On the Spectrum of Curved Quantum Waveguides

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Abstract

The spectrum of the Laplace operator in a curved strip of constant width built along an infinite plane curve, subject to three different types of boundary conditions (Dirichlet, Neumann and a combination of these ones, respectively), is investigated. We prove that the essential spectrum as a set is stable under any curvature of the reference curve which vanishes at infinity and find various sufficient conditions which guarantee the existence of geometrically induced discrete spectrum. Furthermore, we derive a lower bound on the gap between the essential spectrum and the spectral threshold for locally curved strips. The paper is also intended as an overview of some new and old results on spectral properties of curved quantum waveguides.

Contents

1 Introduction

Let Ω be a region in $\mathbb{R}^n$, $n \geq 1$, with sufficiently regular boundary $\partial\Omega$, and consider the corresponding Laplace operator $-\Delta$ on $L^2(\Omega)$ with mixed Dirichlet-Neumann boundary conditions. If Ω is bounded, then it is well known that the spectrum of the Laplacian is purely discrete, and properties of the eigenvalues have been intensively studied. On the other hand, it is easy to see that the spectrum is $[0, \infty)$, *i.e.* purely essential, if Ω is unbounded and sufficiently extended (say, conical) at infinity. Although it was shown already by F. Rellich in 1948, [52], that there exist unbounded regions whose spectrum contains discrete eigenvalues (or it is even purely discrete!), the spectral theory for the eigenvalues has attracted much less attention than in the bounded case.

However, recent advent of mesoscopic physics has given a fresh impetus to study the (discrete) spectrum of the Laplacian in unbounded regions. For, let us recall that the quantum Hamiltonian H of a free spin-less particle of effective mass m^* constrained to a spatial region Ω, *i.e.* $H = -\hbar^2/(2m^*)\Delta$ on $L^2(\Omega)$, represents a reasonable model for the dynamics in various semiconductor structures devised and produced in the laboratory nowadays. Here it is mostly natural to consider the Dirichlet boundary conditions on $\partial\Omega$ corresponding to a large chemical potential barrier, however, other situations modelling the impenetrable walls of Ω may be relevant as well (see *e.g.* [48]) and can in principle model different types of interphase in a solid. We refer to [14, 49, 40] for the physical background and references. An important category of these systems is represented by so-called *quantum waveguides* which are modelled by infinitely stretched tubular regions in $\mathbb{R}^n$ with $n = 2, 3$.

The simplest situation occurs if Ω is an infinite plane strip, *i.e.*, a (tubular neighbourhood of constant width along an infinite curve in $\mathbb{R}^2$. In 1989, P. Exner and P. Šeba demonstrated the existence of discrete spectrum for curved Dirichlet strips which were asymptotically straight and sufficiently thin, [31]. Numerous subsequent studies improved their result and generalised it to space tubes, [39, 53, 14]. For more information and other spectral and scattering properties, see the review paper [14] and references therein. An important improvement was made by J. Goldstone and R. L. Jaffe in 1992, [39]; the authors introduced a variational argument which enables them to demonstrate the existence of discrete eigenvalues without the restriction on the width of the strip. A recent article [45] deals with a more general situation when the strip is not constructed in $\mathbb{R}^2$ but in a two-dimensional Riemannian manifold. As examples of strips with mixed Dirichlet-Neumann boundary conditions, let us mention the works [12, 13]. The evidently more complicated case of Dirichlet layers, *i.e.*, Ω is a tubular neighbourhood about a complete non-compact surface in $\mathbb{R}^3$, was investigated in [15, 16, 28, 8].

A common property of the Dirichlet systems cited above is that a bending of a straight strip or layer generates discrete eigenvalues below the essential spectrum, *i.e.*, geometrically induced quantum *bound states*, which are known to disturb the particle transport. The result is also interesting from the semiclassical point of view because there are no classical closed trajectories in the tubes in question, apart from a zero measure set of initial conditions in the phase space. Hence, this is a pure quantum effect of geometrical origin. The spectral results become richer if one considers more complicated boundary conditions; here the problem is interesting even for some straight waveguides, [12].

Apart from the curved quantum waveguides, the discrete spectrum can be also generated by a local deformation of the boundary $\partial\Omega$ of straight tubes and layers, [5, 4, 36], via introducing an obstacle, [17, 10, 1], or impurities modelled by a Dirac interaction, [19, 27, 29, 30], coupling several waveguides by a window, [33, 34, 35], *etc.* The spectrum of periodically and randomly curved waveguides was investigated in [57, 55] and [42], respectively. Finally, let us mention systems where $\Omega = \mathbb{R}^n$, $n = 2, 3$, and the quantum waveguide is introduced by means of a magnetic field, [21, 26, 22], or a strong Dirac interaction supported by an infinite curve or surface, [20, 37, 24, 25, 18, 23].

