2-Dimensional localization of acoustic waves in random perturbation of periodic media.

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Abstract

In this work we study internal band edges localization of acoustic waves in 2-dimensional space obtained by random perturbation of some periodic media. Our results rely on the study of Lifshitz tails for the integrated density of states for random acoustic operators of the form $A_\omega = -\nabla \frac{1}{\varrho_\omega} \nabla$. Localization is then deduced by the standard multiscale argument.

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1 Introduction

Basic properties of wave propagation in a nonhomogeneous medium eventually boil down to the spectral properties of the relevant self-adjoint differential operator. As far as the acoustic waves are concerned, they are governed by the so called "acoustic operator" [7]. It is a self adjoint operator on $L^2(\mathbb{R}^d)$ and formally defined by:

$$A_\omega = A(\varrho_\omega) = -\nabla \frac{1}{\varrho_\omega} \nabla = -\sum_{i=1}^{d} \partial_{x_i} \frac{1}{\varrho_\omega} \partial_{x_i}, \tag{1.1}$$

where ϱ_ω is a positive and bounded function which represents the mass density or the elasticity contrast of the medium where the wave propagates.In this work, by "***localization***", we mean a pure point spectrum associated with exponential decaying eigenfunctions. In this case, we often say that we have Anderson localization. Localization is said to be a strong ***dynamical localization*** to an order p, on an interval I of the spectrum, when for P_I,the spectral projector onto that interval, and compact set $K \subset \mathbb{R}^d$, we have

$$\mathbb{E}\left\{\sup_{t>0} \left\| |x|^p \, e^{itA_\omega} P_I(A_\omega)\chi_K \right\| \right\} < +\infty.$$

Here $\mathbb{E}$ is the expectation with respect to ω. The last statement is interpreted as the waves with energies in I do not spread on the space. See section 1.1 of [7].

Those two types of localization have been intensively studied during last years for their important physical consequences. We quote localization of classical (for instance, electromagnetic or acoustic) waves in a periodic media perturbed by random impurities. See [7, 8, 9, 10].

The fact that electronic localization in disordered media is of a wave nature has lead to the suggestion that classical waves could be localized in random and inhomogeneous media. Indeed, physical arguments and numerical computations [15, 16], as well as experiments [45] (See research carried out by the group of Zhen Ye) indicate the possibility that this phenomenon occurs in the acoustic case.

The work in searching for localization of classical waves such as acoustic and electromagnetic waves is tremendous. It has drawn intensive attention from both theorists [7, 9] and experimentalists [43, 44, 45].

By means of the transfer matrix method and Kotany theory [18, 29, 41] the localization phenomenon in dimension one is usually characterized by a Lyapunov exponent, the inverse of which is thought to be the characteristic decay length of the amplitude of the waves within the medium and it is known as the localization length. This picture implies that the energy distribution follows decays exponentially.

In spite of the clear similarities between localization of quantum mechanical electron localization and localization of classical waves, there are some significant differences, particularly, classical waves which are harder to localize [17]. Indeed, a local change in a homogenous medium cannot create localized eigenfunction for classical waves operators but it can certainly create localized states for Schrödinger operators i.e quantum mechanical models to study electron waves in disordered media. For random Schrödinger operators it is proved [28] that in the two dimensional case band edge localization occurs. It is natural to ask whether the same kind of phenomenon can appears for classical waves such as acoustic waves. (The answer is the main object of this work and it is positive).

Historically, localization at the edge of the spectrum was conjectured for the first time by the physicists at the end of the fifties [2] for Schrödinger operators. It is widely accepted that random models should exhibit localization near fluctuation boundaries. The latter are those parts of the spectrum which are determined by rather rare events. Such a phenomenon occurs in various cases (large disorder, extreme energies, Lifshitz tails, etc.).

In this context the multiscale analysis is a powerful technique, initially developed for the purpose of proving Anderson localization [6, 7]. It was later shown to also yield dynamical localization (non spreading of the wave packets) [14] and more recently strong dynamical localization (dynamical localization not only with a probability one, but in expectation) up to some order [5]. Strong dynamical localization up to any order and in the Hilbert-Schmidt norm is proved in [13]. See also [3, 4, 5, 7, 22, 23, 40, 47]. In these references, localization near the so called fluctuation boundaries is proved.

The purpose of this paper is to prove that, for $d = 2$, we have localization on the internal spectral edges for acoustic waves, the result relies on the study of the Lifshitz tails on theses edges. Indeed we prove that on this part of the spectrum, the integrated density of states decays exponentially. This will be done without any assumption on the behavior of the integrated density of states of the background operator [32] and independently of the decay nature of the single site potential. The main difference with [7, 40] is that our approach to get localization near the band edges is based on the fact that near those edges the integrated density of states of the divergence operator exhibits Lifshitz tails. This enables us to use a weaker assumption on the probability distribution. Such technique was used for the Schrödinger case in

[28, 47]. Localization is deduced by the standard multiscale argument.

2 The model

Let us start this section by giving the expression of ϱ_ω. We assume that ϱ_ω is a function which satisfies

(**A.0**)

$$\varrho_\omega = \varrho_0\Big(1 + \sum_{\gamma\in\Gamma} \omega_\gamma u_\gamma\Big),$$

where

(i) ϱ_0 is measurable with real values and $\mathbb{Z}^d$-periodic function i.e.

$$\varrho_0(x) = \varrho_0(x+\gamma),\ \forall x \in \mathbb{R}^d, \gamma \in \mathbb{Z}^2.$$

(ii) There exists constants $\varrho_{0,+} > \varrho_{0,-} > 0$ such that for almost all $x \in \mathbb{R}^d$, we have

$$0 < \varrho_{0,-} \leq \varrho_0(x) \leq \varrho_{0,+}. \tag{2.2}$$

(iii) For $\gamma \in \mathbb{Z}^d$, we set $u_\gamma(\cdot) = u(\cdot - \gamma)$. We suppose that u is a real-values function such that there exists $U_+ > 0$:such that for almost all $x \in \mathbb{R}^d$

$$0 \leq U(x) \equiv \sum_{\gamma\in\mathbb{Z}^d} u_\gamma(x) \leq U_+ < \infty. \tag{2.3}$$

(iv) $(\omega_\gamma)_{\gamma\in\mathbb{Z}^2}$ is a family of non constant and positive, independent identically distributed random variables whose common probability measure is noted by $\mathbb{P}_{\omega_0}$ and has a bounded density g. We note the probability space by $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathbb{P}_{\omega_0}$ is supported in $[0, \omega^+]$.

