Among Quadratic Hamiltonians, Bogoliubov Transformations and Non-Regular States on CCRs *-Algebra. I. Pure and Invariant States. (In the Mood for the Manuceau Verbeure Theorems about Quasi-free States and Automorphisma of the CCR Algebra)

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Abstract

The paper's features are these:

1) we discuss **especially** quadratic (alias bilinear) Bose-Hamiltonians, the related Bogoliubov transformations and **especially** quasi-free-like (alias coherent or Fock-like) states

2) we discuss **any** quadratic Bose-Hamiltonians and Bogoliubov transformations, whether diagonalizable or not, whether proper or improper, and **arbitrary** quasi-free-like states, whether regular or non-regular they are

3) we associate notions and terms of the CCRs [1] theory with notions and terms of the indefinite inner product spaces theory. Then, we apply the corresponding 'bilingual dictionary' so as to construct invariant states of some of the quadratic Hamiltonians.

[1] CCRs = Canonical Commutation Relations

1 Introduction

Quadratic (bilinear) **Fermi**-Hamiltonians have a very attractive property. One can diagonalize them and the way is **not unique**, and there exists **not a unique Fock-like invariant** state [2]. Nothing really like this is present in the case of quadratic **Bose**-Hamiltonians. By the contrast, there are cases, e.g., the case of the repulsive oscillator, where one cannot diagonalize the Bose-Hamiltonian and Fock-like invariant states does not exist at all.

Here we shall say more precise: there exist no Fock-like invariant state with **continuous** characteristic function.

But why 'to diagonalize' and even why '**continuous**'? One needs firstly **invariant** states with a definite algebraic structure.

We have tried to compose a suitable constructions and here, in this paper, they are presented.

[2] Fock-like state = even quasi-free state

2 Prerequisites: The Quadratic Hamiltonians, Canonical Commutation Relations, and Bogoliubov Transformations

[3]

This section consists primarily of formal constructions and manipulations as we would like briefly to explain what we mean by '**Quadratic Hamiltonians, CCRs = Canonical Commutation Relations**' and '**Bogoliubov (Canonical)Transformations**.'

The Quadratic Hamiltonian of an N-degree of freedom system [4] is here a formal expression

$$h = \sum\nolimits_{k,l}\left(s_{k\,l}a_k^* a_l - \frac{1}{2}\overline{t_{k\,l}}a_k a_l - \frac{1}{2}t_{k\,l}a_l^* a_k^*\right)$$

where

$$s_{k\,l} = \overline{s_{l\,k}}\ ;\ t_{l\,k} = t_{k\,l}$$

and $a_k^*, a_l; k, l = 1, \ldots, N$ are thought of as elements of an (associative) *-algebra.

We will suppose $a_k^*, a_l; k, l = 1, \ldots, N$ to be subject to the relations

$$[a_k, a_l^*] = (a_k a_l^* - a_l^* a_k) = \delta_{k\,l}\ ; [a_k, a_l] = 0\ ; [a_l^*, a_k^*] = 0$$

The relations are said to be the **Canonical Commutations Relations in the Fock-Dirac form**.

Let us write

$$u := (u_1, ..., u_N)\ ;\ u^+ := (u_1^+, ..., u_N^+)\ ;\ u^- := (u_1^-, ..., u_N^-)\ ;$$

$$\begin{aligned} a^+(u) &:= u_1 a_1^* + \cdots u_N a_N^* \\ a(u) &:= (a^+(u)^* \equiv \overline{u_1}a_1 + \cdots \overline{u_N}a_N \\ a^-(u) &:= a(\overline{u}) \equiv u_1 a_1 + \cdots u_N a_N \equiv a^+(\overline{u})^* \end{aligned}$$

(hence $a^+(u) = a(u)^* = a^-(\overline{u})^*$) and in addition we set

$$A(u^+ \oplus u^-) := a^+(u^+) + a^-(u^-) \equiv u_1^+ a_1^* + \cdots u_N^+ a_N^* + u_1^- a_1 + \cdots u_N^- a_N$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u^{+\prime} \oplus u^{-\prime})\,,$$

where $u^{+\prime} \oplus u^{-\prime}$ is defined by

$$\begin{pmatrix} u^{+\prime} \\ u^{-\prime} \end{pmatrix} = \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix}\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

Here S, T, $\overline{T}$, $\overline{S}$ stand for the operators associated with the matrices $\{s_{kl}\}_{kl}$, $\{t_{kl}\}_{kl}$, $\{\overline{t_{kl}}\}_{kl}$, $\{\overline{s_{kl}}\}_{kl}$:

$$(Su)_k = \sum\nolimits_l s_{k\,l}u_l\ ,\ (Tv)_k = \sum\nolimits_l t_{k\,l}v_l\ ,\ \text{etc.}$$

Note

$$S^* = S\ ;\ T^* = \overline{T}\ ;\ \overline{S}^* = \overline{S}$$

Next, we observe that

$$[A(\cdots), A(\cdots)]$$

[3]Throughout this paper we assume that units of measure are choosen and fixed so that $\hbar = 1$ and so that the momentum quantity, $\mathbf{P}$, and the position quantity, $\mathbf{Q}$, both become dimensionless quantities.

[4]N may be infinite

is scalar-valued (up to the multiplier $I =$ the unity of the *-algebra):

$$[A(u^{+\prime} \oplus u^{-\prime})^*, A(u^+ \oplus u^-)] =< u^{+\prime}, u^+ >_0 - < u^{-\prime}, u^- >_0$$

where $< u', u >_0$ stands for the usual inner product in $\mathbf{C}^N$:

$$< u', u >_0 := \overline{u_1'} u_1 + \overline{u_2'} u_2 + \cdots + \overline{u_N'} u_N$$

Motivated by this, we define:

$$< u^{+\prime} \oplus u^{-\prime}, u^+ \oplus u^- > \quad := \quad [A(u^{+\prime} \oplus u^{-\prime})^*, A(u^+ \oplus u^-)]$$

and we note that $< u^{+\prime} \oplus u^{-\prime}, u^+ \oplus u^- >$ is an indefinite inner product on the "space of coefficients" of a^*, a.

One can write:

$$< u^{+\prime} \oplus u^{-\prime}, u^+ \oplus u^- >= (u^{+\prime} \oplus u^{-\prime}, J_{a^*a} u^+ \oplus u^-)$$

where one sets:

$$J_{a^*a} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Similar formulae hold for the **Heisenberg-Dirac form of CCRs**:

$$i[\mathbf{P}_k, \mathbf{Q}_l] = \delta_{k\,l}\,;\; i[\mathbf{P}_k, \mathbf{P}_l] = 0\,;\; i[\mathbf{Q}_k, \mathbf{Q}_l] = 0\,;\; \mathbf{Q}^* = \mathbf{Q}\,;\; \mathbf{P}^* = \mathbf{P}\,.$$

We will write

$$F(x_p \oplus x_q) := x_{p1}\mathbf{P}_1 + \cdots + x_{pN}\mathbf{P}_N + x_{q1}\mathbf{Q}_1 + \cdots + x_{qN}\mathbf{Q}_N\,.$$

We use this notation whether $x_p \oplus x_q$ is a real-valued vector or complex-valued. In addition, we put

$$\mathbf{P}(x_p) := x_{p1}\mathbf{P}_1 + \cdots + x_{pN}\mathbf{P}_N \quad \mathbf{Q}(x_q) := x_{q1}\mathbf{Q}_1 + \cdots + x_{qN}\mathbf{Q}_N\,.$$

The quadratic Hamiltonian is now a formal expression

$$h := \frac{1}{2}\sum\nolimits_{l\,m} (M_{l\,m}\mathbf{P}_l\mathbf{P}_m - L_{l\,m}(\mathbf{P}_l\mathbf{Q}_m + \mathbf{Q}_m\mathbf{P}_l) + K_{l\,m}\mathbf{Q}_l\mathbf{Q}_m)$$

with $M^T = M\,, K^T = K$. Then

$$i[h, F(x_p \oplus x_q)] = F(x_p' \oplus x_q')$$

where

$$\begin{pmatrix} x_p' \\ x_q' \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

Next, we observe that formally

$$-i[F(x_p' \oplus x_q'), F(x_p \oplus x_q)] = -(x_p', x_q)_0 + (x_q', x_p)_0$$

where $(\cdot,\cdot)_0$ stands for the usual Euclidian-like inner product, alias scalar product of vectors:

$$(x', x)_0 := x_1'x_1 + x_2'x_2 + \cdots + x_N'x_N.$$

So, if we define

$$s(x_p' \oplus x_q', x_p \oplus x_q) \quad := \quad -i[F(x_p' \oplus x_q'), F(x_p \oplus x_q)]$$

then the s becomes a $\mathfrak{C}$-symplectic form, i.e., a bilinear anti-symmetric form on a complex space.

