Leaky quantum wire and dots: a resonance model

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Abstract

We discuss a model of a leaky quantum wire and a family of quantum dots described by Laplacian in $L^2(\mathbb{R}^2)$ with an attractive singular perturbation supported by a line and a finite number of points. The discrete spectrum is shown to be nonempty, and furthermore, the resonance problem can be explicitly solved in this setting; by Birman-Schwinger method it is reformulated into a Friedrichs-type model.

1 Introduction

In this talk we are going to discuss a simple model with the Hamiltonian which is a generalized Schrödinger operator in $L^2(\mathbb{R}^2)$. The interaction is supposed to be supported by a line and a finite family of points, i.e. formally we have

$$-\Delta - \alpha\delta(x-\Sigma) + \sum_{i=1}^{n} \tilde{\beta}_i \delta(x - y^{(i)})\,, \tag{1}$$

where $\alpha > 0$, $\Sigma := \{(x_1, 0);\, x_1 \in \mathbb{R}\}$, and $y^{(i)} \in \mathbb{R}^2 \setminus \Sigma$; coupling constants of the two-dimensional δ potentials will be specified below. First one has to say a few words about a motivation of this problem. Operators of the type (1) or similar have been studied recently as models of nanostructures which are "leaky" in the sense that they do not neglect quantum tunneling. While various results about the discrete spectrum were derived [2]-[7], [9]-[15], much less is known about scattering in this setting, in particular, about resonances.

The simple form of the interaction support, $\Sigma \cup \Pi$ with $\Pi := \{y^{(i)}\}$, will allow us to answer this question for the operator (1). We will achieve that by using the generalized Birman-Schwinger method which makes it possible to convert the original PDE problem into a simpler equation which in the present situation is in part integral, in part algebraic. What is important is that the method works not only for the discrete spectrum but it can be used also to find singularities of the analytically continued resolvent. The problem then boils down to a finite rank perturbation of eigenvalues embedded in the continuous spectrum, i.e. something which calls to mind the celebrated Friedrichs model. To fit into the prescribed volume limit we present here the main results with sketches of the proofs leaving detailed arguments and extensions to a forthcoming paper [8].

2 The Hamiltonian and its resolvent

A proper way to define (1) as a self-adjoint operator is through boundary conditions [1]. Consider functions $f \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\Sigma \cup \Pi)) \cap L^2(\mathbb{R}^2)$ which are continuous on Σ. For small enough $r_i > 0$ the restriction $f\restriction_{r_i}$ of f to the circle $\{x \in \mathbb{R}^2 :\, |x - y^{(i)}| = r_i\}$ makes then sense. We say that such an f belongs to $D(\dot{H}_{\alpha,\beta})$ iff the following limits

$$\Xi_i(f) := -\lim_{r_i \to 0} \frac{1}{\ln r_i} f\restriction_{r_i}\,, \quad \Omega_i(f) := \lim_{r_i \to 0} [f\restriction_{r_i} + \Xi_i(f) \ln r_i]$$

for $i = 1, \ldots, n$, and

$$\Xi_\Sigma(f)(x_1) := \frac{\partial f}{\partial x_2}(x_1, 0+) - \frac{\partial f}{\partial x_2}(x_1, 0-)\,, \quad \Omega_\Sigma(f)(x_1) := f(x_1, 0)$$

are finite and satisfy the relations

$$2\pi\beta_i \Xi_i(f) = \Omega_i(f)\,, \quad \Xi_\Sigma(f)(x_1) = -\alpha \Omega_\Sigma(f)(x_1)\,; \tag{2}$$

we denote $\beta := (\beta_1, ..., \beta_n)$. Then we define $\dot{H}_{\alpha,\beta} :\, D(\dot{H}_{\alpha,\beta}) \to L^2(\mathbb{R}^2)$ acting as

$$\dot{H}_{\alpha,\beta} f(x) = -\Delta f(x) \quad \text{for} \quad x \in \mathbb{R}^2 \setminus (\Sigma \cup \Pi)\,,$$

and $H_{\alpha,\beta}$ as its closure. Modifying the argument of [1] to the present situation one can check that $H_{\alpha,\beta}$ is self-adjoint; an alternative way is to use the method of [16]. We identify it with the formal operator (1). Notice that the β_i's do not coincide with the formal coupling constants in (1), for instance, absence of the point interaction at $y^{(i)}$ means $\beta_i = \infty$.