Let $\mathcal{A}(\varrho_\omega)$ the quadratic form defined as follow:for $u \in H^1(\mathbb{R}^2) = \mathcal{D}(\mathcal{A}(\varrho_\omega))$

$$\mathcal{A}(\varrho_\omega)[u,u] = \int_{\mathbb{R}^2} \frac{1}{\varrho_\omega(x)} \nabla u(x)\overline{\nabla u(x)}dx.$$

$\mathcal{A}(\varrho_\omega)$ is a symmetrical, closed and positive quadratic form. A_ω given by (1.1) is defined to be the self adjoint operator associated to $\mathcal{A}(\varrho_\omega)$ [39].

Assumption **(A.0)** ensures that A_ω is a measurable family of self adjoint operators and ergodic [19, 37]. Indeed, if $\tau_\gamma$refers to the translation by γ, then $(\tau_\gamma)_{\gamma\in\mathbb{Z}^2}$ is a group of unitary operators on $L^2(\mathbb{R}^2)$ and for $\gamma \in \mathbb{Z}^2$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma \omega}.$$

According to [19, 37] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc}closed and non random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp}(respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

2.1 Reference operator

It is convenient to consider A_ω as a perturbation of some periodic operator A_{ω^+}. More precisely, for $\varrho_{\omega^+} = \varrho_0\Big(1+\omega^+ \sum_{\gamma\in\mathbb{Z}^2} u_\gamma\Big)$, where $\omega^+ = \sup(\mathrm{supp}\mathbb{P}_{\omega_0})$ we write:

$$A_\omega = A_{\omega^+} + \Delta A_\omega,$$

with

$$A_{\omega^+} = A(\varrho_{\omega^+}) \text{ and } \Delta \mathrm{A}_\omega = \mathrm{A}_\omega - \mathrm{A}_{\omega^+} = -\nabla \frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega} \nabla \geq 0.$$

We set $V_\omega = \dfrac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega} = \displaystyle\sum_{\gamma\in\mathbb{Z}^2} \frac{(\omega^+ - \omega_\gamma)u_\gamma}{D_\omega}.$

2.1.1 Some Floquet theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references for this material are [30, 39, 42].
As ϱ_{ω^+} is $\mathbb{Z}^2$-periodic, for any $\gamma \in \mathbb{Z}^2$, we have

$$\tau_\gamma A_{\omega^+} \tau_\gamma^* = \tau_\gamma A_{\omega^+} \tau_{-\gamma} = A_{\omega^+}.$$

Let the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ and the dual torus $\mathbb{T}^* = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$. We define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^2) \otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^2 \times \mathbb{T}^* \times \mathbb{Z}^2;\ u(x+\gamma,\theta) = e^{i\gamma\theta} u(x,\theta)\}.$$

Let

$$U : L^2(\mathbb{R}^2) \mapsto \mathcal{H}$$

$$(Uu)(x,\theta) = \sum_{\gamma \in \mathbb{Z}^2} e^{i\gamma\cdot\theta} u(x-\gamma) \tag{2.4}$$

and

$$(U^* v)(x) = \frac{1}{4\pi^2} \int_{\mathbb{T}^{*2}} v(x,\theta) d\theta. \tag{2.5}$$

Then U is a unitary isometry from $L^2(\mathbb{R}^2)$ to $\mathcal{H}$ such that A_{ω^+} admits the Floquet decomposition [30, 42]

$$U A_{\omega^+} U^* = \int_{\mathbb{T}^*}^{\oplus} A_{\omega^+}(\theta) d\theta.$$

Here $A_{\omega^+}(\theta)$ is the operator A_{ω^+} acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^2); \forall \gamma \in \mathbb{Z}^2, u(x+\gamma) = e^{i\gamma\theta} u(x)\}.$$

As A_{ω^+} is elliptic, we know that, $A_{\omega^+}(\theta)$ has a compact resolvent; hence its spectrum is discrete [39]. We denote its eigenvalues, called Floquet eigenvalues of A_{ω^+}, by

$$E_0(\theta) \le E_1(\theta) \le \cdots \le E_n(\theta) \le \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_j)_{j\in\mathbb{N}}$. The functions $(\theta \to E_n(\theta))_{n\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \quad \text{as } n \to +\infty \quad \text{uniformly in } \ \theta. \tag{2.6}$$

The spectrum $\sigma(A_{\omega^+})$ of A_{ω^+} is made of bands (i.e $\sigma(A_{\omega^+}) = \cup_{n\in\mathbb{N}} E_n(\mathbb{T}^*)$.)

2.1.2 Main assumptions

We assume that

(**A**.**1**)

There exists E_+ and $\delta > 0$ such that $\sigma(A_{\omega^+}) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(A_{\omega^+}) \cap (E_+ - \delta, E_+] = \emptyset$.

From (2.6) we see that there exists only finite number, say $n_0 \in \mathbb{N}$; of floquet eigenvalues that riches E_+ on some point $\theta \in \mathbb{T}^2$.

To prove our result, we will need the following assumptions:

(**A**.**2**)

$\bullet \displaystyle\limsup_{\varepsilon\to 0^+} \frac{\log|\log \mathbb{P}_{\omega^0}([\omega^+ - \varepsilon, \omega^+])|}{|\log \varepsilon|} = 0.$

As, $\Delta A_\omega \geq 0$ and ω^+ is in the support of $\mathbb{P}_{\omega_0}$, Σ contains an interval of the form $[E_+, E_+ + a] (a > 0)$ (see [21]).

As we are interested on the study of the neighborhood of the internal spectrum edges we require that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely one requires that the following assumption holds.

(**A**.**3**)

There exists $\delta' > 0$ such that $\Sigma \cap [E_+ - \delta', E_+) = \emptyset$.

Let us end this subsection by making some remarks about our assumptions. Let us start with **(A.1)**. Figotin and Kuchment in [11] studied the possibility of existence of open spectral gaps in the spectrum of certain periodic acoustic operators for $d = 2$ and 3. Indeed the phenomenon of the band-gap structure arises due to the periodicity the underlying operator.

In assumption **(A.1)** we asked that $E_+ > 0$ which excludes the spectral gap $(-\infty, 0)$. Localization is likely to occur at the neighborhood of the fluctuating edges. See [37]. It should be noted that 0 is not a fluctuating edge of the spectrum. It belongs to the spectrum of A_ω independently of the choice of $\varrho_\omega(x)$.

We end with **(A.3)**. By adding a disorder parameter g in the equation which define ϱ_ω i.e.

$$\varrho_\omega = \varrho_0 \Big(1 + g \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma u_\gamma \Big),$$

we can choose g small enough so that the spectral gap in $\sigma(A_{\omega^+})$ will not be closed after the perturbation [7].

