1D Particle, 1D Field, 1D Interaction.

Simple Exactly Solvable Models based on

Finite Rank Perturbations Methods.

III. Linear Friction as Radiation Reaction

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Abstract

This paper is an electronic application to my set of lectures, subject:'Formal methods in solving differential equations and constructing models of physical phenomena'. Addressed, mainly: postgraduates and related readers. Content: a discussion of the simple models of linear friction, the models, that have the mechanism that is based on radiation reaction. The interactions we will deal are based on equation arrays of the kind:

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + f_{compl}(t, q, Q) \\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2 \frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x - x_0)\Big(F_{src}(t, q, Q)\Big) + f_1(t, x) \\
Q(t) &= < l(t)|u >
\end{aligned}$$

Central mathematical points: d'Alembert-Kirchhoff-like formulae. Central physical points: phenomena of Radiation Reaction, Braking Radiation and Friction.

Introduction.

A Harmonic Oscillator Coupled to an One-Dimensional Field.

We will discuss a possible description of a detail of dynamical behaviour of one-dimensional newtonian particle. The detail we are here interested in is named 'friction'. We will focus on only two aspects of this phenomenon: First, if a particle MOVES through a medium, e.g., through water, then a special force arises, the force which acts on the particle so that it brakes the particle's moving, "tries to stop" the particle, and the particle's energy decreases. At the same time, a special medium motion arises: or medium waves of this or that or other kind arise, or the medium becomes more warm, or the medium generates a light... In such cases as these, one associates these two phenomena, thinking of them as reciprocal ones, conceiving them as the result of interplay, and says the particle generates [1] a kind of radiation which brakes the particle's motion; so, one says about the *braking or damping radiation.*

I am now trying to express this point in a language of mathematical formulae, — it is just the subject of this paper.

The Newtonian equation of motion of the particle, that moves under the action of an external force, $F_{external}(t, q, \cdots)$, reads:

$$M\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad F_{external}(t, q, \cdots)$$

where M stands for the mass of the particle. For example, the Newtonian equation of motion of the linear harmonic oscillator, which moves being subjected to an external complementary [2] force $F_{ext,osc}(t, q, \cdots)$, is:

$$M\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad -kq(t) + F_{ext,osc}(t, q, \cdots)$$

This equation is often written as

$$\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad -\Omega^2 q(t) + f_{compl}(t, q, \cdots)$$

where

$$\Omega^2 := k/M\ ,\ f_{compl}(t, q, \cdots) := F_{ext,osc}(t, q, \cdots)/M\ .$$

We will deal with the case, where

$$F_{external}(t, q, \cdots) = -2\gamma_v\frac{\partial q(t)}{\partial t} + F_{compl}(t, q, \cdots)\ ,$$

that is,

$$M\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad -2\gamma_v\frac{\partial q(t)}{\partial t} + F_{compl}(t, q, \cdots)$$

In particular, if we deal with the linear harmonic oscillator, then

$$f_{compl}(t, q, \cdots) = -2\gamma\frac{\partial q(t)}{\partial t} + f_{compl,0}(t, q, \cdots)$$

where

$$\gamma := \gamma_v/M\ ,$$

[1] or, 'is a source of'

[2] external to the oscillator, as a physical system

that is,

$$\frac{\partial^2 q(t)}{\partial t^2} = -2\gamma \frac{\partial q(t)}{\partial t} - \Omega^2 q(t) + f_{compl,0}(t, q, \cdots)\,,$$

It is just $-2\gamma_v \frac{\partial q(t)}{\partial t}$, the term, by means of which one simulates the physical effect that one is used to naming *"linear"* friction. Now then, I want to interpret the appearence of this term as an effect of radiation reaction, an effect of an interaction of the particle with a field. So, I have to declare models of the fields and models of the interactions.

In this paper I will discuss several models of one-dimensional particle coupled to one-dimensional scalar field. Primarily I am interested in the model described by the equation array [3]

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2\Big(q(t) - Q(t)\Big) + f_0(t)\\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x-x_0)\Big(Q(t) - q(t)\Big) + f_1(t,x)\\
Q(t) &= u(t,x_0)
\end{aligned}$$

I am also interested in the model described by the equation array

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + \gamma_1 Q(t) + f_0(t)\\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x-x_0)\Big(\frac{\partial q(t)}{\partial t}\Big) + f_1(t,x)\\
Q(t) &= \alpha_1\frac{\partial u(t,x_0)}{\partial t}
\end{aligned}$$

One can say that these are models of a point interaction. In this paper I will also discuss several modification of these models. All they are described by an equation array of the form

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + f_{compl}(t,q,Q)\\
\frac{\partial^2 u(t,x)}{\partial t^2} &= c^2\frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c\delta(x-x_0)\Big(F_{src}(t,q,Q)\Big) + f_1(t,x)\\
Q(t) &= < l(t)|u >
\end{aligned}$$

where $l(t)$ stands for a functional [4], for any t fixed; in this formula we consider t as a free variable [5] . After indicating **d'Alembert-Kirchhoff-like formulae** for

[3] $4\gamma c\rho = k\,, \Omega^2 = k/M$; the constants of the model mean, e.g.: c = propagating waves velocity, ρ = 'a density' of the field, k = elasticity constant, M = the mass of the particle. Of course, we assume $c > 0$.

[4] I use a P.A.M. Dirac's "bra-ket" syntax and suppose that q and Q are usual (one-dimensional) functions of t :

$$q = q(t)\,, \quad Q = Q(t)\,,$$

[5] thus, we deal with a family $\{l(t)\}_t$ of functionals ; we will normally suppose that every $l(t)$ is linear, for any t fixed. Moreover, we will deal with the case where $< l(t)|u >$ is of the form

$$< l(t)|u > := \alpha_0 u(t,x_0) + \alpha_1\frac{\partial}{\partial t}u(t,x_0)$$

solutions to these systems I obtain insulated effective equations of motion of $q(t)$ and then I briefly compare them.

A few words about the THREE-dimensional particle: the case is very complicated, however we can formally reduce it to the case where ONE THREE-dimensional particle interacts with THREE ONE-dimensional scalar fields, or, ONE ONE-dimensional VECTOR field, e.g.,

$$\begin{aligned}
\frac{\partial^2 q_i(t)}{\partial t^2} &= -\Omega^2 q_i(t) + \gamma_1 Q_i(t) + f_{0,i}(t) \\
\frac{\partial^2 u_i(t,x)}{\partial t^2} &= c^2 \frac{\partial^2 u_i(t,x)}{\partial x^2} - 4\gamma c \delta(x - x_0) \Big(\frac{\partial q_i(t)}{\partial t} \Big) + f_{1,i}(t,x) \\
Q_i(t) &= \alpha_1 \frac{\partial u_i(t,x_0)}{\partial t} \qquad (\text{ here } i = 1,2,3)
\end{aligned}$$

It's all that I want here to say on this difficult topic...

