LOCALIZATION IN INFINITE BILLIARDS: A COMPARISON BETWEEN QUANTUM AND CLASSICAL ERGODICITY

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ABSTRACT. Consider the non-compact billiard in the first quandrant bounded by the positive x-semiaxis, the positive y-semiaxis and the graph of $f(x) = (x+1)^{-\alpha}$, $\alpha \in (1, 2]$. Although the Schnirelman Theorem holds, the quantum average of the position x is finite on any eigenstate, while classical ergodicity entails that the classical time average of x is unbounded.

1. INTRODUCTION AND STATEMENT OF THE RESULT

The purpose of this note is to exhibit a simple example of a chaotic system in which the long time classical and quantum evolutions can be proved to be qualitatively different even though the Schnirelman Theorem (sometimes called quantum ergodicity: see [Sc] and also [CdV], [DGI], [HMR], [Ze], [ZZ]) holds.

Let Q be the planar domain bounded by the positive x-semiaxis, the positive y-semiaxis and the graph of $f(x) = (x+1)^{-\alpha}$, $\alpha \in (1, 2]$. Notice that $\mathrm{Area}(Q) < +\infty$. Let $\phi^t : Q \times S^1 \longrightarrow Q \times S^1$ be the dynamical flow corresponding to the billiard motion in Q. This is the Hamiltonian flow of a particle of energy 1 that moves freely in Q and performs totally elastic collisions at the boundary. By [Le2] ϕ^t is ergodic w.r.t. the normalized Liouville measure $d\nu := (2\pi\, \mathrm{Area}(Q))^{-1}\, dx\, dy\, d\theta$. (Ergodicity there is proved for some Poincaré sections and then extended to the billiard flow; see also [Le3] for a general class of ergodic billiards of the like, of finite or infinite area, including some that qualify as further examples for the present paper.) Moreover, as we show in the Appendix below, the Lyapunov exponent of the flow is positive. Therefore this system is chaotic according to general consensus.

In quantum mechanics, the corresponding system is defined by the Schrödinger operator

$$H := -\hbar^2 \Delta_D, \qquad D(H) = H^2(Q) \cap H_0^1(Q) \subset L^2(Q), \tag{1.1}$$

where Δ_D is the Laplacian with Dirichlet boundary conditions at ∂Q. It is an old result of Rellich that Δ_D is self-adjoint with compact resolvent. Hence $\mathrm{Spec}(H)$ is discrete. (More recent results on $\mathrm{Spec}(\Delta_D)$ include [Ta], [Ro], [Si], [vdB], [Da]). We denote $E_j(\hbar) = \hbar^2 \ell_j$, $j \in \mathbb{N}$, the eigenvalue of H corresponding to the normalized eigenfunction ψ_j. The order is such that $j \mapsto \ell_j$ is non-decreasing and $\{\psi_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system.

The classical time average of the coordinate x is

$$\overline{X}(x,y,\theta) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T X\circ\phi^t(x,y,\theta)\,dt, \tag{1.2}$$

where $X(x,y,\theta) := x$. In principle one does not know whether this limit exists for almost all (x,y,θ), as $X \notin L^1(Q\times S^1)$ and Birkhoff's Theorem does not apply. However, Proposition 1.1 below will override this question.

Tolerating a little abuse of notation, the quantum expectation of the position operator X on any $u \in L^2(Q)$ is instead

$$\langle X\rangle(u) := \langle u, Xu\rangle = \int_Q x|u(x,y)|^2 dxdy, \tag{1.3}$$

The comparison result we want to prove can be formulated as follows.

Proposition 1.1.