The key element in spectral analysis of $H_{\alpha,\beta}$ is finding an expression for its resolvent. Given $z \in \mathbb{C} \setminus [0, \infty)$ we denote by $R(z) := (-\Delta - z)^{-1}$ the free resolvent, which is an integral operator in $L^2 \equiv L^2(\mathbb{R}^2)$ with the kernel $G_z(x, x') = \frac{1}{2\pi} K_0(\sqrt{-z}|x - x'|)$, where $K_0(\cdot)$ is the Macdonald function and $z \mapsto \sqrt{z}$ has a cut on the positive halfline. We also denote by $\mathbf{R}(z)$ the unitary operator defined as $R(z)$ but acting from L^2 to $W^{2,2} \equiv W^{2,2}(\mathbb{R}^2)$. To express the resolvent of $H_{\alpha,\beta}$ we need two auxiliary Hilbert spaces, $\mathcal{H}_0 := L^2(\mathbb{R})$ and $\mathcal{H}_1 := \mathbb{C}^n$, and the corresponding trace maps $\tau_0 : W^{2,2} \to \mathcal{H}_0$ and $\tau_1 : W^{2,2} \to \mathcal{H}_1$ which act as

$$\tau_0 f := f\restriction_\Sigma\,, \quad \tau_1 f := f\restriction_\Pi = (f\restriction_{\{y^{(1)}\}}, ..., f\restriction_{\{y^{(n)}\}})\,,$$

respectively; as before the used symbols means appropriate restrictions. These maps in turn allow us to define canonical embeddings of $\mathbf{R}(z)$ to $\mathcal{H}_i$ by

$$\mathbf{R}_{i,L}(z) = \tau_i R(z) :\, L^2 \to \mathcal{H}_i\,, \quad \mathbf{R}_{L,i}(z) = [\mathbf{R}_{i,L}(z)]^* :\, \mathcal{H}_i \to L^2 \tag{3}$$

and

$$\mathbf{R}_{j,i}(z) = \tau_j \mathbf{R}_{L,i}(z) : \mathcal{H}_i \to \mathcal{H}_j \,. \tag{4}$$

We introduce the operator-valued matrix $\Gamma(z) = [\Gamma_{ij}(z)] : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ with the "blocks" $\Gamma_{ij}(z) : \mathcal{H}_j \to \mathcal{H}_i$ given by

$$\begin{aligned} \Gamma_{ij}(z)g &= -\mathbf{R}_{i,j}(z)g \qquad \text{for} \quad i \neq j \quad \text{and} \quad g \in \mathcal{H}_j \,, \\ \Gamma_{00}(z)f &= \left[\alpha^{-1} - \mathbf{R}_{0,0}(z)\right] f \qquad \text{if} \quad f \in \mathcal{H}_0 \,, \\ \Gamma_{11}(z)\varphi &= \Big(s_\beta(z)\delta_{kl} - G_z(y^{(k)}, y^{(l)})(1-\delta_{kl})\Big)\varphi \qquad \text{for} \quad \varphi \in \mathcal{H}_1 \,, \end{aligned}$$

where $s_\beta(z) = \beta + s(z) := \beta + \frac{1}{2\pi}(\ln \frac{\sqrt{z}}{2i} - \psi(1))$ and the operator in the last row is written explicitly through the components of the corresponding $n \times n$ matrix.

We will see that $\rho(H_{\alpha,\beta})$ coincides with the set of z for which $\Gamma(z)$ has a bounded inverse. The latter is contained in $\mathbb{C} \setminus [-\frac{1}{4}\alpha^2, \infty)$, hence we can define the "reduced determinant"

$$D(z) := \Gamma_{11}(z) - \Gamma_{10}(z)\Gamma_{00}(z)^{-1}\Gamma_{01}(z) \; : \; \mathcal{H}_1 \to \mathcal{H}_1 \,,$$

by means of which the "blocks" of $[\Gamma(z)]^{-1} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$ express as

$$\begin{aligned} [\Gamma(z)]^{-1}_{11} &= D(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{00} &= \Gamma_{10}(z)^{-1}\Gamma_{11}(z)D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{01} &= -\Gamma_{00}(z)^{-1}\Gamma_{01}(z)D(z)^{-1} \,, \\ [\Gamma(z)]^{-1}_{10} &= -D(z)^{-1}\Gamma_{10}(z)\Gamma_{00}(z)^{-1} \,; \end{aligned}$$

we use the natural notation which distinguishes them from the inverses of $\Gamma_{ij}(z)$. Now we can state the sought resolvent formula.