1 Models of a Point Interaction of an only one-dimensional Oscillator with an only one-dimensional Scalar Field

In this paper we fix measure units and let x be dimensionless position parameter, i.e.,

$$\text{physical position coordinate } = [\text{ length unit }] \times x + const\,.$$

Otherwise a confusion can ocurr, in relating to the definition

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x) dx = f(x_0)\,.$$

We assume the standard foramalism, where

$$\delta(x-x_0) = \frac{\partial 1_+(x-x_0)}{\partial x}$$

and where 1_+ stands for a unit step function (Heaviside function):

$$1_+(\xi) = \begin{cases} 1 & , \quad \text{if } \xi \geq 0\,, \\ 0 & , \quad \text{if } \xi < 0\,, \end{cases}$$

1.1 D'Alembert-Kirchhoff-like formulae

Recall that standard D'Alembert-Kirchhoff formulae read: if

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f\,, \quad u = u(t,x)\,, \quad f = f(t,x)\,, \quad (t \geq 0) \tag{$*$}$$

and given initial data, $u(0,\cdot)$ and $\frac{\partial u(t,\xi)}{\partial t})\Big|_{t=0}$, then

$$u = u(t,x) \;=\; \frac{1}{2c}\int_0^t \Big(\tilde{f}(\tau, x + c(t-\tau)) - \tilde{f}(\tau, x - c(t-\tau))\Big) d\tau + u_0(t,x)$$

$$\begin{aligned} u_0(t,x) &= c_+(x+ct) + c_-(x-ct) \\ &= \frac{1}{2}\Big(u(0,x+ct) + u(0,x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} (\frac{\partial u(t,\xi)}{\partial t})\Big|_{t=0} d\xi \end{aligned}$$

and where $\tilde{f}$ stands for any Primitive function of f, i.e.,

$$\frac{\partial \tilde{f}(t,x)}{\partial x} = f(t,x)$$

Note that

$$\tilde{f}(\tau, x + c(t-\tau)) - \tilde{f}(\tau, x - c(t-\tau))$$

does not depend on what a primitive is one has chosen!!! Moreover, we need only $\tilde{f}$ and not f itself.

Now, I specify f. In this paper I will take

$$f = -4\gamma c \delta(x-x_0)\Big(F_{src}(t)\Big) + f_1(t,x)$$

For such an f I conclude that

$$\tilde{f} = -4\gamma c 1_+(x-x_0)\Big(F_{src}(t)\Big) + \tilde{f}_1(t,x),$$

and then I infer that

$$\begin{aligned} u(t,x) &= -2\gamma \int_0^t \Big(1_+(x + c(t-\tau) - x_0) - 1_+(x - c(t-\tau) - x_0)\Big)\Big(F_{src}(\tau)\Big) d\tau \\ &\quad + \frac{1}{2c}\int_0^t \Big(\tilde{f}_1(\tau, x + c(t-\tau)) - \tilde{f}_1(\tau, x - c(t-\tau))\Big) d\tau + u_0(t,x) \end{aligned}$$

Denote now, to be more concise,

$$u_{01}(t,x) := \frac{1}{2c}\int_0^t \Big(\tilde{f}_1(\tau, x + c(t-\tau)) - \tilde{f}_1(\tau, x - c(t-\tau))\Big) d\tau + u_0(t,x)$$

and then rewrite the recent relation as following:

$$\begin{aligned} u(t,x) &= -2\gamma \int_0^t \Big(1_+(x + c(t-\tau) - x_0) - 1_+(x - c(t-\tau) - x_0)\Big)\Big(F_{src}(\tau)\Big) d\tau \\ &\quad + u_{01}(t,x) \end{aligned}$$

Let us now analyse this expression. We have: if $c\tau \neq ct + (x - x_0)$, and if $c\tau \neq ct - (x - x_0)$, then

$$\begin{aligned} & 1_+(x + c(t-\tau) - x_0) - 1_+(x - c(t-\tau) - x_0) \\ &= \left\{ \begin{array}{ccl} 1 & , & c\tau < ct + (x - x_0) \\ 0 & , & ct + (x - x_0) < c\tau \end{array} \right. - \left\{ \begin{array}{ccl} 1 & , & ct - (x - x_0) < \tau \\ 0 & , & c\tau < ct - (x - x_0) \end{array} \right. \\ &= \left\{ \begin{array}{ccl} 1 & , & c\tau < ct - |x - x_0| \\ 0 & , & ct - |x - x_0| < c\tau < ct + |x - x_0| \\ -1 & , & ct + |x - x_0| < c\tau \end{array} \right. \\ &= \left\{ \begin{array}{ccl} 1 & , & \tau < t - |x - x_0|/c \\ 0 & , & t - |x - x_0|/c < \tau < t + |x - x_0|/c \\ -1 & , & t + |x - x_0|/c < \tau \end{array} \right. \end{aligned}$$

Hence, for $t \geq 0$,

$$\begin{aligned} u(t,x) &= -2\gamma \int_0^{t - |x - x_0|/c} \Big(F_{src}(\tau)\Big) d\tau \cdot 1_+(t - |x - x_0|/c) \\ &\quad + u_{01}(t,x) \end{aligned}$$

Finally, this point of the analysis has an interesting consequence:

$$u(t,x_0) - u_{01}(t,x_0) = -2\gamma \int_0^t \Big(F_{src}(\tau)\Big) d\tau$$

and

$$\begin{aligned} u(t,x) &= \left\{ \begin{array}{ccl} u(t - |x - x_0|/c, x_0) - u_{01}(t - |x - x_0|/c, x_0) & , & \text{if } 0 \leq t - |x - x_0|/c \\ 0 & , & \text{if } t - |x - x_0|/c < 0 \leq t \end{array} \right\} \\ &\quad + u_{01}(t,x) \end{aligned}$$

1.2 Oscillator interacting with a scalar field

Recall that a standard relation which one is used to describing one-dimensional harmonic oscillator subjected to an external complementary force $F_{ext,osc}(t)$ is this:

$$\frac{\partial^2 q(t)}{\partial t^2} = -\Omega^2 q(t) + f_{compl}(t)$$

$$f_{compl}(t) = F_{ext,osc}(t)/M \text{ , } M = \text{ the mass of the oscillated particle}$$

Recall also, in this paper I will discuss several systems described by

$$\begin{aligned} \frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + f_{compl}(t, q, Q) \\ \frac{\partial^2 u(t,x)}{\partial t^2} &= c^2 \frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma c \delta(x - x_0)\Big(F_{src}(t, q, Q)\Big) + f_1(t,x) \\ Q(t) &= < l(t)|u > \end{aligned}$$

where $l(t)$ stands for a functional, for any t fixed; in this formula we consider t as a free variable [6] . In this case I rewrite the recent d'Alembert-Kirchhoff relation as following:

$$\begin{aligned} u(t,x) &= -2\gamma \int_0^{t-|x-x_0|/c} \Big(F_{src}(\tau, q, Q)\Big) d\tau \cdot 1_+(t - |x - x_a|/c) \\ &\quad + u_{01}(t,x) \end{aligned}$$