One can write also:

$$s(x'_p \oplus x'_q, x_p \oplus x_q) = (x'_p \oplus x'_q, J_{pq} x_p \oplus x_q)$$

where one sets:

$$J_{pq} := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

The link between a^- , a^+ and $\mathbf{P}$, $\mathbf{Q}$ is this:

$$a^- = \frac{1}{\sqrt{2}}(\mathbf{Q} + i\mathbf{P}) \, , \; a^+ = \frac{1}{\sqrt{2}}(\mathbf{Q} - i\mathbf{P}) \, , \; \mathbf{Q} = \frac{1}{\sqrt{2}}(a^+ + a^-) \, , \; \mathbf{P} = \frac{i}{\sqrt{2}}(a^+ - a^-)$$

Hence,

$$A(u^+ \oplus u^-) = \frac{1}{\sqrt{2}} F(-iu^+ + iu^- \oplus u^+ + u^-)$$

$$F(x_p \oplus x_q) = \frac{1}{\sqrt{2}} A(ix_p + x_q \oplus -ix_p + x_q)$$

The formal calculations show [5] that

$$\begin{aligned} e^{iF(x_p \oplus x_q)} e^{iF(x'_p \oplus x'_q)} &= e^{[iF(x_p \oplus x_q), iF(x'_p \oplus x'_q)]/2} e^{iF(x_p + x'_p \oplus x_q + x'_q)} \\ &= e^{-[F(x_p \oplus x_q), F(x'_p \oplus x'_q)]/2} e^{iF(x_p + x'_p \oplus x_q + x'_q)} \\ &= e^{-is(x_p \oplus x_q, x'_p \oplus x'_q)/2} e^{iF(x_p + x'_p \oplus x_q + x'_q)} \, , \\ \left(e^{iF(x_p \oplus x_q)} \right)^* &= e^{-iF(\overline{x_p} \oplus \overline{x_q})} \, . \end{aligned}$$

These formulae are the so-called **exponential form of the CCRs**. We adopt these formulae as a tenet, as an axiom.

If one restricts himself to the case of real-valued x_p, x_q, then one has the relations

$$\begin{aligned} e^{iF(x_p \oplus x_q)} e^{iF(x'_p \oplus x'_q)} &= e^{-is(x_p \oplus x_q, x'_p \oplus x'_q)/2} e^{iF(x_p + x'_p \oplus x_q + x'_q)} \, , \\ \left(e^{iF(x_p \oplus x_q)} \right)^* &= e^{-iF(x_p \oplus x_q)} \, . \end{aligned}$$

This form of the CCRs is called **Weyl**. [6] [7]

If a Hamiltonian has been given, the standard quantum mechanics practice suggests solving the dynamical equations, in particular

$$\frac{\partial F(x_p \oplus x_q)(t)}{\partial t} = i[h, F(x_p \oplus x_q)(t)]$$

or "equivalently"

$$\frac{\partial A(u^+ \oplus u^-)(t)}{\partial t} = i[h, A(u^+ \oplus u^-)(t)]$$

A formal calculation allows one to rewrite this equation as

$$\frac{\partial}{\partial t} \begin{pmatrix} x_p(t) \\ x_q(t) \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p(t) \\ x_q(t) \end{pmatrix}$$

respectively

$$\frac{\partial}{\partial t} \begin{pmatrix} u^+(t) \\ u^-(t) \end{pmatrix} = i \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} \begin{pmatrix} u^+(t) \\ u^-(t) \end{pmatrix}$$

We will suppose, the solution to the equation exists and let $V(t,s)$ denote the corresponding propagator. [8]

Then, $V(t,s)$ lifts to an *-automorphism $\alpha_{V(t,s)}$ of CCRs by

$$\alpha_{V(t,s)} F(x_p \oplus x_q) := F(V(t,s)(x_p \oplus x_q))$$

[5] see Appendix A

[6] In this case, $e^{iF(x_p \oplus x_q)}$ is unitary,

[7] and $s(\cdot,\cdot)$ is the usual (pre)symplectic form, i.e., the real-valued anti-symmetric form on real space

[8] it means evolution operator

$$\alpha_{V(t,s)} e^{iF(x_p \oplus x_q)} := e^{iF(V(t,s)(x_p \oplus x_q))}$$

This *-automorphism is called either the **linear canonical transformation** or **Bogoliubov transformation** or **quasi-free** automorphism of a Bose system.

Given a "physical system" state ω, e.g., whether a ground state or a state with an interesting energy distribution or a state with the momentum at given exact value ("plane wave state") or something like that, and given an observable $\mathbf{A}$, i.e., something like a momentum, energy, position, spin, particles number, etc,, we will denote the corresponding **expectation value of A at the state** ω by $\omega\mathbf{A}$. The functionals

$$x_p \oplus x_q \mapsto \omega e^{iF(x_p \oplus x_q)}$$

and

$$u^+ \oplus u^- \mapsto \omega e^{A(u^+ \oplus u^-)}$$

are called the **characteristic functionals** or **characteristic functions** of the state ω. If the function

$$\lambda \in \Re \mapsto \omega e^{iF(x_p^0 \oplus x_q^0 + \lambda x_p \oplus x_q)}$$

is continuous, then the state ω is called **regular** . As a rule, one assumes therewith that the coefficients $x_p^0 \oplus x_q^0, x_p \oplus x_q$ are real-valued. If

$$\omega e^{iF(x_p \oplus x_q)} = e^{-\text{ quadratic function of } x_p \oplus x_q}$$

then ω is called **even quasi-free**.

We will slightly extend the class of these states and choose a definition of the **even quasi-free-like** states which emphasizes the latter, algebraic, property and partially deemphasizes the continuity property. The main idea behind the states we will discuss is briefly this: [9] Let us suppose that we consider a one-dimensional system and even quasi-free states given by

$$\omega e^{iF(x_p \oplus x_q)} = \omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-\frac{1}{4}(ax_p^2 + bx_q^2)}, \quad , a > 0, b > 0$$

The uncertainty principle prescribes $ab \geq 1$, and therefore we could not assign $a := 0$, whatever real number b we had chosen. But why not $a := 0, b := +\infty$?" i.e., why not

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} := \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases}$$

Or why not?":

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} := \begin{cases} e^{-bx_q^2/4}, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases}$$

Actually, we may take $\epsilon > 0$ and define states ω_ϵ by

$$\omega_\epsilon e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} := e^{-\frac{1}{4}\left(x_p^2/\epsilon + (b+\epsilon)x_q^2\right)}, \quad b \geq 0$$

Then we may take limit of ω_ϵ as $\epsilon \to +0$ without loss of the main algebraic property of states being positive definite, although we lose partially the continuity property. Thus we obtain just

$$\omega_{+0} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} := \lim_{\epsilon \to +0} \omega_\epsilon e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = \begin{cases} e^{-bx_q^2/4}, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases}$$

and we can refer to such states as abstractions of the usual states, perhaps as artificial abstractions. These states as well as any N-dimensional analogue, we will call such states the **even quasi-free-like** or **quadratic** states.

[9] a precise definition see in the next Section

It's our object.

COMMENT

The more detailed description of the notions in this section, one can find in e.g., [Ber], [BR2], and especially [Fey, Stat.Mech.]. Notice that our definition of $\mathbf{P}$ and $\mathbf{Q}$ slightly differs from the standard. As a rule one sets $\mathbf{P}$ and $\mathbf{Q}$ so as to

$$a = \frac{1}{\sqrt{2}}(\mathbf{Q} + i\mathbf{P})\,,\; a^* = \frac{1}{\sqrt{2}}(\mathbf{Q} - i\mathbf{P})\,,\; \mathbf{Q} = \frac{1}{\sqrt{2}}(a^* + a)\,,\; \mathbf{P} = \frac{i}{\sqrt{2}}(a^* - a)$$

3 Quadratic and Quasi-free States on CCRs Algebra and Quasi-Free-like Atomorphisms

In this section we discuss notions related to that introduced in the previous section.

First, we give an abstract axiomatic definition of the CCRs.

Let Z be a real vector space, $s : Z \times Z \to R$ a bilinear antisymmetric form (s need not be nondegenerated) and $V : Z \to Z$ be a linear operator.

Let $\underline{\mathbf{K}}, <,>, V_{\mathfrak{C}}$ be the standard complexification of $Z, is(,), V$, i.e. $\underline{\mathbf{K}}$ is the standard $\mathfrak{R}$-linear doublication of Z

$$\underline{\mathbf{K}} := \mathfrak{C}Z = Z \oplus_{\mathfrak{R}} Z\,,$$

the multiplication with i is given by

$$i(\mathfrak{f} \oplus \mathfrak{g}) := (-\mathfrak{g}) \oplus \mathfrak{f}\,,$$

and $<,>$ is the standard sesquilinear extention of $is(,)$ and $V_{\mathfrak{C}}$ is the standard $\mathfrak{C}$-linear extention of V .

Often, we will write

$$\mathfrak{f} + i\mathfrak{g} \text{ instead of } \mathfrak{f} \oplus \mathfrak{g}$$

and next the symbol C will denote the natural complex conjugation in $\underline{\mathbf{K}}$:

$$C(\mathfrak{f} + i\mathfrak{g}) := \mathfrak{f} - i\mathfrak{g}\,, \qquad (\mathfrak{f}, \mathfrak{g} \in Z)$$

So, $\underline{\mathbf{K}}, <,>$ is indefinite inner product space with $<,>$-antiunitary involution C:

$$C^2 = I\,, \qquad < C\mathfrak{f}, C\mathfrak{g} > = < \mathfrak{g}, \mathfrak{f} >\,, \qquad (\mathfrak{f}, \mathfrak{g} \in Z)$$

Remark 1.