Theorem 2.1 *For $z \in \rho(H_{\alpha,\beta})$ with $\operatorname{Im} z > 0$ the resolvent of $H_{\alpha,\beta}$ is given by*

$$R_{\alpha,\beta}(z) \equiv (H_{\alpha,\beta} - z)^{-1} = R(z) + \sum_{i,j=0}^{1} \mathbf{R}_{L,i}(z)[\Gamma(z)]^{-1}_{ij}\mathbf{R}_{j,L}(z) \,. \tag{5}$$

Proof: For simplicity we will assume $n = 1$ only, i.e. $\Pi = \{y\}$; extension to the general case is easy. We have to check that $f \in D(H_{\alpha,\beta})$ holds if and only if $f = \tilde{R}_{\alpha,\beta}(z)g$ for some $g \in L^2$, where $\tilde{R}_{\alpha,\beta}(z)$ denotes the operator at the right-hand side of the last equation. Suppose that f is of this form. It belongs obviously to $W^{2,2}_{\text{loc}}(\mathbb{R}^2 \setminus (\Sigma \cup \Pi)) \cap L^2$ because all its components belong to this set. Combining the definitions of $\mathbf{R}_{i,j}$, $[\Gamma(z)]^{-1}_{ij}$, and functionals $\Xi(f) \equiv \Xi_1(f)$, $\Omega_1(f) \equiv \Omega_1(f)$ introduced above with the asymptotic behaviour of Macdonald function, $K_0(\sqrt{-z}\rho) = -2\ln\rho - 4\pi s(z) + \mathcal{O}(\rho)$ as $\rho \to 0$, we arrive at

$$\begin{aligned} 2\pi\Xi(f) &= \sum_{i=0}^{1} [\Gamma(z)]^{-1}_{1i}\mathbf{R}_{i,L}(z)g \,, \\ \Omega(f) &= \mathbf{R}_{1,L}(z)g - \sum_{i=0}^{1} \Gamma_{10}(z)[\Gamma(z)]^{-1}_{0i}\mathbf{R}_{i,L}g - s(z)\sum_{i=0}^{1} [\Gamma(z)]^{-1}_{1i}\mathbf{R}_{i,L}(z)g \,. \end{aligned}$$

Let us consider separately the components of $\Xi(f)$, $\Omega(f)$ coming from the behaviour of g at the point y and on Σ, i.e. $\Xi^i(f) := \frac{1}{2\pi}[\Gamma(z)]^{-1}_{1i}\mathbf{R}_{i,L}g$ and

$$\begin{aligned}\Omega^0(f) &:= \left[-\Gamma_{10}(z)[\Gamma(z)]^{-1}_{00} - s(z)[\Gamma(z)]^{-1}_{10}\right]\mathbf{R}_{0,L}g\,,\\ \Omega^1(f) &:= \left[1-\Gamma_{10}(z)[\Gamma(z)]^{-1}_{01} - s(z)[\Gamma(z)]^{-1}_{11}\right]\mathbf{R}_{1,L}g\,;\end{aligned}$$

using the properties of $[\Gamma_{ij}(z)]$ and its inverse it is straightforward to check that $\Omega^i(f) = 2\pi\beta\Xi^i(f)$ holds for $i=0,1$. Similar calculations yield the relation $\Xi_\Sigma(f) = -\alpha\Omega_\Sigma(f)$ which means that f belongs to $D(H_{\alpha,\beta})$, and the converse statement, namely that any function from $D(H_{\alpha,\beta})$ admits a representation of the form $f = \tilde{R}_{\alpha,\beta}(z)g$. To conclude the proof, observe that for such a function $f\in D(H_{\alpha,\beta})$ which vanishes on $\Sigma\cup\Pi$ we have $(-\Delta - z)f = g$. Consequently, $\tilde{R}_{\alpha,\beta}(z) = R_{\alpha,\beta}(z)$ is the resolvent of the Laplace operator in $L^2(\mathbb{R}^2)$ with the boundary conditions (2). ∎