(a) *The classical average is infinite, i.e., for ν-a.e. initial condition (x,y,θ):*

$$\overline{X}(x,y,\theta) = +\infty. \tag{1.4}$$

(b) *The eigenfunctions of Δ_D are* super-exponentially localized *in the following sense: Denote*

$$\xi_j(x) := \int_0^{f(x)} |\psi_j(x,y)|^2 dy, \tag{1.5}$$

the probability density of the position in the x-direction. Then one has:

$$\forall\gamma > 0, \quad \xi_j(x) = o(e^{-\gamma x}), \tag{1.6}$$

as $x \to +\infty$. Therefore, in particular, the quantum expection of X on every eigenstate is finite:

$$\langle X\rangle(\psi_j) := \int_Q x|\psi_j(x,y)|^2 dxdy = \int_0^{+\infty} x\,\xi_j(x)dx < +\infty. \tag{1.7}$$

(c) *The Schnirelman Theorem holds. As a corollary, there is a density-1 sequence $\{j_n\}$, i.e., a sequence with the property*

$$\lim_{k\to+\infty} \frac{\#\{j_n \le k\}}{k} = 1, \tag{1.8}$$

such that

$$\lim_{n\to+\infty} \langle X\rangle(\psi_{j_n}) = +\infty. \tag{1.9}$$

Remarks

(1) Proposition 1.1,(b) implies that $\langle F(X)\rangle(u) < +\infty$ for any (normalized) state $u := \sum_{j=1}^{\infty} a_j\psi_j$ with $j \mapsto |a_j|$ decaying sufficiently fast. Here $F \in L^1_{loc}(\mathbb{R})$ is any x-dependent observable exponentially bounded as $x \to +\infty$. In fact, a Cauchy-Schwartz inequality shows that

$$\begin{aligned} |\langle u, F(X)u\rangle| &\leq \|F(X)u\| \qquad (1.10) \\ &\leq \sum_{j=1}^{\infty} |a_j| \left(\int_0^{+\infty} |F(x)|^2\, \xi_j(x)dx\right)^{1/2} < +\infty. \end{aligned}$$

If we replace $F(X)$ by the corresponding Heisenberg observable $F(X)(t) := e^{iHt/\hbar}F(X)e^{-iHt/\hbar}$, clearly the bound (1.10) holds uniformly in t:

$$\begin{aligned} |\langle u, F(X)(t)u\rangle| &= |\langle e^{-iHt/\hbar}u, F(X)e^{-iHt/\hbar}u\rangle| \qquad (1.11) \\ &\leq \sum_{j=1}^{\infty} |a_j| \left(\int_0^{+\infty} |F(x)|^2\, \xi_j(x)dx\right)^{1/2} < +\infty. \end{aligned}$$

This last estimate shows that the quantum evolution is also super-exponentially localized.

(2) Therefore, for the same u as above, one also obtains convergence of the time average of the Heisenberg observable $X(t)$:

$$\begin{aligned} \overline{\langle X\rangle}(u) &:= \lim_{T\to\infty} \frac{1}{T}\int_0^T \langle u, e^{iHt/\hbar}Xe^{-iHt/\hbar}u\rangle \\ &= \sum_{\substack{j,k\in\mathbb{N} \\ \ell_j=\ell_k}} a_j^* a_k \langle\psi_j, X\psi_k\rangle < +\infty. \qquad (1.12) \end{aligned}$$

This Heisenberg time average $\overline{\langle X\rangle}$ is the quantity to be directly compared with the classical time average $\overline{X}$.

(3) Assertion (b) shows that quantum mechanics localizes the unbounded classical chaotic motion as soon as $\hbar > 0$. This phenomenon is an example of quantum suppression of classical chaos. Its physical origin is clear: when the quantum particle visits the deepest recesses of the cusp, its y-position is very well determined. Therefore, by the Uncertainty Principle, its y-velocity must be extremely spread out, and this cannot occur at finite energies. (This crucially depends on the Dirichlet boundary conditions: for the Neumann Laplacian the situation is quite different [DS].) One might think of this phenomenon as the opposite of the tunnel effect: the classical particle is much more likely to "penetrate" the cusp than the quantum one.

(4) The classical limit is naturally defined as the joint limit $j \to +\infty$, $\hbar \to 0$, such that $E_j(\hbar) = 1$ (1 being the fixed value of the energy). Since the quantities at hand here do not depend on $\hbar$, the second limit can be forgotten. Obviously, (1.6)-(1.7) are not uniform as $j \to +\infty$.