2.1.3 The main result

The main result of this work is:

Theorem 2.1. *Let $\theta \in \mathbb{R}^2$ and $E_+ > 0$ a band edge of the spectrum of A_ω. then for any $\alpha > 0$, there exists integer $p > 0$ such that for $k \in \mathbb{N}$ sufficiently large, one has*

$$(\mathbf{P1})\ \mathbb{P}\big(\big\{ dist\big(\sigma(A^\theta_{\omega,\Lambda_{k^\alpha}}), E_+\big) \leq \frac{1}{k} \big\} \big) \leq \frac{1}{k^p}.$$

Where Λ_k is the box centered in 0 of side length $2k+1$ and $A^\theta_{\omega,\Lambda_k}$ is the *operator A_ω restricted to this box with θ-quasiperiodic boundary condition i.e with boundary condition $\varphi(x+\gamma) = e^{i\gamma\cdot\theta}\varphi(x)$ for any $\gamma \in (2k+1)\mathbb{Z}^2$.*

As when the single site decrease polynomially, i.e

$$u \simeq \frac{1}{(1+|x|)^m}, \text{ for some m} \geq 1; \tag{2.7}$$

A_ω satisfies a Wegner estimate [7, 35] i.e for some $\alpha > 0$ and $n > 0$ for $E \in \mathbb{R}$ for $k \geq 1$ and $\varepsilon > 0$, there exists $C(E) > 0$ such that one has

$$(\mathbf{P2})\ \mathbb{P}\big(\big\{dist(\sigma(A^\theta_{\omega,\Lambda_{k^\alpha}}), E_+) \leq \varepsilon\big\}\big) \leq C(E)\cdot|\Lambda_k|^\alpha\cdot\varepsilon^n. \tag{2.8}$$

So for a band edge energy E_+ using Theorem 2.1 for $\theta = 0$ we obtain the initial estimate to start multi-scale analysis. This proves that the spectrum of A_ω is exponentially localized in some interval around the energy E_+ i.e that in some neighborhood of E_+ eigenfunction associated to energies in that interval are exponentially localized. More precisely we have

Theorem 2.2. *Let A_ω defined by (1.1). We assume that* **(A.1)**, **(A.2)** *and* **(A.3)** *hold and the single site decrease polynomially. There exists $\varepsilon_0 > 0$ such that*
(i) $\Sigma \cap [E_+, E_+ + \varepsilon_0] = \Sigma_{pp} \cap [E_+, E_+ + \varepsilon]$.
(ii) an eigenfunction corresponding to an eigenvalue in $[E_+, E_+ + \varepsilon_0]$ decays exponentially.
(iii) there exist $p > 0$ such that

$$\mathbb{E}\Big\{\sup_{t>0}\Big|\Big|\Big|X\Big|^p e^{itA_\omega}P_{[E_+,E_++\varepsilon_0]}(A_\omega)\chi_K\Big|\Big|\Big\} < +\infty.$$

Here $P_I(A_\omega)$ is the spectral projection on the interval I, χ_K is the characteristic function of K, K is a compact of $\mathbb{R}^d$ and X is the position operator.

The result of Theorem 2.1 and so that of Theorem 2.2 can be related to the behavior of the integrated density of states in the neighborhood of the so-called fluctuation boundary E_+[27, 28, 35, 36]. For this, we recall that the integrated density of states is defined as follows:we note by $A_{\omega,\Lambda}$ the restriction of A_ω to Λ with self adjoint boundary conditions. As A_ω is elliptic, the resolvent of $A_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $A_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity [39]. We define

$$N_\Lambda(E) = \frac{1}{|\Lambda|} \cdot \#\{\text{eigenvalues of } \mathrm{A}_{\omega,\Lambda} \leq \mathrm{E}\}. \tag{2.9}$$

Here vol(Λ) is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^2$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of A_ω (IDS for the short form). See [19].

For what concern the IDS, we prove that an operator defined by (1.1) exhibits Lifshitz tails in the neighborhood of E_+, this was done for various families of random operators (See [21, 25, 26, 32, 33, 34, 36]). For acoustic operators this is proved in [33] under the assumption of non-degeneracy of the IDS of the back-ground operator A_0 which is relaxed here.

Theorem 2.3. *Let A_ω the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *hold. Then*

$$\limsup_{\varepsilon\to 0,\varepsilon>0} \frac{\log|\log\left(N(E_+ + \varepsilon) - N(E_+)\right)|}{\log\varepsilon} < 0. \tag{2.10}$$

Theorem 2.3 says that the IDS decays at least exponentially fast at E_+ without any information on the rate of decay. Notice that when the single site potential u is of long range type, then we have already a better result. Indeed in this particular case we have the the estimation of the rate [34].

Theorem 2.4. *Assume that the single site u is compactly supported. Then there exists $0 < \alpha \leq +\infty$ such that*

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\log |\log \big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log \varepsilon} \leq -\alpha \tag{2.11}$$

From (2.11) we deduce that roughly speaking,

$$N(E_+ + \varepsilon) - N(E_+) \lesssim e^{-\varepsilon^{-\alpha}} \text{ as } \varepsilon \downarrow 0.$$

So the IDS is very thin near E_+. Notice that the equality holds under other assumptions on u. The exponent α is called the *Lifshitz exponent*. In [33] it is proved that $\alpha = \frac{d}{2}$ if and only if

$$\lim_{\varepsilon \to 0^+} \frac{\log \big(n(E_+ + \varepsilon) - n(E_+)\big)}{\log \varepsilon} = \frac{d}{2}.$$

Where $n(E)$ is the IDS of the back ground periodic operator A_{ω^+}. In the general case the computation of α is not obvious. We end this section by a remark.

Remark 2.5. *Theorems 2.1 ; 2.2; 2.3 and 2.4 are stated for upper bands edges of spectral gap. Under adequate assumption their results are also true for lower bands edges.*
Notice that the exponent α in Theorem 2.4 can be determined independently of the behavior of $n(E)$ which refutes the conjuncture that $N(E)$ behaves roughly as $e^{-\frac{1}{n(E)}}$, if one takes the double logarithm of both terms. (see [36]).

2.2 Lifshitz tails

2.2.1 The proof of Theorem 2.4

The proof of Theorem 2.4 will be done on three steps. Notice in advance that the first and the second steps are independents of the dimension in spite of the third one.

