ as the element of T_γ given by $D^-F(\gamma,x){:=}D_x^-F(\gamma){:=}F(\gamma\setminus x)-F(\gamma)$, $x\in\mathbb{R}^d$. (Evidently, $D^-F(\gamma)$ indeed belongs to T_γ.)

Let μ be a Gibbs measure as in Proposition 2.1. We will preserve the notation $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ for the set of all μ-classes of functions from $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$. The set $\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ is dense in $L^2(\Gamma,\mu)$. We now define

$$\begin{aligned}\mathcal{E}(F,G){:=}&\int_\Gamma (D^-F(\gamma),D^-G(\gamma))_{T_\gamma}\,\mu(d\gamma)\\ =&\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}\gamma(dx)\,D_x^-F(\gamma)\,D_x^-G(\gamma),\qquad F,G\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma).\end{aligned} \tag{3.2}$$

Notice that, for any $F\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$, there exists $f\in\mathcal{D}$ such that $|D_x^-F(\gamma)|\le f(x)$ for all $\gamma\in\Gamma$ and $x\in\gamma$. Hence, by (2.4), the right hand side of (3.2) is well defined. By (2.1), we also get, for $F,G\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$,

$$\mathcal{E}(F,G)=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}z\,dx\,\exp\left[-E(x,\gamma)\right]D_x^+F(\gamma)\,D_x^+G(\gamma), \tag{3.3}$$

where $D_x^+F(\gamma){:=}F(\gamma)-F(\gamma\cup x)$.

Lemma 3.1 *We have: $\mathcal{E}(F,G)=0$ for all $F,G\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$ such that $F=0$ μ-a.e.*

Proof. Let $F\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma)$, $F=0$ μ-a.e. Denote $B(r){:=}\{x\in\mathbb{R}^d:|x|<r\}$. Then, by (2.1),

$$\begin{aligned}0=&\int_\Gamma\mu(d\gamma)\int_{B(r)}\gamma(dx)\,|F(\gamma)|\\ =&\int_\Gamma\mu(d\gamma)\int_{B(r)}z\,dx\,\exp\left[-E(x,\gamma)\right]|F(\gamma\cup x)|.\end{aligned}$$

Hence, $F(\gamma\cup x)=0$ for $\mu(d\gamma)\,z\,dx\,\exp\left[-E(x,\gamma)\right]$-a.e. $(\gamma,x)\in\Gamma\times\mathbb{R}^d$. Therefore, by (3.3), the lemma follows. $\square$

Thus, $(\mathcal{E},\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma))$ is a well-defined bilinear form on $L^2(\Gamma,\mu)$.

Proposition 3.1 *We have:*

$$\mathcal{E}(F,G)=\int_\Gamma HF(\gamma)G(\gamma)\,\mu(d\gamma),\qquad F,G\in\mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma), \tag{3.4}$$

where

$$HF(\gamma) = \int_{\mathbb{R}^d} z\,dx\,\exp\left[-E(x,\gamma)\right] D_x^+ F(\gamma) - \int_{\mathbb{R}^d} \gamma(dx)\, D_x^- F(\gamma) \tag{3.5}$$

and $HF \in L^2(\Gamma,\mu)$. *The bilinear form* $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is closable on* $L^2(\Gamma,\mu)$ *and its closure will be denoted by* $(\mathcal{E}, D(\mathcal{E}))$. *The operator* $(H, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *in* $L^2(\Gamma,\mu)$ *has Friedrichs' extension, which we denote by* $(H, D(H))$.

Proof. Equations (3.4), (3.5) easily follow from (2.1) and (3.2). Let us show that $HF \in L^2(\Gamma,\mu)$. By (2.4), the inclusion $\int_{\mathbb{R}^d} \gamma(dx)\, D_x^- F(\gamma) \in L^2(\Gamma,\mu)$ is trivial. Next, we can find a compact $\Lambda \subset \mathbb{R}^d$ and $C_1 > 0$ such that $|D_x^+ F(\gamma)| \le C_1 \chi_\Lambda(x)$ for all $\gamma \in \Gamma$ and $x \in \mathbb{R}^d$. Here, χ_Λ denotes the indicator of Λ. Hence, by (2.1) and (2.2),

$$\begin{aligned}
&\int_\Gamma \left(\int_{\mathbb{R}^d} z\,dx\,\exp\left[-E(x,\gamma)\right] D_x^+ F(\gamma) \right)^2 \mu(d\gamma) \\
&\quad \le C_1^2 \int_\Gamma \mu(d\gamma) \int_\Lambda z\,dx \int_\Lambda z\,dy\,\exp\left[-E(x,\gamma) - E(y,\gamma) - \phi(x-y)\right] \exp[\phi(x-y)] \\
&\quad = 2C_1^2 \int_\Gamma \sum_{\{x,y\} \in \gamma_\Lambda} \exp[\phi(x-y)]\, \mu(d\gamma) \\
&\quad = C_1^2 \int_{\Lambda^2} \exp[\phi(x-y)] k_\mu^{(2)}(x,y)\,dx\,dy < \infty,
\end{aligned} \tag{3.6}$$

since $k_\mu^{(2)}(x,y) \le C_2 \exp[-\phi(x-y)]$ for all $x, y \in \mathbb{R}^d$, $C_2 > 0$, cf. [2, Eq. (4.29)]. Therefore, H is the $L^2(\Gamma,\mu)$-generator of the bilinear form $\mathcal{E}$. The rest of the proposition now follows from e.g. [20, Theorem X.23]. $\square$

For the notion of a "Dirichlet form," appearing in the following lemma, we refer to e.g. [14, Chap. I, Sect. 4].

Lemma 3.2 $(\mathcal{E}, D(\mathcal{E}))$ *is a Dirichlet form on* $L^2(\Gamma,\mu)$.

Proof. On $D(\mathcal{E})$ consider the norm $\|F\|_{D(\mathcal{E})} := \left(\|F\|^2_{L^2(\mu)} + \mathcal{E}(F)\right)^{1/2}$, $F \in D(\mathcal{E})$. Here, we denoted $\mathcal{E}(F) := \mathcal{E}(F,F)$. For any $F, G \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, we define

$$S(F,G)(x,\gamma) := D_x^- F(\gamma)\, D_x^- G(\gamma), \qquad x \in \mathbb{R}^d,\ \gamma \in \Gamma.$$