In a similar way one can compare $R_{\alpha,\beta}(z)$ to the resolvent $R_\alpha(z)$ of the operator H_α with the point interactions absent using the operators $\mathbf{R}_{\alpha;1,L}(z)$, $\mathbf{R}_{\alpha;L,1}(z)$ mapping between L^2 and $\mathcal{H}_1$, and $\mathbf{R}_{\alpha;1,1}(z)\equiv\Gamma_{\alpha;11}(z)$ on $\mathcal{H}_1$ defined in analogy with (3) and (4); the latter is

$$\Gamma_{\alpha;11}(z)\varphi = \left(s^{(\alpha)}_{\beta,k}(z)\delta_{kl} - G^{(\alpha)}_z(y^{(k)},y^{(l)})(1-\delta_{kl})\right)\varphi \qquad \text{for } \varphi\in\mathcal{H}_1\,,$$

where $s^{(\alpha)}_{\beta,k}(z) := \beta - \lim_{\eta\to 0}\left(G^{(\alpha)}_z(y^{(k)},y^{(k)}+\eta) + \frac{1}{2\pi}\ln|\eta|\right)$ and $G^{(\alpha)}_z$ is the integral kernel of the operator $R_\alpha(z)$. Using the standard Krein-formula argument mimicking [1] we find that the two resolvents differ by $\mathbf{R}_{\alpha;L,1}(z)[\Gamma_{\alpha;11}(z)]^{-1}\mathbf{R}_{\alpha;1,L}(z)$. This can be simplified further: we have $R_\alpha(z) = R(z) + R_{L,0}(z)\Gamma_{00}(z)^{-1}R_{0,L}(z)$ for $z\in\rho(H_\alpha) = \mathbb{C}\setminus[-\frac14\alpha^2,\infty)$, and taking into account the asymptotic behaviour of Macdonald function we get

$$s^{(\alpha)}_{\beta,k}(z) = s_\beta(z) - (\mathbf{R}_{1,0}(z)\Gamma_{00}(z)^{-1}\mathbf{R}_{0,1}(z))_{kk}\,.$$

Taken together, these considerations mean that $\Gamma_{\alpha;1,1}(z) = D(z)$, or in other words

Proposition 2.2 *For $z\in\rho(H_{\alpha,\beta})$ with $\operatorname{Im} z>0$ the resolvent of $H_{\alpha,\beta}$ is given by*

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \mathbf{R}_{\alpha;L,1}(z)D(z)^{-1}\mathbf{R}_{\alpha;1,L}(z)\,.$$

3 Spectral properties

Before addressing our main question about resonances in this model, let us describe spectral properties of $H_{\alpha,\beta}$. The spectrum of H_α is found easily by separation of variables; using Proposition 2.2 in combination with Weyl's theorem and [17, Thm. XIII.19] we find that

$$\sigma_{\rm ess}(H_{\alpha,\beta}) = \sigma_{\rm ac}(H_{\alpha,\beta}) = [-\frac14\alpha^2,\infty)\,.$$

Less trivial is the discrete spectrum. An efficient way to determine it is provided by the generalized Birman-Schwinger principle, which in view of Theorem 2.1 reads

$$z \in \sigma_{\rm disc}(H_{\alpha,\beta}) \quad \Leftrightarrow \quad 0 \in \sigma_{\rm disc}(\Gamma(z))\,, \quad \dim\ker\Gamma(z) = \dim\ker(H_{\alpha,\beta}-z)\,, \tag{6}$$

$$H_{\alpha,\beta}\phi_z = z\phi_z \quad \Leftrightarrow \quad \phi_z = \sum_{i=0}^{1} \mathbf{R}_{L,i}(z)\eta_{i,z} \quad \text{for } z \in \sigma_{\rm disc}(H_{\alpha,\beta})\,, \tag{7}$$

where $(\eta_{0,z}, \eta_{1,z}) \in \ker\Gamma(z)$ – cf. [16]. Moreover, it is clear from the explicit form of $[\Gamma(z)]^{-1}$ that $0 \in \sigma_{\rm disc}(\Gamma(z)) \Leftrightarrow 0 \in \sigma_{\rm disc}(D(z))$; this reduces the task to an algebraic problem.