(5) In part (c) the Schnirelman Theorem is formulated for the first time for systems whose classical energy surface is not compact (although one does not really need more than [ZZ] to prove it in our case—see below). As is well known, the Schnirelman Theorem is a purely asymptotic statement. The present example shows moreover that this asymptotic statement cannot exclude that quantization can turn a classical behavior at infinity into a behavior of a completely different nature. Hence, in general, it might appear physically misleading to call quantum ergodicity the simple validity of the Schnirelman Theorem.

2. Proof of the Proposition

(a) Since clearly

$$\int_{Q\times S^1} X \, d\nu = \frac{1}{\mathrm{Area}(Q)} \int_0^{+\infty} x f(x) dx = +\infty. \tag{2.1}$$

the assertion follows follows directly from classical ergodicity [Le2] and the following easy lemma.

Lemma 2.1. *Let $(\mathcal{P}, \nu)$ be a probability space and ϕ^t a flow on $\mathcal{P}$ that preserves the measure ν. If the measurable function g is bounded below and $(\mathcal{P}, \phi^t, \nu)$ is ergodic then, for ν-a.e. $z \in \mathcal{P}$,*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T g \circ \phi^t(z) \, dt = \int_{\mathcal{P}} g \, d\nu, \tag{2.2}$$

whether g is integrable or not.

Proof. If g is integrable there is nothing to prove by ergodicity. Otherwise $\int_{\mathcal{P}} g \, d\nu = +\infty$, because g is bounded below. In this case, for $m \in \mathbb{N}$, set $g_m := i_m \circ g$, with $i_m(x) := x$ for $x \leq m$ and $i_m(x) := m$ for $x > m$. By the Birkhoff theorem, there is a full-measure set B_m over which the time average of g_m exists and is equal to its phase average. Therefore, for $z \in B := \bigcap_{m\in\mathbb{N}} B_m$ (still a full-measure set),

$$\begin{aligned}\liminf_{T\to+\infty} \frac{1}{T}\int_0^T g\circ\phi^t(z)\,dt, &\geq \liminf_{T\to+\infty}\frac{1}{T}\int_0^T g_m\circ\phi^t(z)\,dt \\ &= \int_{\mathcal{P}} g_m\,d\nu. \end{aligned} \tag{2.3}$$

By monotonic convergence, the sup of the r.h.s. in m is $+\infty$, while the l.h.s. does not depend on m. This proves at once that the limit in T exists and is $+\infty$. □

(b) For the proof of this part, let us drop the subscript j from all notation. Hence ψ is continuous on Q, infinitely smooth in its interior, and such that

$$-(\partial_x^2+\partial_y^2)\psi(x,y) = \ell\psi(x,y), \tag{2.4}$$

with $\psi|_{\partial Q} = 0$. Without loss of generality ψ is real. Recalling definition (1.5) one easily checks, by repeated differentiation inside the integral, that

$$\xi''(x) = \int_0^{f(x)} \partial_x^2\psi^2 dy \geq 2\int_0^{f(x)} \psi\partial_x^2\psi dy. \tag{2.5}$$

Now, think of $-\partial_y^2$ as the 1−dimensional Laplacian on $[0,a]$ with Dirichlet boundary conditions. Then, in the sense of the quadratic forms, $-\partial_y^2 \geq (\pi/a)^2$. Therefore, multiplying (2.4) by ψ, integrating over y, and using that lower bound, we obtain

$$\begin{aligned}-\int_0^{f(x)}\psi\partial_x^2\psi dy &= \ell\int_0^{f(x)}\psi^2 dy + \int_0^{f(x)}\psi\partial_y^2\psi \\ &\leq \left[\ell - \left(\frac{\pi}{f(x)}\right)^2\right]\int_0^{f(x)}\psi^2 dy. \end{aligned} \tag{2.6}$$

Plugging into (2.5),

$$\xi''(x) \geq 2\left[\left(\frac{\pi}{f(x)}\right)^2 - \ell\right]\xi(x) \geq \gamma^2\xi(x), \tag{2.7}$$

for any $\gamma > 0$, provided x is large enough depending on γ. This means that either $\xi(x) \geq Ce^{\gamma x}$ or $\xi(x) \leq Ce^{-\gamma x}$. Since ψ is an eigenfunction, the first possibility cannot occur. (1.6) is thus proved.