Using the Cauchy–Schwarz inequality, we conclude that S extends to a bilinear continuous map from $(D(\mathcal{E}), \|\cdot\|_{D(\mathcal{E})}) \times (D(\mathcal{E}), \|\cdot\|_{D(\mathcal{E})})$ into $L^1(\mathbb{R}^d \times \Gamma, \tilde{\mu})$, where $\tilde{\mu}(dx, d\gamma) := \gamma(dx)\mu(d\gamma)$. Let $F \in D(\mathcal{E})$ and consider any sequence $F_n \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$, $n \in \mathbb{N}$, such that $F_n \to F$ in $(D(\mathcal{E}), \|\cdot\|_{D(\mathcal{E})})$ as $n \to \infty$. Then, $F_n(\gamma) \to F(\gamma)$ as $n \to \infty$ for μ-a.e. $\gamma \in \Gamma$ (if necessary, take a subsequence of $(F_n)_{n \in \mathbb{N}}$ with this property). Furthermore, for any $r > 0$, we have, analogously to (3.6):

$$\int_{B(r)\times\Gamma} |F_n(\gamma \setminus x) - F(\gamma \setminus x)|\, \tilde{\mu}(dx, d\gamma)$$

$$
\begin{aligned}
&= \int_\Gamma \mu(d\gamma) \int_{B(r)} z\, dx\, \exp\left[-E(x,\gamma)\right] |F_n(\gamma) - F(\gamma)| \\
&\le \left(\int_\Gamma |F_n(\gamma) - F(\gamma)|^2 \,\mu(d\gamma)\right)^{1/2} \\
&\quad \times \left(\int_\Gamma \left(\int_{B(r)} z\, dx\, \exp\left[-E(x,\gamma)\right]\right)^2 \mu(d\gamma)\right)^{1/2} \to 0 \qquad (3.7)
\end{aligned}
$$

as $n \to \infty$. Therefore, $F_n(\gamma \setminus x) \to F(\gamma \setminus x)$ for $\tilde{\mu}$-a.e. $(x,\gamma) \in \mathbb{R}^d \times \Gamma$. Thus, $D_x^- F_n(\gamma) \to D_x^- F(\gamma)$ as $n \to \infty$ for $\tilde{\mu}$-a.e. $(x,\gamma) \in \mathbb{R}^d \times \Gamma$, which yields:

$$
S(F,G)(x,\gamma) = D_x^- F(\gamma)\, D_x^- G(\gamma), \qquad \tilde{\mu}\text{-a.e. } (x,\gamma) \in \mathbb{R}^d \times \Gamma,\ F,G \in D(\mathcal{E}). \qquad (3.8)
$$

Hence,

$$
\mathcal{E}(F,G) = \int_{\mathbb{R}^d \times \Gamma} D_x^- F(\gamma) D_x^- G(\gamma)\, \tilde{\mu}(dx, d\gamma), \qquad F,G \in D(\mathcal{E}). \qquad (3.9)
$$

Define $\mathbb{R} \ni x \mapsto g(x) := (0 \vee x) \wedge 1$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions from $C_{\mathrm{b}}^\infty(\mathbb{R})$ such that, for all $x \in \mathbb{R}$: $0 \le g_n(x) \le 1$, $0 \le g_n'(x) \le 2$, $g_n(x) \to g(x)$ as $n \to \infty$. We again fix any $F \in D(\mathcal{E})$ and and let $(F_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ such that $F_n \to F$ in $(D(\mathcal{E}), \|\cdot\|_{D(\mathcal{E})})$. Consider the sequence $(g_n(F_n))_{n\in\mathbb{N}}$. We evidently have: $g_n(F_n) \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ for each $n \in \mathbb{N}$ and $g_n(F_n) \to g(F)$ as $n \to \infty$ in $L^2(\Gamma,\mu)$. Next, by the above argument, we have: $D_x^- g_n(F_n(\gamma)) \to D_x^- g(F(\gamma))$ as $n \to \infty$ for $\tilde{\mu}$-a.e. (x,γ). Furthermore, the sequence $(D_x^- g_n(F_n(\gamma)))_{n\in\mathbb{N}}$ is $\tilde{\mu}$-uniformly square-integrable, since so is the sequence $(D_x^- F_n(\gamma))_{n\in\mathbb{N}}$. Therefore, the sequence $(D_x^- g_n(F_n(\gamma)))_{n\in\mathbb{N}}$ converges to $D_x^- g(F(\gamma))$ in $L^2(\mathbb{R}^d \times \Gamma, \tilde{\mu})$. This yields: $g(F) \in D(\mathcal{E})$.

For any $x,y \in \mathbb{R}$, we evidently have $|g(x) - g(y)| \le |x-y|$. By (3.9), we then finally have: $\mathcal{E}(g(F)) \le \mathcal{E}(F)$, which means that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form. $\square$

We will now need the bigger space $\ddot{\Gamma}$ consisting of all $\mathbb{Z}_+$-valued Radon measures on $\mathbb{R}^d$ (which is Polish, see e.g. [9]). Since $\Gamma \subset \ddot{\Gamma}$ and $\mathcal{B}(\ddot{\Gamma}) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}, \mathcal{B}(\ddot{\Gamma}))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma},\mu)$.

For the notion of a"quasi-regular Dirichlet form," appearing in the following lemma, we refer to [14, Chap. IV, Sect. 3].

Proposition 3.2 *$(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(\ddot{\Gamma},\mu)$.*

Proof. By [15, Proposition 4.1], it suffices to show that there exists a bounded, complete metric ρ on $\ddot{\Gamma}$ generating the vague topology such that, for all $\gamma_0 \in \ddot{\Gamma}$, $\rho(\cdot,\gamma_0) \in D(\mathcal{E})$ and $\int_{\mathbb{R}^d} S(\rho(\cdot,\gamma_0))(x,\gamma)\,\gamma(dx) \le \eta(\gamma)$ μ-a.e. for some $\eta \in L^1(\ddot{\Gamma},\mu)$ (independent of γ_0). Here, $S(F) := S(F,F)$.

Let us recall some well-known facts about cylinder functions on the configuration space (see e.g. [11] for details). Let $\mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$ denote the set of all open, relatively compact sets in $\mathbb{R}^d$. For $\Lambda \in \mathcal{O}_{\mathrm{c}}(\mathbb{R}^d)$, we have: $\Gamma_\Lambda = \bigsqcup_{n=0}^\infty \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)}$ denotes the set of all n-point subsets of Λ, $n \in \mathbb{N}$, and $\Gamma_\Lambda^{(0)} = \{\varnothing\}$. For $n \in \mathbb{N}$, we can naturally identify $\Gamma_\Lambda^{(n)}$ with

$\tilde{\Lambda}^n/S_n$, where $\tilde{\Lambda}^n{:=}\{(x_1,\dots,x_n)\in\Lambda^n : x_i\ne x_j \text{ if } i\ne j\}$ and S_n denotes the group of permutations of $\{1,\dots,n\}$ that acts on $\tilde{\Lambda}^n$ by permuting the numbers of the coordinates. Furthermore, the trace σ-algebra of $\mathcal{B}(\Gamma)$ on $\Gamma_\Lambda^{(n)}$ coincides with the σ-algebra $\mathcal{B}_{\mathrm{sym}}(\tilde{\Lambda}^n)$ of all symmetric Borel subsets of $\tilde{\Lambda}^n$ (again under a natural isomorphism). Finally, any measurable function F_Λ on Γ_Λ may be identified with a measurable cylinder function F on Γ by setting $\Gamma\ni\gamma\mapsto F(\gamma){:=}F_\Lambda(\gamma_\Lambda)$.