2 Scope of the Paper

The main aim of the present paper is to study the geometrically induced (discrete) spectrum of the quantum Hamiltonian H^ι of a free non-relativistic particle living in the infinite planar curved strip (the index ι will distinguish different boundary conditions considered here). Removing physical constants, H^ι is identified with the Laplacian $-\Delta$ on $L^2(\Omega)$ where Ω is a (one-sided) tubular neighbourhood of a fixed width $d > 0$ along an infinite plane curve Γ of curvature k, see Figure 1. The boundary $\partial\Omega$ consists of two parallel connected curves which are supposed to be of class C^2. We compare three different types of boundary conditions representing impenetrable walls of the strip in the sense that there is no probability current through the boundary. In particular, we consider the recently widely investigated *Dirichlet* boundary condition ($\iota := D$), the *Neumann* boundary condition ($\iota := N$) and the simplest *combination* of the both just mentioned ($\iota := DN$): the Dirichlet boundary condition imposed on one connected component of $\partial\Omega$ and the Neumann condition on the other one.

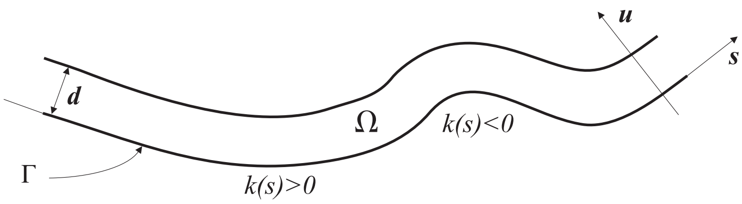

Figure 1: Configuration space Ω defined as a strip over an infinite curve Γ in $\mathbb{R}^2$.

If the reference curve Γ is a *straight* line, then it is rather a textbook exercise to analyse the operator H^ι by means of a separation of variables and conclude that its spectrum is purely absolutely continuous and equals the interval $[E_1^\iota, \infty)$, where the non-negative value E_1^ι is determined by the respective boundary conditions, see (3.10). However, the spectral problem for H^ι becomes always difficult whenever Γ is *curved*, and two basic questions arise in this context:

1. Which geometry preserves the essential spectrum $[E_1^\iota, \infty)$?

2. Which geometry produces a spectrum below E_1^ι?

These questions represent ultimate concern of this paper. We try to make a survey of known answers and contribute to the problem by our own results. Furthermore, if the spectrum below E_1^ι exists, we establish various estimates of the spectral threshold $\inf \sigma(H^\iota)$. It should be stressed here that the existence of *discrete* spectrum, *i.e.* the issue mentioned in Introduction, is proved whenever the considered geometry is in accordance with *both* the above questions (because then the spectrum below E_1^ι consists of isolated eigenvalues of finite multiplicity only).

Concerning the first question, we show that the essential spectrum of a curved strip coincides with the spectrum of the straight one provided the reference curve Γ is straight asymptotically in the sense that its curvature vanishes at infinity, *cf* Theorem 4.1. Although this sufficient condition is very natural and in perfect accordance with the intuition, it is for the first time in this paper when the essential spectrum is localised without imposing any additional conditions (*e.g.*, about the decay of the derivatives of curvature at infinity, *cf* [14, 53, 13]). The progress has become possible due to a general characterisation of essential spectrum adopted from [11] (see our Lemma 5.1), which is for our purposes more suitable than the classical Weyl criterion. On the other hand, periodic strips are discussed as an illustration of asymptotically non-straight geometry which does change the essential spectrum.

The answer to the second question depends substantially on the choice of boundary conditions. First of all, notice that the question does not make sense for the Neumann strips because $E_1^N = 0$, *cf* Theorem 4.2. A characteristic property of the Dirichlet strips is that any bending of the reference curve Γ pushes the infimum of the spectrum below the spectral threshold $E_1^D > 0$ of the corresponding straight strip, *cf* Theorem 4.3. This property was shown first in [31] for sufficiently thin strips and the proof for general cases was introduced in [39]. On the other hand, the case of combined Dirichlet-Neumann boundary condition was investigated in the recent letter [13]. The authors established the existence of spectrum below $E_1^{DN} > 0$ provided the total bending angle of Γ (*i.e.*, the integral of curvature, see (4.1)) has a suitable sign. In this paper, we generalise this result and add two new sufficient conditions, *cf* Theorem 4.4. We also derive an interesting result on the number of discrete eigenvalues of H^{DN}, *cf* Proposition 4.1

