REMARKS ON THE SPECTRUM OF THE NEUMANN PROBLEM WITH MAGNETIC FIELD IN THE HALF SPACE

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Abstract

We consider a Schrdinger operator with a constant magnetic field in a half 3-dimensional space, with Neumann type boundary conditions. It is known from the works by Lu-Pan and Helffer-Morame that the lower bound of its spectrum is less than the intensity b of the magnetic field, provided that the magnetic field is not normal to the boundary.

We prove that the spectrum under b is a finite set of eigenvalues (each of infinite multiplicity).

In the case when the angle between the magnetic field and the boundary is small, we give a sharp asymptotic expansion of the number of these eigenvalues.

1 Introduction

Let us consider, for (t, x, s) in the half space $E = I\!R_+ \times I\!R^2$, the Neumann realization of the operator with magnetic field

$$H = (D_t - A_1)^2 + (D_x - A_2)^2 + (D_y - A_3)^2$$

where $D_s = -i(\frac{\partial}{\partial s})$.

We will assume that the magnetic field $B = dA$, seen as a 3-dimensional vector field, is not tangent to the boundary ∂E, and denote by θ the angle between B and the plane $t = 0$ and by b the norm of B.

This implies that a suitable choice for the gauge A is the 1-form

$$A = b(x \sin\theta - t \cos\theta)dy$$

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(so that $A_1 = A_2 = 0$), since the condition $B = dA$ leads to the 2-form

$$B = b\sin\theta \; dx \wedge dy - b\cos\theta \; dt \wedge dy, \quad (\theta \in \left[0, \frac{\pi}{2}\right]) \,.$$

Now the operator H can be written as

$$H_\theta = D_t^2 + D_x^2 + (D_y - b(x\sin\theta - t\cos\theta))^2 \tag{1.1}$$

When $\theta = 0$, it is easy to see that the spectrum of the Neumann operator H_0 is absolutely continuous. More precisely one has :

$$\sigma(H_0) = \sigma_{ac}(H_0) = [b\mu_0, +\infty[\;; \tag{1.2}$$

$$\mu_0 = \inf_{\xi \in I\!\!R} \; \mu(\xi) \;, \tag{1.3}$$

where $\mu(\xi)$ denotes the first eigenvalue of the Neumann operator
$Q_\xi = D_t^2 + (t-\xi)^2$ on $L^2(I\!\!R_+)$:

$$\mu(\xi) = \inf \; \sigma(Q_\xi) = \inf_{\|f\|_{L^2(I\!\!R_+)}=1} \int_{I\!\!R_+} \left[|D_t f|^2 + (t-\xi)^2|f|^2\right] dt \,. \tag{1.4}$$

It is also easy to see that, if $\theta = \frac{\pi}{2}$, the spectrum of $H_{\frac{\pi}{2}}$ is absolutely continuous:

$$\sigma(H_{\frac{\pi}{2}}) = \sigma_{ac} = [b, +\infty[\;. \tag{1.5}$$

When $\theta \in \left]0, \frac{\pi}{2}\right[$, the spectrum of H_θ is no longer absolutely continuous as proved by K. Lu and X-B. Pan [LuPa], (see also [HeMo2]).
We are precisely interested in that case:

$$0 < \theta < \frac{\pi}{2} \,. \tag{1.6}$$

First, we observe that

$$\sigma(H_\theta) = \bigcup_{\tau \in I\!\!R} \sigma(H_{\theta,\tau}) \;, \tag{1.7}$$

where $H_{\theta,\tau}$ denotes the Neumann realization in the half plane
$F = I\!\!R_+ \times I\!\!R$ of the operator

$$H_{\theta,\tau} = D_t^2 + D_x^2 + (\tau - b(x\sin\theta - t\cos\theta))^2 \,. \tag{1.8}$$

Furthermore using for any τ the change of coordinates $x \to x - \frac{\tau}{b\sin\theta}$,
we see that $\sigma(H_{\theta,\tau}) = \sigma(H_{\theta,0})$,
and then the spectrum of H_θ is essential and given by :

$$\sigma(H_\theta) = \sigma_{ess}(H_\theta) = \sigma(H_{\theta,0}) = b \times \sigma(P_\theta) \;, \tag{1.9}$$

if $P_\theta = D_t^2 + D_x^2 + (t\cos\theta - x\sin\theta)^2$ is the Neumann operator on the half plane $F = I\!\!R_+ \times I\!\!R$.

In [LuPa], (see also [HeMo2]), it was proved that

$$\inf\ \sigma(P_\theta) = \nu(\theta) < 1 = \inf\ \sigma_{ess}(P_\theta)\ ; \tag{1.10}$$

so there exists a countable set of eigenvalues of P_θ, $(\nu_j(\theta))_{j\in I}$, $(I \subset I\!N)$, in $[\nu(\theta), 1[$. Each eigenvalue is of finite multiplicity, so we will assume that each one is repeated according to its multiplicity. The associated orthonormalized sequence of eigenfunctions will be denoted by $(\psi_{\theta,j})_{j\in I}$:

$$\begin{aligned} \nu(\theta) &= \nu_1(\theta) \le \nu_2(\theta) \le \dots \nu_j(\theta) \le \nu_{j+1}(\theta) \le \dots < 1, \qquad (1.11)\\ P_\theta \psi_{\theta,j} &= \nu_j(\theta)\psi_{\theta,j},\\ \langle \psi_{\theta,j} | \psi_{\theta,k}\rangle &= \delta_{jk},\\ E_{\big]-\infty,1\big[}(P_\theta) f &= \sum_j \langle \psi_{\theta,j} | f\rangle \psi_{\theta,j}\ ; \end{aligned}$$

($\langle g\ | f\rangle = \int_F \overline{g} f dt dx$ and $E_J(P_\theta)$ denotes the spectral projection of P_θ on J). So

$$\sigma(H_\theta) \cap \big] -\infty, b[= \{b\nu_1(\theta),\ b\nu_2(\theta), \dots,\ b\nu_j(\theta),\ b\nu_{j+1}(\theta), \dots\}\ , \tag{1.12}$$

(each $b\nu_j(\theta)$ is an eigenvalue of infinite multiplicity of H_θ).