Lemma 3.3 *Let $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$. Any measurable bounded function F on Γ_Λ such that $F\restriction\Gamma_\Lambda^{(n)}\equiv 0$ for all $n\ge N$, $N\in\mathbb{N}$, belongs to $D(\mathcal{E})$.*

Proof. We take arbitrary, open, disjoint subsets $O_1,\dots,O_n$ of Λ. Consider functions $g_1,g_2\in C_{\mathrm{b}}^\infty(\mathbb{R})$ such that $g_1(1)=1$, $g_2(0)=1$ and $g_1(x)=0$ if $|x-1|>1/2$, $g_2(x)=0$ if $|x|>1/2$. Approximating the indicator functions χ_{O_i}, $i=1,\dots,n$, and $\chi_{\Lambda\setminus(O_1\cup\dots\cup O_n)}$ by functions from $\mathcal{D}$, we easily conclude that the statement of the lemma holds for the function

$$\begin{aligned}F(\gamma)&=g_1(\langle\chi_{O_1},\gamma\rangle)\cdots g_1(\langle\chi_{O_n},\gamma\rangle)g_2(\langle\chi_{\Lambda\setminus(O_1\cup\dots\cup O_n)},\gamma\rangle)\\&=\chi_{\{\,|\gamma_{O_1}|=1,\dots,|\gamma_{O_n}|=1,\,|\gamma_{\Lambda\setminus(O_1\cup\dots\cup O_n)}|=0\,\}}(\gamma).\end{aligned}$$

Hence, $F(\gamma)$ may be identified with the indicator function $\chi_{S_n(O_1\times\dots\times O_n)}(x_1,\dots,x_n)$ on $\tilde{\Lambda}^n/S_n$. Using a monotone class argument, we get the statement for any indicator function, and then for any measurable bounded function on $\Gamma_\Lambda^{(n)}$. $\square$

Lemma 3.4 *Let $f:\mathbb{R}^d\to\mathbb{R}$ be measurable, bounded, and with compact support. Let $\zeta\in C_{\mathrm{b}}^\infty(\mathbb{R})$ and $a\in\mathbb{R}$. Then, $\zeta(|\langle f,\cdot\rangle-a|)\in D(\mathcal{E})$.*

Proof. Let $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ be such that $\operatorname{supp} f\subset\Lambda$. Define

$$F_n(\gamma){:=}\zeta(|\langle f,\gamma\rangle-a|)\,\chi_{\{|\gamma_\Lambda|\le n\}}(\gamma),\qquad \gamma\in\Gamma,\ n\in\mathbb{N}.$$

By Lemma 3.3, $F_n\in D(\mathcal{E})$, $n\in\mathbb{N}$. We evidently have: $F_n\to F$ in $L^2(\ddot{\Gamma},\mu)$. Furthermore, using Proposition 2.1 and the majorized convergence theorem, we get: $\mathcal{E}(F_n-F_m)\to 0$ as $n,m\to\infty$. From here the statement follows. $\square$

The rest of the proof of Proposition 3.2 is quite analogous to the proof of [15, Proposition 4.8]. So, we only outline the main changes needed.

Let $E_k{:=}B(k)\subset\mathbb{R}^d$, $\delta_k=1/2$, $\gamma_0\in\ddot{\Gamma}$ and let g_{E_k,δ_k}, ϕ_k, and ζ be defined as in [15], and we additionally demand that $\zeta'(x)\in[0,1]$ on $[0,\infty)$. Since $\phi_k g_{E_k,\delta_k}$ is bounded and has a compact support, we have by Lemma 3.4:

$$\zeta(|\langle\phi_k g_{E_k,\delta_k},\cdot\rangle-\langle\phi_k g_{E_k,\delta_k},\gamma_0\rangle|)\in D(\mathcal{E}).$$

(Notice that $\langle\phi_k g_{E_k,\delta_k},\gamma_0\rangle$ is a constant.) Furthermore, taking to notice that $\zeta'(x)\in[0,1]$, we get from (3.8) and the mean value theorem:

$$S(\zeta(|\langle\phi_k g_{E_k,\delta_k},\cdot\rangle-\langle\phi_k g_{E_k,\delta_k},\gamma_0\rangle|)(x,\gamma)\le(\phi_k g_{E_k,\delta_k})^2(x)\le\chi_{B(k+1/2)}(x).\tag{3.10}$$

Set

$$c_k := \left(1 + \int_{B(k+1/2)} k_\mu^{(1)}(x)\,dx\right)^{-1/2} 2^{-k/2}.$$

Using estimate (3.10) and the numbers c_k, we now easily obtain the statement of the proposition absolutely analogously to the proof of [15, Lemma 4.11 and Proposition 4.8]. $\square$

For the notion of an "$\mathcal{E}$-exceptional set," appearing in the next proposition, we refer to e.g. [14, Chap. III, Sect. 2].

Proposition 3.3 *The set* $\ddot{\Gamma} \setminus \Gamma$ *is* $\mathcal{E}$*-exceptional.*

Proof. We modify the proof of [21, Proposition 1 and Corollary 1] according to our situation.

It suffices to prove the result locally, that is, to show that, for every fixed $a \in \mathbb{N}$, the set

$$N := \left\{ \gamma \in \ddot{\Gamma} : \sup(\gamma(\{x\}) : x \in [-a, a]^d) \geq 2 \right\}$$

is $\mathcal{E}$-exceptional. By [21, Lemma 1], we need to prove that there exists a sequence $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, such that each u_n is a continuous function on $\ddot{\Gamma}$, $u_n \to \mathbf{1}_N$ pointwise as $n \to \infty$, and $\sup_{n\in\mathbb{N}} \mathcal{E}(u_n) < \infty$.

Let $f \in C_0(\mathbb{R})$ be such that $\mathbf{1}_{[0,1]} \leq f \leq \mathbf{1}_{[-1/2,3/2)}$. For any $n \in \mathbb{N}$ and $i = (i_1, \dots, i_d) \in \mathbb{Z}^d$, define a function $f_i^{(n)} \in \mathcal{D}$ by

$$f_i^{(n)}(x) := \prod_{k=1}^{d} f(nx_k - i_k), \qquad x \in \mathbb{R}^d.$$

Let also

$$I_i^{(n)}(x) := \prod_{k=1}^{d} \mathbf{1}_{[-1/2,3/2)}(nx_k - i_k), \qquad x \in \mathbb{R}^d,$$

and note that $f_i^{(n)} \leq I_i^{(n)}$.