I have now seen: given q and u_{01}, then,

$$\boxed{\textbf{in order to obtain } u(t,x) \textbf{ I need to obtain ONLY } Q(t) \equiv < l(t)|u >}$$

After this observation use the last formula for $u(t,x)$ and then obtain

$$\begin{aligned} Q(t) &= \Big\langle l(t) \Big| \Big\{ -2\gamma \int_0^{t-|x-x_0|/c} \Big(F_{src}(\tau, q, Q)\Big) d\tau \cdot 1_+(t - |x - x_0|/c) \\ &\quad + u_{01}(t,x) \Big\}_{(\text{ we have to consider this expression as a function of } t, x \text{ })} \Big\rangle \end{aligned}$$

We restrict ourselves to the case, where

$$< l(t)|u > := \alpha_0 u(t, x_0) + \alpha_1 \frac{\partial}{\partial t} u(t, x_0)$$

i.e.,

$$\begin{aligned} Q(t) &= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Big(-2\gamma \int_0^{t-|x_0-x_0|/c} \Big(F_{src}(\tau, q, Q)\Big) d\tau \cdot 1_+(t - |x_0 - x_0|/c) \\ &\quad + u_{01}(t, x_0)\Big)\Big|_{x=x_0} \end{aligned}$$

[6] thus, we deal with a family $\{l(t)\}_t$ of functionals ; we will normally suppose that $l(t)$ is linear, for any t fixed. Moreover, we will deal with the case where $< l(t)|u >$ is of the form

$$< l(t)|u > := \alpha_0 u(t, x_0) + \alpha_1 \frac{\partial}{\partial t} u(t, x_0)$$

Hence

$$Q(t) \quad = \quad \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma \int_0^t \Big(F_{src}(\tau, q, Q)\Big) d\tau + u_{01}(t, x_0)\Bigg)$$

and we have obtained:

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + f_{compl}(t, q, Q) \\
Q(t) &= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma \int_0^t \Big(F_{src}(\tau, q, Q)\Big) d\tau + u_{01}(t, x_0)\Bigg) \\
Q(t) &= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg\{ u(t, x) \Bigg\}\Bigg|_{x=x_0}
\end{aligned}$$

$$\begin{aligned}
u(t, x) \quad &= \quad -2\gamma \int_0^{t-|x-x_0|/c} \Big(F_{src}(\tau)\Big) d\tau \cdot 1_{+}(t - |x - x_a|/c) \\
&\qquad + u_{01}(t, x) \qquad , \text{ if } t \geq 0\,.
\end{aligned}$$

The specific F_{src} and f_{compl} we will discuss are:

(A)

$$F_{src}(\tau, q, Q) = -\gamma_2 q(\tau) + \gamma_3 \frac{\partial}{\partial \tau} q(\tau)\,,\ f_{compl}(t, q, Q) = \gamma_1 Q(t) + f_0(t)$$

(B)

$$F_{src}(\tau, q, Q) = \gamma_0 (Q(\tau) - q(\tau))\,,\ f_{compl}(t, q, Q) = \Omega^2 Q(t) + f_0(t)$$

2 the Models

2.1 Effective Equation of Motion of the Particle in the Case of (A), i.e., in the Case where

$$F_{src}(\tau,q,Q) = -\gamma_2 q(\tau)+\gamma_3\frac{\partial}{\partial\tau}q(\tau)\ ,\ f_{compl}(t,q,Q) = \gamma_1 Q(t)+f_0(t)$$

In this case the formulae

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + f_{compl}(t,q,Q) \\
Q(t) &= \Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\bigg(- 2\gamma\int_0^t \Big(F_{src}(\tau,q,Q)\Big)d\tau + u_{01}(t,x_0)\bigg) \\
Q(t) &= \Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Big\{u(t,x)\Big\}\Big|_{x=x_0}
\end{aligned}$$

$$u(t,x) = \left\{ \begin{array}{ccl} -2\gamma\displaystyle\int_0^{t-|x-x_0|/c} \Big(F_{src}(\tau,q,Q)\Big)d\tau & , & \text{if } 0\le t-|x-x_0|/c \\ 0 & , & \text{if } t-|x-x_0|/c < 0 \le t \end{array} \right\}$$

$$+\,u_{01}(t,x)$$

become

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + \gamma_1 Q(t) + f_0(t) \\
Q(t) &= \Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\bigg(- 2\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial\tau}\Big)q(\tau)d\tau + u_{01}(t,x_0)\bigg) \\
Q(t) &= \Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Big\{u(t,x)\Big\}\Big|_{x=x_0}
\end{aligned}$$

$$u(t,x) = \left\{ \begin{array}{ccl} -2\gamma\displaystyle\int_0^{t-|x-x_0|/c} \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial\tau}\Big)q(\tau)d\tau & , & \text{if } 0\le t-|x-x_0|/c \\ 0 & , & \text{if } t-|x-x_0|/c < 0 \le t \end{array} \right\}$$

$$+\,u_{01}(t,x)$$

As a consequence,

$$\frac{\partial^2 q(t)}{\partial t^2} = -\Omega^2 q(t) + \gamma_1\Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau + u_{01}(t,x_0)\Bigg) + f_0(t)$$

and then there follow quite regular transformations:

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + \gamma_1\Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau\Bigg)\\ &\quad + \gamma_1\Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Bigg(u_{01}(t,x_0)\Bigg) + f_0(t)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + \gamma_1\alpha_0\Bigg(- 2\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau\Bigg)\\ &\quad + \gamma_1\alpha_1\frac{\partial}{\partial t}\Bigg(- 2\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau\Bigg)\\ &\quad + \gamma_1\Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)\Bigg(u_{01}(t,x_0)\Bigg) + f_0(t)\end{aligned}$$

If γ is a constant in t , then

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) - 2\gamma_1\alpha_0\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau\\ &\quad - 2\gamma_1\alpha_1\gamma\Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial t}\Big)q(t)\\ &\quad + \gamma_1\Big(\alpha_0 + \alpha_1\frac{\partial}{\partial t}\Big)u_{01}(t,x_0) + f_0(t)\end{aligned}$$

We have just obtained an effective equation of motion of the particle subject to the model (A), and now, let us now try to solve this equation. We restrict ourselves to the case where the field is initially not excited:

$$u_{01}(t,x_0) = 0.$$

So, we now deal with the case where

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) - 2\gamma_1\alpha_0\gamma\int_0^t \Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial \tau}\Big)q(\tau)d\tau\\ &\quad - 2\gamma_1\alpha_1\gamma\Big(- \gamma_2 + \gamma_3\frac{\partial}{\partial t}\Big)q(t) + f_0(t)\end{aligned}$$

and we see, in the relation written down, the term that one writes tending to express the idea of linear friction: it is

$$-2\gamma_1\alpha_1\gamma\gamma_3\frac{\partial}{\partial t}q(t)\,.$$

But another detail attracts attention: this relation is not an ORDINARY differential relation whenever

$$\alpha_0 \neq 0\,.$$

It is because of the term

$$-2\gamma_1\alpha_0\gamma\int_0^t\Big(-\gamma_2+\gamma_3\frac{\partial}{\partial\tau}\Big)q(\tau)d\tau$$