We take it as known that V is a homomorphism of $s(,)$, i.e.

$$s(V\mathfrak{f}, V\mathfrak{g}) = s(\mathfrak{f}, \mathfrak{g}) \qquad (\forall \mathfrak{f}, \mathfrak{g} \in Z)$$

iff and $V_{\mathfrak{C}}$ is a $<,>$-isometric operator. Similarly V is an automorphism of $s(,)$ i.e.

$$V \text{ is bijective and } s(V\mathfrak{f}, V\mathfrak{g}) = s(\mathfrak{f}, \mathfrak{g}) \qquad (\forall \mathfrak{f}, \mathfrak{g} \in Z)$$

iff $V_{\mathfrak{C}}$ is a $<,>$-unitary operator.

Definition 1.

Abstract Weyl *-algebra, we denote it by $W_{Z,s}$, is here a free *-algebra on the symbols $\epsilon^{\mathfrak{f}}, \mathfrak{f} \in Z$ subject to the relations

$$(\epsilon^{\mathfrak{f}})^* = \epsilon^{-\mathfrak{f}}\,,\ \epsilon^{\mathfrak{f}}\epsilon^{\mathfrak{g}} \ = \ e^{-is(\mathfrak{f},\mathfrak{g})/2}\epsilon^{\mathfrak{f}+\mathfrak{g}}\,.$$

Remark 2. (cf. e.g., [MV],[BR])

If V is an automorphism of s, then the correspondence $\epsilon^{\mathfrak{f}} \mapsto \epsilon^{V\mathfrak{f}}$ induces a *-automorphism; this *-automorphism is called **quasi-free**, often, **Bogoliubov *-automorphism** , alias **Bogoliubov transformation**. We will denote it by α_V

If χ is a *-character of the additive group Z i.e. if

$$\chi(\mathfrak{f} + \mathfrak{g}) = \chi(\mathfrak{f})\chi(\mathfrak{g}), \chi(\mathfrak{f})^* = \chi(-\mathfrak{f}), \forall \mathfrak{f}, \mathfrak{g} \in Z$$

then the correspondence $\epsilon^{\mathfrak{f}} \mapsto \chi(\mathfrak{f})\epsilon^{V\mathfrak{f}}$ extends to a **gauge-like *-automorphism**, alias **coherent *-automorphism**; we denote it by α_χ .

If

$$\chi(\mathfrak{f}) = e^{il(\mathfrak{f})}$$

where l is a real-valued $mod 2\pi$-additive function on Z, then we prefer to write α_l instead of α_χ.

The automorphisms of the form

$$\alpha_{V,\chi} := \alpha_V \alpha_\chi, \qquad (\alpha_{\chi,V} := \alpha_\chi \alpha_V)$$

are called **quasi-free-like**.

The rest of this section until Example 1 is an extending modificaton of the Manuceau Verbeure Theory of quasi-free states.

It will be convenient to change (equivalently!) the usual definition of positive quadratic form.

Definition 2.

We will say that $q : Z \to [0, \infty]$ is **quadratic** iff

$$q(\mathfrak{f} + \mathfrak{g}) + q(\mathfrak{f} - \mathfrak{g}) = 2[q(\mathfrak{f}) + q(\mathfrak{g})]$$

$$q(k\mathfrak{f}) \;=\; k^2 q(\mathfrak{f}) \quad (\mathfrak{f}, \mathfrak{g} \in Z,\; k \in R)$$

(hereafter $0 \cdot \infty \;=\; 0$, $\infty \;+\; \infty \;=\; \infty$ and so on)

The set

$$Q(q) := \{\mathfrak{f} \in Z | q(\mathfrak{f}) < \infty\} \equiv \{\mathfrak{f} \in Z | q(\mathfrak{f}) \neq \infty\}$$

is called the **form domain** or the **domain** of q. If $Q(q) = Z$, then q is called **finite**.

Given two quadratic $q_1,\ q_2$, we write

$$q_1 \leq q_2 \qquad \text{iff} \qquad q_1(\mathfrak{f}) \leq q_2(\mathfrak{f}) \qquad (\forall \mathfrak{f} \in Z)$$

Remark 3.

Given a quadratic q, the form domain of q is linear, and q is associated with a unique symmetric bilinear positive (if Z is over $\mathfrak{R}$) or symmetric sesquilinear positive form (if Z is over $\mathfrak{C}$), we denote it by $q(\cdot, \cdot)$; this form can be recovered from the q by the **polarization identity**

$$q(\mathfrak{f}, \mathfrak{g}) = \frac{1}{2}(q(\mathfrak{f} + \mathfrak{g}) + q(\mathfrak{f} - \mathfrak{g})) \qquad (\text{if } Z \text{ is over } \mathfrak{R})$$

resp.

$$q(\mathfrak{f}, \mathfrak{g}) = \frac{1}{4}(q(\mathfrak{f} + \mathfrak{g}) + q(\mathfrak{f} - \mathfrak{g}) - iq(\mathfrak{f} + i\mathfrak{g}) + iq(\mathfrak{f} - i\mathfrak{g})) \qquad (\text{if } Z \text{ is over } \mathfrak{C})$$

Definition 3.

We say q is a quadratic-like **majorant of** s, iff

$$2|s(\mathfrak{f}, \mathfrak{g})| \leq q(\mathfrak{f}) + q(\mathfrak{g}) \qquad (\mathfrak{f}, \mathfrak{g} \in Z)$$

and if, of course, q in itself is quadratic.

Proposition 1.

For any majorant q, there exists a minimal quadratic-like majorant, say q_0, such that $q_0 \leq q$. Hereafter, we mean by 'q_0 is a **minimal majorant***' that, if $q_1 \leq q_0$ for a quadratic majorant q_1, then $q_1 = q_0$.*

Definition 4.cf. [Oks])

We say that a linear *-functional ω on $W_{Z,s}$ is **quadratic (alias even quasi-free-like, generalized even quasi-free)** iff

$$\omega\epsilon^{\mathfrak{f}} = e^{-q(\mathfrak{f})/4} \quad (e^{-\infty} \;=\; 0)$$

for a quadratic q .

Theorem 1.

(i) *A quadratic ω is a state iff the associated q is a quadratic-like majorant of s.*

(ii) *A quadratic ω is a pure state iff the associated q is a minimal quadratic-like majorant of s.*

Example 1.

Put $q(\mathfrak{f}) := 0$ at $\mathfrak{f} = 0$ and $q(\mathfrak{f}) := +\infty$ otherwise, i.e., define a linear functional δ_0 on $W_{Z,s}$ so that

$$\delta_0 \epsilon^{\mathfrak{f}} := \begin{cases} 1, & \text{if } \mathfrak{f} = 0 \\ 0, & \text{if } \mathfrak{f} \neq 0 \end{cases}$$

Then, the q is a quadratic-like majorant, called **trivial** *, and δ_0 is* **trivial** *state. Notice* (e.g., [BR2, p.79], EXAMPLE 5.3.2)*, the δ_0 is a trace-state on $W_{Z,s}$.* [10] *In addition, this state is invariant under all Bogoliubov transformations.*

Definition 5.

Given a quadratic q, we denote its standard complexification by $q_{\mathfrak{C}}$. We define it so:

$$q_{\mathfrak{C}}(\mathfrak{f} + i\mathfrak{g}) := q(\mathfrak{f}) + q(\mathfrak{g})$$

A similar notation is given to the complexification of an arbitrary linear $\mathcal{T} : Z \to Z$:

$$\mathcal{T}_{\mathfrak{C}}(\mathfrak{f} + i\mathfrak{g}) := \mathcal{T}\mathfrak{f} + i\mathcal{T}\mathfrak{g} \, .$$

Remark 4.

For the complex space case the definition of the sentence 'a functional, e.g., $q_{\mathfrak{C}}$, is quadratic' is to be modified:

$$q_{\mathfrak{C}}(kz) = |k|^2 q_{\mathfrak{C}}(z) \, .$$

The rest of the definition remains as befor.

Observation 1.

Given a linear $\tilde{\mathcal{T}} : \underline{\mathbf{K}} \to \underline{\mathbf{K}}$, it is of the form $\tilde{\mathcal{T}} = \mathcal{T}_{\mathfrak{C}}$ for a suitable $\mathcal{T} : Z \to Z$ iff

$$C\tilde{\mathcal{T}}C = \tilde{\mathcal{T}}.$$

Similarly, for any quadratic $\tilde{q} : \underline{\mathbf{K}} \to [0, \infty]$, there is a quadratic $q : Z \to [0, \infty]$ such that $\tilde{q} = q_{\mathfrak{C}}$ iff

$$\tilde{q}C = \tilde{q}$$

Theorem 2.

(i) *q is a quadratic-like majorant of s iff $q_{\mathfrak{C}}$ is a quadratic-like majorant of $<,>$*

(ii) *q is a minimal quadratic-like majorant of s iff $q_{\mathfrak{C}}$ is a minimal quadratic-like majorant of $<,>$*

Remark 5.

We will deal, first and foremost, with *quadratic-like* majorants. So, if no confusion can ocurr, we will omit the particle '-like' or the whole word 'quadratic-like', although occasionally we will repeat the whole term 'quadratic-like majorant' for emphasis.

If a new object is declared, the first question is whether this object does exist. Of course, automorphisms, majorants and invariant (under Bogoliubov transformation) quadratic states, all these objects do exist. Interestingly enough, a **finite** majorant

[10] and this is the unique trace-state if $s(\cdot, \cdot)$ is nondegenerate

need not exist; a quasi-free *-automorphism **need not** have an **invariant non-trivial** quadratic state.