Let $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R})$ be such that $\mathbf{1}_{[2,\infty)} \leq \psi \leq \mathbf{1}_{[1,\infty)}$ and $0 \leq \psi' \leq 2\,\mathbf{1}_{(1,\infty)}$. Set $\mathcal{A}_n := \mathbb{Z}^d \cap [-na, na]^d$ and define continuous functions

$$\ddot{\Gamma} \ni \gamma \mapsto u_n(\gamma) := \psi\left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle\right), \qquad n \in \mathbb{N}.$$

Evidently, $u_n \to \mathbf{1}_N$ pointwise as $n \to \infty$. Furthermore, by an appropriate approximation of the function $\mathbb{R}^{|\mathcal{A}_n|} \ni (y_1, \dots, y_{|\mathcal{A}_n|}) \mapsto \sup_{i=1,\dots,|\mathcal{A}_n|} y_i$ by $C_{\mathrm{b}}^\infty(\mathbb{R}^{|\mathcal{A}_n|})$ functions, we conclude that, for each $n \in \mathbb{N}$, $u_n \in D(\mathcal{E})$. We now have:

$$S(u_n)(x, \gamma) = \left(\psi\left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma - \varepsilon_x\rangle\right) - \psi\left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle\right)\right)^2 \quad \tilde{\mu}\text{-a.e.}$$

By the mean value theorem, we get, for $\tilde{\mu}$-a.e. $(x, \gamma) \in \mathbb{R}^d \times \ddot{\Gamma}$,:

$$S(u_n)(x, \gamma) = \psi'(T_n(\gamma, x))^2 \left(\sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma - \varepsilon_x\rangle - \sup_{i\in\mathcal{A}_n} \langle f_i^{(n)}, \gamma\rangle\right)^2, \tag{3.11}$$

where $T_n(\gamma,x)\in\mathbb{R}$ is a point between $\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma-\varepsilon_x\rangle$ and $\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma\rangle$. It is easy to see that the following estimate holds, for any $\gamma\in\ddot\Gamma$ and $x\in\mathbb{R}^d$:

$$\begin{aligned}\left|\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma-\varepsilon_x\rangle-\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma\rangle\right| &\le \sup_{i\in\mathcal{A}_n}|\langle f_i^{(n)},\gamma-\varepsilon_x\rangle-\langle f_i^{(n)},\gamma\rangle|\\ &=\sup_{i\in\mathcal{A}_n} f_i^{(n)}(x)\\ &\le \sup_{i\in\mathcal{A}_n} I_i^{(n)}(x)\\ &\le \mathbf{1}_{[-a-1,a+1]^d}(x).\end{aligned}\tag{3.12}$$

We evidently have, for each $\gamma\in\ddot\Gamma$ and $x\in\operatorname{supp}(\gamma)$:

$$\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma-\varepsilon_x\rangle\le T_n(\gamma,x)\le\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\gamma\rangle.$$

Hence,

$$\begin{aligned}\psi'(T_n(\gamma,x))^2 &\le 4\,\mathbf{1}_{\{\sup_{i\in\mathcal{A}_n}\langle f_i^{(n)},\cdot\rangle>1\}}(\gamma)\\ &\le 4\,\mathbf{1}_{\{\sup_{i\in\mathcal{A}_n}\langle I_i^{(n)},\cdot\rangle\ge 2\}}(\gamma)\\ &\le 4\sum_{i\in\mathcal{A}_n}\mathbf{1}_{\{\langle I_i^{(n)},\cdot\rangle\ge 2\}}(\gamma),\end{aligned}\tag{3.13}$$

where we used the fact that $I_i^{(n)}$ is integer-valued. By (3.11)–(3.13), we have, for $\tilde\mu$-a.e. $(x,\gamma)\in\mathbb{R}^d\times\ddot\Gamma$:

$$S(u_n)(x,\gamma)\le 4\sum_{i\in\mathcal{A}_n}\mathbf{1}_{\{\langle I_i^{(n)},\cdot\rangle\ge 2\}}(\gamma)\,\mathbf{1}_{[-a-1,a+1]^d}(x).$$

Therefore, by the Cauchy–Schwarz inequality and (3.9),

$$\mathcal{E}(u_n)\le 4\sum_{i\in\mathcal{A}_n}(\mu(\{\langle I_i^{(n)},\cdot\rangle\ge 2\}))^{1/2}\left(\int_{\ddot\Gamma}\langle\gamma,\mathbf{1}_{[-a-1,a+1]^d}\rangle\,\mu(d\gamma)\right)^{1/2}.\tag{3.14}$$

By using [23, Theorem 5.5], we easily conclude that there exists a constant $C_3>0$, independent of i and n, such that, for all $i\in\mathcal{A}_n$ and $n\in\mathbb{N}$,

$$\mu(\{\langle I_i^{(n)},\cdot\rangle\ge 2\})\le C_3\left(\int_{\mathbb{R}^d} I_i^{(n)}(x)\,dx\right)^2=C_3\left(\frac{2}{n}\right)^{2d}.\tag{3.15}$$

Since $|\mathcal{A}_n|=(2na+1)^d$, we get from (2.2), (3.14), and (3.15):

$$\mathcal{E}(u_n)\le 4C_3^{1/2}(2na+1)^d\left(\frac{2}{n}\right)^d\left(\int_{[-a-1,a+1]^d}k_\mu^{(1)}(x)\,dx\right)^{1/2},\qquad n\in\mathbb{N}.$$

Therefore, there exists a constant $C_4>0$, independent of n, such that $\mathcal{E}(u_n)\le C_4$ for all $n\in\mathbb{N}$. $\square$

We now have the main result of this section.

Theorem 3.1 1) *Suppose that the conditions of Proposition* 2.1 *are satisfied. Then, there exists a Hunt process*

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}(t))_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\Gamma})$$

on Γ (*see e.g.* [14, p. 92]) *which is properly associated with* $(\mathcal{E}, D(\mathcal{E}))$, *i.e., for all* ($\mu$*-versions of*) $F \in L^2(\Gamma,\mu)$ *and all* $t > 0$ *the function*

$$\Gamma \ni \gamma \mapsto p_t F(\gamma) := \int_{\mathbf{\Omega}} F(\mathbf{X}(t))\, d\mathbf{P}_\gamma \tag{3.16}$$

is an $\mathcal{E}$*-quasi-continuous version of* $\exp(-tH)F$, *where* H *is the generator of* $(\mathcal{E}, D(\mathcal{E}))$. $\mathbf{M}$ *is up to* μ*-equivalence unique* (*cf.* [14, Chap. IV, Sect. 6]). *In particular,* $\mathbf{M}$ *is* μ*-symmetric* (*i.e.,* $\int G\, p_t F\, d\mu = \int F\, p_t G\, d\mu$ *for all* $F, G : \Gamma \to \mathbb{R}_+$, $\mathcal{B}(\Gamma)$*-measurable*) *and has* μ *as an invariant measure.*

2) $\mathbf{M}$ *from* 1) *is up to* μ*-equivalence* (*cf.* [14, Definition 6.3]) *unique between all Hunt processes* $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}'(t))_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\Gamma})$ *on* Γ *having* μ *as an invariant measure and solving the martingale problem for* $(-H, D(H))$, *i.e., for all* $G \in D(H)$

$$\widetilde{G}(\mathbf{X}'(t)) - \widetilde{G}(\mathbf{X}'(0)) + \int_0^t (HG)(\mathbf{X}'(s))\, ds, \qquad t \geq 0,$$

is an $(\mathbf{F}'_t)$*-martingale under* $\mathbf{P}'_\gamma$ *for* $\mathcal{E}$*-q.e.* $\gamma \in \Gamma$. (*Here,* $\widetilde{G}$ *denotes a quasi-continuous version of* G, *cf.* [14, Ch. IV, Proposition 3.3].)