First step:

First we do a cut-off in energy for the density of states. More precisely, this means that we construct a bounded random operator A_ω^0 (actually $A_\omega^0 = \Pi_0 A_\omega \Pi_0$. Here Π_0, is the spectral projection for the background operator A_{ω^+} at the energy band starting at E_+.) that also has E_+ as spectral edge and such that, up to some scaling factor in energy, the IDS of the new random operator controls the behavior of $N(E) - N(E_+)$ near E_+. So we have the following result:

Theorem 2.6. *Let A_ω be defined by (1.1). We assume that **(A.1)** and **(A.2)** hold. There exists $E_0 > E_+$ and $C > 1$such that, for $E_+ \leq E \leq E_0$ we have*

$$N(E) - N(E_+) \leq N_{\mathcal{E}_0}\Big(C \cdot (E - E_+) + E_+\Big). \tag{2.12}$$

Where $N_{\mathcal{E}_0}$ is the IDS of $A_\omega^0 = \Pi_0 A_\omega \Pi_0$.

The proof of the last statement is already given in [33] and it is based on the use of the technique of the periodic approximations [25]. For $k \in \mathbb{N}^*$, we define a new periodic operator $A_{\omega,k}$ by

$$A_{\omega,k} = -\nabla \frac{1}{\varrho_{\omega,k}} \nabla,$$

where the function $\varrho_{\omega,k}$ is defined by

$$\varrho_{\omega,k} = \varrho_0 \left(1 + \sum_{\gamma \in \Lambda_k \cap \mathbb{Z}^2} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^2} u_{\gamma+\beta} \right),$$

Λ_k is the cube

$$C_k = \{x \in \mathbb{R}^2; \forall 1 \leq j \leq 2,\ -\frac{2k+1}{2} < x_j \leq \frac{2k+1}{2}\}.$$

$A_{\omega,k}$ is $(2k+1)\mathbb{Z}^2$-periodic and essentially self adjoint operator. We denote by $N_{\omega,k}$ the IDS of $A_{\omega,k}$.

First we approach the IDS of A_ω by the IDS of periodic approximations, then we show that for these periodic periodic approximation, we may localize the IDS in energy; that is we control the behavior of the IDS near E_+ by the behavior of the IDS of some reduced operator (which is a bounded 2-dimensional Jacobi matrix). We then compute the limit of the IDS of this reduced operator to get the convergence to the IDS ; $N_{\mathcal{E}_0}$ and thus to get the announced control on the IDS of A_ω.

As in [33] we shift the energy i.e instead of considering A^0_ω we consider $\tilde{A}^0_\omega = A^0_\omega - E_+ \cdot I_{n_0}$, where, I_{n_0} is the $n_0 \times n_0$ identity matrix considered as a multiplication operator on $L^2(\mathbb{T}^2) \otimes \mathbb{C}^{n_0}$. The new operator $\tilde{A}^0_\omega$ have 0 as a bottom of his spectrum. In the following, we identify $\tilde{A}^0_\omega$ to A^0_ω.

<u>Second step:</u>

To study the behavior of $N_{\mathcal{E}_0}(E)$ we shed light on A^0_ω. Indeed using the Wannier basis for $\mathcal{E}_0 = \Pi_0(L^2(\mathbb{R}))$ we can construct an orthonormal system of vectors that span $\mathcal{E}_0$ and which are analytic [25, 33]. A^0_ω is unitary equivalent to a random Jacobi matrix acting on $L^2(\mathbb{T}^2) \otimes \mathbb{C}^{n_0}$ which will be denoted by

$\mathcal{A}^0_\omega(\theta)$ and it is written $\mathcal{A}^0_{\omega^+}(\theta)+\Delta\mathcal{A}^0_\omega(\theta)$, with

$$\mathcal{A}^0_{\omega^+}(\theta)=\begin{pmatrix}\langle v_1(\cdot,\theta),A_{\omega^+}(\theta)v_1(\cdot,\theta)\rangle_{L^2([0,1[^2)} & \dots & \langle v_1(\cdot,\theta),A_{\omega^+}(\theta)v_{n)}(\cdot,\theta)\rangle_{L^2([0,1[^2)}\\ \vdots & \ddots & \vdots\\ \langle v_{j(\theta^0)}(\cdot,\theta),A_{\omega^+}(\theta)v_1(\cdot,\theta)\rangle_{L^2([0,1[^2)} & \dots & \langle v_n(\cdot,\theta),A_{\omega^+}(\theta)v_{j(\theta)}(\cdot,\theta)\rangle_{L^2([0,1[^2)}\end{pmatrix};$$

and $\Delta\mathcal{A}^0_\omega(\theta)$ is the operator acting on $L^2(\mathbb{T}^2)\otimes\mathbb{C}^{n_0}$ defined for $t\in L^2(\mathbb{T}^*)\otimes$ $\mathbb{C}^{n_0}$ by

$$\langle\Delta\mathcal{A}^0_\omega t,t\rangle=\sum_{\gamma\in\mathbb{Z}^d}\omega^+_\gamma\langle V^0_\gamma t,t\rangle.$$

where $\omega^+_\gamma=\omega^+-\omega_\gamma$ and V^0_γ is an operator matrix whose coefficients $\big(v_{i,j}(\gamma,\theta,\theta')\big)_{1\le i,j\le n_0}$are operators with the following kernels

$$v_{i,j}(\gamma,\theta,\theta')=\int_{\mathbb{R}^d}\frac{u_\gamma(x)}{D_\omega(x)}\nabla v_j(x,\theta)\overline{\nabla v_{j'}(x,\theta')}dx=\sum_{i=1}^2\int_{\mathbb{R}^d}\frac{u_\gamma(x)}{D_\omega(x)}\partial_{x_i}v_j(x,\theta)\overline{\partial_{x_i}v_{j'}(x,\theta')}dx.$$

For the unitary correspondence between $\Pi_0(L^2(\mathbb{R}))$ and $L^2(\mathbb{T}^2)\otimes\mathbb{C}^{n_0}$ it suffice to consider $\mathcal{W}:\Pi_0(L^2(\mathbb{R}))\mapsto L^2(\mathbb{T}^2)\otimes\mathbb{C}^{n_0}$ defined by

$$\mathcal{W}(u)=\big(\langle(Uu)(\cdot,\theta),v_j(\cdot,\theta)\rangle_{L^2([0,1[^2)}\big)_{1\le j\le n}\text{ and }\mathcal{W}^*(u)=\sum_{j=1}^{n_0}U^*\Big(u_j(\theta)v_j(\cdot,\theta)\Big).$$

Where U and U^* are given by (2.4) and (2.5).