Finally, when Γ is curved only locally, we derive an upper bound on the spectral threshold, *i.e.* $\inf \sigma(H^\iota)$, for the Dirichlet strip and the one with combined Dirichlet-Neumann boundary condition, *cf* Theorem 4.5. In particular, we find important qualitative differences between these two respective results. Making the curvature small, the leading term in the estimate of the difference $\inf \sigma(H^D) - E_1^D$ is proportional to the fourth power of the total bending angle, while it is the second power what one obtains for the Dirichlet-Neumann case, *cf* Remark 6.4. Another interesting difference appears, when we are shrinking the width of the strip to zero, *cf* Remark 6.5. These estimates are new in the theory of curved quantum waveguides. We can only compare them with the eigenvalue asymptotics for mildly curved, respectively thin, Dirichlet strips established in [14]. Let us note that a similar estimate for straight, window-coupled waveguides was given in [34, 35], see also [3].

All our proofs of the statements concerning the existence and properties of the spectrum below E_1^ι are based on a variational strategy. The corner stone of them, *i.e.* the construction of a suitable trial function, follows the idea of [39], see also [14, 53].

The paper is organised as follows. Section 3 is devoted to some preliminary material in order to be able to state precisely the main results of the paper, *i.e.* Theorems 4.1–4.5, in the subsequent Section 4. The proofs and discussions of the Theorems are presented in Sections 5 and 6. We conclude the paper by Section 7, where some open problems and directions of a future research are mentioned.

3 Preliminaries

3.1 Configuration Space

Let Γ be a unit-speed infinite plane curve *i.e.* the (image of the) C^2-mapping $\Gamma : \mathbb{R} \to \mathbb{R}^2$ satisfying $|\dot{\Gamma}(s)| = 1$ for all $s \in \mathbb{R}$ (the arc-length parameter of the curve). The function $N := (-\dot{\Gamma}^2, \dot{\Gamma}^1)$ defines a unit normal vector field and the couple $(\dot{\Gamma}, N)$ gives a distinguished Frenet frame, *cf* [43, Chap. 1]. The curvature is defined through the Frenet-Serret formulae by $k := \det(\dot{\Gamma}, \ddot{\Gamma})$. We note that k is a continuous function and the sign of $k(s)$ is defined uniquely up to the re-parameterisation $s \mapsto -s$ of the arc-length parameter. It is also worth to notice that the curve Γ is fully determined (except for its position and orientation in the plane) by the curvature function k only, *cf* [46, Sec. II. 20].

Let $d > 0$, $I := (0, d)$ and $\Omega_0 := \mathbb{R} \times I$ be a straight strip of width d. A curved strip of the same width based on Γ is defined via $\Omega := \mathcal{L}(\Omega_0)$, where

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2 : \{(s, u) \mapsto \Gamma(s) + u\, N(s)\}\,. \tag{3.1}$$

Then $s \mapsto \mathcal{L}(s, u)$ for u fixed traces out a parallel curve at a distance $|u|$ from Γ. Through all the paper, we shall always assume that

$$\langle \mathsf{H} \rangle \qquad \Omega \text{ is not self-intersecting} \quad \text{and} \quad k \in L^\infty(\mathbb{R}) \ \text{ with } \ d\,\|k_+\|_\infty < 1,$$

where $k_\pm := \max\{0, \pm k\}$. Then the mapping $\mathcal{L} : \Omega_0 \to \Omega$ is a C^1-diffeomorphism and its inverse determines a system of natural "coordinates" (s, u) in a neighbourhood of Γ. We remark that under our assumption $\langle \mathsf{H} \rangle$ the curve $\mathcal{L}(\mathbb{R} \times \{u\})$ is of class C^2 for any fixed $u \in \overline{I}$, in particular, this claim holds true for both the boundary curves.

Remark 3.1. In this paper, we adopt the standard component notation of the tensor analysis together with the repeated indices convention. The range of indices is $1, 2$ and they are associated with the above mentioned coordinates via $(1, 2) \leftrightarrow (s, u)$. The partial derivatives are marked by a comma with the index. □

By virtue of the Frenet-Serret formulae, the metric tensor of Ω in these coordinates, *i.e.* $G_{ij} := \mathcal{L}_{,i} \cdot \mathcal{L}_{,j}$ where "$\cdot$" denotes the scalar product in $\mathbb{R}^2$, has the following diagonal form

$$\big(G_{ij}(s, u)\big) = \begin{pmatrix} (1 - uk(s))^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