For any $d \le 1$ let us denote by $N(d,\ P_\theta)$ the number of eigenvalues of P_θ in $\big] -\infty, d[$:

$$N(d,\ P_\theta) = Tr(E_{\big]-\infty,d\big[}(P_\theta) = \sharp\{j;\ \nu_j(\theta) < d\}\ . \tag{1.13}$$

The aim of this work is first to prove that for any $\theta\ \in \big]0, \frac{\pi}{2}\big[$, the number of eigenvalues of P_θ in $\big]-\infty,\ 1[$ is finite. This is the purpose of section 2. Another interesting question is to get the asymptotic behaviour of $N(d,\ P_\theta)$ as θ goes to zero, when $d < 1$. This is done in section 4. Section 3 is devoted to a survey of preliminary results about the function $\mu(\xi)$ defined in (1.4) , which are required in the computation of the asymptotics in section 4.

2 Finiteness of the discrete spectrum

The purpose of this section is to prove the following Theorem.

Theorem 2.1 *There exists a constant $C \ge 1$ such that, for any $\theta\ \in \big]0, \frac{\pi}{2}\big[$,*

$$N(1,\ P_\theta) \le \frac{C}{\sin\theta}\ . \tag{2.1}$$

Proof

Convention 2.2 $\theta\ \in \big]0, \frac{\pi}{2}\big[$ *is fixed.*

Convention 2.3 *From now on, any constant depending only on θ will be denoted invariably C_θ .*

If the constant does not depend on θ, it will be denoted invariably C.

Let us denote by q_θ the quadratic form associated to P_θ :

$$q_\theta(u) = \int_F \big[[D_t u|^2 + |D_x u|^2 + (t\cos\theta - x\sin\theta)^2 |u|^2\big] dtdx \ , \tag{2.2}$$

$\forall u \in H^1(F) \bigcap L^2(F;\ (t\cos\theta - x\sin\theta)^2 dtdx)$; $\quad (F = I\!R_{+,t} \times I\!R_x\ )$.

There exists a partition of unity $(\chi_0(t),\ \chi_1(t))$ satisfying :

$$\begin{aligned} &\chi_0(t) = 1 \text{ if } t < 1, \\ &\chi_0(t) = 0 \text{ if } t > 2, \\ &\chi_0^2(t) + \chi_1^2(t) = 1\ . \end{aligned} \tag{2.3}$$

Let $R > 1$ be fixed. We consider the following covering of F :

$$\begin{aligned} O_{0,R} &= \{(t,x) \in I\!R_+ \times I\!R,\ 0 < t < 2R\} \\ O_{1,R} &= \{(t,x) \in I\!R_+ \times I\!R,\ R < t\} \end{aligned} \tag{2.4}$$

We define the partition of unity $(\chi_{0,R}(t),\ \chi_{1,R}(t))$ by :

$$\chi_{j,R}(t) = \chi_j(\frac{t}{R})\ . \tag{2.5}$$

Let us recall that

$$q_\theta(u) = \sum_j q_\theta(\chi_{j,R} u) - \sum_j \|\chi'_{j,R} u\|^2\ . \tag{2.6}$$

We define the following quadratic forms:

$$q_{\theta,0}(u) = \int_{O_{0,R}} \big[D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2\big] dtdx\ , \tag{2.7}$$

$\forall u \in H^1(O_{0,R}) \bigcap L^2(O_{0,R}; x^2 dtdx),\ u/\{t = 2R\} = 0$, with $V_R(t) = \sum_j |\chi'_{j,R}(t)|^2$, and

$$q_{\theta,1}(u) = \int_{O_{1,R}} \big[D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2\big] dtdx\ , \tag{2.8}$$

$\forall\ u\ \in\ H^1(O_{1,R}) \bigcap L^2(O_{1,R}; (t\cos\theta - x\sin\theta)^2 dtdx),\ u/\{t = R\} = 0$.

By min-max principle, we have

$$N(1,\ q_\theta) \le N(1,\ q_{\theta,0}) + N(1,\ q_{\theta,1})\ . \tag{2.9}$$

This estimate remains if we change $O_{1,R}$ into $I\!R^2$ in the definition of $q_{\theta,1}$:

$$q_{\theta,1}(u) = \int_{I\!R^2} \big[D_t u|^2 + |D_x u|^2 + (\ (t\cos\theta - x\sin\theta)^2 - V_R(t)\ )|u|^2\big] dtdx\ , \tag{2.10}$$

$\forall\ u\ \in\ H^1(I\!R^2)\bigcap L^2(I\!R^2;(t\cos\theta-x\sin\theta)^2dtdx)$.