Consider again the case $n = 1$ with the point interaction placed at $(0, a)$ with $a > 0$. In absence of the line, the operator $H_{0,\beta}$ has a single eigenvalue $\epsilon_\beta = -4\mathrm{e}^{2(-2\pi\beta+\psi(1))}$; we will show that $\sigma_{\rm disc}(H_{\alpha,\beta})$ is nonempty for any $\alpha > 0$. More specifically, we claim that

Theorem 3.1 *For any $\alpha > 0$ and $\beta \in \mathbb{R}$ the operator $H_{\alpha,\beta}$ has one isolated eigenvalue $-\kappa_a^2$ with the eigenvector given in terms of the Fourier transform*

$$\text{const} \int_{\mathbb{R}^2} \left(\frac{\mathrm{e}^{-ip_2 a}}{2\pi} + \frac{\alpha \mathrm{e}^{-(p_1^2+\kappa_a^2)^{1/2}a}}{(2(p_1^2+\kappa_a^2)^{1/2}-\alpha)} \right) \frac{\mathrm{e}^{ipx}}{p^2+\kappa_a^2}\, \mathrm{d}p\,,$$

where $p = (p_1, p_2)$. The function $a \mapsto -\kappa_a^2$ is continuously increasing in $(0,\infty)$ and satisfies $\lim_{a\to\infty}(-\kappa_a^2) = \min\left\{\epsilon_\beta,\, -\frac{1}{4}\alpha^2\right\}$, while the opposite limit $-\kappa_0^2 := \lim_{a\to 0}(-\kappa_a^2)$ is finite.

Proof: One has to find z for which $\ker D(\cdot)$ is nontrivial. We put $z = -\kappa^2$ with $\kappa > 0$ and introduce $\breve{D}(\kappa) := D(-\kappa^2)$, and similarly for other quantities. By a straightforward calculation we find that $\breve{D}(\kappa)$ acts as a multiplication by $\breve{\gamma}_a(\kappa) := \breve{s}_\beta(\kappa) - \breve{\phi}_a(\kappa)$, where

$$\breve{\phi}_a(\kappa) = \frac{\alpha}{4\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{-2(p^2+\kappa^2)^{1/2}a}}{(2(p^2+\kappa^2)^{1/2}-\alpha)(p^2+\kappa^2)^{1/2}}\, \mathrm{d}p \tag{8}$$

and $\breve{s}_\beta(\kappa) = \frac{1}{2\pi}\left[\ln\frac{\kappa}{2} - \psi(1)\right]$. It is straightforward to check that $\kappa \to \breve{\gamma}_a(\kappa)$ is continuous, strictly increasing, and tends to $\pm\infty$ as $\kappa \to \infty$ and $\kappa \to \frac{1}{2}\alpha+$, respectively. Hence the equation $\breve{\gamma}_a(\kappa) = 0$ has a unique solution κ_a in $(\frac{1}{2}\alpha, \infty)$. Evaluating $\breve{\mathbf{R}}_{L,1}(\kappa)$ we get the eigenfunction from (7). Moreover, using (8) we find that for a fixed κ the function $a \mapsto \breve{\phi}_a(\kappa)$ is decreasing; combining this with the fact that $\breve{s}_\beta(\cdot)$ is increasing we conclude that $a \mapsto \kappa_a$ is decreasing. Next we employ the relation $\lim_{a\to\infty}\breve{\phi}_a(\kappa) = 0$, which is easily seen to be valid pointwise; in combination with $\breve{s}_\beta(\sqrt{-\epsilon_\beta}) = 0$ it yields the sought limit for $a \to \infty$. To finish the proof, recall that (8) is bounded from above by $\breve{\phi}_0(\kappa)$ and the equation $\breve{s}_\beta(\kappa) - \breve{\phi}_0(\kappa) = 0$ has a unique finite solution κ_0. ∎

If $n > 1$ the structure of the spectrum becomes more complicated. For instance, it is clear that $H_{\alpha,\beta}$ can have embedded eigenvalues provided the sets

Π and β have a mirror symmetry w.r.t. Σ and $\sigma_{\rm disc}(H_{0,\beta}) \cap \left(-\frac14\alpha^2, 0\right) \neq \emptyset$. In this short paper we restrict ourselves to quoting the following general result, referring to [8] for proof and more details.

Theorem 3.2 *For any $\alpha > 0$ and $\beta = (\beta_1, \dots, \beta_n) \subset \mathbb{R}^n$ the operator $H_{\alpha,\beta}$ has N isolated eigenvalues, where $1 \leq N \leq n$. In particular, if all the point interactions are strong enough, i.e. the numbers $-\beta_i$ are sufficiently large, we have $N = n$.*