Only if $\alpha_0 = 0$ we see a relation which construction is habitual:

$$\frac{\partial^2 q(t)}{\partial t^2} \;=\; -\Omega^2 q(t) - 2\gamma_1\alpha_1\gamma\Big(-\gamma_2+\gamma_3\frac{\partial}{\partial t}\Big)q(t) + f_0(t)$$

In other cases we deal with the particle's motion that one is used to qualifying as motion (or, evolution) with memory. Even if

$$\gamma_2 = 0$$

the equation of motion is not ORDINARY differential equation because of $q(0)$ in

$$-2\gamma_1\alpha_0\int_0^t\gamma_3\frac{\partial}{\partial\tau}q(\tau)d\tau = -2\gamma_1\alpha_0\gamma_3\Big(q(t)-q(0)\Big) \qquad (\text{ if } \gamma_3 = const)$$

One can say we have models of dynamics with "on only one instant concentrated memory". For a contrast, the case where

$$\gamma_2 \neq 0\,,\; \gamma_3 = 0\,,$$

can be referred to as a case of "wide memory"

Let us now consider some particular cases that represent (as we think) the most typical properties of the general case. For simplicity, we assume all α-s and γ-s to be positive, and constant in t.

"Habitual" case:

$$\frac{\partial^2 q(t)}{\partial t^2} \;=\; -\Omega^2 q(t) - 2\gamma_1\alpha_1\gamma\Big(-\gamma_2+\gamma_3\frac{\partial}{\partial t}\Big)q(t) + f_0(t)\,.$$

There is no surprise, with the possible exception of the case, where

$$(\gamma_1\alpha_1\gamma\gamma_3)^2 - \Omega^2 + 2\gamma_1\alpha_1\gamma\gamma_2 \geq 0\,,$$

$$-\gamma_1\alpha_1\gamma\gamma_3 + \sqrt{(\gamma_1\alpha_1\gamma\gamma_3)^2 - \Omega^2 + 2\gamma_1\alpha_1\gamma\gamma_2} \geq 0$$

In such a case we observe self-acceleration of the oscillator, a phenomenon discussed and been discussing at least in electrodynamics.

The second case is the case of "on only one instant concentrated memory":

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} \;&=\; -\Omega^2 q(t) - 2\gamma_1\alpha_0\gamma\gamma_3\Big(q(t)-q(0)\Big)\\ &\quad - 2\gamma_1\alpha_1\gamma\gamma_3\frac{\partial}{\partial t}q(t) + f_0(t)\end{aligned}$$

As we have recently pointed up, this equation is not ordinary differential one. However, the machinery of the **ordinary** differential equations does here **quite for**. We illustrate it by the example, where

$$f_0(t) = 0\,,$$

i.e., where

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) - 2\gamma_1\alpha_0\gamma\gamma_3\Big(q(t) - q(0)\Big)\\ &\quad - 2\gamma_1\alpha_1\gamma\gamma_3\frac{\partial}{\partial t}q(t)\end{aligned}$$

We emphasise: this equation is linear *homogeneous* in $q(t)$, but write it as

$$\frac{\partial^2 q(t)}{\partial t^2} + 2\gamma_1\alpha_1\gamma\gamma_3\frac{\partial q(t)}{\partial t} + (\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3)q(t) \quad = \quad 2\gamma_1\alpha_0\gamma\gamma_3 q(0)$$

and consider this equation as *inhomogeneous* one.

Thus, we infer that

$$q(t) = \frac{2\gamma_1\alpha_0\gamma\gamma_3}{\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3}q(0) + e^{-\gamma_1\alpha_1\gamma\gamma_3 t}(C_c\cos(\Omega_g t) + C_s\sin(\Omega_g t))$$

where

$$\Omega_g := \sqrt{\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3 - (\gamma_1\alpha_1\gamma\gamma_3)^2}$$

At first, we can calculate C_c. Actually, we have

$$q(0) = \frac{2\gamma_1\alpha_0\gamma\gamma_3}{\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3}q(0) + C_c$$

i.e.,

$$C_c = q(0) - \frac{2\gamma_1\alpha_0\gamma\gamma_3}{\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3}q(0)$$

Thus we have calculated C_c. As for C_s, it can be similarly calculated.

An interesting detail is: if $t \to +\infty$ then $q(t)$ has a limit, in this sense $q(t)$ behaves as the usual damped oscillator. But

$$q(t) \to \frac{2\gamma_1\alpha_0\gamma\gamma_3}{\Omega^2 + 2\gamma_1\alpha_0\gamma\gamma_3}q(0) \neq 0 \qquad \text{every time that } q(0) \neq 0 \text{ and } \gamma_1\alpha_0\gamma\gamma_3 \neq 0 \ \textbf{!!!}$$

We observe an element of a plastic behaviour!

The last particular case we wish to discuss is the case of "wide memory":

$$\begin{aligned}\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + 2\gamma_1\alpha_0\gamma\gamma_2\int_0^t q(\tau)d\tau\\ &\quad + 2\gamma_1\alpha_1\gamma\gamma_2 q(t) + f_0(t)\end{aligned}$$

Technically, this is the most complicated case, with the possible exception of the general one, for that reason we restrict ourselves to case where

$$f_0(t) = 0\,,\ \alpha_1 = 0\,,$$

i.e., where

$$\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad -\Omega^2 q(t) + 2\gamma_1\alpha_0\gamma\gamma_2 \int_0^t q(\tau)d\tau$$

and concentrate only on the system's behaviour at large t . In order to estimate the asymptotic behaviour of $q(t)$ as $t \to +\infty$, let us handle with the characteristic polynomials. It is:

$$\lambda^2 + \Omega^2 - 2\gamma_1\alpha_0\gamma\gamma_2\frac{1}{\lambda} = (\lambda^3 + \Omega^2\lambda - 2\gamma_1\alpha_0\gamma\gamma_2)/\lambda$$

The situation is dramatic. The polynomial

$$\lambda^3 + \Omega^2\lambda - 2\gamma_1\alpha_0\gamma\gamma_2$$

has one pure real root λ_1 and two complex-conjugated ones: $\lambda_2\,, \lambda_3\,,\ \lambda_3 = \overline{\lambda_2}$. Since

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\,,\ \lambda_1\lambda_2\lambda_3 = 2\gamma_1\alpha_0\gamma\gamma_2$$

we have

$$\lambda_1 + 2Re\lambda_2 = 0\,,\ \lambda_1|\lambda_2|^2 = 2\gamma_1\alpha_0\gamma\gamma_2$$

and hence

$$\lambda_1 > 0\ \ (!!!)\ \ , Re\lambda_2 = Re\lambda_3 < 0\,.$$

Thus, we expect an EXPONENTIAL GROWTH of the oscillator amplitude, as $t \to +\infty$. Does this exponential growth really exist? Actually, any expression

$$C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + C_3e^{\overline{\lambda_2} t}\,,$$

as a function of t, satisfies the relation

$$\frac{\partial^3 q(t)}{\partial t^3} \quad = \quad -\Omega^2\frac{\partial q(t)}{\partial t} + 2\gamma_1\alpha_0\gamma\gamma_2 q(t)\,.$$

