CONVEX REPLICA SIMMETRY BREAKING FROM POSITIVITY AND THERMODYNAMIC LIMIT

Pierluigi Contucci, Sandro Graffi

Dipartimento di Matematica

Università di Bologna, 40127 Bologna, Italy

e-mail:contucci@dm.unibo.it, graffi@dm.unibo.it

Abstract

Consider a correlated Gaussian random energy model built by successively adding one particle (spin) into the system and imposing the positivity of the associated covariance matrix. We show that the validity of a recently isolated condition ensuring the existence of the thermodynamic limit forces the covariance matrix to exhibit the Parisi replica symmetry breaking scheme with a convexity condition on the matrix elements.

The existence of the thermodynamic limit has been recently proved for the Sherrington-Kirckpatrick (SK) model [1], and more generally for any correlated Gaussian random energy (CGREM) model including the Derrida REM and the Derrida-Gardner GREM [2]. In this letter we point out that the proof of [2] may shed some light on the origin of Parisi's algebraic ansatz for the replica symmetry breaking (RSB) scheme which lies at the basis his solution [3] of the SK model.

Algebraically, Parisi's ansatz may be described as follows: start from the one by one matrix $q(0) > 0$. Take $q(1)$ as $0 < q(0) < q(1) < 1$ first replicate the system

$$R(q(0)) = \begin{pmatrix} q(0) & q(0) \\ q(0) & q(0) \end{pmatrix}, \tag{1}$$

then break the symmetry

$$BR(q(0)) = \begin{pmatrix} q(1) & q(0) \\ q(0) & q(1) \end{pmatrix}. \tag{2}$$

If we iterate n times the operation BR we end up with the Parisi RSB matrix of order n, $[BR]^n(q(0)) = Q^{(n)}$. Example: for $p = 3$ we get the 8×8 matrix

$$Q^{(3)} = \begin{pmatrix} q(3) & q(2) & q(1) & q(1) & q(0) & q(0) & q(0) & q(0) \\ q(2) & q(3) & q(1) & q(1) & q(0) & q(0) & q(0) & q(0) \\ q(1) & q(1) & q(3) & q(2) & q(0) & q(0) & q(0) & q(0) \\ q(1) & q(1) & q(2) & q(3) & q(0) & q(0) & q(0) & q(0) \\ q(0) & q(0) & q(0) & q(0) & q(3) & q(2) & q(2) & q(2) \\ q(0) & q(0) & q(0) & q(0) & q(2) & q(3) & q(2) & q(2) \\ q(0) & q(0) & q(0) & q(0) & q(2) & q(2) & q(3) & q(2) \\ q(0) & q(0) & q(0) & q(0) & q(2) & q(2) & q(2) & q(3) \end{pmatrix}, \tag{3}$$

As is well known, the RSB method consists in solving the saddle point equations for the SK free energy after n replicas assuming that the saddle point matrix has the form $Q^{(n)}$; the maximum is to be found among the functions on $[0,1]$ taking the values $q(k)$ at the prescribed points m_k at step k of RSB. Performing first the limit $k \to \infty$ and then the limit $n \to 0$ one gets:

$$\lim_{n\to 0}\lim_{k\to\infty}\frac{1}{n}\sum_{a,b=1}^{n} Q^k_{a,b} = -\int_0^1 q(x)\,dx \tag{4}$$

$q(x)$ is continuous on $[0,1]$ and is the (Parisi) order parameter of the local square magnetization because its inverse function $x(q)$ is such that $x(q) = \int_0^q P(s)\,ds$ where $P(q)$ is the overlap probability distribution between pure magnetization states. Hence $q(x(q)) = q$ and the elements $q(1),\ldots,q(k)$ are overlaps.

Even though it has been recently proved that the Parisi free energy is at least a lower bound of the SK free energy[4], the RSB is still far from a mathematical understanding. A major puzzle is still represented by the origin of the algebraic structure of the Parisi ansatz, which lies at the basis also of the Derrida-Gardner GREM[5].

In this letter we point out that the algebraic Parisi ansatz and, equivalently, the Derrida-Gardner GREM are generated by a precise prescription to reach monotonically the thermodynamic limit with a family of correlated gaussian random energy models (CGREM).