As the operator $P_{\theta,0}$, associated to $q_{\theta,0}$ has compact resolvent, and

$$q_{\theta,0}(u)\geq\int_{O_{0,R}}\Big[D_tu|^2+|D_xu|^2+\Big(\frac{1}{2}x^2\sin^2\theta-4R^2\cos^2\theta-\frac{C}{R^2}\Big)|u|^2\Big]dtdx\ ,$$

we get easily

$$N(1,\ q_{\theta,0})\leq\frac{CR}{\sin\theta}\big[1+R^2\cos^2\theta\big]^{3/2}\ . \tag{2.11}$$

Using the orthonormal change of coordinates :

$(t,x)\rightarrow(s,y)$ with $s=t\cos\theta-x\sin\theta$, and $y=t\sin\theta+x\cos\theta$,
we can take for $q_{\theta,1}$ the following expression :

$$q_{\theta,1}(u)=\int_{I\!R^2}\big[D_su|^2+|D_yu|^2+(s^2-V_R(s,y)\ )|u|^2\big]dsdy\ , \tag{2.12}$$

$\forall\ u\ \in\ H^1(I\!R^2)\bigcap L^2(I\!R^2;\ s^2dsdy)$, with

$$V_R(s,y)=\frac{1}{R^2}\sum_j\Big|\chi_j'\Big(\frac{s\cos\theta+y\sin\theta}{R}\Big)\Big|^2\ . \tag{2.13}$$

Let us consider the orthogonal projections

$$\begin{aligned}\Pi_1(u)(s,y)\quad&=\quad e^{-s^2/2}\int_{I\!R}u(\tau,y)e^{-\tau^2/2}\frac{d\tau}{\sqrt{\pi}}\\ \Lambda_1u\quad&=\quad u-\Pi_1u\ ,\end{aligned} \tag{2.14}$$

so that, for any $u\ \in\ L^2(I\!R^2)$, we get : $\|u\|^2=\|\Pi_1u\|^2+\|\Lambda_1u\|^2$.

Writing : $\Pi_1u(s,y)=\frac{e^{-s^2/2}}{\pi^{1/4}}\psi(y)$ and

$$W_R(y)=\frac{1}{\sqrt{\pi}}\int_{I\!R}e^{-s^2}V_R(s,y)ds\ ,$$

we obtain that :

$$q_{\theta,1}(\Pi_1u)=\int_{I\!R}\big[|D_y\psi|^2+(1-W_R(y))|\psi|^2\big]dy\ . \tag{2.15}$$

We have also

$$q_{\theta,1}(\Lambda_1u)\geq\int_{I\!R^2}\Big[|D_y\Lambda_1u|^2+\Big(3-\frac{C}{R^2}\Big)|\Lambda_1u|^2\Big]dsdy\ . \tag{2.16}$$

But

$$q_{\theta,1}(u)=q_{\theta,1}(\Pi_1u)+q_{\theta,1}(\Lambda_1u)-2\mathcal{R}e\int_{I\!R^2}V_R(s,y)\Pi_1u.\overline{\Lambda_1u}dsdy\ ; \tag{2.17}$$

so, for any $\epsilon \in \left]0,1\right]$,

$$q_{\theta,1}(u) \geq q_{\theta,1}(\Pi_1 u) - \frac{1}{\epsilon}\int_{I\!R^2} V_R^2(s,y)|\Pi_1 u|^2 ds dy \, + \, q_{\theta,1}(\Lambda_1 u) - \epsilon\|\Lambda_1 u\|^2 \; . \quad (2.18)$$

Thanks to (2.16), we can take $\epsilon = 1$ and R large enough such that

$$q_{\theta,1}(\Lambda_1 u) - \epsilon\|\Lambda_1 u\|^2 > \|\Lambda_1 u\|^2 \;, \quad (2.19)$$

for example R satisfying : $3 - \frac{C}{R^2} - \epsilon > 1$.
Then, by (2.15) (2.17) and (2.19), we get that

$$N(1, \; q_{\theta,1}) \leq N(0, \; q_{\theta,1,0}) \;, \quad (2.20)$$

if

$$q_{\theta,1,0}(\psi) = \int_{I\!R} \left[|D_y\psi|^2 - W_{R,1}(y)|\psi|^2\right] dy \;, \quad (2.21)$$

$\forall\, \psi \in H^1(I\!R)$, with

$$W_{R,1}(y) = \frac{1}{\sqrt{\pi}}\int_{I\!R} e^{-s^2}\left[V_R(s,y) + V_R^2(s,y)\right] ds \;.$$

¿From (2.3) and the formula (2.13), the following bound holds :

$$0 \leq W_{R,1}(y) \leq \frac{C}{R^2}\int_{\frac{R - y\sin\theta}{\cos\theta}}^{\frac{2R - y\sin\theta}{\cos\theta}} e^{-s^2} ds \;,$$

(we used the fact that, for any fixed y, $V_R(s,y) = 0$ for s outside the interval defined by $R < s\cos\theta + y\sin\theta < 2R$), so

$$\begin{aligned} 0 &\leq W_{R,1}(y) \\ &\leq \frac{C}{R^2}\chi_{\left[\frac{R}{2\sin\theta},\; \frac{3R}{\sin\theta}\right]}(y) + \frac{C}{R\cos\theta}\left[e^{-\frac{(y\sin\theta - R)^2}{\cos^2\theta}} + e^{-\frac{(y\sin\theta - 2R)^2}{\cos^2\theta}}\right]. \end{aligned} \quad (2.22)$$