We need also the following result:

Theorem 2.7. *[28] There exists*

1- $f:\mathbb{T}^2\to\mathbb{R}$ an analytique and positive function.

2- $G:\mathbb{T}^2\times\mathbb{T}^2\mapsto\mathcal{M}_n(\mathbb{C})$ a smooth function such that, for any $\theta\in\mathbb{T}^2, G(\theta,\theta)$ is a positive and definite matrix, and some constant $C>1$ such that:

$$\frac{1}{C}\cdot f\cdot I_n\le\mathcal{A}^0_{\omega^+}\quad\text{and}\quad\frac{1}{C}\cdot(f\cdot I_{n_0}+G_\omega)\le\mathcal{A}_\omega.\tag{2.13}$$

Here $\mathcal{M}_{n_0}(\mathbb{C})$ is the space of $n_0 \times n_0$ complex matrices and G_ω is the operator with the following kernel

$$G_\omega(\theta, \theta') = \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma^+ G(\theta, \theta') e^{i\gamma(\theta - \theta')}. \tag{2.14}$$

The proof of the last result is based on the study of the geometric properties of the set of points of $\mathbb{T}^2$ where we lose analyticity of $\mathcal{A}_\omega^0(\theta)$. Roughly this set is the meeting points of two or more Folquet eigenvalues. For 1 it suffice to take f as the determinant of some analytic and positive matrix obtained from $\mathcal{A}_\omega^0(\theta)$ by unitary analytical transformations.
The positivity of G_ω it is based on the unique continuation principe [31] and the smoothness of G is a consequence of the fact that u is compactly supported. More details of the proof are given in [28].
By Theorem 2.7 it is sufficient to consider the new operator $h_\omega = f \cdot I_{n_0} + G_\omega$ acting on $L^2(\mathbb{T}^2) \otimes \mathbb{C}^{n_0}$. This represents several advantages. Indeed the main gain here is that $h_\omega$is a regular operator i.e. that has a smooth kernel when considered acting on $L^2(\mathbb{T}^2) \otimes \mathbb{C}^{n_0}$ and so his composition with Fourier transform which we denote by $\hat{h}_\omega$ has off-diagonal coefficients that are rapidly decaying when it is considered acting on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^{n_0}$.
Let $L \in \mathbb{N}$ and $\Lambda_L \subset \mathbb{Z}^2$ be the cube centered at 0 of side length $2L$. Define $\hat{h}_{\omega,L} = \Pi_{\Lambda_L} \hat{h}_\omega \Pi_{\Lambda_L}$. Let $\hat{N}(E)$ be the IDS of $\hat{h}_\omega$ and $\mathbf{E}_{\omega,\varepsilon}$ be the events:

$$\{\hat{h}_{\omega,L} \text{ admits an eigenvalue} \leq \varepsilon\}.$$

It is known [36] that, with a good enough precision, we can estimate the IDS of the reduced operator between 0 and ε by the probability that the reduced operator restricted to some large cube has an eigenvalues in the interval $[0, \varepsilon]$. Precisely we have the following result

Proposition 2.8. *[28]*

$$\lim_{\varepsilon\to 0^+}\frac{\log|\log\left(\hat{N}(\varepsilon)-\hat{N}(0)\right)|}{\log\varepsilon}=\lim_{\varepsilon\to 0^+}\frac{\log|\log\left(\mathbb{P}(\mathbf{E}_{\omega,\varepsilon})\right)|}{\log\varepsilon}. \tag{2.15}$$

Third step: From what it was said previously we deal only with the computation of the right hand side of (2.15). This is essentially a question in random Fourier series. It is the subject of the following paragraph.

Notice that as we are interested on the behavior of the IDS of $\hat{h}_\omega$ near 0 and both f and G are non negatives so both must be smaller than ε. For this we have have to take into account the set of points where f vanish.

Let f be an analytic function on $\mathbb{T}^2$. Define $\mathcal{S}=\{\theta\in\mathbb{T}^2;\ f(\theta)=0\}$.

Let $\theta_0\in\mathcal{S}$ and consider the Newton diagram of f at θ_0; i.e.,

a) Express f as a Taylor series expansion at θ_0,

$$f(\theta^1,\theta^2)=\sum_{ij}a_{ij}(\theta^1-\theta_0^1)^i(\theta^2-\theta_0^2)^j,\ \theta=(\theta^1,\theta^2).$$

b) The Newton polygon of f at θ_0 is defined to be the convex hull of the union of all quadrants in $\mathbb{R}^2_{\geq 0}$ with corners at the points (i,j) with $a_{ij}\neq 0$.

c) The Boundary of the polygon is called the Newton diagram.

Extend each line segment of he Newton diagram to a complete line and intersect it with the diagonal line $\theta_1=\theta_2$. this gives a collection of points (a_k,a_k), one for each boundary segment. Take the reciprocal of the largest a_k and call this number $\tilde{\alpha}(f,\theta_0)$.

Define $\alpha(f,\theta_0)=\min\{\tilde{\alpha}(f\circ T_0,\theta_0):T_0(\cdot)=\theta_0+T(\cdot-\theta_0),T\in SL(2,\mathbb{R})\}$.

Similarly, define $\alpha(f,\theta)$ if θ is any other point in $\mathcal{S}$, the zero set f. Then, one defines the Newton decay rate α as

$$\alpha=\min_{\theta\in\mathcal{S}}\alpha(f,\theta).$$

Let G be a smooth function on $\mathbb{T}^2 \times \mathbb{T}^2$ taking values in the $n_0 \times n_0$ complex matrices, and assume that G is positive on the diagonal and $G(\theta, \theta)$ is positive definite for each $\theta \in \mathbb{T}^2$.