Remark 3.1 In fact, the statement of Theorem 3.1 remains true for any Gibbs measure $\mu \in \mathcal{G}(z,\phi)$ whose correlation functions satisfy the Ruelle bound.

Proof of Theorem 3.1. The first part of the theorem follows from Propositions 3.2, 3.3 and [14, Chap. IV, Theorem 3.5 and Chap. V, Proposition 2.15]. The second part follows directly from (the proof of) [3, Theorem 3.5]. □

In the above theorem, $\mathbf{M}$ is canonical, i.e., $\mathbf{\Omega}$ is the set of all *cadlag* functions $\omega : [0,\infty) \to \Gamma$ (i.e., ω is right continuous on $[0,\infty)$ and has left limits on $(0,\infty)$), $\mathbf{X}(t)(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}$, $(\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [7, Section 4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

4 Selfadjointness of the generator

In what follows, we will always suppose that the potential ϕ is positive.

Theorem 4.1 *Suppose that conditions* (P), (I) *are satisfied and* $\mu \in \mathcal{G}(z,\phi)$, $z > 0$. *Then, the operator* $(H, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *is essentially selfadjoint in* $L^2(\Gamma,\mu)$. *In particular, Friedrichs' extension of* $(H, \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma))$ *coincides with its closure.*

Proof. Let $(\tilde{H}, D(\tilde{H}))$ denote the closure of $(H, \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D},\Gamma))$, which exists since the latter operator is a Hermitian one. We have to show that $(\tilde{H}, D(\tilde{H}))$ is selfadjoint. Since $\tilde{H} \geq 0$, by the Nussbaum theorem, it is enough to show that there exists a set $\mathcal{S} \subset \bigcap_{n=1}^{\infty} D(\tilde{H}^n)$ which is total in $L^2(\Gamma,\mu)$ and each $F \in \mathcal{S}$ satisfies:

$$\sum_{n=0}^{\infty} \frac{\|\tilde{H}^n F\|_{L^2(\mu)}}{(2n)!}\, t^n < \infty$$

for some $t > 0$, see e.g. [20, Theorem X.40].

We have the following lemma, whose proof is completely analogous to that of Lemma 3.3.

Lemma 4.1 *Suppose that the conditions of Theorem* 4.1 *are satisfied. Let a function F be as in the formulation of Lemma* 3.3. *Then, $F \in D(\tilde{H})$ and the action of $\tilde{H}$ on F is given by the right hand side of* (3.5).

We denote by $\mathcal{P}$ the set of all continuous polynomials on Γ, i.e., the set of all finite sums of functions of the form $F(\gamma) = \prod_{i=1}^{n}\langle\gamma,\varphi_i\rangle$, $\varphi_i \in \mathcal{D}$, $i = 1,\dots,n$, $n \in \mathbb{N}$, and constants. We preserve the notation $\mathcal{P}$ for the set of all μ-classes of functions from $\mathcal{P}$. Using (2.3) and e.g. [5], we see that $\mathcal{P}$ is a dense subset in $L^2(\Gamma,\mu)$. Furthermore, any function from $\mathcal{P}$ is cylinder, and we can easily conclude from Lemma 4.1 that its statement remains true for any function $F \in \mathcal{P}$.

We will now show that $\mathcal{P} \subset \bigcap_{n=1}^{\infty} D(\tilde{H}^n)$. We first make some informal calculations. So, we define:

$$\begin{aligned} H_1F(\gamma) &:= \int z\, dx \exp\left[-E(x,\gamma)\right] D_x^+ F(\gamma),\\ H_2F(\gamma) &:= \int \gamma(dx)\, D_x^- F(\gamma), \end{aligned}$$

so that $H = H_1 - H_2$. Then,

$$H^n = (H_1 - H_2)^n = \sum_{I \subset \{1,\dots,n\}} (-1)^{n-|I|} \mathcal{H}_I^{(n)}, \tag{4.1}$$

where

$$\begin{aligned} \mathcal{H}_I^{(n)} &:= \mathcal{H}_{I,1}\mathcal{H}_{I,2}\cdots\mathcal{H}_{I,n},\\ \mathcal{H}_{I,i} &:= \begin{cases} H_1, & i \in I,\\ H_2, & i \notin I, \end{cases} \qquad i = 1,\dots,n. \end{aligned} \tag{4.2}$$

Furthermore, by induction, we conclude:

$$\mathcal{H}_I^{(n)} = \sum_{J \subset \left\{1,\dots,\max\{i:\, i\in I\}-1\right\}} \mathcal{H}_{I,J}^{(n)}, \tag{4.3}$$

where

$$
\begin{aligned}
\mathcal{H}_{I,J}^{(n)}F(\gamma) &= \bigg(\int m_{I,n}(dx_n)\, U_{I,J,n,x_n} \int m_{I,n-1}(dx_{n-1}) U_{I,J,n-1,x_{n-1}} \\
&\quad \cdots \int m_{I,1}(dx_1)\, U_{I,J,1,x_1} \prod_{i\in I} \exp\Bigg[- \sum_{u\in\eta\setminus\{x_s:\, s\in I^c,\, s>i\}} \phi(x_i - u) \Bigg] \\
&\quad \times \prod_{j=1}^{n} G_{I,J,j}(x_1,\dots,x_n) F(\gamma) \bigg) \bigg|_{\eta=\gamma}, \\
I^c &:= \{1,\dots,n\} \setminus I, \\
m_{I,i}(dx_i) &:= \begin{cases} z\, dx_i, & i\in I, \\ \gamma(dx_i), & i \in I^c, \end{cases} \\
U_{I,J,i,x_i} &:= \begin{cases} D^+_{x_i}, & i\in I,\ i\in J^c, \\ D^-_{x_i}, & i\in I^c,\ i\in J^c, \\ \mathrm{id}, & i\in J, \end{cases} \\
G_{I,J,j}^{(n)}(x_1,\dots,x_n) &:= \begin{cases} \exp\Big[-\sum_{r\in I,\, r<j} \phi(x_j - x_r)\Big], & j\in I,\ j\in J^c, \\ 1-\exp\Big[-\sum_{r\in I,\, r<j} \phi(x_j - x_r)\Big], & j\in J, \\ 1, & j\in I^c,\ j\in J^c, \end{cases} \\
i,j &= 1,\dots,n.
\end{aligned} \tag{4.4}
$$