Example 2. ([Bog, p.62-63, **Example 3.2**])
Let $\underline{\mathbf{H}}$ *be the vector space of those doubly infinite numerical sequences where only a finite number of terms with negative index is different from zero, and for* $\mathfrak{f} = \{\xi_j\}_{j\in\mathbf{Z}} \in \underline{\mathbf{H}}$, $\mathfrak{g} = \{\eta_j\}_{j\in\mathbf{Z}} \in \underline{\mathbf{H}}$ *let*

$$< \mathfrak{f}, \mathfrak{g} >:= \sum_{j=-\infty}^{\infty} \overline{\xi_j}\eta_{-j-1}$$

Then $< \cdot, \cdot >$ *cannot have a norm majorant.* [11]

We are interested in the symplectic space case and translate the previous Example 2 into obvious:

Example 3.
For this Example, let $\mathbf{S_0}$ *denote the linear space of those real-valued sequences* $\mathfrak{f} : \mathbf{N} \to \mathbf{R}$ *such that*

$$\mathfrak{f}(n) = 0 \text{ for all but a finite number of } n$$

and $\mathbf{S_{all}}$ *denote the linear space of all real-valued sequences. Finally, define*

$$Z := \mathbf{S_0} \oplus \mathbf{S_{all}}$$

and

$$s(\mathfrak{f}_1 \oplus \mathfrak{f}_2, \mathfrak{g}_1 \oplus \mathfrak{g}_2) := \sum_n (\mathfrak{f}_1(n)\mathfrak{g}_2(n) - \mathfrak{f}_2(n)\mathfrak{g}_1(n))$$

Then the symplectic form $s(\cdot, \cdot)$ *cannot have a norm majorant.*

Example 4. ([Ch4])
Let $\mathbf{R}[\mathbf{Z}]$ *be the free real *-algebra on the symbols* $u[n], n \in \mathbf{Z}$ *subject to the relations* [12]

$$u[n]u[m] := u[n+m], u[n]^* := u[-n] \qquad (n, m \in \mathbf{R}[\mathbf{Z}]) \quad ^{13}$$

Let f *be a linear functional defined by*

$$fu[n] := e^{\sqrt{|n|}} - e^{-\sqrt{|n|}}$$

Next, define

$$K_f := \{K \in \mathbf{R}[\mathbf{Z}] | \quad (\forall A \in \mathbf{R}[\mathbf{Z}]) \quad f(A^*K) = 0\}$$

and

$$A_f := A + K_f \qquad (A \in \mathbf{R}[\mathbf{Z}]) \quad ^{14}$$

Then, the bilinear anti-symmetric form

$$A, B \mapsto f(A^*B)$$

lifts to a symplectic form, s, *on the quotient space* $\mathbf{R}[\mathbf{Z}]/K_f$ *and the map*

$$A \in \mathbf{R}[\mathbf{Z}] \mapsto u[1]A \in \mathbf{R}[\mathbf{Z}] \qquad (A \in \mathbf{R}[\mathbf{Z}])$$

[11] consequently, cannot have a finite quadratic majorant
[12] we will deal with the so-called group *-algebra of $\mathbf{Z}$ over $\mathbf{R}$
[13] thus $u[n]^*u[n] = u[n]u[n]^* = u[0] = 1$
[14] actually, $K_f = \{0\}$ and in essence $A_f = A$, however we ignore it

lifts to a symplectic automorphism $V : \mathbf{R}[\mathbf{Z}]/K_f \to \mathbf{R}[\mathbf{Z}]/K_f$ *; they are correctly defined by*

$$s(A_f, B_f) := f(A^*B) \qquad (A, B \in \mathbf{R}[\mathbf{Z}])$$

$$VA_f := (u[1]A)_f \qquad (A \in \mathbf{R}[\mathbf{Z}])$$

Now then, there is no non-trivial V*-invariant majorant of* s *and there is no non-trivial quadratic* α_V*-invariant state on* $W_{Z,s}$ *where* $Z := \mathbf{R}[\mathbf{Z}]/K_f$*.*

COMMENT

A state ω is said to be regular iff the function $x \in \mathfrak{R} \to \omega\epsilon^{x\mathfrak{f}+\mathfrak{g}}$ is continuous whatever $\mathfrak{f}$ and $\mathfrak{g}$.

Manuceau and Verbeure [MV] discussed only regular states and therefore only *finite* quadratic forms and the corresponding states. As for non-regular states, one can confer the approach in this section with one of [FS], [Gru], [LMS], [CMS], and especially with that of [Oks]. Recently Halvorson [Hal] proposed a very interesting standpoint which is reminiscent of some of the papers of Antonets, Shereshevski, first of all [AS].

The 'non-regular' part of this section is based entirely on [Ch1], [Ch2], [Ch3] and the summarizing [Ch4].

In Example 4, we have applied a GNS-like [15] construction. For detailes of such constructions, see e.g., [Schatz] or/and [BD] (or [Ch 2–4], if one deals with the objects discussed in this Section).

[15] GNS = Gelfand–Naimark–Segal

4 The Case of Regular Spaces

In the previous section we discussed relatively general spaces and forms. So, the statements were 'in general'.

With stronger hypothesis on Z, $\underline{\mathbf{K}}$ and forms $s(\cdot,\cdot)$, $< \cdot,\cdot >$ one can obtain a stronger conclusion. We start with two restricting definitions which one finds among the primary definitions of two different theories. We mean the standard theory of the quasi-free states (e.g., [BR2]) and, as for the second definition, the so-called Krein spaces theory (e.g., [Bog])

Definition 1.

Z, s is said to be **regular** iff there is a linear $J : Z \to Z$ such that

1) $s(J\mathfrak{f}, J\mathfrak{g}) = s(\mathfrak{f}, \mathfrak{g}) \qquad \forall \mathfrak{f}, \mathfrak{g} \in Z;$

2) $J^2 = -I;$

3) $s(\mathfrak{f}, J\mathfrak{f}) \geq 0 \qquad \forall \mathfrak{f} \in Z\ ,$

4) Z is a real pre-Hilbert space with respect to the scalar product $\mathfrak{f}, \mathfrak{g} \to s(\mathfrak{f}, J\mathfrak{g}) \qquad (\mathfrak{f}, \mathfrak{g} \in Z)\ .$

Definition 2. (e.g., [Bog])

Let $\mathcal{K}$, $,<,>$ be an inner product space.

$\mathcal{K}$, $,<,>$ is said to be **regular indefinite inner product space** iff there is a linear $\mathfrak{J} : \mathcal{K} \to \mathcal{K}$ such that

1) $< \mathfrak{J}z, \mathfrak{J}w >=< z, w > \qquad \forall z, w \in \mathcal{K};$

2) $\mathfrak{J}^2 = I;$

3) $< z, \mathfrak{J}z >\geq 0 \qquad \forall z \in \mathcal{K}\ ,$

4) $\mathcal{K}$ is a pre-Hilbert space with respect to the scalar product $z, w \to< z, Jw > \qquad (z, w \in \mathcal{K})\ .$

If $\mathcal{K}$ is complete, then $\mathcal{K}$ is said to be **a Krein space**.

We see one definition is very much like another. We state it in the mathematical terms as an

Observation 1.

Z, s is regular if and only if the corresponding standard complexification of Z, s, i.e., $\underline{\mathbf{K}}$, $<,>$ in the sense of the previous section, is a regular indefinite inner product space. For the corresponding $\mathfrak{J}$, we may take

$$\mathfrak{J} := iJ_{\mathfrak{C}}\text{ , recall that } J_{\mathfrak{C}} := \textit{ standard complexification of } J\,.$$

The idea behind the constructions we will discuss is very simple. If we see that some of the primary definitions of two different theories are similar, then we expect it may be well worth stating the similarity between the results of these theories. Thus we need to elaborate a machinery so that we could translate the statements of the one theory into the language of another.

First, we consider what is common in both languages and we start to do it by introducing the general notations of the basic terms.

The symbol $\underline{\mathbf{H}}$ will denote a linear space, real or complex, and b be a bilinear or sesquilinear form respectively. In addition, we suppose that b is whether symmetric (hermitian for $\mathfrak{C}$) or antisymmetric (antihermitian for $\mathfrak{C}$).

The symbol Ɉ will denote a linear operator which has either the properties

$$\begin{matrix} \text{Ɉ}^2 := -I \\ b(\text{Ɉ}\mathfrak{f}, \text{Ɉ}\mathfrak{g}) = b(\mathfrak{f}, \mathfrak{g}) \end{matrix} \qquad \text{case of antisymmetric (antihermitan) } b\ ,$$

or

$$\begin{matrix} \text{Ɉ}^2 := I \\ b(\text{Ɉ}z, \text{Ɉ}w) = b(z, w) \end{matrix} \qquad \text{case of symmetric (hermitian) } b.$$

In addition, we define, unless otherwise specified, that

$$\text{Ɉ}^* := -\text{Ɉ} \qquad \text{case of antisymmetric (antihermitian) } b\ ,$$

$$\text{Ɉ}^* := \text{Ɉ} \qquad \text{case of symmetric (hermitan) } b\ .$$

It is unlikely that this definition can produce any confusion: we will deal, typically, with *nondegenerated* forms s, $<,>$; in these cases Ɉ^* will coincide with standard s- or $<,> -$ adjoint of Ɉ respectively.

There are two classes of regular spaces which we will discuss. The first class is given by:

Example 1.