As the operators on $L^2(I\!R)$, $\frac{1}{3}D_y^2 - \frac{C}{R\cos\theta}e^{-\frac{(y\sin\theta-R)^2}{\cos^2\theta}}$, $\frac{1}{3}D_y^2 - \frac{C}{R\cos\theta}e^{-\frac{(y\sin\theta-2R)^2}{\cos^2\theta}}$ and $\frac{\sin^2\theta}{3\cos^4\theta}D_y^2 - \frac{C}{R\cos\theta}e^{-y^2}$ have the same spectrum, we get from (2.20)–(2.22) that

$$N(1, \; q_{\theta,1}) \leq 2N(0, \; q_{\theta,1,1}) \; + \; N(0, \; q_{\theta,1,2}) \;, \quad (2.23)$$

if

$$q_{\theta,1,1}(\psi) = \int_{I\!R}\left[\frac{\sin^2\theta}{3\cos^4\theta}|D_y\psi|^2 - \frac{C}{R\cos\theta}e^{-y^2}|\psi|^2\right] dy \;, \quad (2.24)$$

and

$$q_{\theta,1,2}(\psi) = \int_{I\!R}\left[\frac{1}{3}|D_y\psi|^2 - \frac{C}{R^2}\chi_{\left[\frac{R}{2\sin\theta},\; \frac{3R}{\sin\theta}\right]}(y)|\psi|^2\right] dy \;, \quad (2.25)$$

$\forall\, \psi \in H^1(I\!R)$, for some $R > 1$ independent of $\theta \in \left]0, \frac{\pi}{2}\right[$.

It is well known that $N(0, \; q_{\theta,1,1}) \leq C\frac{\cos^{3/2}\theta}{\sin\theta}$
and $N(0, \; q_{\theta,1,2}) \leq \frac{C}{\sin\theta}$, so

$$N(1, \; q_{\theta,1}) \leq \frac{C}{\sin\theta} \;. \quad (2.26)$$

We conclude from (2.9), (2.11) and (2.26) that the estimate (2.1) is valid.

3 Some properties of $\mu(\xi)$

The properties of the first eigenvalue, $\mu(\xi)$, of the Neumann operator on $L^2(I\!R_+)$, $Q_\xi = D_t^2 + (t-\xi)^2$,
can be found in [DaHe], or in [BeSt], or in [HeMo1].
The main one is

$$
\begin{array}{l}
\mu \in C^\infty(I\!R), \\
\mu'(\xi) \neq 0 \ \text{ if } \xi \neq \xi_0, \\
\mu(\xi) > 1, \ \text{ if } \ \xi < 0, \\
\mu(\xi) < 1, \ \text{ if } \ \xi > 0, \\
\lim_{\xi \to -\infty} \mu(\xi) = +\infty, \ \ \lim_{\xi \to +\infty} \mu(\xi) = 1 \ ;
\end{array}
\tag{3.1}
$$

($\xi_0 > 0$ is such that $\xi_0^2 = \mu(\xi_0)$).
Let φ_ξ be a normalized eigenfunction associated to $\mu(\xi)$:

$$
\|\varphi_\xi\|_{L^2(I\!R_+)} = 1, \ \ \varphi'_\xi(0) = 0, \quad Q_\xi \varphi_\xi = \mu(\xi)\varphi_\xi \ , \tag{3.2}
$$

then

$$
\mu'(\xi) = -(\mu(\xi) - \xi^2)\varphi_\xi^2(0) \ . \tag{3.3}
$$

It is easy to see that φ_ξ is exponentially decreasing. More precisely we have

Lemma 3.1 *For any* $\xi > 1$ *and* $\eta \in \,]0,1[$,

$$
\int_{I\!R_+} \big[\eta(t-\xi)^2 - \mu(\xi)\big]_+ e^{(1-\eta)^{1/2}(t-\xi)^2} |\varphi_\xi(t)|^2 dt \leq \mu(\xi) e^{\mu(\xi)/\eta} \ ; \tag{3.4}
$$

consequently, there exists $C_0 > 1$ *such that*

$$
1 - C_0 e^{-|\xi|^2/C_0} \leq \mu(\xi) \ . \tag{3.5}
$$

More over, if $\mu_2(\xi)$ *is the second eigenvalue of* Q_ξ *, then we have also*

$$
3 - C_0 e^{-|\xi|^2/C_0} \leq \mu_2(\xi) \tag{3.6}
$$

and then

$$
\|\frac{\partial}{\partial \xi}\varphi_\xi(\,.\,)\|_{L^2(I\!R_+)} \leq C_0, \ \ \forall\, \xi > 1 \ . \tag{3.7}
$$

Proof We proceed as in [He1] to get (3.4). For any Lipschitz and real function Φ, with compact support,

$$
\|D_t(e^\Phi \varphi_\xi)\|^2_{L^2(I\!R_+)} = \langle \big[\mu(\xi) - (t-\xi)^2 + (\Phi')^2\big] e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle
$$

so

$$
\langle \big[(t-\xi)^2 - \mu(\xi) - (\Phi')^2\big]_+ e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle \ \leq \ \langle \big[\mu(\xi) - (t-\xi)^2 + (\Phi')^2\big]_+ e^\Phi \varphi_\xi | e^\Phi \varphi_\xi \rangle \ .
$$

This estimate is still valid for Φ with non compact support, provided that the right hand side of the inequality is finite; so we can take