More precisely, we will build by recurrence over N, increasing the size by 1 at each step (equivalently, adding a spin) a family $E_\sigma(N)$ of CGREM for which we require:

1. $E_\sigma(N)$ (N-spins) has to be a subsystem of $E_\sigma(N+1)$ ($N+1$-spins);

2. The sequence $E_\sigma(N)$ fulfills the conditions ensuring the existence of the thermodynamic limit.

The two previous conditions imply that the matrix sequence C_N fulfills the RSB scheme, in the sense that for each N the correlation matrix C_N fulfills the Parisi ansatz at step N. In particular $E_\sigma(N)$ can be identified with the Derrida-Gardner GREM with a convex growth scheme.

For $N = 0$ the system is the *vacuum* (no particles) with random energy

$$E_0 = \xi_0(0) \tag{5}$$

ξ_0 is a centered Gaussian variable with variance $c(0)$, which we can consider as a one-by-one matrix. We include the system of 0 spins into a new one-spin system (for short: we add a spin) assuming at first that the one-spin Hamiltonian doesn't depend on the added spin

$$\tilde{E}_1(\sigma_1) = \xi_0(0) \; ; \tag{6}$$

the covariance matrix of this process is

$$\tilde{C}(1) = \begin{pmatrix} c(0) & c(0) \\ c(0) & c(0) \end{pmatrix} \; ; \tag{7}$$

This construction was called *lifting* in [2].

The one-spin system is a CGREM of size one only if its covariance matrix is a (non-degenerate) positive definite matrix. The elimination of the degenarcy requires the introduction in the Hamiltonian of a new Gaussian variable ξ_1 parametrizing the dependence on the added spin:

$$E_1(\sigma_1) = \xi_0(1) + \xi_1(1)\sigma_1 \; ; \tag{8}$$

the covariance matrix element is

$$Av(E_1(\sigma_1)E_1(\tau_1)) = Av(\xi_0(1)^2) + Av(\xi_1(1)^2)\sigma_1\tau_1 \ ; \tag{9}$$

Since

$$\sigma\tau = 2\delta_{\sigma,\tau} - 1 \tag{10}$$

we have

$$Av(E_1(\sigma_1)E_1(\tau_1)) = Av(\xi_0(1)^2) - Av(\xi_1(1)^2) + 2Av(\xi_1(1)^2)\delta_{\sigma_1,\sigma_2} \tag{11}$$

which we will write as

$$Av(E_1(\sigma_1)E_1(\tau_1)) = a_0(1) + a_1(1)\delta_{\sigma_1,\sigma_2} \ . \tag{12}$$

Defining $c(l) = \sum_{k=0}^{l} a_k$ we have

$$C(1) = \begin{pmatrix} c(1) & c(0) \\ c(0) & c(1) \end{pmatrix}, \tag{13}$$

with $c(1) > c(0)$

To iterate the procedure, let us first describe the second step, i.e. the addition of a second spin to build a 2-spin system. As before, we first assume independence on the newly added spin variable. The covariance matrix turns then out to be (with the lexicographic order of the spin configurations)

$$\tilde{C}(2) = \begin{pmatrix} c(1) & c(1) & c(0) & c(0) \\ c(1) & c(1) & c(0) & c(0) \\ c(0) & c(0) & c(1) & c(1) \\ c(0) & c(0) & c(1) & c(1) \end{pmatrix} . \tag{14}$$

Again, this matrix is the covariance of a CGREM process only if it positive definite and non-degenerate. As above, this requires the dependence on the second spin variable. Among the possible ways to parametrize this dependence we choose the *minimal* one, namely the preceding one which only modifies the subprincipal diagonals:

$$\bar{C}(2) = \begin{pmatrix} c(1) & \bar{q}_2 & c(0) & c(0) \\ \bar{q}_2 & c(1) & c(0) & c(0) \\ c(0) & c(0) & c(1) & \bar{q}_2 \\ c(0) & c(0) & \bar{q}_2 & c(1) \end{pmatrix}, \tag{15}$$

with $c(0) < \bar{q}_2 < c(1)$. We relabel the elements $c(1) \to c(2)$ and $\bar{q}_2 \to c(1)$ obtaining the final 2-spin covariance matrix