For example, let $n=9$, $I=\{3,4,5,8\}$, $J=\{6,7\}$. Then,

$$
\begin{aligned}
\mathcal{H}_{I,J}^{(n)}F(\gamma) = \bigg(&\int \gamma(dx_9)\, D^-_{x_9} \int z\, dx_8\, D^+_{x_8} \int \gamma(dx_7) \int \gamma(dx_6) \int z\, dx_5\, D^+_{x_5} \\
&\times \int z\, dx_4\, D^+_{x_4} \int z\, dx_3\, D^+_{x_3} \int \gamma(dx_2)\, D^-_{x_2} \int \gamma(dx_1)\, D^-_{x_1} \\
&\times \exp\Bigg[- \sum_{u\in\eta\setminus\{x_9\}} \phi(x_8 - u) \Bigg] \\
&\times \exp\Bigg[- \sum_{u\in\eta\setminus\{x_6,x_7,x_9\}} \big(\phi(x_5-u)+\phi(x_4-u)+\phi(x_3-u)\big) \Bigg] \\
&\times \exp\big[-\phi(x_8-x_5) - \phi(x_8-x_4) - \phi(x_8-x_3) \big] \\
&\times \big(1-\exp\big[-\phi(x_7-x_5) - \phi(x_7-x_4) - \phi(x_7-x_3)\big]\big) \\
&\times \big(1-\exp\big[-\phi(x_6-x_5) - \phi(x_6-x_4) - \phi(x_6-x_3)\big]\big) \\
&\times \exp\big[-\phi(x_5-x_4) - \phi(x_5-x_3) - \phi(x_4-x_3)\big] F(\gamma) \bigg) \bigg|_{\eta=\gamma}.
\end{aligned}
$$

Lemma 4.2 *Suppose that the conditions of Theorem* 4.1 *are satisfied. We then have* $\mathcal{P} \subset \bigcap_{n=1}^{\infty} D(\tilde{H}^n)$, *and for any* $F \in \mathcal{P}$, $\tilde{H}^n F$ *is given by formulas* (4.1)–(4.4) (*in which* H *is replaced by* $\tilde{H}$).

Proof. This statement follows from (2.3), Lemma 4.1 and formulas (4.1)–(4.4). Indeed, replace in formulas (4.1)–(4.4) $\phi(x)$ by the function $\phi_n(x) := \phi(x)\chi_{B(n)}(x)$, $n \in \mathbb{N}$, and take $F \in \mathcal{P}$. Then, the obtained functions become cylindrical. Approximate these by functions as in Lemma 4.1 and let $n \to \infty$. The rest then easily follows. $\square$

Lemma 4.3 *Suppose that the conditions of Theorem* 4.1 *are satisfied. Then, for any* $F(\gamma) = \prod_{i=1}^{l} \langle \gamma, \varphi_i \rangle$, $\varphi_i \in \mathcal{D}$, $i = 1, \dots, l$, $l \in \mathbb{N}$, *there exists* $t > 0$ *such that* $\sum_{n=0}^{\infty} \|\tilde{H}^n F\|_{L^2(\mu)} t^n/(2n)! < \infty$.

Proof. We first derive some estimates.

Let $f_1, \dots, f_k$ be bounded integrable functions on $\mathbb{R}^d$. Consider $G(\gamma) := \prod_{i=1}^{k} \langle \gamma, f_i \rangle$. From (2.2), we conclude that

$$\int_\Gamma G(\gamma)\,\mu(d\gamma) = \sum_{i=1}^{k} \sum_{\substack{(A_1,\dots,A_i):\, \varnothing \neq A_j \subset \{1,\dots,k\},\, j=1,\dots,i \\ A_j\text{'s disjoint, } A_1 \cup \dots \cup A_i = \{1,\dots,k\}}}$$

$$\times \int_{(\mathbb{R}^d)^i} g^{(k)}(\underbrace{x_1, \dots, x_1}_{|A_1| \text{ times}}, \underbrace{x_2, \dots, x_2}_{|A_2| \text{ times}}, \dots, \underbrace{x_i, \dots, x_i}_{|A_i| \text{ times}}) k_\mu^{(i)}(x_1, \dots, x_i)\, dx_1 \cdots dx_i,$$

where $g^{(k)}(x_1, \dots, x_k) = (1/k!) \sum_{\sigma \in S_k} f_1(x_{\sigma(1)}) \cdots f_k(x_{\sigma(k)})$. By induction, we prove

$$\sum_{i=1}^{k} \sum_{\substack{(A_1,\dots,A_i):\, \varnothing \neq A_j \subset \{1,\dots,k\},\, j=1,\dots,i \\ A_j\text{'s disjoint, } A_1 \cup \dots \cup A_i = \{1,\dots,k\}}} 1 \leq 2^{k-1} k!, \qquad k \in \mathbb{N}.$$

Therefore, by (2.3),

$$\int_\Gamma |G(\gamma)|\,\mu(d\gamma) \leq 2^{k-1} \max\{1, \xi\}^k\, k! \prod_{i=1}^{k} \max\{\|f_i\|_{L^1}, \|f_i\|_{L^\infty}\}. \tag{4.5}$$

Note that, for μ-a.e. $\gamma \in \Gamma$,

$$D_x^+ G(\gamma) = -\sum_{i=0}^{k-1} \binom{i}{k} \int \gamma(dy_1) \cdots \gamma(dy_i)\, g^{(k)}(y_1, \dots, y_i, \underbrace{x, \dots, x}_{(k-i) \text{ times}}),$$

$$D_x^- G(\gamma) = \sum_{i=0}^{k-1} \binom{i}{k} (-1)^{k-i} \int \gamma(dy_1) \cdots \gamma(dy_i)\, g^{(k)}(y_1, \dots, y_i, \underbrace{x, \dots, x}_{(k-i) \text{ times}}). \tag{4.6}$$

We next easily get the following identity:

$$\sum_{k_1=0}^{n}\binom{k_1}{n}\sum_{k_2=0}^{k_1}\binom{k_2}{k_1}\sum_{k_3=0}^{k_2}\binom{k_3}{k_2}\cdots\sum_{k_m=0}^{k_{m-1}}\binom{k_m}{k_{m-1}}=m^n,\qquad m,n\in\mathbb{N}. \tag{4.7}$$

Recall also the standard estimate

$$(a_1+\cdots+a_n)^2\le n(a_1^2+\cdots+a_n^2),\qquad a_1,\dots,a_n\in\mathbb{R},\ n\in\mathbb{N}. \tag{4.8}$$

Finally, using condition (P), we get, for any $x,y_1,\dots,y_k\in\mathbb{R}^d$ and $k\in\mathbb{N}$,

$$1-\exp\left[-\sum_{i=1}^{k}\phi(x-y_i)\right]\le\sum_{i=1}^{k}(1-\exp[-\phi(x-y_i)]). \tag{4.9}$$

Let now $F(\gamma)$ be as in the formulation of the lemma. Fix $\Lambda\in\mathcal{O}_c(\mathbb{R}^d)$ and $C_5>0$ such that

$$|f_1(x_1)\cdots f_l(x_l)|\le C_5\chi_{\Lambda^l}(x_1,\dots,x_l),\qquad x_1,\dots,x_l\in\mathbb{R}^d. \tag{4.10}$$

For any sets $I,J\subset\{1,\dots,n\}$ as in (4.3), we define

$$n_1{:=}|I\cap J|,\quad n_2{:=}|I^c\cap J|,\quad n_3{:=}|I\cap J^c|,\quad n_4{:=}|I^c\cap J^c|.$$