(c) The version of the Schnirelman Theorem that can be verified in our case is the following: For any pseudodifferential operator A of order 0, compactly supported in the position variables, there exists a density-1 sequence $\{j_n\}$ such that

$$\lim_{n\to+\infty}\langle\psi_{j_n}, A\,\psi_{j_n}\rangle = \int_{Q\times S^1} a\,d\nu, \tag{2.8}$$

where a is the principal symbol of A. The proof is a strightforward check that all arguments of [ZZ] valid for compact billiards hold true in this case too. We limit ourselves to remark that the crucial fact is that Area$(Q) < +\infty$, and in particular that the system is recurrent (see [Le1] and references therein for better results in this direction).

To prove (1.9) we use an argument that is somewhat similar to Lemma 2.1. Let $\{X_m\}_{m\in\mathbb{N}}$ be an increasing sequence of functions on $Q \times S^1$ such that: X_m is smooth and depends only on x, $\mathrm{supp}(X_m) \subseteq \{x \le m\}$, $X_m \le X$ and, pointwise, $X_m \to X$ for $m \to +\infty$. With the usual abuse of notation, we denote X_m also the corresponding multiplication operator on $L^2(Q)$.

For every m there is a density-1 sequence $\{j_n^{(m)}\}_{n\in\mathbb{N}} =: \sigma^{(m)}$ such that (2.8) holds for $A = X_m$ (thus $a = X_m$), using that sequence. Now fix $p_1 = 0$ and, for $m \ge 2$, consider the following recursive definition: Select a sufficiently large $p_m > p_{m-1}$ so that

$$\forall j_n^{(m)} \ge p_m, \qquad \left| \langle \psi_{j_n^{(m)}}, X_m\, \psi_{j_n^{(m)}} \rangle - \int_{Q\times S^1} X_m \, d\nu \right| \le \frac{1}{m}, \tag{2.9}$$

and

$$\frac{\#\left(\sigma^{(m-1)} \cap [p_{m-1}, p_m)\right)}{p_m - p_{m-1}} \ge \frac{1}{m} \tag{2.10}$$

(here we have identified $\sigma^{(k)}$ with its image). (2.10) can always be verified because $\sigma^{(k)} \cap [p_k, +\infty)$ has density 1.

Now set $\sigma := \bigcup_{m\in\mathbb{N}} \left(\sigma^{(m)} \cap [p_m, p_{m+1})\right)$. Think of σ as a sequence and label its elements $\{j_n\}$. By (2.10), σ has density 1. Moreover, by (2.9),

$$\langle \psi_{j_n}, X\, \psi_{j_n} \rangle \ge \langle \psi_{j_n}, X_m\, \psi_{j_n} \rangle \ge \int_{Q\times S^1} X_m \, d\nu - \frac{1}{m}, \tag{2.11}$$

if $j_n \in [p_m, p_{m+1})$. Therefore the lim inf of the l.h.s. in n corresponds to the lim inf of the r.h.s. in m: the latter is infinity by monotonic convergence.

3. Appendix: Positivity of the Lyapunov exponent

In this section we prove that, as claimed in the Introduction, ϕ^t has a positive Lyapunov exponent λ (and thus a negative exponent $-\lambda$, as ϕ^t is volume-preserving and the exponent in the direction of the flow is obviously 0).

For $z := (x, y, \theta) \in Q \times S^1$, denote $E_z^c \subset \mathrm{T}_z(Q \times S^1)$ the one-dimensional subspace of the tangent space at z generated by $\partial_t \phi^t(z)$, i.e., the direction of the flow in z. (The inconsequential complications arising when $(x, y) \in \partial Q$ are ignored). It is known that $D\phi^t$ preserves the splitting $E^c \oplus (E^c)^\perp$ (see [SC]—this corresponds to the fact that the wave-front of a light wave is always orthogonal to its rays). Therefore the stable and unstable directions at z (denoted E_z^s and E_z^u, respectively) are to be looked for within $(E^c_z)^\perp$. To verify their existence we use the results of [Le2] about $\mathcal{M}$, the Poincaré section given by all unit vectors (*line elements*, in the language of [S]) that are based in ∂Q and point towards the interior of Q. We know that T, the Poincaré map of $\mathcal{M}$, has stable and unstable directions, denoted $F^{s(u)}$: we will use these to define $E^{s(u)}$ everywhere (more precisely, in a full-measure subset of $Q \times S^1$).