Let Z_0 be a real or complex Hilbert or pre-Hilbert space. Put

$$Z := Z_0 \oplus Z_0$$

and

$$\text{Ɉ} := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

This choice of Z and Ɉ corresponds to the case where we adopt a definition of CCRs phrased in terms of $\mathbf{P}, \mathbf{Q}$, i.e., in terms of momentum and position operators.

Another class of regular spaces is:

Example 2.

Let H_0 be a complex Hilbert or pre-Hilbert space. Put $H_+ := H_0$, $H_- := H_0$,

$$\underline{\mathbf{K}} := H_+ \oplus H_-$$

and

$$\text{Ɉ} := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

This case corresponds to that, when one adapts himself to the CCRs phrased in terms of a^*, a, i.e., in terms of creation and annihilation operators.

The connection between these classes is simple:

Observation 2.

If

$$J_{a^*a} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad iJ_{pq} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

then

$$J_{a^*a} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} iJ_{pq}$$

$$iJ_{pq} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} J_{a^*a}$$

$$\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$$

We now return to quadratic forms and majorants and henceforth in this section we **assume that the above spaces Z_0 and H_0 are complete.**

Definition 3.

We say a form q is **closed** if q is closed as a usual quadratic form on Hilbert space

$$\overline{Q(q)} := \text{ closure of } Q(q) \text{ in } \underline{\mathbf{H}} \text{ with respect to } \|\cdot\|\ ,$$

i.e. is closed in the sense adopted in [RS1].

Theorem 1.
Let q be a majorant.
If q is minimal, then q is closed.

Definition 4.
We say a form q has an **operator representative** if there exists an $\mathcal{T} : D_{\mathcal{T}} \subset \underline{\mathbf{H}} \to \underline{\mathbf{H}}$ such that

$$D_T = Q(q) \text{ and } q(\mathfrak{f}) = \|\mathcal{T}\mathfrak{f}\|^2 \qquad (\mathfrak{f} \in Q(q))\,.$$

Theorem 2.
If q is closed, then q has an operator representative.
In addition, there is a unique self-adjoint operator $Q : D_Q \subset \overline{Q(q)} \to \overline{Q(q)}$ such that

$$D_{Q^{1/2}} = Q(q) \text{ and } q_Q := \|Q^{\frac{1}{2}}x\|^2.$$

Proof. Straightforward from Definitions 3, 4 and Theorem 1, using [RS1]. □

Definition 5.
In the situation of the Theorem 2, we say Q is the **operator of** q and write $q = q_Q$. Thus, we isolate a class of majorants. We will call these majorants **operator majorants**.
Let Q be the operator of q. Then we write

$$P := \text{ orthogonal projection of } \underline{\mathbf{H}} \text{ onto } \overline{Q(q)},$$
$$R := (I+Q)^{-1}P.$$

Remark 1.
It is evident that

$$0 \le R \le I, \qquad R = R^*$$

and that Q (and q) can be recovered from the R by the formulae

$$Q = R^{-1} - I; D_Q = Ran\,R\,.$$

We now characterize the operator majorants by means of the above Q and R.

Theorem 3.
(i) *q_Q is a majorant iff*

$$R + \mathfrak{J}^* R \mathfrak{J} \le I\,;$$

(ii) *q_Q is a minimal majorant iff*

$$R + \mathfrak{J}^* R \mathfrak{J} = I.$$

If we handle indefinite inner product space, we can say more:

Observation 3.
Let $\underline{\mathbf{H}} = \underline{\mathbf{K}}$ i.e., let us assume $\mathfrak{J}^* = \mathfrak{J}$. Then,

$$R + \mathfrak{J}^* R \mathfrak{J} = I, \qquad 0 \le R \le I$$

if and only if

$$R = \frac{1}{2}\begin{pmatrix} I & K^* \\ K & I \end{pmatrix},$$

where R is thought of as an operator from $H_+ \oplus H_-$ into $H_+ \oplus H_-$,
and where K is an operator such that $\|K\| \leq 1$, or,
equivalently,

$$R = \frac{1}{4}\begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix}$$

where R is thought of as an operator from $Z_0 \oplus Z_0$ into $Z_0 \oplus Z_0$,
and where K is the same operator as above.

With this Observation 3, Theorem 3 implies

Corollary 1.
Let $\underline{\mathbf{H}} = \underline{\mathbf{K}}$. Then q_Q is a minimal majorant if and only if

$$R = \frac{1}{2}\begin{pmatrix} I & K^* \\ K & I \end{pmatrix} \text{ with respect to } R : H_+ \oplus H_- \to H_+ \oplus H_-$$

and with a K such that $\|K\| \leq 1$ or, equivalently,

$$R = \frac{1}{4}\begin{pmatrix} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{pmatrix} \text{ with respect to } R : Z_0 \oplus Z_0 \to Z_0 \oplus Z_0$$

and with the same K.

Finally, we turn to the question: What about **complexificated** majorants and operators?
The theorems are:

Theorem 4. (cf. Observation 3.1)
In the situation described in Example 1, let

$$\underline{\mathbf{K}}_0 := \mathfrak{C} Z_0 = \text{ the standard complexification of } Z_0$$

$$C_0 := \text{ the corresponding complex conjugation operator on } \underline{\mathbf{K}}_0\,, \quad C := C_0 \oplus C_0$$

Then, a minimal majorant $\tilde{q}$ on $\underline{\mathbf{K}} = \underline{\mathbf{K}}_0 \oplus \underline{\mathbf{K}}_0$ is a complexification of a (minimal majorant) q on $Z = Z_0 \oplus Z_0$ i.e., $\tilde{q}$ is of the form $\tilde{q} = q_{\mathfrak{C}}$ if and only if

$$\tilde{q}C = \tilde{q}$$

or, equivalently,

$$R = CRC$$

or, equivalently,

$$K^* = \overline{K} := C_0 K C_0$$

Theorem 5. (cf. Observation 3.1)
A linear $\tilde{\mathcal{T}} : \underline{\mathbf{K}}_0 \oplus \to \underline{\mathbf{K}}_0 \oplus \underline{\mathbf{K}}_0$ is of the form $\tilde{\mathcal{T}} = \mathcal{T}_{\mathfrak{C}}$ for a suitable $\mathcal{T} : Z_0 \oplus Z_0 \to Z_0 \oplus Z_0$ if and only if

$$\tilde{\mathcal{T}} = C \tilde{\mathcal{T}} C$$

or, equivalently, $\tilde{\mathcal{T}}$ is a **cross-matrix** *i.e., $\tilde{\mathcal{T}}$ is of the form*

$$\tilde{\mathcal{T}} = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \text{ with respect to the decomposition } H_+ \dot{+} H_- \to H_+ \dot{+} H_-$$

Now, what about **invariant** majorant?
The theorem is:

Theorem 6.

Let $V : \underline{\mathbf{H}} \to \underline{\mathbf{H}}$ be a linear bounded invertible operator, and let q be an **operator** *majorant of $b(\cdot,\cdot)$.*

The following conditions are equivalent:

(i)
$$qV = q$$

(ii)
$$qV^{-1} = q$$

(iii)
$$(I-R)VR = RV^{*-1}(1-R)$$

(iv)
$$(I-R)V^{-1}R = RV^{*}(1-R)$$

In addition, if q is a minimal majorant, and if V is an automorphism of $b(\cdot,\cdot)$, i.e., if

$$b(V\mathfrak{f}, V\mathfrak{g}) = b(\mathfrak{f},\mathfrak{g})\,, \quad \forall \mathfrak{f},\mathfrak{g} \in \underline{\mathbf{H}}\,,$$

and if $b(\cdot,\cdot)$ is symmetric (it means that the situation is the same as one in Example 2), then all conditions (i)-(iv) *are equialent to:*

(v)
$$V\begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} V \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}$$

Let us discuss the above Theorem, especially, the condition (v) of this Theorem. In discussing them we will indicate at least three factors which have to be taken into account.

First, we observe that the operator

$$\mathcal{P} := \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}$$

is a projection operator because $\mathcal{P}^2 = \mathcal{P}$. Therefore the condition (v) is a condition for $\mathcal{P}\underline{\mathbf{H}}$ to be a V-invariant subspace. We emphasize, it is true for any operator V even if V is not a Bogoliubov transformation. As for the sort of the subspace, some authors refer to such subspaces as the graph subspaces because one may consider

$$\mathcal{P}\underline{\mathbf{H}} = \{x_+ \oplus Kx_+ |\, x_+ \in H_+\}$$

as the graph of the operator K. In this case, K is called the **angular operator** of $\mathcal{P}\underline{\mathbf{H}}$ with respect to H_+.

The second factor is that $\|K\| \le 1$. This inequality means, in particular, that whatever $x \in \mathcal{P}\underline{\mathbf{H}}$, the value of $b(x,x)$ is positive:

$$\begin{aligned} b(x,x) &= b(x_+ \oplus Kx_+, x_+ \oplus Kx_+) = (x_+ \oplus Kx_+, \mathfrak{J}(x_+ \oplus Kx_+)) \\ &= \|x_+\|^2 - \|Kx_+\|^2 \ge 0 \end{aligned}$$

For such a sort of subspaces, there are special terms:

A subspace, L, is called **$\mathfrak{J}$-positive** or **b-positive** or mere **positive** if

$$b(x,x) \ge 0\,, \qquad (\forall x \in L)$$

Given a positive L, one says 'L is **maximal positive**', if L is 'set'-maximal among positive subspaces. In other words, a maximal positive subspace is a positive subspace L such that: whatever positive L_1 is given, $L_1 \supset L$ implies that $L_1 = L$.