But it in itself does not mean that such an expression satisfies the proper relation

$$\frac{\partial^2 q(t)}{\partial t^2} \quad = \quad -\Omega^2 q(t) + 2\gamma_1\alpha_0\gamma\gamma_2 \int_0^t q(\tau)d\tau\,.$$

A priori, all that we can now assert is that if a function, F of t , is of the form

$$F(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + C_3e^{\overline{\lambda_2} t}\,,$$

then

$$\frac{\partial^2 F(t)}{\partial t^2} \quad = \quad -\Omega^2 F(t) + 2\gamma_1\alpha_0\gamma\gamma_2 \int_0^t F(\tau)d\tau + a\ Constant\ \ .$$

Fortunately, the problem, that has just arisen, is not difficult. Of course, every proper $q(t)$ is of the form

$$C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + C_3e^{\overline{\lambda_2} t}\,.$$

Moreover,

$$q(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + \overline{C_2}e^{\overline{\lambda_2} t}$$

because $q(t)$ is a particle position, hence, $q(t)$ is real. Thus, if we take into account that

$$\begin{aligned}
&\int_0^t C_1 e^{\lambda_1 \tau} + C_2 e^{\lambda_2 \tau} + C_2 e^{\overline{\lambda_2} \tau} d\tau \\
&\qquad = \frac{C_1}{\lambda_1} e^{\lambda_1 t} + \frac{C_2}{\lambda_2} e^{\lambda_2 t} + \overline{\Big(\frac{C_2}{\lambda_3}\Big)} e^{\overline{\lambda_2} t} - \Big(\frac{C_1}{\lambda_1} + \frac{C_2}{\lambda_2} + \overline{\Big(\frac{C_2}{\lambda_3}\Big)}\Big) \\
&\qquad = \frac{C_1}{\lambda_1} e^{\lambda_1 t} + \frac{C_2}{\lambda_2} e^{\lambda_2 t} + \overline{\Big(\frac{C_2}{\lambda_3}\Big)} e^{\overline{\lambda_2} t} - \Big(\frac{C_1}{\lambda_1} + 2Re\Big(\frac{C_2}{\lambda_2}\Big)\Big)
\end{aligned}$$

we can conclude: if

$$\frac{C_1}{\lambda_1} + 2Re\Big(\frac{C_2}{\lambda_2}\Big) = 0$$

then

$$\begin{aligned}
q(t) &:= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \overline{C_2} e^{\overline{\lambda_2} t}\,, \\
&\equiv -2Re\Big(\frac{C_2}{\lambda_2}\Big) e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \overline{C_2} e^{\overline{\lambda_2} t}
\end{aligned}$$

satisfies the proper relation. Thus, if we take C_2 so that

$$2Re\Big(\frac{C_2}{\lambda_2}\Big) \neq 0\,,$$

e.g., $C_2 = \lambda_2$, then

$$q(t) \asymp -2Re\Big(\frac{C_2}{\lambda_2}\Big) e^{\lambda_1 t} \text{ as } t \to +\infty\,.$$

Thus, we really observe an EXPONENTIAL GROWTH of the oscillator amplitude, as $t \to +\infty$, a factor that can throw Physicist's mind into confusion. We cannot here hope we have simply confounded the 'time directions'. If we had, we would have **two** roots with strictly positive real parts! In any case, we cannot escape from phenomenon of "self-acceleration". We defer the more detailed discussion on this subject and notice only, that a related phenomenon is known in electrodynimics, see Abraham-Lorentz-Dirac equations.

****

Now then, we have obtained an effective equation of motion of the particle subject to the model (A) and discussed properties of this model, and we turn now to the model (B) with the same intention.

2.2 Effective Equation of Motion of the Particle in the Case of (B), i.e., in the Case where

$$F_{src}(\tau,q,Q)=\gamma_0(Q(\tau)-q(\tau))\,,\ f_{compl}(t,q,Q)=\Omega^2 Q(t)+f_0(t)$$

In this case the formulae

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t)+f_{compl}(t,q,Q)\\
Q(t) &= \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\Bigg(-2\gamma\int_0^t\Big(F_{src}(\tau,q,Q)\Big)d\tau+u_{01}(t,x_0)\Bigg)\\
Q(t) &= \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\Big\{u(t,x)\Big\}\Big|_{x=x_0}
\end{aligned}$$

$$\begin{aligned}
u(t,x) &= \left\{\begin{array}{ccl} -2\gamma\displaystyle\int_0^{t-|x-x_0|/c}\Big(F_{src}(\tau,q,Q)\Big)d\tau & , & \text{if } 0\le t-|x-x_0|/c\\ 0 & , & \text{if } t-|x-x_0|/c<0\le t\end{array}\right\}\\
&\quad + u_{01}(t,x)
\end{aligned}$$

become

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t)+\Omega^2 Q(t)+f_0(t)\\
Q(t) &= \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\Bigg(-2\gamma\int_0^t\Big(\gamma_0(Q(\tau)-q(\tau))\Big)d\tau+u_{01}(t,x_0)\Bigg)\\
Q(t) &= \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\Big\{u(t,x)\Big\}\Big|_{x=x_0}
\end{aligned}$$

$$\begin{aligned}
u(t,x) &= \left\{\begin{array}{ccl} -2\gamma\displaystyle\int_0^{t-|x-x_0|/c}\Big(\gamma_0(Q(\tau)-q(\tau))\Big)d\tau & , & \text{if } 0\le t-|x-x_0|/c\\ 0 & , & \text{if } t-|x-x_0|/c<0\le t\end{array}\right\}\\
&\quad + u_{01}(t,x)
\end{aligned}$$

Let us focus our attention firstly on

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} &= -\Omega^2 q(t) + \Omega^2 Q(\tau) + f_0(t) \\
Q(t) &= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Big(- 2\gamma \int_0^t \Big(\gamma_0 (Q(\tau) - q(\tau))\Big) d\tau + u_{01}(t, x_0)\Big)
\end{aligned}$$

Write it as

$$\begin{aligned}
\frac{\partial^2 q(t)}{\partial t^2} - f_0(t) &= -\Omega^2 (q(t) - Q(\tau)) \\
Q(t) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \Big(\gamma_0 Q(\tau)\Big) d\tau &= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Big(2\gamma \int_0^t \Big(\gamma_0 q(\tau)\Big) d\tau + u_{01}(t, x_0)\Big)
\end{aligned}$$