For $u, v \in (L^2(\mathbb{T}), \mathbb{C}^{n_0})$ we define $B(u, v) : \mathbb{Z}^d \longrightarrow \mathbb{C}$ by

$$B(u,v)(n) = \int_{\mathbb{T}^2 \times \mathbb{T}^2} \langle G(\theta, \theta')u(\theta), v(\theta')\rangle e^{in\cdot(\theta-\theta')} d\theta d\theta'. \tag{2.16}$$

Let $\varepsilon > 0$ and $M_\varepsilon < \infty$ be parameters. Let $\mathcal{P}(M_\varepsilon)$ be the set of $\mathbb{C}^2$valued functions on $\mathbb{T}^2$ whose Fourier transforms are supported in the square Q_{M_ε} centered at the origin with side length M_ε, and

$$H(\omega, u) = \int_{\mathbb{T}^2} f(\theta)|u(\theta)|^2 d\theta + \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma^+ B(u,u)(\gamma).$$

$$\begin{aligned} h(M_\varepsilon, \omega) &= \inf_{\substack{u \in \mathcal{P}(M_\varepsilon) \\ \|u\|_2 = 1}} H(\omega, u) \\ \mathbb{P}(\mathbf{E}_{\omega,\varepsilon}) &= \mathbb{P}(h(M_\varepsilon, \omega) < \varepsilon). \end{aligned} \tag{2.17}$$

Let $\mathcal{S}$ be a random subset of the square $[-M_\varepsilon, M_\varepsilon] \times [-M_\varepsilon, M_\varepsilon]$. As f is positive and G_ω is definite positive, we can express $\mathbb{P}(\mathbf{E}_{\omega,\varepsilon})$ to be the probability that there is a function $u \in L^2(\mathbb{T}^2)$such that

$$\int_{\mathbb{T}^2} f(\theta)|u(\theta)|^2 d\theta \leq \varepsilon \|u\|_2^2 \ \text{ and } \ \text{supp}(\hat{\mathrm{u}}) \subset \mathcal{S}.$$

Thus the Fourier coefficient of u are to be supported on the random set, and u is to be mostly supported near the zero set of f. So we have to study the probability $\mathbb{P}(\mathbf{E}_{\omega,\varepsilon})$ as ε goes to zero and M_ε goes to infinity. The behavior of $\mathbb{P}(\mathbf{E}_{\omega,\varepsilon})$ is a question of analytical geometry and estimation of oscillatory integrals [38, 46]. It turns out that $\mathbb{P}(\mathbf{E}_{\omega,\varepsilon})$ behaves roughly as $e^{-\varepsilon^{-\alpha}}$. See section 3 of [28]. This is end the proof of Theorem 2.4.

2.2.2 The proof of Theorem 2.3

When u is not compactly supported, first we bring ourselves to the case when u is compactly supported. Precisely for $R > 0$ large and χ_R be the characteristic function of the ball of center 0 and radius R we define the operator
$A_\omega^R = A_{\omega^+} + \Delta A_\omega^R$, where

$$\Delta A_\omega^R = -\nabla V_\omega^R \nabla \text{ and } V_\omega^R = \sum_{\gamma \in \mathbb{Z}^2} \omega_\gamma^+ \frac{u_\gamma^R}{D_\omega},$$

with $u^R = u \cdot \chi_R$. As $A_\omega^R \leq A_\omega$ one has

$$N(E) \leq N_R(E), \ \forall E \in \mathbb{R}.$$

Here N_R is the IDS of A_ω^R.
Notice also that A_ω^R converges to A_ω in the resolvent sense uniformly in $(\omega_\gamma)_{\gamma \in \mathbb{Z}^2}$. So for R large enough and $E \in (E_+ - \delta, E_+)$,

$$N(E) = N_R(E_+).$$

Hence, we get

$$N(E_+ + \varepsilon) - N(E_+) \leq N_R(E_+ + \varepsilon) - N_R(E_+).$$

But for A_ω^RTheorem 2.4 is true. This ends the proof of Theorem 2.3.

2.3 Localization for the acoustic waves:the proof of Theorems 2.1 and 2.2

All the proofs of localization, except in the discrete case [1] for the multi-dimensional case, use the method of the multiscale analysis. This method was used for the first time by Fröhlich and Spencer, at the beginning of the eighties [12] and it knew many extensions and simplifications to lead to the form described in the section below. This analysis makes possible to obtain information on the operator in the hole space, starting from information on the operator restricted to cubes of finished size [6]. We omit the detail, we note that the Combes-Thomas estimate and the decomposition of resolvent in the multiscale argument works for our case with essentially no modification. We refer the reader to [6, 20, 23, 40] for detailed description of results mentioned below.

2.4 Initial estimate:Theorem 2.1

We start by verifying **(P1)**. This already has been done in [3, 4, 5, 22, 23] for Schrödinger operators and in [40] for the model given by the form of (1.1) under the assumption of a special mode of disorder. More precisely their approach requires a sufficiently fast decay of the probability distribution $\mathbb{P}_{\omega_0}$ at the approach to the edges of the support. Indeed, they assumed that there exists some $\tau > 0$ ($\tau > \frac{3d}{2}$ in [3]; $\tau > d$ in [22] and $\tau > \frac{d}{2}$ in [23]):such that $\forall \varepsilon > 0$,

$$\mathbb{P}_{\omega_0}([0, \varepsilon]) = \int_0^{\varepsilon} g(s)(s)ds \leq \varepsilon^{\tau},$$

or

$$\mathbb{P}_{\omega_0}([\omega^+ - \varepsilon, \omega^+]) = \int_{1-\varepsilon}^{1} g(s)ds \leq \varepsilon^{\tau}.$$

According to whether one considers the higher or lower edges of the gap. Those conditions are relaxed here.

In what follows, to get **(P1)** one will adapt an other approach which uses the fact that at the internal band edges, the IDS of A_ω exhibits Lifshitz tails. See [24, 33, 47, 34].

To get **(P1)**, in an interval of energy which contains E_+ one will compare the number of states of A_ω to that of $A_{0,k}$ (here $A_{0,k}$ is A_ω restricted to the cube Λ_k). This will be done by a comparison between $N(A_\omega, E)$ and $N(A_{0,k}, E)$. At the bottom of the spectrum, such technique was used with the Čebiševe inequality by Klopp, in [24], and by Kirsch, Stollman and Stolz in [22]. For the internal band edges one refers to the work of Veselić in [28, 47]. All results quoted here are given in the Schrödinger case.

Notice that (2.10) is a limit when one approaches the band edge. We ask for an estimate to hold on an interval. This is the purpose of the following computation.

Lemma 2.9. *For all $n \in \mathbb{N}$ and $\alpha \in]0,1[$; there exists $k_2 \in \mathbb{N}$ such that if $k \geq k_2$ we have*

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq k^{-n}. \tag{2.18}$$

Here we recall that $N_{\omega,k}$ denotes the IDS of the periodic operator $A_{\omega,k}$.

Proof. Lemma 2.9 is a consequence of Theorem 2.3 and the fact that the left side of (2.18) is exponentially approximated by the IDS (See Theorem 7.1 of [28] which is true in our case). $\square$

We need also the two following lemmas. See [35] for the proof.

Lemma 2.10. *[47] For all $\varepsilon > 0$, we have*

$$\int_{\mathbb{T}_k^*} \mathbb{P}\Big(\{\omega | \sigma(A_{\omega,k}(\theta)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}\Big) d\theta \leq (2\pi)^2 \mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big).$$

Where $\mathbb{T}_k^* = \mathbb{R}^2/(2k+1)\pi\mathbb{Z}^2$

Let us note the j^{th} Floquet eigenvalue of $A_{\omega,k}(\theta)$ by $E_j(A_{\omega,k}(\theta))$.