$\Phi(t) = (1-\eta)^{1/2}(t-\xi)^2/2$ to get (3.4).
Now, let χ be a smooth cut-off function on $I\!R$:

$$\begin{array}{l} \chi \in C^\infty(I\!R), \\ \chi(t) = 1, \quad \text{if} \quad -1 < t < 1, \\ \chi(t) = 0, \quad \text{if} \quad |t| > 2, \\ 0 \le \chi \le 1 \,. \end{array} \tag{3.8}$$

If $\xi > 1$, we define the function $\widetilde{\varphi}_{1,\xi}(t) = \chi(4\frac{t-\xi}{\xi})\varphi_\xi(t)$.
So

$$\|D_t\widetilde{\varphi}_{1,\xi}\|^2 + \|(t-\xi)\widetilde{\varphi}_{1,\xi}\|^2 = \mu(\xi))\|\widetilde{\varphi}_{1,\xi}\|^2 + \frac{16}{\xi^2}\|\chi'(4\frac{t-\xi}{\xi})\widetilde{\varphi}_\xi\|^2 \,.$$

As $\widetilde{\varphi}_{1,\xi}$ is of compact support and the first eigenvalue of $D_t^2 + (t-\xi)^2$ on $L^2(I\!R)$ is 1 , then

$$\|\widetilde{\varphi}_{1,\xi}\|^2 \le \mu(\xi))\|\widetilde{\varphi}_{1,\xi}\|^2 + \frac{16}{\xi^2}\|\chi'(4\frac{t-\xi}{\xi})\varphi_\xi\|^2 \,;$$

then we use the estimate (3.4) to see that, for some constant $C > 1$, $1 \le \mu(\xi) + Ce^{-\xi^2/C}$: the estimate (3.5) follows.

If $\mu_2(\xi)$ is the second eigenvalue of Q_ξ and $\varphi_{2,\xi}$ the associated normalized eigenfunction, then we have in the same way, for any real function Φ,

$$\begin{aligned} &\langle \big[(t-\xi)^2 - \mu_2(\xi) - (\Phi')^2\big]_+ e^\Phi\varphi_{2,\xi}|e^\Phi\varphi_{2,\xi}\rangle \\ &\qquad\qquad \le \langle \big[\mu_2(\xi) - (t-\xi)^2 + (\Phi')^2\big]_+ e^\Phi\varphi_{2,\xi}|e^\Phi\varphi_{2,\xi}\rangle; \end{aligned}$$

so $\varphi_{2,\xi}$ is exponentially decreasing as φ_ξ, and if $\widetilde{\varphi}_{2,\xi}(t) = \chi(4\frac{t-\xi}{\xi})\varphi_{2,\xi}(t)$, then

$$\|D_t\widetilde{\varphi}_{2,\xi}\|^2 + \|(t-\xi)\widetilde{\varphi}_{2,\xi}\|^2 \le \mu_2(\xi) + Ce^{-\xi^2/C}$$

and

$$\begin{array}{l} |\ \|\widetilde{\varphi}_{2,\xi}\| - 1| \le Ce^{-\xi^2/C}, \\ |\langle\widetilde{\varphi}_{1,\xi}|\widetilde{\varphi}_{2,\xi}\rangle| \le Ce^{-\xi^2/C} \,. \end{array} \tag{3.9}$$

Those estimates, (3.5) and the min-max principle show that $\mu_2(\xi) + Ce^{-\xi^2/C}$ is greater then the second eigenvalue of $D_t^2 + (t-\xi)^2$ on $L^2(I\!R)$,
so $3 \le \mu_2(\xi) + Ce^{-\xi^2/C}$: this ends the proof of the Lemma.

We recall that some elementary technique of perturbation shows that

$$\frac{\partial}{\partial\xi}\varphi_\xi(t) = 2\big[Q_\xi - \mu(\xi)\big]^{-1}\psi_\xi \,, \tag{3.10}$$

with

$$\psi_\xi(t) = (t-\xi)\varphi_\xi(t) - \langle(t-\xi)\varphi_\xi|\varphi_\xi\rangle\varphi_\xi(t) \,.$$

As $\|D_t^2\varphi_\xi\| + \|(t-\xi)\varphi_\xi\| = \mu(\xi)\|\varphi_\xi\| = \mu(\xi)$, we get easily (3.7) from (3.10).

Remark 3.2 *As ξ goes to $-\infty$, the following asymptotic expansions hold*

$$\mu(\xi) = \xi^2 \ + \ (-2\xi)^{2/3}\rho_1 \ + \ \mathbf{O}(|\xi|^{-2/3}) \tag{3.11}$$

and

$$\mu_2(\xi) = \xi^2 \ + \ (-2\xi)^{2/3}\rho_2 \ + \ \mathbf{O}(|\xi|^{-2/3}) \tag{3.12}$$

if (ρ_j) is the increasing sequence of eigenvalues of the Neumann operator on $L^2(I\!R_+)$ associated to $D_t^2 + t$.

As a matter of fact, if $\xi < -1$, we have easily $\sigma(Q_\xi) = \xi^2\big[1 \ + \ \sigma(Q_{\xi,2})\big]$, if $Q_{\xi,2}$ is the Neumann operator on $L^2(I\!R_+)$ associated to $h^2D_t^2 + 2t + t^2$ with $h = \xi^{-2}$.

Using semi-classical method as in [Si1] or in [HeSj1] (or in [He1]), we get easily that the N-first eigenvalues of $Q_{\xi,2}$ are equal to $(2h)^{2/3}\rho_1, \ \ldots, \ (2h)^{2/3}\rho_N$ modulo a $\mathbf{O}(h^{4/3})$.