We have now obtained an equation array for Q and q. The next step is to form an **insulated** equation for q. [7]For this purpose, we apply, at first, the operator

$$\Big(I + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 \cdot d\tau\Big)$$

to the former equation, i.e., to the equation

$$\frac{\partial^2 q}{\partial t^2} - f_0 = -\Omega^2 (q - Q)$$

Then we infer

$$\begin{aligned}
&\Big(\frac{\partial^2 q(t)}{\partial t^2} - f_0(t)\Big) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 \Big(\frac{\partial^2 q(\tau)}{\partial \tau^2} - f_0(t)\Big) d\tau \\
&\qquad = -\Omega^2 \Big(q(t) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 q(\tau) d\tau\Big) \\
&\qquad\quad + \Omega^2 \Big(Q(t) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 Q(\tau) d\tau\Big)
\end{aligned}$$

$$\begin{aligned}
&\Big(\frac{\partial^2 q(t)}{\partial t^2} - f_0(t)\Big) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 \Big(\frac{\partial^2 q(\tau)}{\partial \tau^2} - f_0(t)\Big) d\tau \\
&\qquad = -\Omega^2 \Big(q(t) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 q(\tau) d\tau\Big) \\
&\qquad\quad + \Omega^2 \Big(\Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 q(\tau) d\tau + u_{01}(t, x_0)\Big)
\end{aligned}$$

$$\boxed{\begin{aligned}
&\Big(\frac{\partial^2 q(t)}{\partial t^2} - f_0(t)\Big) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 \Big(\frac{\partial^2 q(\tau)}{\partial \tau^2} - f_0(t)\Big) d\tau \\
&\qquad = -\Omega^2 \Big(q(t) + u_{01}(t, x_0)\Big)
\end{aligned}}$$

[7] as for an **insulated** equation for Q, one can find it in Appendix A.

or,

$$\Big((I+K)(\ddot{q}-f_0)\Big)(t) \quad = \quad -\Omega^2\Big(q(t)+u_{01}(t,x_0)\Big)$$

where

$$K := \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)2\gamma\int_0^t \gamma_0\cdot d\tau\,, \qquad \ddot{q}(t) := \frac{\partial^2 q(t)}{\partial t^2}$$

[8]

We can stop at this last equation, or, observing that

$$\int_0^t \frac{\partial^2 q(\tau)}{\partial \tau^2}d\tau = \frac{\partial q(t)}{\partial t} - \frac{\partial q(t)}{\partial t}\Big|_{t=0}\,,$$

we can stop at that:

$$\begin{aligned}(1+2\gamma\gamma_0\alpha_1)\Big(\frac{\partial^2 q(t)}{\partial t^2}-f_0(t)\Big)& \\ =\;& -2\gamma\gamma_0\alpha_0\frac{\partial q(t)}{\partial t}-\Omega^2 q(t)+2\gamma\gamma_0\alpha_0\frac{\partial q(t)}{\partial t}\Big|_{t=0} \\ & +\Omega^2 u_{01}(t,x_0)+2\gamma\gamma_0\alpha_0\int_0^t f_0(\tau)d\tau\end{aligned}$$

Some people prefer to write such an equation as following:

$$\begin{aligned}&\Big((1+2\gamma\gamma_0\alpha_1)\frac{\partial^2}{\partial t^2}+2\gamma\gamma_0\alpha_0\frac{\partial}{\partial t}+\Omega^2\Big)q(t) \\ &\quad = \quad 2\gamma\gamma_0\alpha_0\frac{\partial q(t)}{\partial t}\Big|_{t=0}+\Omega^2 u_{01}(t,x_0)+(1+2\gamma\gamma_0\alpha_1)f_0(t)+2\gamma\gamma_0\alpha_0\int_0^t f_0(\tau)d\tau\end{aligned}$$

We have just obtained an effective equation of motion of the particle subject to the model (B), and now, as in the previous subsection, we are trying to solve the resulting equation. As in the previous subsection, we restrict ourselves to the case where the field is initially in unexcited state:

$$u_{01}(t,x_0)=0.$$

In addition, for simplicity, we take

$$\gamma_0 := 1\,,\ \alpha_0 := 1\,,\ \alpha_1 := 0\,.$$

[8]general abstractions are evident!

So, we now deal with the case where

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)q(t) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} + f_0(t) + 2\gamma\int_0^t f_0(\tau)d\tau$$

We start out emphasising that the homogeneous equation, connected to this equation is exactly

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)q(t) \quad = \quad 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

and NOT

$$\Big(\frac{\partial^2}{\partial t^2} + 2\gamma\frac{\partial}{\partial t} + \Omega^2\Big)q(t) \quad = \quad 0$$

In the previous subsection [9], we have already discussed the similar factor, and we are using the similar machinery. The difference between the equations we have just now written is the **rank one** term $2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$ [10]. This detail allows us to apply the usual machinery of the finite rank perturbations theory. Thus, having put

$$\Omega_\gamma := \sqrt{\Omega^2 - \gamma^2}$$

and having taken into account the reasons of the previous subsections, we can show that

$$\begin{aligned} q(t) \quad = \quad & e^{-\gamma t}\Big(cos(\Omega_\gamma t) + \gamma\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}\Big)\Big(q(0) - \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big) + e^{-\gamma t}\frac{sin(\Omega_\gamma t)}{\Omega_\gamma}\frac{\partial q(t)}{\partial t}\Big|_{t=0} \\ & + \frac{2\gamma}{\Omega^2}\frac{\partial q(t)}{\partial t}\Big|_{t=0}, \end{aligned}$$

of course, in the case where

$$f_0(t) = 0\,.$$

If we now concentrate on the system's behaviour at large t , and where again, for simplicity, $f_0(t) = 0$, a mathematical detail calls attention. We observe:

$$q(t) \to 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0} \text{ as } t \to +\infty\,.$$

Again, as in the previous subsection, we observe an element of plastic behaviour. A surprising detail in the new situation is: the limit function

$$q_\infty(t) = 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

is NO solution to

$$\frac{\partial^2 q(t)}{\partial t^2} = -2\gamma\frac{\partial q(t)}{\partial t} - \Omega^2 q(t) + 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$$

[9] and in subsection 1.1, as well!

[10] This term, as a function of t , is a FIXED function of t, in this context a constant non-zero function, e.g. 1, multiplied by a CONSTANT depended on q, i.e., by a fixed functional of q. Using the Dirac's syntax, one can write this term as $|a>< b|$ with $|a>= 1$ and $< b|q >= 2\gamma\frac{\partial q(t)}{\partial t}\Big|_{t=0}$.

every time that

$$2\gamma \frac{\partial q(t)}{\partial t}\Big|_{t=0} \neq 0\,,$$

because the $q = q_\infty$ is a constant, hence its derivative is zero:

$$\frac{\partial q(t)}{\partial t}\Big|_{t=0} = \frac{\partial q_\infty}{\partial t} = \frac{\partial}{\partial t}\Big(2\gamma \frac{\partial q(t)}{\partial t}\Big|_{t=0}\Big) = 0\,.$$

Similar phenomena, one can detect them in the electrodynamics of moving charges.

****

Now then, we have obtained effective equations of motion of the particles subject to the model (A) and, resp., to the model (B), and discussed properties of these models.