We can state now: if a quadratic-like majorant is minimal, than the associated subspace is positive. The property of L being a positive subspace, it, in itself, does not implies that this subspace is of the form

$$L = \{x_+ \oplus Kx_+ |\, x_+ \in H_+\}\,, \quad \|K\| \le 1$$

but if one replaces 'being a positive' by 'being a maximal positive', it does.

As the result:

1) if a quadratic-like majorant is minimal, then the associated subspace is maximal positive;

2) every maximal positive subspace is a subspace associated with a unique minimal quadratic-like majorant.

One of the additional factors which have to be taken into account in discussing the above Theorem 6, is that a minimal majorant is regular [16] if and only if $\|K\| < 1$, and this is exactly the case if the corresponding maximal positive subspace is **uniformly positive**:

$$(\exists\gamma > 0)(\forall x \in \mathcal{P}\underline{\mathbf{H}}) \qquad b(x,x) \geq \gamma b(x, \mathfrak{J}x)$$

or in other words, which are more usual for the Krein spaces theory,

$$(\exists\gamma > 0)(\forall x \in \mathcal{P}\underline{\mathbf{H}}) \qquad < x, x > \geq \gamma\|x\|^2$$

The third factor which have to be taken into account is that $\mathcal{P}\underline{\mathbf{H}}$ is to be a V-invariant subspace.

We will often work with the operator matrix representation of V,

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \text{ with respect to the decomposition } H_+ \oplus H_- \to H_+ \oplus H_-$$

In this case the mentioned condition (v) will look like this:

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ K & 0 \end{pmatrix}$$

One can straightforwardly verify that this condition is exactly equivalent to

$$V_{21} + V_{22}K = K(V_{11} + V_{12}K)$$

It is the equation which is placed among the most singular equations of the Krein spaces theory [17] and it is just the equation which we will sistematically exploit when discussing Examples.

We conclude this section with a dictionary, paralleling the most explicit notions of the Krein spaces theory with the theory of quadratic Bose Hamiltonians.

[16] it means that $Q = R^{-1} - I$ exists and is bounded as an operator acting on the whole space $\underline{\mathbf{H}}$

[17] if, of course, one is interested in solving invariant subspaces problems

symplectic operator or form, which has the matrix $\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix}$	$\longleftrightarrow$	the quadratic Hamiltonian $h := \frac{1}{2}\sum_{l\,m}\Big(M_{l\,m}\mathbf{P}_l\mathbf{P}_m - L_{l\,m}(\mathbf{P}_l\mathbf{Q}_m + \mathbf{Q}_m\mathbf{P}_l) + K_{l\,m}\mathbf{Q}_l\mathbf{Q}_m\Big)$
J-symmetric operator or form, which has the matrix $\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix}$	$\longleftrightarrow$	the quadratic Hamiltonian $h = \sum_{k,l}\left(s_{k\,l}a_k^*a_l - \frac{1}{2}\overline{t_{k\,l}}a_ka_l - \frac{1}{2}t_{k\,l}a_l^*a_k^*\right)$
J-unitary operator with the matrix $\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$	$\longleftrightarrow$	invertible Bogoliubov transformation (quasi-free automorphism) with the same matrix
maximal positive subspace with the angular operator K such that $K^* = \overline{K}$	$\longleftrightarrow$	pure quadratic-like state
maximal uniformly positive subspace with the angular operator K such that $K^* = \overline{K}$	$\longleftrightarrow$	regular pure quadratic-like state, i.e., pure even quasi-free state
maximal positive invariant subspace with the angular operator K such that $K^* = \overline{K}$	$\longleftrightarrow$	pure quadratic-like invariant state

Now then, it is time to Examples.

COMMENT

About linear canonical transformations and hamiltonians, see e.g., [W1,2,3] for $dim < \infty$ and e.g., [Ber], [BR2], [RS2] for the quantum case. Theorem 5 see in [Ber], see also [DK], [K]. The standard point is concentrated on the questions "how diagonalize a given hamiltonian or automophism ?" and "does there exist a **regular** invariant state ?" We interested in any invariant states no matter whether they are regular or not and any hamiltonians no matter whether they are diagonalizable or not.

For terms 'angular operator', 'positive subspace', 'maximal positive subspace' and for other details of the Krein spaces theory, see, e.g., [Bog], [DR].

5 Examples

5.1 Example 1. Oscillator

In terms of $\mathbf{P}, \mathbf{Q}$, the Hamiltonian is written as

$$h := \frac{1}{2}\mathbf{P}^2 + \frac{1}{2}\Omega_0^2\mathbf{Q}^2$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x_p' \oplus x_q')$$

where

$$\begin{pmatrix} x_p' \\ x_q' \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega_0^2 & 0 \end{pmatrix}$$

$$V_t := e^{t\begin{pmatrix} 0 & 1 \\ -\Omega_0^2 & 0 \end{pmatrix}} = \begin{pmatrix} cos(\Omega_0 t) & \Omega_0^{-1} sin(\Omega_0 t) \\ -\Omega_0 sin(\Omega_0 t) & cos(\Omega_0 t) \end{pmatrix}$$

In terms of a^*, a , the Hamiltonian is rewritten as

$$\begin{aligned} h &:= \frac{1}{2}\mathbf{P}^2 + \frac{1}{2}\Omega_0^2\mathbf{Q}^2 \\ &= \frac{1}{2}(\frac{i}{\sqrt{2}}(a^* - a))^2 + \frac{1}{2}\Omega_0^2(\frac{1}{\sqrt{2}}(a^* + a))^2 \\ &= \frac{1+\Omega_0^2}{2}a^*a - \frac{1-\Omega_0^2}{4}a^{*2} - \frac{1-\Omega_0^2}{4}a^2 + const \end{aligned}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u^{+'} \oplus u^{-'}),$$

where $u^{+'} \oplus u^{-'}$ is defined by

$$\begin{pmatrix} u^{+'} \\ u^{-'} \end{pmatrix} = \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} = \begin{pmatrix} \frac{1+\Omega_0^2}{2} & \frac{1-\Omega_0^2}{2} \\ -\frac{1-\Omega_0^2}{2} & -\frac{1+\Omega_0^2}{2} \end{pmatrix}$$

$$-\frac{1-\Omega_0^2}{2} - \frac{1+\Omega_0^2}{2}K = K\frac{1+\Omega_0^2}{2} + K\frac{1-\Omega_0^2}{2}K$$

$$-(1-\Omega_0^2) = 2K(1+\Omega_0^2) + (1-\Omega_0^2)K^2$$

Recall that $\|K\| \le 1$ and assume $\Omega_0 > 0$. Then

$$K = -\frac{1-\Omega_0}{1+\Omega_0}$$

is a **unique** solution. As for the corresponding R, Q, q, ω, we have in terms of $\mathbf{P}, \mathbf{Q}$,

$$R = \frac{1}{4}\begin{pmatrix} 2-K-K^* & iK-iK^* \\ iK-iK^* & 2+K+K^* \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\Omega_0} & 0 \\ 0 & \frac{\Omega_0}{1+\Omega_0} \end{pmatrix}$$

$$Q = \begin{pmatrix} \Omega_0 & 0 \\ 0 & \frac{1}{\Omega_0} \end{pmatrix}$$

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-q(x_p \oplus x_q)/4} = e^{-(\Omega_0 x_p^2 + \frac{1}{\Omega_0} x_q^2)/4}$$

A few detailes of asymptotic behaviour of $\alpha_t := \alpha_{V_t}$ are the folllowing: Consider the standard Fock state, i.e., the state, ω_F, defined by

$$\omega_F e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-|x_p \oplus x_q|^2/4} = e^{-(x_p^2 + x_q^2)/4}$$

Then

$$\begin{aligned} &\omega_F \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} \\ &\quad = \quad e^{-|V_t(x_p \oplus x_q)|^2/4} \\ &\quad = \quad e^{-((cos(\Omega_0 t)x_p + \Omega_0^{-1} sin(\Omega_0 t)x_q)^2 + (-\Omega_0 sin(\Omega_0 t)x_p + cos(\Omega_0 t)x_q)^2)/4} \end{aligned}$$

We see, this quantity has no usual limit, neither as $t \to +\infty$ nor as $t \to -\infty$, whenever $\Omega_0 \neq \pm 1$.