Notice that $n_1+n_2+n_3+n_4=n$. Estimating all the multipliers of the form $e^{-\phi(\cdot)}$ by 1, and using (4.5)–(4.10), we get from (4.4)

$$\begin{aligned}\|\mathcal{H}^{(n)}_{I,J}F\|^2_{L^2(\mu)}&\le\max\{1,z\}^{2n}C_5^2\max\left\{1,\int_\Lambda dx\right\}^{2(n+l)}\max\{1,\xi\}^{2(n+l)}\\&\times\max\left\{1,\int_{\mathbb{R}^d}(1-\exp[-\phi(x)])\right\}^{2n}\\&\times 2^{2(l+n_2)-1}(2(l+n_2))!\,(n_1+n_3)!\,(n_1+n_3)^{n_2}(n_3+n_4)^{l+n_2},\end{aligned} \tag{4.11}$$

where the factor $2^{2(l+n_2)-1}(2(l+n_2))!$ is connected with estimate (4.5) and the fact that we get monomials of order $\le l+n_2$, the factor $(n_1+n_3)!$ is connected with the application of (4.9) to the terms connected with the set I, the factor $(n_1+n_3)^{n_2}$ is connected with the application of (4.9) to the terms connected with $I^c\cap J$, and the factor $(n_3+n_4)^{l+n_2}$ is connected with the application of (4.6), (4.7) to a monomial of order $\le(l+n_2)$. Using the estimate $(2k)!\le 4^k(k!)^2$, $k\in\mathbb{N}$, we conclude from (4.11) that there exists a constant $C_6>0$, independent of n,I,J and thus depending only on F, such that

$$\|\mathcal{H}^{(n)}_{I,J}F\|^2_{L^2(\mu)}\le C_6^n(n!)^2n^{2n}.$$

Hence, by (4.1)

$$\|\tilde{H}^nf\|_{L^2(\mu)}\le(2C_6)^{n/2}n!\,n^n. \tag{4.12}$$

Estimate (4.12), together with Stirling's formula, easily implies the statement of the lemma, and hence the statement of the theorem. □

5 Spectral gap of the generator

We first prove a coercivity identity for the gradient D^-. We note that, for any $\gamma \in \Gamma$ and $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma)$, $(D^-)^2 F(\gamma)$ is the element of the Hilbert space $T_\gamma^{\otimes 2} = L^2((\mathbb{R}^d)^2, \gamma^{\otimes 2})$ given by $(D^-)^2 F(\gamma, x, y) = D_x^- D_y^- F(\gamma)$, $x, y \in \mathbb{R}^d$. Furthermore, for any $x, y \in \gamma$:

$$D_x^- D_y^- F(\gamma) = \begin{cases} F(\gamma \setminus \{x, y\}) - F(\gamma \setminus x) - F(\gamma \setminus y) + F(\gamma), & x \neq y, \\ F(\gamma) - F(\gamma \setminus x) = -D_x^- F(\gamma), & x = y. \end{cases} \tag{5.1}$$

Through the natural identification of the elements of $T_\gamma^{\otimes 2}$ with linear continuous operators in T_γ, we get:

$$\mathrm{Tr}(D^-)^2 F(\gamma)((D^-)^2 F(\gamma))^* = \sum_{x, y \in \gamma} (D_x^- D_y^- F(\gamma))^2. \tag{5.2}$$

Here, Tr denotes the trace of an operator and $((D^-)^2 F(\gamma))^*$ is the adjoint operator of $(D^-)^2 F(\gamma)$.

Lemma 5.1 (Coercivity identity) *Suppose that the conditions of Theorem* 4.1 *are satisfied. Then, for any $F \in \mathcal{F}C_{\mathrm{b}}^{\infty}(\mathcal{D}, \Gamma)$, we have*

$$\int_\Gamma (HF(\gamma))^2 \, \mu(d\gamma) = \int_\Gamma \bigg[\mathrm{Tr}(D^-)^2 F(\gamma)((D^-)^2 F(\gamma))^* + \sum_{x, y \in \gamma,\, x \neq y} (\exp[\phi(x-y)] - 1)$$
$$\times (F(\gamma \setminus \{x, y\}) - F(\gamma \setminus x))(F(\gamma \setminus \{x, y\}) - F(\gamma \setminus y)) \bigg] \, \mu(d\gamma).$$

Proof. Analogously to (3.6), we get from (2.1):

$$\begin{aligned} &\int_\Gamma \mu(d\gamma) \left(\int_{\mathbb{R}^d} z \, dx \, \exp\left[-E(x, \gamma)\right] D_x^+ F(\gamma) \right)^2 \\ &\quad = \int_\Gamma \mu(d\gamma) \sum_{x, y \in \gamma,\, x \neq y} \exp[\phi(x-y)] (F(\gamma \setminus \{x, y\}) - F(\gamma \setminus x)) \\ &\qquad \times (F(\gamma \setminus \{x, y\}) - F(\gamma \setminus y)). \end{aligned} \tag{5.3}$$

Next,

$$\begin{aligned} &-2 \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z \, dx \, \exp\left[-E(x, \gamma)\right] D_x^+ F(\gamma) \int_{\mathbb{R}^d} \gamma(dy) \, D_y^- F(\gamma) \\ &\quad = -2 \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) \, (F(\gamma \setminus x) - F(\gamma)) \\ &\qquad \times \int_{\mathbb{R}^d} (\gamma \setminus x)(dy) \, (F(\gamma \setminus \{x, y\}) - F(\gamma \setminus x)) \\ &\quad = -\int_\Gamma \mu(d\gamma) \sum_{x, y \in \gamma,\, x \neq y} \big[(F(\gamma \setminus x) - F(\gamma))(F(\gamma \setminus \{x, y\}) - F(\gamma \setminus x)) \end{aligned}$$

$$- (F(\gamma\setminus y)-F(\gamma))(F(\gamma\setminus\{x,y\})-F(\gamma\setminus y))\big]. \tag{5.4}$$

Finally,

$$\begin{aligned}
&\int_\Gamma \mu(d\gamma)\left(\int_{\mathbb{R}^d}\gamma(dx)\,D_x^- F(\gamma)\right)^2\\
&\qquad=\int_\Gamma \mu(d\gamma)\sum_{x\in\gamma}(F(\gamma\setminus x)-F(\gamma))^2\\
&\qquad\quad+\int_\Gamma \mu(d\gamma)\sum_{x,y\in\gamma,\,x\neq y}(F(\gamma\setminus x)-F(\gamma))(F(\gamma\setminus y)-F(\gamma))
\end{aligned} \tag{5.5}$$

By (3.5) and (5.1)–(5.5), the lemma follows. $\square$

Theorem 5.1 *Suppose that* (P) *holds*, $z>0$, *and*

$$\delta:=\int_{\mathbb{R}^d}(1-\exp[-\phi(x)])\,z\,dx<1. \tag{5.6}$$

Let $\mu\in\mathcal{G}(z,\phi)$. *Then, the set* $(0,1-\delta)$ *does not belong to the spectrum of* H.