4 The case of small θ

We are still investigating the spectrum of the operator P_θ, defined in the introduction as following :

$$P_\theta = D_t^2 + D_x^2 + (t\cos\theta - x\sin\theta)^2 \ .$$

Performing the scaling $(t,x) \to (t\sqrt{\cos\theta}, -\frac{x sin\theta}{\sqrt{\cos\theta}})$, we observe that this operator has the same spectrum as :

$$P_\theta = \cos\theta\big[D_t^2 + (t-x)^2\big] + \frac{\sin^2\theta}{\cos\theta}D_x^2$$

(we keep on the same notation for simplification).

It has been proved [HeMo2] that for small values of $\theta > 0$ the following asymptotics hold :

$$\inf \ \sigma(P_\theta) \sim \mu_0 + \sum_{j\geq 1} c_j\theta^j \ .$$

Therefore let us consider a set

$$I_d = \ \big] - \infty, \ d\big[\ \text{ with } \ d \in \big]\,\mu_0, 1\,\big[\ .$$

The goal of this section is to get information about

$$N(d, P_\theta) = \sharp\sigma(P_\theta)\cap\big] - \infty, \ d\big[\,, \tag{4.1}$$

which denotes the number of eigenvalues of P_θ included in the set I_d.

Here may be one can apply the technique of Balazard-Konlein [Bal] to get the asymptotics of $N(d, P_\theta)$, but the result will be rough, compared to our result in Theorem 4.2: our remainder is an $\mathbf{O}(1)$ and the result of [Bal] would

give $\mathbf{O}(\sin^{-\rho}\theta)$ with $\rho > 1/2$.
More over the assumptions in [Bal] are not satisfied in our case.

For a fixed $a > 1$ let us consider the following sets

$$J_0 = \big[-a, +\infty\big[, \qquad J_1 = \big] -\infty, -\frac{a}{2}\big] \ ,$$

and a partition of unity :

$$\chi_0^2(x) + \chi_1^2(x) = 1, \quad \text{support } (\chi_j) \subset J_j, \ \sum_j |\chi_j'(x)|^2 < C \ .$$

For $j = 0, 1$ let us denote by Ω_j the domains $I\!R_+ \times J_j$:
$\Omega_0 = I\!R_+ \times \big] -a, +\infty\big[$ and $\Omega_1 = I\!R_+ \times \big] -\infty, -\frac{a}{2}\big[$.
We take now the realization of the operators P_θ^j on each domain Ω_j, associated to the quadratic form q_{Ω_j}, with Neumann conditions on $\Gamma_N = \{0\} \times J_j$ and Dirichlet conditions on $\Gamma_D = I\!R_+ \times \partial J_j$.

The quadratic forms are defined as follows :

$$q_{\Omega_j}(u) = \int_{\Omega_j} \{\cos\theta\big[|D_t u|^2 + (t-x)^2|u|^2\big] + \frac{\sin^2\theta}{\cos\theta}|D_x u|^2 - \frac{\sin^2\theta}{\cos\theta}\sum_{j=0}^{1}|\chi_j'(x)|^2|u|^2\}dtdx$$

Let us first explain why q_{Ω_1} will not give any contribution to the term $N(d, P_\theta)$.

According to section 3 we know that $\mu(x)$, the first eigenvalue of $D_t^2 + (t-x)^2$ is decreasing on J_1 , so we have :

$$q_{\Omega_1}(u) \geq \big[\mu(-\frac{a}{2})\cos\theta - C\ \frac{\sin^2\theta}{\cos\theta}\big]\|u\|^2_{L^2(\Omega_1)} \ .$$

But $\mu(-\frac{a}{2}) > 1$, so for small θ the preceding minoring ensues

$$q_{\Omega_1}(u) \ \geq \ \|u\|^2_{L^2(\Omega_1)} \ , \quad \text{if} \quad 0 < \theta < \theta_0 \ , \tag{4.2}$$

for some $\theta_0 \ \in \big]0, \frac{\pi}{4}\big[$.

In order to study the form q_{Ω_0}, it is convenient to use the normalized eigenfunction φ_x, associated to $\mu(x)$, in the following way.

Let us denote by $\Pi_0(u)$ the orthogonal projection on the set

$$F_0 = \{\varphi_x(t)\psi(x); \quad \psi \in L^2(J_0)\} \ , \tag{4.3}$$

defined by

$$\Pi_0(u) = \varphi_x(t)\big(\int_{I\!R_+} u(s,x)\varphi_x(s)ds\big) \ , \tag{4.4}$$

and by $F_1 = (F_0)^\perp$ the orthogonal set of F_0 . The corresponding orthogonal projection is

$$\Pi_1 = I - \Pi_0 \ .$$

A direct computation gives :

$$\partial_x(\Pi_0 u) = \Pi_0(\partial_x u) + R(u) \ ,$$

where R is defined by

$$R(u) = \varphi_x(t)\Big(\int_{\mathbb{R}_+} u(s,x)\partial_x\varphi_x(s)ds\Big) + \partial_x\varphi_x(t)\Big(\int_{\mathbb{R}_+} u(s,x)\varphi_x(s)ds\Big) \ . \quad (4.5)$$