4 Resonances

4.1 Poles of the continued resolvent

For simplicity we consider again a single point interaction placed at $y = (0, a)$ with $a > 0$. In addition we have to assume that if the tunneling between y and the line is neglected, the point interaction eigenvalue is embedded into the continuous spectrum of H_α, in other words, that $\epsilon_\beta > -\frac14\alpha^2$. As usual analyzing resonances means to investigate singularities in the analytical continuation of $R(\cdot)$ from the "physical sheet" across the cut $[-\frac14\alpha^2, \infty)$. Our main insight is that the constituents of the operator at the right-hand side of (5) can be separately continued analytically. Consequently, one can extend the Birman-Schwinger principle to the complex region and to look for zeros in the analytic continuation of $D(\cdot)$. A direct calculation shows that $D(z)$ acts for $z \in \mathbb{C} \setminus [-\frac14\alpha^2, \infty)$ as a multiplication by

$$d_a(z) := s_\beta(z) - \phi_a(z) = s_\beta(z) - \int_0^\infty \frac{\mu(z,t)}{t - z - \frac14\alpha^2}\,\mathrm{d}t\,, \tag{9}$$

where

$$\mu(z,t) := \frac{i\alpha}{16\pi}\,\frac{(\alpha - 2i(z-t)^{1/2})\,\mathrm{e}^{2ia(z-t)^{1/2}}}{t^{1/2}(z-t)^{1/2}}\,.$$

We shall construct the continuation of d_a to a region Ω_- of the other sheet which has the interval $(-\frac14\alpha^2, 0)$ as a part of its boundary at the real axis. To this aim we need more notation. Put $\mu^0(\lambda,t) := \lim_{\varepsilon\to0}\mu(\lambda+i\varepsilon, t)$ and for $\lambda \in (-\frac14\alpha^2, 0)$ introduce the symbol

$$I(\lambda) := \mathcal{P}\int_0^\infty \frac{\mu^0(\lambda,t)}{t-\lambda-\frac14\alpha^2}\,\mathrm{d}t$$

with the integral understood as its corresponding principal value. Finally, we denote

$$g_{\alpha,a}(z) := \frac{i\alpha}{4}\,\frac{\mathrm{e}^{-\alpha a}}{(z+\frac14\alpha^2)^{1/2}} \quad \text{for } z \in \Omega_- \cup (-\frac14\alpha^2, 0)\,.$$

Lemma 4.1 *The function $z \mapsto \phi_a(z)$ defined in (9) can be continued analytically across $(-\frac{1}{4}\alpha^2, 0)$ to a region Ω_- of the second sheet as follows,*

$$\phi_a^0(\lambda) = I(\lambda) + g_{\alpha,a}(\lambda) \qquad \text{for} \qquad \lambda \in (-\frac{1}{4}\alpha^2, 0)\,,$$
$$\phi_a^-(z) = -\int_0^\infty \frac{\mu(z,t)}{t - z - \frac{1}{4}\alpha^2}\,\mathrm{d}t - 2g_{\alpha,a}(z) \qquad \text{for} \qquad z \in \Omega_-,\ \mathrm{Im}\, z < 0\,.$$

Proof: By a direct if tedious computation – cf. [8] – one can verify the relations

$$\lim_{\varepsilon \to 0^+} \phi_a^\pm(\lambda \pm i\varepsilon) = \phi_a^0(\lambda)\,, \qquad -\frac{1}{4}\alpha^2 < \lambda < 0\,,$$

where $\phi_a^+ \equiv \phi_a$; so the claim of the lemma follows from the edge-of-the-wedge theorem. ∎

Notice that apart of fixing a part of its boundary, we have imposed no restrictions on the shape of Ω_-. The lemma allows us in turn to construct the analytic continuation of $d_a(\cdot)$ across the same segment of the real axis. It is given by the function $\eta_a : M \mapsto \mathbb{C}$, where $M = \{z : \mathrm{Im}\, z > 0\} \cup (-\frac{1}{4}\alpha^2, 0) \cup \Omega_-$ acting as

$$\eta_a(z) = s_\beta(z) - \phi_a^{l(z)}(z)\,,$$

where $l(z) = \pm$ if $\pm\mathrm{Im}\, z > 0$ and $l(z) = 0$ if $z \in (-\frac{1}{4}\alpha^2, 0)$, respectively. The problem at hand is now to show that $\eta_a(\cdot)$ has a second-sheet zero, i.e. for some $z \in \Omega_-$. To proceed further it is convenient to put $\varsigma_\beta := \sqrt{-\epsilon_\beta}$, and since we are interested here primarily in large distances a, to make the following reparametrization,

$$b := \mathrm{e}^{-a\varsigma_\beta} \quad \text{and} \quad \tilde{\eta}(b, z) := \eta_a(z) :\ [0, \infty) \times M \mapsto \mathbb{C}\,;$$