Proof. We fix any $F\in\mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$. By (5.1) and (5.2), we have:

$$\operatorname{Tr}(D^-)^2F(\gamma)((D^-)^2F(\gamma))^*\ge\sum_{x\in\gamma}(D_x^-D_x^-F(\gamma))^2=\sum_{x\in\gamma}(D_x^-F(\gamma))^2,\qquad \gamma\in\Gamma. \tag{5.7}$$

Using (P), (2.1), (5.6), and the Cauchy–Schwarz inequality, we next have

$$\begin{aligned}
&\bigg|\int_\Gamma\sum_{x,y\in\gamma,\,x\neq y}(\exp[\phi(x-y)]-1)(F(\gamma\setminus\{x,y\})-F(\gamma\setminus x))\\
&\qquad\times(F(\gamma\setminus\{x,y\})-F(\gamma\setminus y))\,\mu(d\gamma)\bigg|\\
&\quad\le\int_\Gamma\sum_{x,y\in\gamma,\,x\neq y}(\exp[\phi(x-y)]-1)(F(\gamma\setminus\{x,y\})-F(\gamma\setminus y))^2\,\mu(d\gamma)\\
&\quad=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}\gamma(dy)\int_{\mathbb{R}^d}(\gamma\setminus y)(dx)\,(\exp[\phi(x-y)]-1)\\
&\qquad\times(F(\gamma\setminus\{x,y\})-F(\gamma\setminus y))^2\\
&\quad=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}z\,dy\,\exp[-E(y,\gamma)]\int_{\mathbb{R}^d}\gamma(dx)\,(\exp[\phi(x-y)]-1)\\
&\qquad\times(F(\gamma\setminus x)-F(\gamma))^2\\
&\quad=\int_\Gamma\mu(d\gamma)\int_{\mathbb{R}^d}z\,dy\int_{\mathbb{R}^d}\gamma(dx)\,\exp[-E(y,\gamma\setminus x)]\\
&\qquad\times(1-\exp[-\phi(x-y)])(F(\gamma\setminus x)-F(\gamma))^2
\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dy \int_{\mathbb{R}^d} \gamma(dx)\, (1-\exp[-\phi(x-y)])(F(\gamma\setminus x)-F(\gamma))^2 \\
&= \int_{\mathbb{R}^d} (1-\exp[-\phi(y)])\, z\, dy \times \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, (D_x^- F(\gamma))^2 \\
&= \delta\, (HF, F)_{L^2(\mu)}. \qquad (5.8)
\end{aligned}
$$

Using Lemma 5.1, (5.7), and (5.8), we get, for each $F \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$:

$$
(HF, HF)_{L^2(\mu)} \ge (1-\delta)(HF, F)_{L^2(\mu)}. \qquad (5.9)
$$

From Theorem 4.1, we then conclude that (5.9) holds true for each $F \in D(H)$. Therefore, denoting by $(E_\lambda)_{\lambda\ge 0}$ the resolution of the identity of the operator H, we have:

$$
\int_{[0,\infty)} \lambda(\lambda-(1-\delta))\, d(E_\lambda F, F)_{L^2(\mu)} \ge 0, \qquad F \in D(H).
$$

From here the statement of the theorem trivially follows. $\square$

Corollary 5.1 (Poincaré inequality) *Suppose* (P) *and* (5.6) *hold and suppose* μ *is an extreme point of the convex set* $\mathcal{G}(z,\phi)$. *Then,*

$$
\mathcal{E}(F,F) \ge (1-\delta) \int_\Gamma (F(\gamma) - \langle F\rangle_\mu)^2\, \mu(d\gamma), \qquad F \in D(\mathcal{E}), \qquad (5.10)
$$

where $\langle F\rangle_\mu := \int_\Gamma F(\gamma)\, \mu(d\gamma)$.

Remark 5.1 The Poincaré inequality (5.10) means that, in addition to the fact that the set $(0, 1-\delta)$ does not belong to the spectrum of H, we also have that the kernel of H consists only of the constants.

Proof of Corrolary 5.1. Since μ is extreme in $\mathcal{G}(z,\phi)$, analogously to proof of the part (i)⇒(ii) of [2, Theorem 6.2], we conclude:

$$
\begin{aligned}
&\{\nu \in \mathcal{G}(z,\phi) \mid \nu = \rho\cdot\mu \text{ for some bounded, } \mathcal{B}(\Gamma)\text{-measurable function } \rho : \Gamma \to \mathbb{R}_+\} \\
&\quad = \{\mu\}. \qquad (5.11)
\end{aligned}
$$

Let $G \in D(\mathcal{E})$ be such that $\mathcal{E}(G) = 0$. It suffices to prove that $G = \text{const}$. By the proof of [2, Lemma 6.1], without loss of generality, we can suppose that the function G is bounded.

Now, we modify the proof of the part (ii)⇒(iii) of [2, Theorem 6.2]. Since $1 \in \mathcal{F}C_{\mathrm{b}}^\infty(\mathcal{D},\Gamma)$ and $D^-1 = 0$, replacing G by $G - \operatorname{ess\,inf} G$, we may suppose that $G \ge 0$, and, in addition, that $\int G\, d\mu = 1$. Define $\nu := G\cdot\mu$. Since $\mathcal{E}(G) = 0$, by (3.9), we have that $G(\gamma\setminus x) - G(\gamma) = 0$ $\tilde{\mu}$-a.e. Since $\mu \in \mathcal{G}(z,\phi)$, we have, for any measurable function $F : \Gamma\times\mathbb{R}^d \to [0,+\infty]$,

$$
\begin{aligned}
\int_\Gamma \nu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, F(\gamma,x) &= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, G(\gamma) F(\gamma,x) \\
&= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\, G(\gamma\setminus x) F(\gamma,x)
\end{aligned}
$$

$$= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx\, \exp[-E(x,\gamma)] G(\gamma) F(\gamma \cup x, x)$$
$$= \int_\Gamma \nu(d\gamma) \int_{\mathbb{R}^d} z\, dx\, \exp[-E(x,\gamma)] F(\gamma \cup x, x).$$

Hence, $\nu \in \mathcal{G}(z,\phi)$ and, by (5.11), $G = 1$ μ-a.e. $\square$

Let us suppose that the potential ϕ satisfies the following condition:

(LAHT) (*Low activity-high temperature regime*)

$$\delta = \int_{\mathbb{R}^d} (1 - \exp[-\phi(x)])\, z\, dx < \exp(-1).$$

Under (P) and (LAHT), there exists a unique Gibbs measure $\mu \in \mathcal{G}(z,\phi)$, see [22] and [13, Theorem 6.2] (notice that (2.1) and (P) imply that the correlation functions of any $\mu \in \mathcal{G}(z,\phi)$ satisfy the Ruelle bound (2.3) with $\xi = 1$, and hence we can take the constant C_R in [13, Theorem 6.2] to be equal to e). The following statement now immediately follows from Corollary 5.1.

Corollary 5.2 (Poincaré inequality in the LAHT regime) *Assume that* (P) *and* (LAHT) *are satisfied and consider the unique Gibbs measure* $\mu \in \mathcal{G}(z,\phi)$*. Then,* (5.10) *holds.*

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