Lemma 2.11. *For all $\theta, \theta' \in \mathbb{T}_k^*$ and $j \in \mathbb{N}$ there exists $M_{j,k} > 0$ such that*

$$|E_j(A_{\omega,k}(\theta) - E_j(A_{\omega,k}(\theta')| \leq M_{j,k}|\theta - \theta'|. \tag{2.19}$$

Now using Lemma 2.10 and Lipshitz continuity in θ of the Floquet eigenvalues, we will be able to estimate the probability to find an eigenvalue of $A_{0,k}(\theta^0)$ in the interval $[E_+, E_+ + \varepsilon]$. Here $A_{0,k}(\theta^0) = A_{0,k}|_{\Lambda_k}^{\theta^0}$ is the operator $A_{0,k}$ restricted to Λ_k with θ^0-periodic conditions. More precisely we have

Lemma 2.12. *Let $\theta^0 \in \mathbb{T}_k^*$. For all $\varepsilon > 0$ we have*

$$\mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k}(\theta^0)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}\Big)$$
$$\leq \frac{(2\pi)^2}{\mathrm{vol}(\mathbb{T}_k^*)}\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon + ck^{-1}) - N_{\omega,k}(E_+)\Big). \tag{2.20}$$

Proof. From Lemma 2.11 we know that there exists $M_{j,k} > 0$ such that for all $j \in \mathbb{N}$ we have

$$|E_j(A_{0,k}(\theta)) - E_j(A_{0,k}(\theta'))| \leq M_{j,k}|\theta - \theta'|.$$

As we are interested only in the eigenvalues in $[E_+, E_+ + \varepsilon]$, the $M_{j,k}$ can be chosen uniformly in j and k. So there exists $M > 0$ such that

$$M \geq M_{j,k}, \quad \forall j, k \in \mathbb{N}.$$

Now we will use this to estimate

$$\mathbb{P}(\{\omega|\sigma(A_{0,k}(\theta^0)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}).$$

For this it is enough to estimate

$$\mathbb{P}(\{\omega|\exists\ j \in \mathbb{N};\ E_j(A_{0,k}(\theta^0)) \in [E_+, E_+ + \varepsilon]\}) =$$

$$\frac{1}{\mathrm{vol}(\mathbb{T}_k^*)}\int_{\mathbb{T}_k^*}\mathbb{P}(\{\omega|\exists\ j\in\mathbb{N};\ E_j(A_{0,k}(\theta^0))\in[E_+,E_++\varepsilon]\})d\theta. \tag{2.21}$$

Thus if $E_j(A_{0,k}(\theta^0)) \in [E_+, E_+ + \varepsilon]$, the Lipshitz continuity implies that $E_j(A_{0,k}(\theta)) \in [E_+, E_+ + \varepsilon + M\mathrm{diam}(\mathbb{T}_k^*)],\ \forall\theta \in \mathbb{T}_k^*$ with $\mathrm{diam}(\mathbb{T}_k^*) \leq ck^{-1}$. So we get that

$$\begin{aligned}
(2.21) &\leq \frac{1}{\mathrm{vol}(\mathbb{T}_k^*)}\int_{\mathbb{T}_k^*}\mathbb{P}(\{\omega|\exists\ j\in\mathbb{N};\ E_j(A_{0,k}(\theta))\in[E_+,E_++\varepsilon+ck^{-1}]\})d\theta \\
&= \frac{1}{\mathrm{vol}(\mathbb{T}_k^*)}\int_{\mathbb{T}_k^*}\mathbb{P}(\{\omega|;\sigma(A_{0,k}(\theta))\cap[E_+,E_++\varepsilon+ck^{-1}]\neq\emptyset\})d\theta \\
&\leq \frac{(2\pi)^d}{\mathrm{vol}(\mathbb{T}_k^*)}\mathbb{E}\Big[N_{\omega,k}(E_++\varepsilon+ck^{-1})-N_{\omega,k}(E_+)\Big].
\end{aligned}$$

This proves Lemma 2.12. □

For $\alpha < 1$, (α is given in Lemma 2.9) and $\varepsilon = k^{-\alpha} > 0$ there exists $k_3 \in \mathbb{N}$such that if $k > k_3$, we have $k^{-\alpha} + ck^{-1} \leq 2k^{-\alpha}$where k_3 depends on α and c. Then we obtain that for $k > k_3$;

$$\mathbb{E}\Big(N_{\omega,k}(E_++2k^{-\alpha})-N_{\omega,k}(E_+)\Big)\leq ck^{-n}.$$

So Lemma 2.12 gives:

$$\mathbb{P}\Big(\{\omega|\sigma(A_{0,k}(\theta^0))\cap[E_+,E_++k^{-\alpha}]\neq\emptyset\}\Big)\leq ck^{-n+2}.$$

Let $p > 2$ and n given in Lemma 2.9 such that $n > p+2$. There exists $k_4 \in \mathbb{N}$ such that if $k \geq k_4$we have $ck^{-n+2} \leq k^{-p}$. This implies that for $k > k_4$we have

$$\mathbb{P}\Big(\{\omega|\sigma(A_{0,k}(\theta^0))\cap[E_+,E_++k^{-\alpha}]\neq\emptyset\}\Big)\leq k^{-p}. \tag{2.22}$$

This ends the proof of the Theorem 2.1 and from what it was said in the beginning of section 2.3 Theorem 2.2 is proven. □

References

[1] M. Aizenman and Molchanov: *Localization at large disorder and at extreme energies:an elementary derivation.* Commun. Math. Phys. **157** (1993) p 245-278.

[2] P.W. Anderson: *Absence of diffusion in certain random lattices.* Physical Reviews **109** (1959) p 1492-1501.

[3] J. M. Barbaroux, J. M Combes and P.D Hislop : *Localization near band edges for random Schrödinger operators.* Helv. phys. Acta. **70** (1997) N° 1-2 p 16-43.

[4] J. M. Combes P.D. Hislop and A. Tip: *Band edge localization and the density of sates for acoustic and electromagnetic waves in random media.* Ann. I.A.P **70** (1999) N° 4 p 381-428.

[5] D. Damanik and P. Stollmann: *Multiscale Analysis implies Strong Dynamical Localization.* Geom. Funct. Anal. **26** (2001) N° 3 p 411-428.