$$C(2) = \begin{pmatrix} c(2) & c(1) & c(0) & c(0) \\ c(1) & c(2) & c(0) & c(0) \\ c(0) & c(0) & c(2) & c(0) \\ c(0) & c(0) & c(1) & c(2) \end{pmatrix}, \tag{16}$$

with $c(2) > c(1) > c(0) > 0$. This last condition implies that the matrix is positive definite because the 4 principal minors are

$$\Delta_1 = c(2) > 0 \ , \tag{17}$$

$$\Delta_2 = c(2)^2 - c(1)^2 > 0 \ , \tag{18}$$

$$\Delta_3 = [(c(2) - c(1)][c(2)(c(1) + c(2)) - 2c(0)^2] > 0 \tag{19}$$

and

$$\Delta_4 = [c(2) - c(1)]^2[((c(1) + c(2))^2 - 4c(0)^2] > 0 \tag{20}$$

Corrispondigly with the spin representation we would have

$$E_2(\sigma_1, \sigma_2) = \xi_0(2) + \xi_1(2)\sigma_1 + \xi_2(2)\sigma_2 + \xi_{1,2}\sigma_1\sigma_2 \ , \tag{21}$$

and

$$\begin{aligned} Av(E_2(\sigma_1, \sigma_2)E_2(\tau_1, \tau_2)) &= Av(\xi_0(2)^2) - Av(\xi_1(2)^2) - Av(\xi_2(2)^2) + Av(\xi_{1,2}(2)^2) \\ &+ \delta_{\sigma_1,\tau_1}[2Av(\xi_1(2)^2) - 2Av(\xi_{1,2}(2)^2)] \\ &+ \delta_{\sigma_2,\tau_2}[2Av(\xi_2(2)^2) - 2Av(\xi_{1,2}(2)^2)] \\ &+ \delta_{\sigma_1,\tau_1}\delta_{\sigma_2,\tau_2}4Av(\xi_{1,2}(2)^2) \ ; \end{aligned}$$

and since we choose

$$Av(\xi_2(2)^2) = 2Av(\xi_{1,2}(2)^2) \tag{22}$$

the covariance matrix exibit the RSB scheme:

$$Av(E_1(\sigma_1, \sigma_2)E_1(\tau_1\tau_2)) = a_0(2) + a_1(2)\delta_{\sigma_1,\sigma_2} + a_2(2)\delta_{\sigma_1,\tau_1}\delta_{\sigma_2,\tau_2} \ . \tag{23}$$

In general we will assume that our construction is done adding at each step the N-th spin variable and the newly added interaction terms are indipendent realization of the same Gaussian distribution or, in other terms, the distribution of the Gaussian random variables depends only on the last index. The general scheme is the described by a correlated Gaussian process

$$E_N(\sigma) = \xi_0(N) + \sum_{1\leq i\leq N} \xi_i(N)\sigma_i + \sum_{1\leq i<j\leq N} \xi_{i,j}(N)\sigma_i\sigma_j + \ldots + \xi_{1,2,\ldots,N}\sigma_1\cdots\sigma_N \tag{24}$$

in which the distribution of each Gaussian variable depends only on the last index; its covariance matrix:

$$Av(E_N(\sigma)E_N(\tau)) = \sum_{k=0}^{N} a_k(N) \prod_{i=0}^{k} \delta_{\sigma_i,\tau_i} \ , \tag{25}$$

turns out ot fulfill the RSB structure An important observation is that we may, within the positivity conditions, chose a subset of c's that fulfill the conditions for the existence of the thermodynamical limit as shown in [2]. Here we observe that those conditions translate into a convexity property for the matrix elements. In fact introducing the two complementary liftings (labelled as left l and right r) of the one-spin system:

$$C_l(2) = \begin{pmatrix} c(2) & q(2) & c(0) & c(0) \\ c(2) & c(2) & c(0) & c(0) \\ c(0) & c(0) & c(2) & c(2) \\ c(0) & c(0) & c(2) & c(2) \end{pmatrix}, \tag{26}$$