we look then for zeros of the function $\tilde{\eta}$ for small values of b. With this notation we have

$$\mu^0(\lambda, t) = \frac{\alpha}{16\pi} \frac{(\alpha + 2(t-\lambda)^{1/2})\, b^{2(t-\lambda)^{1/2}/\varsigma_\beta}}{t^{1/2}(t-\lambda)^{1/2}}\,, \quad g_{\alpha,a(b)}(\lambda) = \frac{i\alpha}{4} \frac{b^{\alpha/\varsigma_\beta}}{(\lambda + \frac{1}{4}\alpha^2)^{1/2}}\,, \tag{10}$$

for $\lambda \in (-\frac{1}{4}\alpha^2, 0)$, and similarly for the other constituents of $\tilde{\eta}$. This yields our main result.

Theorem 4.2 *Assume $\epsilon_\beta > -\frac{1}{4}\alpha^2$. For any b small enough the function $\tilde{\eta}(\cdot,\cdot)$ has a zero at a point $z(b) \in \Omega_-$ with the real and imaginary part, $z(b) = \mu(b) + i\nu(b)$, $\nu(b) < 0$, which in the limit $b \to 0$, i.e. $a \to \infty$, behave in the following way,*

$$\mu(b) = \epsilon_\beta + \mathcal{O}(b)\,, \quad \nu(b) = \mathcal{O}(b)\,. \tag{11}$$

Proof: By assumption we have $\varsigma_\beta \in (0, \frac{1}{2}\alpha)$. Using formulae (10) together with the similar expressions of $\mu(z,t)$ and $g_{\alpha,a}(z)$ in terms of b one can check that for a fixed $b \in [0, \infty)$ the function $\tilde{\eta}(b, \cdot)$ is analytic in M while with respect to both variables $\tilde{\eta}$ is just of the C^1 class in a neighbourhood of the point $(0, \epsilon_\beta)$. Moreover, it is easy to see that $\tilde{\eta}(0, \epsilon_\beta) = 0$ and $\partial_z \tilde{\eta}(0, \epsilon_\beta) \neq 0$. Thus by the implicit function theorem there exists a neighbourhood U_0 of zero and a unique function $z(b) : U_0 \mapsto \mathbb{C}$ such that $\tilde{\eta}(b, z(b)) = 0$ holds for all $b \in U_0$. Since $H_{\alpha,\beta}$ is self-adjoint, $\nu(b)$ cannot be positive, while $z(b) \in (-\frac{1}{4}\alpha^2, 0)$ for $b \neq 0$ can be excluded by inspecting the explicit form of $\tilde{\eta}$. Finally, by smoothness properties of $\tilde{\eta}$ both the real and imaginary part of $z(b)$ are of the C^1 class which yields the behaviour (11). ∎

Remark 4.3 Since Ω_- can be arbitrarily extended to the lower complex half-plane and all the quantities involved depend analytically on a, it is natural to ask what happens with the pole for other values of a. Using Lemma 4.1 one can check that in the limit $\mathrm{Im}\, z \to -\infty$ we have $|\phi_a^-(z)| \to 0$ uniformly in a and $|s_\beta(z)| \to \infty$. Thus the imaginary part of the solution $z(a)$ to $s_\beta(z) - \phi_a^-(z) = 0$ is bounded as a function of a, and in particular, the resonance pole survives as $a \to 0$. On the other hand, this argument says nothing about the residue.

4.2 Scattering

Let us consider now the same problem from the viewpoint of scattering in the system $(H_{\alpha,\beta}, H_\alpha)$. In view of Proposition 2.2 and Birman-Kuroda theorem the wave operators exist and are complete; our aim is to find the on-shell S-matrix in the interval $(-\frac{1}{4}\alpha^2, 0)$, i.e. the corresponding transmission and reflection amplitudes. Using the notation introduced above and Proposition 2.2 we can write the resolvent for $\mathrm{Im}\, z > 0$ as