The additional fact that

$$\partial_x(\Pi_1 u) = \Pi_1(\partial_x u) - R(u)$$

yields the following bounds :

$$\begin{aligned}(1-\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] + 2(1-\frac{1}{\epsilon})\|R(u)\|^2_{L^2(\Omega_0)} \\ \leq \|\partial_x u\|^2_{L^2(\Omega_0)} \\ \leq (1+\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] + 2(1+\frac{1}{\epsilon})\|R(u)\|^2_{L^2(\Omega_0)} \ .\end{aligned}$$

Using the result (3.10) in the Lemma 3.1, we get the following bound :

Lemma 4.1

$$\exists\, C_0 > 0, \ s.t. \ \ \forall\, u \ \in \ L^2(\Omega_0), \quad \|R(u)\|_{L^2(\Omega_0)} \leq C_0\|u\|_{L^2(\Omega_0)} \ . \quad (4.6)$$

Let us sketch the proof of the lemma 4.1. Taking norms in (4.5) we have :

$$\begin{aligned}\|R(u)\|^2_{L^2(\Omega_0)} &\leq 2\int_{\Omega_0} |u(s,x)(\partial_x\varphi_x(s))|^2 ds dx + 2\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|u\|^2_{L^2(\Omega_0)} \\ &\leq 4\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|u\|^2_{L^2(\Omega_0)} \ .\end{aligned}$$

The lemma will then be proved if we show that $\sup_{x\in J_0}\|\partial_x\varphi_x(t)\|^2_{L^2(\mathbb{R}_+)}\|$ is finite.

Going back to the relation (3.10)

$$\frac{\partial}{\partial x}\varphi_x(t) = 2\big[Q_x - \mu(x)\big]^{-1}\psi_x \ , \quad (4.7)$$

with

$$\psi_x(t) = (t-x)\varphi_x(t) - \langle (t-x)\varphi_x|\varphi_x\rangle\varphi_x(t) \ ,$$

and using :

$$\|D_t^2\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} + \|(t-x)\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} = \mu(x)\|\varphi_x\|^2_{L^2(\mathbb{R}^2_+)} = \mu(x) \ ,$$

we get that :

$$\|(t-x)\varphi_x\|_{L^2(\mathbb{R}^2_+)} \leq \sqrt{\mu(x)}$$

and then

$$|\langle (t-x)\varphi_x | \varphi_x \rangle_{L^2(\mathbb{R}^2_+)}| \leq \sqrt{\mu(x)}\ ,$$

so

$$\|\psi_x\|_{L^2(\mathbb{R}^2_+)} \leq 2\sqrt{\mu(x)}\quad .$$

Since ψ_x lives on the orthogonal space of φ_x , let us consider the norm N_x of the restriction of $\big[Q_x - \mu(x)\big]^{-1}$ to this orthogonal space. It is given by :

$$N_x = \frac{1}{\mu_2(x) - \mu_1(x)} = \frac{1}{\mu_2(x) - \mu(x)}\quad ,$$

where $(\mu_j(x))_j$ is the increasing sequence of the eigenvalues of Q_x .

According to (3.1) and (3.6) , $\mu(x)$ and N_x are uniformly bounded on J_0 , so there exists $c_0 > 0$ such that

$$\sup_{x\in J_0} \|\partial_x \varphi_x(t)\|_{L^2(\mathbb{R}_+)} \leq 2 \sup_{x\in J_0} \frac{\sqrt{\mu(x)}}{\mu_2(x)-\mu(x)} \leq C_0\ ,$$

so the Lemma 4.1 follows.

¿From Lemma 4.1, we see that we can find a constant $C_1 > 0$, such that, for any $\epsilon \in]0,1[$,

$$(1-\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] - \frac{C_1}{\epsilon}\|u\|^2_{L^2(\Omega_0)}$$

$$\leq \|\partial_x u\|^2_{L^2(\Omega_0)}$$

$$\leq (1+\epsilon)\big[\|\partial_x(\Pi_0 u)\|^2_{L^2(\Omega_0)} + \|\partial_x(\Pi_1 u)\|^2_{L^2(\Omega_0)}\big] + \frac{C_1}{\epsilon}\|u\|^2_{L^2(\Omega_0)}$$

¿From that we obtain the corresponding bounds on the quadratic form q_{Ω_0} :

$$q^{\epsilon,-}_{\Omega_0}(\Pi_0 u) + q^{\epsilon,-}_{\Omega_0}(\Pi_1 u) \leq q_{\Omega_0}(u) \leq q^{\epsilon,+}_{\Omega_0}(\Pi_0 u) + q^{\epsilon,+}_{\Omega_0}(\Pi_1 u) \tag{4.8}$$

where we used the natural notations :

$$q^{\epsilon,-}_{\Omega_0}(u) = \int_{\Omega_0} \Big\{\cos\theta\big[|D_t u|^2 + (t-x)^2|u|^2\big] + (1-\epsilon)\frac{\sin^2\theta}{\cos\theta}|D_x u|^2\Big\} dtdx$$
$$- \frac{\sin^2\theta}{\cos\theta}\Big[C + \frac{C_1}{\epsilon}\Big]\|u\|^2_{L^2(\Omega_0)}$$

and

$$q^{\epsilon,+}_{\Omega_0}(u) = \int_{\Omega_0} \Big\{\cos\theta\big[|D_t u|^2 + (t-x)^2|u|^2\big] + (1+\epsilon)\frac{\sin^2\theta}{\cos\theta}|D_x u|^2\Big\} dtdx$$
$$+ \frac{\sin^2\theta}{\cos\theta}\frac{C_1}{\epsilon}\|u\|^2_{L^2(\Omega_0)}$$