5.2 Example 2. Free Evolution on Line

In terms of $\mathbf{P}, \mathbf{Q}$, the Hamiltonian is written as

$$h := \frac{1}{2}\mathbf{P}^2$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x'_p \oplus x'_q)$$

where

$$\begin{pmatrix} x'_p \\ x'_q \end{pmatrix} = \left(\begin{array}{cc} L & M \\ -K & -L^T \end{array} \right) \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\left(\begin{array}{cc} L & M \\ -K & -L^T \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$

$$V_t := e^{t\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)} = \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right)$$

In terms of a^*, a , the Hamiltonian is rewritten as

$$\begin{aligned} h &:= \frac{1}{2}\mathbf{P}^2 \\ &= \frac{1}{2}(\frac{i}{\sqrt{2}}(a^* - a))^2 \\ &= \frac{1}{2}a^*a - \frac{1}{4}a^{*2} - \frac{1}{4}a^2 + const \end{aligned}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u^{+'} \oplus u^{-'}),$$

where $u^{+'} \oplus u^{-'}$ is defined by

$$\begin{pmatrix} u^{+'} \\ u^{-'} \end{pmatrix} = \left(\begin{array}{cc} S & T \\ -\overline{T} & -\overline{S} \end{array} \right) \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\left(\begin{array}{cc} S & T \\ -\overline{T} & -\overline{S} \end{array} \right) = \left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{array} \right)$$

$$-\frac{1}{2} - \frac{1}{2}K = K\frac{1}{2} + K\frac{1}{2}K$$

$$-1 = 2K + K^2$$

Then

$$K = -1$$

is a **unique** solution. As for the corresponding R, Q, q, ω, we have in terms of $\mathbf{P}, \mathbf{Q}$,

$$R = \frac{1}{4}\left(\begin{array}{cc} 2 - K - K^* & iK - iK^* \\ iK - iK^* & 2 + K + K^* \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$$

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-q(x_p \oplus x_q)/4} = e^{-\infty \cdot x_q^2/4} = \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases}$$

A few detailes of asymptotic behaviour of $\alpha_t := \alpha_{V_t}$ are the folllowing: Consider the standard Fock state, i.e., the state, ω_F, defined by

$$\omega_F e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-|x_p \oplus x_q|^2/4} = e^{-(x_p^2 + x_q^2)/4}$$

Then

$$\begin{aligned}\omega_F\alpha_t e^{ix_p\mathbf{P}+ix_q\mathbf{Q}} &= e^{-|V_t(x_p\oplus x_q)|^2/4}\\ &= e^{-((x_p+tx_q)^2+x_q^2)/4}\end{aligned}$$

We see, this quantity has a limit as $t\to+\infty$ and as $t\to-\infty$ as well and these limits are equal:

$$\begin{aligned}\lim_{t\to\pm\infty}\omega_F\alpha_t e^{ix_p\mathbf{P}+ix_q\mathbf{Q}} &= \lim_{t\to\pm\infty} e^{-((x_p+tx_q)^2+x_q^2)/4}\\ &= \begin{cases} e^{-x_p^2/4}, & \text{if } x_q=0\\ 0, & \text{if } x_q\neq 0\end{cases}\end{aligned}$$

Notice,

$$\lim_{t\to\pm\infty}\omega_F\alpha_t e^{ix_p\mathbf{P}+ix_q\mathbf{Q}}$$

is not a charactiristic functional of a **pure** state.

5.3 Example 3. $h := \frac{1}{2}(\mathbf{PQ} + \mathbf{QP})$

In terms of $\mathbf{P}, \mathbf{Q}$, the Hamiltonian is written as

$$h := \frac{1}{2}(\mathbf{PQ} + \mathbf{QP})$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x'_p \oplus x'_q)$$

where

$$\begin{pmatrix} x'_p \\ x'_q \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$V_t := e^{t\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

In terms of a^*, a , the Hamiltonian is rewritten as

$$\begin{aligned} h &:= \frac{1}{2}(\mathbf{PQ} + \mathbf{QP}) \\ &= \frac{1}{2}\left(\frac{i}{\sqrt{2}}(a^* - a) \cdot \frac{1}{\sqrt{2}}(a^* + a) + \frac{i}{\sqrt{2}}(a^* + a) \cdot \frac{1}{\sqrt{2}}(a^* - a) \right) \\ &= \frac{i}{2}a^{*2} - \frac{i}{2}a^2 \end{aligned}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u^{+\prime} \oplus u^{-\prime}) \,,$$

where $u^{+\prime} \oplus u^{-\prime}$ is defined by

$$\begin{pmatrix} u^{+\prime} \\ u^{-\prime} \end{pmatrix} = \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$-i = K \cdot (-i) \cdot K$$

Then there are **two (!)** solutions:

$$K = K_{+1} = 1$$

$$K = K_{-1} = -1$$

As for the corresponding $R_{+1}, Q_{+1}, q_{+1}, \omega_{+1}$, and $R_{-1}, Q_{-1}, q_{-1}, \omega_{-1}$, we have in terms of $\mathbf{P}, \mathbf{Q}$,

$$R_{+1} = \frac{1}{4}\begin{pmatrix} 2 - K_{+1} - K_{+1}^* & iK_{+1} - iK_{+1}^* \\ iK_{+1} - iK_{+1}^* & 2 + K_{+1} + K_{+1}^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-q_{+1}(x_p \oplus x_q)/4} = e^{-\infty \cdot x_p^2/4} = \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases}$$

$$R_{-1} = \frac{1}{4}\begin{pmatrix} 2 - K_{-1} - K_{-1}^* & iK_{-1} - iK_{-1}^* \\ iK_{-1} - iK_{-1}^* & 2 + K_{-1} + K_{-1}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-q_{-1}(x_p \oplus x_q)/4} = e^{-\infty \cdot x_q^2/4} = \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases}$$

If we confer these expressions with that in Example 1,

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-q(x_p \oplus x_q)/4} = e^{-(\Omega_0 x_p^2 + \frac{1}{\Omega_0} x_q^2)/4}$$

we can infer that ω_{+1} is an approximation of the ground state of an oscillator with the extremely high frequency Ω_0 whereas ω_{-1} is an approximation of the ground state of an oscillator with the extremely low frequency Ω_0.

Finally, we have

$$\begin{aligned} \lim_{t\to+\infty} \omega_F \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= \lim_{t\to+\infty} e^{-(e^{-2t}x_p^2 + e^{2t}x_q^2)/4} \\ &= \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases} \\ &= \omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} \end{aligned}$$

$$\begin{aligned} \lim_{t\to-\infty} \omega_F \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= \lim_{t\to-\infty} e^{-(e^{-2t}x_p^2 + e^{2t}x_q^2)/4} \\ &= \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} \\ &= \omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} \end{aligned}$$

This situation is typical. Whatever **regular** quadratic state ω,

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-(q_{11}x_p^2 + 2q_{22}x_p x_q + q_{22}x_q^2)/4},$$

we have choosen, the result is:

$$\begin{aligned} \lim_{t\to+\infty} \omega \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= \begin{cases} 1, & \text{if } x_q = 0 \\ 0, & \text{if } x_q \neq 0 \end{cases} \\ &= \omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} \end{aligned}$$

$$\begin{aligned} \lim_{t\to-\infty} \omega \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= \begin{cases} 1, & \text{if } x_p = 0 \\ 0, & \text{if } x_p \neq 0 \end{cases} \\ &= \omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} \end{aligned}$$

5.4 Example 4. Repulsive Oscillator

In terms of $\mathbf{P}, \mathbf{Q}$, the Hamiltonian is written as

$$h := \frac{1}{2}\mathbf{P}^2 - \frac{1}{2}\Omega_0^2\mathbf{Q}^2$$

Then

$$i[h, F(x_p \oplus x_q)] = F(x_p' \oplus x_q')$$

where

$$\begin{pmatrix} x_p' \\ x_q' \end{pmatrix} = \begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} \begin{pmatrix} x_p \\ x_q \end{pmatrix}$$

$$\begin{pmatrix} L & M \\ -K & -L^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Omega_0^2 & 0 \end{pmatrix}$$

$$V_t := e^{t\begin{pmatrix} 0 & 1 \\ -\Omega_0^2 & 0 \end{pmatrix}} = \begin{pmatrix} cos(i\Omega_0 t) & (i\Omega_0)^{-1} sin(i\Omega_0 t) \\ -i\Omega_0 sin(i\Omega_0 t) & cos(i\Omega_0 t) \end{pmatrix}$$

$$V_t := e^{t\begin{pmatrix} 0 & 1 \\ -\Omega_0^2 & 0 \end{pmatrix}} = \frac{1}{2}\begin{pmatrix} e^{\Omega_0 t} + e^{-\Omega_0 t} & \Omega_0^{-1}(e^{\Omega_0 t} - e^{-\Omega_0 t}) \\ \Omega_0(e^{\Omega_0 t} - e^{-\Omega_0 t}) & e^{\Omega_0 t} + e^{-\Omega_0 t} \end{pmatrix}$$

In terms of a^*, a , the Hamiltonian is rewritten as

$$\begin{aligned} h &:= \frac{1}{2}\mathbf{P}^2 - \frac{1}{2}\Omega_0^2\mathbf{Q}^2 \\ &= \frac{1}{2}(\frac{i}{\sqrt{2}}(a^* - a))^2 - \frac{1}{2}\Omega_0^2(\frac{1}{\sqrt{2}}(a^* + a))^2 \\ &= \frac{1 - \Omega_0^2}{2}a^* a - \frac{1 + \Omega_0^2}{4}a^{*2} - \frac{1 + \Omega_0^2}{4}a^2 + const \end{aligned}$$