[6] A. von Dreifus and A. Klein: *A new proof of localization in the Anderson tight binding model.* Commun. Math. Phys. **124** (1989) p 285-299.

[7] A. Figotin and A. Klein: *Localization of Classical Waves I :Acoustic Waves.* Commun. Math. Phys. **180** (1996) p 439-482.

[8] A. Figotin and A. Klein: *Localization of Classical Waves II :electromagnetic Waves.* Commun. Math. Phys. **184** N°2 (1997) p 411-441.

[9] A. Figotin and A. Klein: *Localization of Classical Waves created by defects.* J. Stat. Phys. **86** (1997) p 165-177.

[10] A. Figotin and A. Klein: *Localization of light in lossless inhomogeneous dielectrics.* J. Opt. Soc. Amer. **A 15** N° 6 (1998) p 1748-1773.

[11] A. Figotin and P. Kuchment: *Band-Gap Structure of Spectra of Periodic Dielectric and Acoustic Media. I Scalar Model.* SIAM Jour. App. Math. **56** N° 1-02 (1996) p 1-23.

[12] J. Fröhlich and T. Spencer: *Absence of diffusion in the Anderson tight binding model.* Commun. Math. Phys. (1983) **88** p 151-184.

[13] F. Germinet and A. Klein: *Bootstrap Multiscale Analysis and Localization in random media.* Commun. Math. Phys. N° 2 **222** (2001) p 415-448.

[14] F. Germinet and S. De Bièvre: *Dynamical localization for discrete and continuous random Schrödinger operators.* Commun. Math. Phys. **194** (1998) p 323-341.

[15] K. M. Ho, C. T. Chan and C. M. Soukoulis: *Existence of Photonic Gap in Periodic Dielectric Structures.* Phy. Rev. Lett. **62** (1989) p 575-797.

[16] S. John and R. Rangavajan: *Optimal structures for classical wave localization:An alternative to the Ioffe-Regei criterion.* Phy. Rev. **B,38** (1988) p 1010-1043.

[17] S. John : *The localization of light in photonic gaps and localization.* NATO ASI Serie B:Physical (1993) p 308-321.

[18] W. Kirsch: *On a class of random Schrödinger operators* Adv. in Appl. Math. **6** (1985) p 495-507.

[19] W. Kirsch: *Random Schrödinger operators* A Course Lecture Notes In Phys. **345** (1989)Springer-Verlag, Berlin p 264-370.

[20] W. Kirsch: *Wegner estimates and Anderson localization for alloy-type potentials.* Math. Z. **221** (1996) p 507-512.

[21] W. Kirsch and F. Martinelli: *On the the Spectrum Of Schrödinger operators with a Random Potential.* Commun. Math. Phys. **85** (1982) p 329-350.

[22] W. Kirsch, P. Stollmann and G. Stolz: *Localization for random perturbation of periodic Schrodinger operators.* Random Oper. Stochastic Equations **6** (1998) p 241-268.

[23] W. Kirsch, P. Stollmann and G. Stolz: *Anderson localization for random Schrodinger operators with long range interactions.* Commun. Math. Phys. **195** (1998) p 495-507.

[24] F. Klopp: *Localization for some continus Random Schrödinger operators.* Commun. Math. Phys. **167** (1995)

[25] F. Klopp: *Internal Lifshitz Tails For Random Perturbations Of Periodic Schrödinger Operators.* Duke Math. Jour. **98** (1999) N° 2 p 335-396.

[26] F. Klopp: *Internal Lifshitz Tails For long range single site potentials.* Jour. Math. Phys. **43** (2002) N° 6 p 2948-2958.

[27] F. Klopp: *Weak disorder localization and Lifshitz Tails:continuous hamiltonian.* Anna. I.A.P **3** (2002) N° 4 p 711-735.

[28] F. Klopp and T. Wolff: *Lifshitz tails for 2-dimensional random Schrödinger operators.* to appear.

[29] S. Kotani: *Lyapunov indices determine absolutely continues spectra of stationary one dimensional Schrödinger operators.* Stochastic Analysis (1984) p 225-247.

[30] P. Kuchment: *Floquet theory for partial differential equations* volume 60 of Operator Theory:Advances and applications Birkhäser, Basel, 1983.

[31] H. Najar : *Asymptotique de Lifshitz pour les opérateurs acoustiques aléatoires.* Thèse de doctorat, Université de Paris 13, Villetaneuse, November 2000.

[32] H. Najar: *Asymptotique de la densité d'états intégrée des opérateurs acoustiques aléatoires.* C. R. Acad. Sci. Paris, **333** I, p 191-194 (2001).

[33] H. Najar: *Lifshitz tails for random acoustic operators.* Jour. Math. Phys. **44** N° 4 (2003) p 1842-1867.

[34] H. Najar: *Asymptotic of the integrated density of states of acoustic operator with long range random perturbations.* (2002) Submitted.

[35] H. Najar: *Localization for divergence operators with long range random perturbations.* In preparation (2003).

[36] L. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators.* Springer-Verlag.

[37] L. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators* Springer-Verlag Heidelberg 1992.

[38] D. H. Phong and E. M. Stein: *The Newton polyhedron and oscillatory integral operators* Acta. Math. **179** N° 1 (1997) p 105-152.

[39] M. Reed and B. Simon: *Methods of Modern Mathematical vol IV:Analysis of Operators.* Academic Press, New York, 1978.

[40] P. Stollmann: *Strategies In localization Proofs for One-dimensional Random Schrödinger operators.* preprint.

[41] G. Stolz: *Localization for acoustic waves in random perturbation of periodic media.* Israel Jour. Math. **107** (1998) p 125-139.

[42] J. Sjöstrand: *Microlocal analysis for the periodic magnetic Schrödinger equation and related questions.* Lecture Notes in Math 1495, Springer-Verlag, Berlin (1991) p 237-332.

[43] M. P. Van Albada and A. Lagendijk: *Observation of Weak Localization of Light in Random Medium.* Phys. Rev. Lett. **55** (1985) p 2692-2712.

[44] E. Yablonovitch and T. J. Gmitter: *Photonic Band Structure:The FaceCentered-Cubic Case* Phys. Rev. Lett. **63** (1989) p 259-278.

[45] Z. Ye: *Wave localization in two Dimensions* Chinese Jour. Phy. **39** N°3 (2001) p 207-210.

[46] V. Varchenko: *Newton polyhedra and estimations of oscillatory integrals.* Funct. Anal. Appl **8** (1976) p 175-196.

[47] I. Veselić: *Localization for random perturbation of periodic Schrödinger operators with regular Floquet eigenvalues.* Technical report Univ. Bochum.