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \eta_a(z)^{-1}(\cdot, v_z)v_z,$$

where $v_z := R_{\alpha;L,1}(z)$. We apply this operator to $\omega_{\lambda+i\varepsilon}(x) := \mathrm{e}^{i(\lambda+i\varepsilon+\alpha^2/4)^{1/2}x_1}\, \mathrm{e}^{-\alpha|x_2|/2}$ and take the limit $\varepsilon \to 0+$ in the sense of distributions; then a straightforward if tedious calculation shows that $H_{\alpha,\beta}$ has a generalized eigenfunction which for large $|x_1|$ behaves as

$$\psi_\lambda(x) \approx \mathrm{e}^{i(\lambda+\alpha^2/4)^{1/2}x_1}\, \mathrm{e}^{-\alpha|x_2|/2} + \frac{i}{4}\, \alpha \eta_a(\lambda)^{-1}\, \frac{\mathrm{e}^{-\alpha a}}{(\lambda + \frac{1}{4}\alpha^2)^{1/2}} \mathrm{e}^{i(\lambda+\alpha^2/4)^{1/2}|x_1|} e^{-\alpha|x_2|/2}$$

for each $\lambda \in (-\frac{1}{4}\alpha^2, 0)$. This yields the sought quantities.

Proposition 4.4 *The reflection and transmission amplitudes are given by*

$$\mathcal{R}(\lambda) = \mathcal{T}(\lambda) - 1 = \frac{i}{4}\, \alpha \eta_a(\lambda)^{-1}\, \frac{\mathrm{e}^{-\alpha a}}{(\lambda + \frac{1}{4}\alpha^2)^{1/2}}\,;$$

they have the same pole in the analytical continuation to Ω_- as the continued resolvent.

4.3 Resonances induced by broken symmetry

If $n \geq 2$ the resonance structure may become more complicated. A new feature is the occurrence of resonances coming from a violation of mirror symmetry. We will illustrate it on the simplest example of a pair of point interactions placed at $x_1 = (0, a)$ and $x_2 = (0, -a)$ with $a > 0$ and coupling $\beta_b := (\beta, \beta + b)$, where b is the symmetry-breaking parameter. We choose α, a, β in such a way that the Hamiltonian H_{0,β_0} with two identical point interactions spaced by $2a$ has two eigenvalues, the larger of which – called ϵ_2 – exceeds $-\frac{1}{4}\alpha^2$. As we have pointed out, H_{α,β_0} has then in view of antisymmetry the same eigenvalue ϵ_2 embedded in the negative part of its continuous spectrum.

Modifying the argument which led us to Theorem 2.1 we have now to continue analytically the 2×2 matrix $D(z)$ and find zeros of its determinant. This yields the equation

$$s_\beta(z)(s_\beta(z)+b) - K_0(2a\sqrt{-z})^2 - (2s_\beta(z)+b)\phi_a^{l(z)}(z) - 2K_0(2a\sqrt{-z})\phi_a^{l(z)}(z) = 0\,, \tag{12}$$

where $\phi_a^{l(z)}(\cdot)$ is defined in Lemma 4.1 and the left-hand side can be understood as a function $\hat{\eta}(b, z) : \mathbb{R} \setminus \{0\} \times M \to \mathbb{C}$. We denote also $\kappa_2 = \sqrt{-\epsilon_2}$, $\tilde{g}(\lambda) := -ig_{\alpha,a}(\lambda)$ and put $\breve{s}'_\beta(\cdot)$, $K'_0(\cdot)$ for corresponding derivatives; then we have the following result.

Theorem 4.5 *Suppose that $\epsilon_2 \in (-\frac{1}{4}\alpha^2, 0)$, then for all nonzero b small enough the equation (12) has a solution $z_2(b) \in \Omega_-$ with the real and imaginary part, $z_2(b) = \mu_2(b) + i\nu_2(b)$, which are real-analytic functions with the following expansions,*

$$\begin{aligned}
\mu_2(b) &= \epsilon_2 + \frac{\kappa_2}{\breve{s}'_\beta(\kappa_2) + 2aK'_0(2a\kappa_2)}\, b + \mathcal{O}(b^2)\,, \\
\nu_2(b) &= -\frac{\kappa_2 \tilde{g}(\epsilon_2)}{2(\breve{s}'_\beta(\kappa_2) + 2aK'_0(2a\kappa_2))|\breve{s}_\beta(\kappa_2) - \phi_a^0(\epsilon_2)|^2}\, b^2 + \mathcal{O}(b^3)\,.
\end{aligned}$$

Proof: As in Theorem 4.2 we rely on the implicit function theorem, but $\tilde{\eta}$ is now jointly analytic, so is z_2. Since $\breve{s}'_\beta(\kappa_2) + 2aK'_0(2a\kappa_2) > 0$ the leading term of $\nu_2(b)$ is negative. ∎

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