and

$$C_r(2) = \begin{pmatrix} c(2) & q(0) & c(2) & c(0) \\ c(0) & c(2) & c(0) & c(2) \\ c(0) & c(2) & c(2) & c(0) \\ c(2) & c(0) & c(0) & c(2) \end{pmatrix}, \tag{27}$$

the conditions developed in [2] are, for the symmetric sub-division of the system into 2 subsystems of the same size,

$$C(2) \leq \frac{1}{2}[C_l(2) + C_r(2)] \tag{28}$$

(component-wise) which imposes

$$c(1) \leq \frac{1}{2}[c(0) + c(2)] \; . \tag{29}$$

For general N the conditions in [2] turn out to be a set of 2^N relations between the matrix elements:

$$Q(N) \leq \frac{1}{2}[C_l(N) + C_r(N)] \tag{30}$$

With respect to the size $N-1$ the only new relationships required only involve the quantity $c(N), c(N-1), c(N-2)$. Indeed at step N the only new inequalities (with respect to those at step $N-1$) are those coming from the four by four principal minors of the three matrices. After the operation of re-enumeration these inequalities become (element wise)

$$Q^{(4)}(N) \leq \frac{1}{2}[C_l^{(4)}(N) + C_r^{(4)}(N)] \tag{31}$$

with

$$Q^{(4)}(N) = \begin{pmatrix} c(N) & c(N-1) & c(N-2) & c(N-2) \\ c(N-1) & c(N) & c(N-2) & c(N-2) \\ c(N-2) & c(N-2) & c(N) & c(N-1) \\ c(N-2) & c(N-2) & c(N-1) & c(N) \end{pmatrix}, \tag{32}$$

$$C_l^{(4)}(N) = \begin{pmatrix} c(N) & q(N) & c(N-2) & c(N-2) \\ c(N) & c(N) & c(N-2) & c(N-2) \\ c(N-2) & c(N-2) & c(N) & c(N) \\ c(N-2) & c(N-2) & c(N) & c(N) \end{pmatrix}, \tag{33}$$

and

$$C_r^{(4)}(N) = \begin{pmatrix} c(N) & q(N-2) & c(N) & c(N-2) \\ c(N-2) & c(N) & c(N-2) & c(N) \\ c(N) & c(N-2) & c(N) & c(N-2) \\ c(N-2) & c(N) & c(N-2) & c(N) \end{pmatrix}, \tag{34}$$

from which we have

$$c(N-1) \leq \frac{1}{2}[c(N) + c(N-2)]\ . \tag{35}$$

Equivalently in the case of general partition of the N-particle systems into two subsystems of size N_1 and N_2 the existence of the thermodynamic limit will be assured by

$$c(N-1) \leq \frac{N_1}{N} c(N) + \frac{N_2}{N} c(N-2) \ , \tag{36}$$

so that any positive increasing sequence $c(i)$, $i = 1, ..., N$ with the convexity property

$$c(i) \leq \alpha_i c(i+1) + (1-\alpha_i) c(i-1) \ , \tag{37}$$

for suitable chosen $0 < \alpha_i < 1$ provides a CGREM with RSB structure whose thermodynamic limit is reached monotonically.
Remark: introducing [5] the ultrametric overlap

$$d_N(\sigma,\tau) \ = \ \frac{1}{N} \min(i|\sigma_i \neq \tau_i) \tag{38}$$

one can see that the covariance matrix (25) is a growing function of d_N for every choice of the a's. Obviously at zero temperature the overlap (38) coincides with the standard GREM overlap among magnetization states.

Acknowledgements

We want to thank Francesco Guerra for many enlightening discussions.

References

[1] F.Guerra and F.Toninelli, Commun.Math.Phys, **230**, 71-79 (2002):

[2] P.Contucci, M.Degli Esposti, C.Giardina' and S.Graffi, Commun.Math.Phys., **236**, 55-63, (2003)

[3] M.Mézard, G.Parisi and M.Virasoro, *Spin Glasses and Beyond*, World Scientific, Singapore, 1986

[4] B.Derrida and F.Gardner, Jou. Phys. C, **19**, 2253-2274, (1986)

[5] A.Bovier, I.Kurkova, Preprint 727, U. Paris 6, (2002)