We have seen that the linear friction can actually be described as a result of radiation reaction. In addition we have seen a very simple model of plastic behaviour of dynamical system (section 2.1) and a little more complicated model displaying the same effects (section 2.2). But the specific properties of these two models are different. In the first model memory is a function of initial position, whereas in the second model we rather deal with a function of initial velocity. The memory effects in the two models have an interesting specificity: the moving particle 'keeps in its memory' only initial data and 'forgets' the rest ones, with the possible exception of the 'past immediate': really, one needs this 'past' to calculate the derivatives of the position, i.e., velocity and acceleration! . . . We have also seen a model of dynamical system with 'wide memory' and self-acceleration . . .

Of course, there are other interesting properties of the models, which are presented, and many interesting abstractions and generalizations are possible. Nevertheless, it does not form the subject of this paper.

****

3 Appendix A. An insulated equation to $Q(t)$

After $q(t)$ is found, we can determine $Q(t)$, at least formally, by solving

$$Q(t) = \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma \int_0^t \Big(\gamma_0(Q(\tau) - q(\tau))\Big) d\tau + u_{01}(t, x_0)\Bigg)$$

or

$$\frac{\partial Q(t)}{\partial t} = \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma\Big(\gamma_0(Q(t) - q(t))\Big) + u_{01}(t, x_0)\Bigg),$$

$$Q(0) = \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) u_{01}(t, x_0)\Bigg|_{t=0} + \alpha_1 \Bigg(- 2\gamma\Big(\gamma_0(Q(0) - q(0))\Big)\Bigg)$$

Thus we have already reduced our model, a model of an oscillator coupled the a scalar field, to a pair of linear 'ordinary' differential equations. Nevertheless we want to continue to analyse the matter and we now go searching for another relationships, which would simplify calculations of q, Q and u.

At first, we will obtain another insulated equation for Q, differently and in a different form.

We have

$$\frac{\partial^2 q(t)}{\partial t^2} - f_0(t) = -\Omega^2(q(t) - Q(\tau))$$

$$Q(t) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \Big(\gamma_0 Q(\tau)\Big) d\tau$$

$$= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(2\gamma \int_0^t \Big(\gamma_0 q(\tau)\Big) d\tau + u_{01}(t, x_0)\Bigg),$$

i.e.,

$$\frac{\partial^2 q(t)}{\partial t^2} - f_0(t) = -\Omega^2(q(t) - Q(\tau))$$

$$Q(t) - u_{01}(t, x_0) + \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big) 2\gamma \int_0^t \gamma_0 \Big(Q(\tau) - u_{01}(\tau, x_0)\Big) d\tau$$

$$= \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(2\gamma \int_0^t \gamma_0 \Big(q(\tau) - u_{01}(\tau, x_0)\Big) d\tau\Bigg)$$

Denote now, to be more concise,

$$Q_d(t) := Q(t) - u_{01}(t, x_0) \quad , \quad D_{01} := \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)$$

Then we infer

$$\Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big)\Big(Q_d(t) + D_{01} 2\gamma \int_0^t \gamma_0 Q_d(\tau) d\tau\Big)$$

$$= \Big(\frac{\partial^2}{\partial t^2} + \Omega^2\Big) D_{01} \Bigg(2\gamma \int_0^t \gamma_0 \Big(q(\tau) - u_{01}(\tau, x_0)\Big) d\tau\Bigg)$$

and

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q_d(t)+D_{01}2\gamma\int_0^t \gamma_0 Q_d(\tau)d\tau\Big)\\&\quad= \Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(D_{01}2\gamma\int_0^t \gamma_0 q(\tau)d\tau\Big)-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(D_{01}2\gamma\int_0^t \gamma_0 u_{01}(\tau,x_0)d\tau\Big)\end{aligned}$$

Note

$$\frac{\partial^2}{\partial t^2}\int_0^t \gamma_0 q(\tau)d\tau=\frac{\partial}{\partial t}\gamma_0 q(t)=\int_0^t \frac{\partial^2}{\partial \tau^2}\gamma_0 q(\tau)d\tau+\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0},$$

$$\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)D_{01}=D_{01}\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q_d(t)+2\gamma\int_0^t \gamma_0 Q_d(\tau)d\tau\Big)\\&\quad= D_{01}\left(2\gamma\int_0^t \Big(\frac{\partial^2}{\partial \tau^2}+\Omega^2\Big)\gamma_0 q(\tau)d\tau+2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_0^t \gamma_0 u_{01}(\tau,x_0)d\tau\Big)\right)\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q_d(t)+D_{01}2\gamma\int_0^t \gamma_0 Q_d(\tau)d\tau\Big)\\&\quad= D_{01}\left(2\gamma\int_0^t \gamma_0\Big(\Omega^2 Q(\tau)+f_0(\tau)\Big)d\tau+2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(2\gamma\int_0^t \gamma_0 u_{01}(\tau,x_0)d\tau\Big)\right)\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)\Big(Q_d(t)+D_{01}2\gamma\int_0^t \gamma_0 Q_d(\tau)d\tau\Big)\\&\quad= D_{01}\left(2\gamma\int_0^t \gamma_0\Big(\Omega^2 Q(\tau)-\Omega^2 u_{01}(\tau,x_0)\Big)d\tau+2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\frac{\partial^2}{\partial t^2}\Big(2\gamma\int_0^t \gamma_0 u_{01}(\tau,x_0)d\tau\Big)\right.\\&\qquad\left.+2\gamma\int_0^t \gamma_0 f_0(\tau)d\tau\right)\end{aligned}$$

Then

$$\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)Q_d(t)+D_{01}\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)2\gamma\int_0^t \gamma_0 Q_d(\tau)d\tau\\&\quad= D_{01}\left(2\gamma\int_0^t \gamma_0\Big(\Omega^2 Q(\tau)-\Omega^2 u_{01}(\tau,x_0)\Big)d\tau+2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\frac{\partial^2}{\partial t^2}\Big(2\gamma\int_0^t \gamma_0 u_{01}(\tau,x_0)d\tau\Big)\right.\\&\qquad\left.+2\gamma\int_0^t \gamma_0 f_0(\tau)d\tau\right)\end{aligned}$$

Use now that

$$Q(t)-u_{01}(t,x_0)=:Q_d(t)$$

Then

$$\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)Q_d(t)+D_{01}\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)2\gamma\int_0^t\gamma_0 Q_d(\tau)d\tau$$
$$= D_{01}\left(2\gamma\int_0^t\gamma_0\Omega^2 Q_d(\tau)d\tau+2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\frac{\partial^2}{\partial t^2}\Big(2\gamma\int_0^t\gamma_0 u_{01}(\tau,x_0)d\tau\Big)\right.$$
$$\left.+2\gamma\int_0^t\gamma_0 f_0(\tau)d\tau\right)$$

Then

$$\Big(\frac{\partial^2}{\partial t^2}+\Omega^2\Big)Q_d(t)+D_{01}\frac{\partial^2}{\partial t^2}2\gamma\int_0^t\gamma_0 Q_d(\tau)d\tau$$
$$= D_{01}\left(2\gamma\frac{\partial}{\partial t}\gamma_0 q(t)\Big|_{t=0}-\frac{\partial^2}{\partial t^2}\Big(2\gamma\int_0^t\gamma_0 u_{01}(\tau,x_0)d\tau\Big)+2\gamma\int_0^t\gamma_0 f_0(\tau)d\tau\right)$$