Consider a non-singular element $z \in \mathcal{M}$, that is, a line element whose future orbit never hits ∂Q tangentially or intersects the vertex at $(0,1)$. (Singular line elements can be neglected because they form a measure-zero set in $\mathcal{M}$, and thus in $Q \times S^1$, relatively to their respective invariant measures). Since z is non-singular, $E^c_z \not\subset \mathrm{T}_z\mathcal{M}$ and so $\Pi_z : \mathrm{T}_z\mathcal{M} \longrightarrow (E^c_z)^\perp$, the projection in the direction of E^c_z, is non-degenerate. We define $E^{s(u)}_z := \Pi_z F^{s(u)}_z$. Finally, for $t > 0$, $D\phi^t E^{s(u)}_z$ define the stable and unstable directions at all points in the future orbit of z. It is easy to see that this definition is unambiguous.

For a non-singular z (not necessarily in $\mathcal{M}$), take $v^u \in E^u_z$. The sought Lyapunov exponent is

$$\lambda(z) := \lim_{T\to+\infty} \frac{1}{T} \log \frac{|D\phi^T_z(v^u)|}{|v^u|} = \lim_{T\to+\infty} \frac{1}{T} \log \left\| D\phi^T_z\big|_{E^u_z} \right\|. \tag{3.1}$$

Let us introduce the following notation: $\tau(z) := \min \left\{ t \geq 0 \,\middle|\, \phi^t(z) \in \partial Q \times S^1 \right\}$ is the (forward) *free path* of the line element z, $\tau^-(z)$ is the backward free path, analogously defined, and $z' = z'(z) := \phi^{-\tau^-(z)}(z)$ is the closest line element in the past orbit of z that is based in the boundary. Define

$$g(z) := \frac{1}{\tau(z')} \log \left\| D\phi^{\tau(z')+}_{z'}\Big|_{E^u_{z'}} \right\|, \tag{3.2}$$

where $D\phi^{t+}$ means $\lim\limits_{\varepsilon\to0^+} D\phi^{t+\varepsilon}$. (This is needed because ϕ^t is discontinuous at $\partial Q \times S^1$.) By construction, g is constant along the rectilinear parts of a given orbit. Furthermore, due to the composition properties of the differential,

$$\lambda(z) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T g \circ \phi^t(z)\, dt. \tag{3.3}$$

In fact, in (3.1) and (3.3), the two functions that are averaged in T can only differ in the first and last segment of of $\{\phi^t(z)\}_{t=0}^T$. This difference gets washed away as $T \to +\infty$.

We claim that $g > 0$. In order to see this, let us go back to the Poincaré map T. For $z \in \mathcal{M}$, we have seen that

$$D\phi^{\tau(z)+}_z\big|_{(E^c_z)^\perp} = \Pi_{T(z)}\, DT_z\, \Pi_z^{-1}. \tag{3.4}$$

But it is a fact that $DT|_{F^u}$ is strictly expanding w.r.t. to a certain metric in $\mathrm{T}\mathcal{M}$. This metric is called in [Le2] the *increasing metric* and is defined as $|v|_{\mathrm{inc}} := |\Pi v|$. This fact and (3.4) show that $D\phi^{\tau+}|_{E^u}$ is strictly expanding which, in view of (3.2), proves the claim.

Finally we apply Lemma 2.1 to conclude that, for ν-a.e. $z \in Q \times S^1$,

$$\lambda(z) = \int_{Q\times S^1} g \, d\nu > 0. \tag{3.5}$$

Acknowledgments. We thank N. Burq, G. Forni, J. Wunsch and M. Zworski for some very useful discussions.

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