Formal calculations show that

$$[h, A(u^+ \oplus u^-)] = A(u^{+'} \oplus u^{-'}) ,$$

where $u^{+'} \oplus u^{-'}$ is defined by

$$\begin{pmatrix} u^{+'} \\ u^{-'} \end{pmatrix} = \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} = \begin{pmatrix} \frac{1-\Omega_0^2}{2} & \frac{1+\Omega_0^2}{2} \\ -\frac{1+\Omega_0^2}{2} & -\frac{1-\Omega_0^2}{2} \end{pmatrix}$$

$$-\frac{1 + \Omega_0^2}{2} - \frac{1 - \Omega_0^2}{2}K = K\frac{1 - \Omega_0^2}{2} + K\frac{1 + \Omega_0^2}{2}K$$

$$-(1 + \Omega_0^2) = 2K(1 - \Omega_0^2) + (1 + \Omega_0^2)K^2$$

Recall that $\|K\| \leq 1$ and assume $\Omega_0 > 0$. Then there are **two (!)** solutions:

$$K = K_{+1} = -\frac{1 - i\Omega_0}{1 + i\Omega_0}$$

$$K = K_{-1} = -\frac{1 + i\Omega_0}{1 - i\Omega_0}$$

As for the corresponding $R_{+1}, Q_{+1}, q_{+1}, \omega_{+1}$, and $R_{-1}, Q_{-1}, q_{-1}, \omega_{-1}$, we have in terms of $\mathbf{P}, \mathbf{Q}$,

$$R_{+1} = \frac{1}{4}\begin{pmatrix} 2 - K_{+1} - K_{+1}^* & iK_{+1} - iK_{+1}^* \\ iK_{+1} - iK_{+1}^* & 2 + K_{+1} + K_{+1}^* \end{pmatrix} = \frac{1}{1+\Omega_0^2}\begin{pmatrix} 1 & \Omega_0 \\ \Omega_0 & \Omega_0^2 \end{pmatrix}$$

$$\begin{aligned} \omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= e^{-q_{+1}(x_p \oplus x_q)/4} \\ &= e^{-\infty \cdot (-\Omega_0 x_p + x_q)^2/4} = \begin{cases} 1, & \text{if } -\Omega_0 x_p + x_q = 0 \\ 0, & \text{if } -\Omega_0 x_p + x_q \neq 0 \end{cases} \end{aligned}$$

$$R_{-1} = \frac{1}{4}\begin{pmatrix} 2 - K_{-1} - K_{-1}^* & iK_{-1} - iK_{-1}^* \\ iK_{-1} - iK_{-1}^* & 2 + K_{-1} + K_{-1}^* \end{pmatrix} = \frac{1}{1+\Omega_0^2}\begin{pmatrix} 1 & -\Omega_0 \\ -\Omega_0 & \Omega_0^2 \end{pmatrix}$$

$$\begin{aligned} \omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} &= e^{-q_{-1}(x_p \oplus x_q)/4} \\ &= e^{-\infty \cdot (\Omega_0 x_p + x_q)^2/4} = \begin{cases} 1, & \text{if } \Omega_0 x_p + x_q = 0 \\ 0, & \text{if } \Omega_0 x_p + x_q \neq 0 \end{cases} \end{aligned}$$

Finally, one can verify that

$$\lim_{t\to+\infty} \omega_F \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = \omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}}$$

$$\lim_{t\to-\infty} \omega_F \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = \omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}}$$

This situation is typical as in the previous Example. Whatever **regular** quadratic state ω,

$$\omega e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = e^{-(q_{11}x_p^2 + 2q_{22}x_p x_q + q_{22}x_q^2)/4},$$

we have choosen, the result is:

$$\lim_{t\to+\infty} \omega \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = \omega_{-1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}}$$

$$\lim_{t\to-\infty} \omega \alpha_t e^{ix_p\mathbf{P} + ix_q\mathbf{Q}} = \omega_{+1} e^{ix_p\mathbf{P} + ix_q\mathbf{Q}}$$

5.5 Example 5.

consider the approximating (Bogoliubov) Hamiltonian

$$H'_B = \int dp \left\{ \omega(p)a^*(p)a(p) + \frac{1}{2}\Delta_B(p)\left[a(p)^*a(-p)^* + a(-p)a(p)\right]\right\}$$

Formal calculations show that

$$[H'_B, A(u^+ \oplus u^-)] = A(u^{+'} \oplus u^{-'}),$$

where $u^{+'} \oplus u^{-'}$ is defined by

$$\begin{pmatrix} u^{+'} \\ u^{-'} \end{pmatrix} = \begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix}(p,p') = \begin{pmatrix} \omega(p)\delta(p-p') & -\Delta_B(p)\delta(p+p') \\ \Delta_B(p)\delta(p+p') & -\omega(p)\delta(p-p') \end{pmatrix}$$

$$\begin{pmatrix} S & T \\ -\overline{T} & -\overline{S} \end{pmatrix} = \begin{pmatrix} \hat{\omega} & -\hat{\Delta}_B J_0 \\ J_0\hat{\Delta}_B & -\hat{\omega} \end{pmatrix}$$

$$J_0\hat{\Delta}_B - \hat{\omega}K = K\hat{\omega} - K\hat{\Delta}_B J_0 K$$

$$-\hat{\Delta}_B + J_0\hat{\omega}J_0K = -J_0K\hat{\omega} + J_0K\hat{\Delta}_B J_0 K$$

If we take into account that $\omega(-p) = \omega(p)$, i.e., $J_0\hat{\omega}J_0 = \hat{\omega}$ and if we restrict ourselves to the case where J_0K commutes with the multiplications by functions, i.e., if K is of the form

$$K(p,p') = \delta(p+p')k_0(p)$$

for a function k_0, then we obtain:

$$-\Delta_B(p) = -2\omega(p)k_0(p) + \Delta_B(p)k_0(p)^2\,,\ |k_0(p)| \le 1\,.$$

Thus

$$k_0(p) = \begin{cases} 0, & \text{if } p \text{ is such that} \\ & \Delta_B(p) = 0, \omega(p) \neq 0 \\ arbitrary, & \text{if } p \text{ is such that} \\ & \omega(p) = 0, \Delta(p) = 0 \\ \dfrac{\omega(p) - sgn(\omega(p))\sqrt{-\Delta_B(p)^2 + \omega(p)^2}}{\Delta_B(p)}, & \text{if } p \text{ is such that} \\ & -\Delta_B(p)^2 + \omega(p)^2 \ge 0, \Delta(p) \neq 0 \\ \dfrac{\omega(p) - i\epsilon(p)\sqrt{\Delta_B(p)^2 - \omega(p)^2}}{\Delta_B(p)}, & \text{if } p \text{ is such that} \\ \text{where } \epsilon(p)^2 = 1 & -\Delta_B(p)^2 + \omega(p)^2 \le 0, \Delta(p) \neq 0 \end{cases}$$

These relationships can be transformed as follows:

$$k_0(p) = \begin{cases} arbitrary, & \text{if } p \text{ is such that} \\ & \omega(p) = 0, \Delta(p) = 0 \\ \dfrac{\Delta_B(p)}{\omega(p) + sgn(\omega(p))\sqrt{-\Delta_B(p)^2 + \omega(p)^2}}, & \text{if } p \text{ is such that} \\ & -\Delta_B(p)^2 + \omega(p)^2 \ge 0 \\ \dfrac{\Delta_B(p)}{\omega(p) + i\epsilon(p)\sqrt{\Delta_B(p)^2 - \omega(p)^2}}, & \text{if } p \text{ is such that} \\ \text{where } \epsilon(p)^2 = 1 & -\Delta_B(p)^2 + \omega(p)^2 \le 0 \end{cases}$$

In particular, if there are infinitely many p such that $-\Delta_B(p)^2 + \omega(p)^2 \le 0$, then there are infinitely many invariant pure quadratic-like states.

6 Appendix A: The Formal Calculations of e^{A+B}

Assume

$$[[A,B],B]=0\,,\ [[A,B],A]=0\,.$$

Now let us calculate

$$U:=e^{t(A+B)}.$$

The definition of

$$U$$

formally implies

$$dU/dt=(A+B)U\ ,\ U(0)=I\,.$$

Let

$$U:=e^{tA}V$$

Hence,

$$dV/dt=e^{-tA}Be^{tA}V=(B-t[A,B])V\ ,\ V(0)=I\,.$$

Let

$$V:=e^{tB}C\,.$$

Then,

$$dC/dt=-e^{-tB}t[A,B]e^{tB}=-t[A,B]C\ ,\ C(0)=I\,.$$

Hence

$$C=e^{-\frac{t^2}{2}[A,B]}\,.$$

the Result is:

$$e^{t(A+B)}=e^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]}$$

$$e^{i(A+B)}=e^{iA}e^{iB}e^{\frac{i}{2}[A,B]}$$

$$e^{i(A+B)}=e^{iA}e^{iB}e^{\frac{1}{2}[A,B]}$$

$$e^{A}e^{B}=e^{\frac{1}{2}[A,B]}e^{A+B}$$

$$e^{iA}e^{iB}=e^{-\frac{1}{2}[A,B]}e^{i(A+B)}$$

Remark .

The usual form of the CCRs is motivated by the Schrödinger representation of the position and momentum operators:

$$\mathbf{Q}=\hat{x}\quad,\quad\mathbf{P}=\frac{\hbar}{i}\frac{\partial}{\partial x}\,.$$

Hence,

$$\frac{i}{\hbar}[\mathbf{P},\mathbf{Q}]=1\,,\ [\mathbf{Q},\mathbf{P}]=i\hbar\,,\ [\frac{1}{2}\mathbf{P}^2,\mathbf{Q}]=\frac{1}{2}\cdot 2\cdot\mathbf{P}\cdot\frac{\hbar}{i}=\frac{\hbar}{i}\mathbf{P}\,.$$

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