Then, finally,

$$\boxed{\begin{aligned}&\Big(\frac{\partial^2}{\partial t^2}+D_{01}2\gamma\frac{\partial}{\partial t}\gamma_0+\Omega^2\Big)Q_d(t)\\ &\qquad = D_{01}\left(2\gamma\frac{\partial\gamma_0 q(t)}{\partial t}\Big|_{t=0}-2\gamma\frac{\partial\gamma_0 u_{01}(t,x_0)}{\partial t}+2\gamma\int_0^t\gamma_0 f_0(\tau)d\tau\right)\\ &\qquad\text{where } Q_d(t):=Q(t)-u_{01}(t,x_0)\quad,\quad D_{01}:=\Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\end{aligned}}$$

On the surface, this equation appears to be a second order **ordinary** differential equation. It is not exactly the case. We may not arbitrary take the initial data for $Q(t)$ because

$$Q(t) = \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\left(-2\gamma\int_0^t\Big(\gamma_0(Q(\tau)-q(\tau))\Big)d\tau+u_{01}(t,x_0)\right)$$

and

$$\frac{\partial Q(t)}{\partial t} = \Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)\left(-2\gamma\Big(\gamma_0(Q(t)-q(t))\Big)+u_{01}(t,x_0)\right),$$
$$Q(0)=\Big(\alpha_0+\alpha_1\frac{\partial}{\partial t}\Big)u_{01}(t,x_0)\Big|_{t=0}+\alpha_1\left(-2\gamma\Big(\gamma_0(Q(0)-q(0))\Big)\right)$$

The proper ones are these:

$$\left.\frac{\partial Q(t)}{\partial t}\right|_{t=0} = \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\left(- 2\gamma\Big(\gamma_0(Q(t) - q(t))\Big) + u_{01}(t, x_0)\right)\Bigg|_{t=0} ,$$
$$Q(0) = \Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)u_{01}(t, x_0)\Bigg|_{t=0} + \alpha_1\left(- 2\gamma\Big(\gamma_0(Q(0) - q(0))\Big)\right)$$

Thus, we have already obtained two simple 'ordinary' differential equations for

$$q(t)\,, \quad Q(t)$$

If we have found these quantities, we can try to find $u(t, x)$ by

$$u(t, x) = \left\{ \begin{array}{ccc} -2\gamma \int_0^{t-|x-x_0|/c} \Big(F_{src}(\tau, q, Q)\Big) d\tau & , & \text{if } 0 \leq t - |x - x_0|/c \\ 0 & , & \text{if } t - |x - x_0|/c < 0 \leq t \end{array} \right\}$$
$$+\, u_{01}(t, x)$$

4 Appendix B. Remark. Complete Reflection

Suppose, we DO have a situation where $Q(t) \equiv 0$. In that case

$$\Big(\alpha_0 + \alpha_1 \frac{\partial}{\partial t}\Big)\Bigg(- 2\gamma \int_0^t \Big(F_{src}(\tau, q, Q)\Big) d\tau + u_{01}(t, x_0)\Bigg) = 0$$

and then

$$-2\gamma \int_0^t \Big(F_{src}(\tau, q, Q)\Big) d\tau + u_{01}(t, x_0) = const_0 e^{-\alpha_{01} t}$$

for suitable constants $const_0, \lambda_0$. Thus, we have

$$\begin{array}{rcl} u(t,x) & = & const_0 e^{-\alpha_{01}(t-|x-x_0|/c)} \\ & & -\, u_{01}(t-|x-x_0|/c, x_0) + u_{01}(t, x_0) \\ & & \qquad \text{if } 0 \leq t - |x-x_0|/c \end{array}$$

and, of course,

$$u(t,x) \quad = \quad u_{01}(t,x) \quad , \quad \text{if } t - |x-x_0|/c < 0 \leq t$$

Suppose in addition, that $u_{01}(t,x)$ is of the form $u_+(t + x/c)$, i.e., $u_{01}(t,x)$ is a "wave moving from right to left". Then

$$\begin{array}{rcl} u(t,x) & = & const_0 e^{-\alpha_{01}(t-|x-x_0|/c)} \\ & & -\, u_+(t-|x-x_0|/c + x_0/c) + u_+(t + x/c) \\ & & \qquad \text{if } 0 \leq t - |x-x_0|/c \end{array}$$

Hence

$$u(t,x) \quad = \quad \left\{ \begin{array}{l} const_0 e^{-\alpha_{01}(t-|x-x_0|/c)} \\ const_0 e^{-\alpha_{01}(t-|x-x_0|/c)} - u_+(t - x/c + 2x_0/c) + u_+(t + x/c) \\ u_+(t + x/c) \end{array} \right\}$$

if

$$\left\{ \begin{array}{c} 0 \leq t - |x-x_0|/c\,,\; x \leq x_0 \\ 0 \leq t - |x-x_0|/c\,,\; x \geq x_0 \\ t - |x-x_0|/c < 0 \leq t \end{array} \right\},$$

respectively. It is because

$$\begin{array}{rll} -u_+(t-|x-x_0|/c + x_0/c) + u_+(t + x/c) = 0 & \text{if} & x \leq x_0 \\ -u_+(t-|x-x_0|/c + x_0/c) = -u_+(t - x/c + 2x_0/c) & \text{if} & x \geq x_0\,. \end{array}$$

In this situation we can say, that the incident wave, $u_{01}(t,x) = u_+(t + x/c)$, is **completely reflected, rejected,** by the oscillator.

Some interesting details:

For the F_{src}, f_{compl} we wish to discuss, the conditions

$$Q = 0\,,\; f_0 = 0$$

mean that

$$\frac{\partial^2 q(t)}{\partial t^2} = -\Omega^2 q(t)\,.$$

That is,

$$q(t) = A_s \sin(\Omega t) + A_c \cos(\Omega t)$$

where A_s, A_c are suitable constants. Therefore, the complete reflection occurs [11] , where

$$u_{01}(t, x_0) = \tilde{A}_s \sin(\Omega t) + \tilde{A}_c \cos(\Omega t) + \tilde{A}_0 + const_0 e^{-\alpha_{01} t}$$

for suitable constants $\tilde{A}_s, \tilde{A}_c, \tilde{A}_0$. Thus, we have seen: The complete reflection occurs only where $u_{01}(t, x_0)$ is trigonometric up to $const_0 e^{-\alpha_{01} t}$; moreover, —only where the spectrum of the TRIGONOMETRIC PART OF $u_{01}(t, x_0)$ contains only one non-zero (and real) frequence; moreover, —only where this FREQUENCE COINCIDES WITH THE EIGENFREQUENCE OF THE OSCILLATOR.

[11] recall, $F_{src}(\tau, q, Q) = const_1 q(\tau) + const_2 \frac{\partial}{\partial \tau} q(\tau) + const_3 Q(\tau)$

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