Writing

$$h = \frac{\sin\theta}{\sqrt{\cos\theta}}\ , \tag{4.9}$$

taking into account (4.3), we define

$$W(x) = \int_{\mathbb{R}_+} |\frac{\partial}{\partial x}\varphi_x(t)|^2 dt,$$

and we get, using (4.4) that :

$$\begin{aligned} q^{\epsilon,-}_{\Omega_0}(\Pi_0 u) &= q^{\epsilon,-}(\psi) \\ &= \int_{J_0} \{\big[\mu(x)\ \cos\theta + (1-\epsilon)h^2 W(x)\big]|\psi(x)|^2 \\ &\qquad + (1-\epsilon)h^2|D_x\psi(x)|^2 - h^2\Big[C + \frac{C_1}{\epsilon}\Big]|\psi(x)|^2\}dx. \end{aligned}$$

In the same way we have :

$$\begin{aligned} q^{\epsilon,+}_{\Omega_0}(\Pi_0 u) &= q^{\epsilon,+}(\psi) \\ &= \int_{J_0} \{\big[\mu(x)\ \cos\theta + (1+\epsilon)h^2 W(x)\big]|\psi(x)|^2 \\ &\qquad + (1+\epsilon)h^2|D_x\psi(x)|^2 + h^2\frac{C_1}{\epsilon}|\psi(x)|^2\}dx. \end{aligned}$$

Now we have to deal with the terms involving the second projection $\Pi_1 u$. But the definition of $\Pi_1 u$, the min-max principle and the estimate $1-\cos\theta \leq h^2 C$ give the following lower bound :

$$q^{\epsilon,\pm}_{\Omega_0}(\Pi_1 u) \geq \big[\inf_{x\in J_0} \mu_2(x) - h^2\Big(C + \frac{C_1}{\epsilon}\Big)\|\Pi_1 u\|_{L^2(\Omega_0)}$$

where $\mu_2(x)$ denotes the second eigenvalue defined in the Lemma 3.1. This eigenvalue has to be greater than the first eigenvalue of the corresponding Dirichlet problem, so

$$\mu_2(x) > 1\ .$$

If we take for example $\epsilon = h$, we get that

$$N(d, q^{\epsilon,\pm}_{\Omega_0}) = N(d, q^{\epsilon,\pm})\ .$$

Let us take an extension $\widetilde{\mu}(x)$ of $\mu(x)$ outside of

$$J_{0,d} = \{x \in J_0,\ \mu(x) < d + (1-d)/2\}\ , \tag{4.10}$$

such that

$$\begin{array}{l} \widetilde{\mu}(x) = \mu(x),\ \text{if}\ x\ \in\ J_{0,d} \\ \widetilde{\mu}(x) \geq (1+d)/2\ ,\ \forall\, x \notin\ J_{0,d} \\ \widetilde{\mu}(x) = 1 \text{ if } |x| > C_d \end{array} \tag{4.11}$$

for some constant $C_d > 0$. Let us define

$$q_0^{\epsilon,\pm}(\psi) = \int_{I\!R} \left[\widetilde{\mu}(x)\ |\psi(x)|^2 + (1 \pm \epsilon) h^2 |D_x \psi(x)|^2\right] dx\ .$$

The exponential localization of eigenfunctions of $q^{\epsilon,\pm}$ associated to eigenvalues in $\left]-\infty, d\right[$, (see [He1]), and the polynomial bound of $N(d, q^{\epsilon,\pm})$ and $N(d + hC, q_0^{\epsilon,\pm})$ obtained by Theorem 2.1, show as in [He1] that

$$N(d - hC, q_0^{\epsilon,\pm}) \leq N(d, q^{\epsilon,\pm}) \leq N(d + hC, q_0^{\epsilon,\pm})\ , \tag{4.12}$$

(we have used that $W(x)$ is bounded in J_0, thanks to (3.7)).

Applying a classical estimate of $N(d, q_0^{\epsilon,\pm})$, (see for example D. Robert's book [Ro]), (Theorem V-11, page 263), we have that, for any $\lambda < (1 + d)/2$, there exists $C_\lambda > 0$ such that, for any $h \in \left]0, 1\right[$ and any $\epsilon \in \left]0, 1/2\right[$,

$$\left| N(\lambda, q_0^{\epsilon,\pm})\ -\ \frac{1}{2\pi h\sqrt{1 \pm \epsilon}} \int_{I\!R} \left[\lambda - \mu(x)\right]_+^{1/2} dx \right| \leq C_\lambda\ . \tag{4.13}$$

Taking $\epsilon = h$, we get from (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) with $\lambda = d \pm hC$, that there exists $C_d > 0$, depending only on d, such

$$\left| N(d, q_{\Omega_0})\ -\ \frac{1}{2\pi \sin\theta} \int_{I\!R} \left[d - \mu(x)\right]_+^{1/2} dx \right| \leq C_d\ . \tag{4.14}$$

We get easily from the above discussion the following theorem :

Theorem 4.2 *For any* $d \in \left]\mu_0,\ 1\right[$, *there exists* $C_d > 0$ *such that*

$$\left| N(d, P_\theta)\ -\ \frac{1}{2\pi \sin\theta} \int_{I\!R} \left[d - \mu(x)\right]_+^{1/2} dx \right| \leq C_d\ . \tag{4.15}$$

Remark 4.3 *The condition* $\theta < \theta_0$ *(4.2) can be removed since* $N(d, P_\theta)$ *is finite for fixed* θ *according to Theorem 2.1.*

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