Its determinant, $G := \det(G_{ij})$, defines through $d\Omega := G(s, u)^{\frac{1}{2}} ds du$ the area element of the strip. By virtue of the second part of the assumption $\langle \mathsf{H} \rangle$, it is clear that the metric (3.2) is uniformly elliptic. In particular, we have the following useful estimates:

$$\forall (s, u) \in \Omega_0 : \quad C_- \leq 1 - uk(s) \leq C_+ \qquad \text{with} \qquad C_\pm := 1 \pm d\,\|k_\mp\|_\infty. \tag{3.3}$$

3.2 Hamiltonian

The infinite strip-like region Ω models the configuration space of a non-relativistic particle in a curved quantum waveguide. Putting $\hbar^2/(2m) = 1$, the particle Hamiltonian could be identified with the Laplace operator, $-\Delta$ on $L^2(\Omega)$. The dynamics is then well defined by means of the boundary conditions imposed on $\partial\Omega$. Using the mapping (3.1), we identify the Hilbert space $L^2(\Omega)$ with $\mathcal{H} := L^2(\Omega_0, d\Omega)$ and consider three different situations.

Dirichlet case. The easiest case is given by the Dirichlet boundary conditions. The corresponding Hamiltonian H^D is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form

$$Q^D[\psi] := \left(\psi_{,i}, G^{ij}\psi_{,j}\right), \tag{3.4}$$

$$\operatorname{Dom} Q^D := W_0^{1,2}(\Omega_0, d\Omega). \tag{3.5}$$

Here and in what follows, (G^{ij}) stands for the inverse of (G_{ij}) and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$; the induced norm will be denoted by $\|\cdot\|$.

Neumann case. The Hamiltonian H^N corresponding to the Neumann boundary conditions imposed on $\partial\Omega$ is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form Q^N which acts like Q^D in (3.4), however, on a different domain:

$$\operatorname{Dom} Q^N := W^{1,2}(\Omega_0, d\Omega). \tag{3.6}$$

Dirichlet-Neumann case. The last situation considered here is given by the Dirichlet boundary condition imposed on the reference curve $\mathcal{L}(\mathbb{R}\times\{0\})$ and the Neumann one imposed on the opposite boundary $\mathcal{L}(\mathbb{R}\times\{d\})$. More precisely, the Hamiltonian H^{DN} in such a case is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form Q^{DN} which acts like Q^D in (3.4), however, with

$$\operatorname{Dom} Q^{DN} := \left\{\psi \in W^{1,2}(\Omega_0, d\Omega) \mid \psi(s,0) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\}. \tag{3.7}$$

Here $\psi(\cdot,0)$ denotes the trace of the function ψ on the boundary part $\mathcal{L}(\mathbb{R}\times\{0\})$.

Remark 3.2. Since the metric (G_{ij}) is uniformly elliptic due to (3.3), it is not necessary to take into account the measure $d\Omega$ in (3.5), (3.6) and (3.7). □

Sometimes it will be convenient to consider two or all of the three different situations simultaneously and employ the common superscript ι in order to unify the notation. Hereafter, whenever we use this abridged notation without specifying the range of ι explicitly, we shall assume $\iota \in \{D, N, DN\}$.

Remark 3.3 (Operators associated with Q^ι). We have

$$H^\iota = -G^{-\frac{1}{2}}\partial_i G^{\frac{1}{2}} G^{ij}\partial_j\,, \tag{3.8}$$

which is a general expression for the Laplace operator in a manifold equipped with a metric (G_{ij}). The equality in (3.8) must be understood in the form sense if the curvature k is not differentiable (which is the case we are particularly concerned to deal with in this paper). Nevertheless, assuming that the reference curve Γ is, say, C^3-smooth, then the metric is differentiable and, putting (3.2) into (3.8), we can write

$$H^\iota = -\frac{1}{(1-uk(s))^2}\,\partial_s^2 - \frac{u\dot{k}(s)}{(1-uk(s))^3}\,\partial_s - \partial_u^2 + \frac{k(s)}{1-uk(s)}\,\partial_u$$

in the strong sense on $\operatorname{Dom} H^\iota$. Moreover, we can give the explicit form of the operator domain $\operatorname{Dom} H^\iota$, see [47]:

$$\begin{aligned}
\operatorname{Dom} H^D &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi(s,0) = \psi(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\},\\
\operatorname{Dom} H^N &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi_{,2}(s,0) = \psi_{,2}(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\},\\
\operatorname{Dom} H^{DN} &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi(s,0) = \psi_{,2}(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\}.
\end{aligned}$$

□

3.3 Straight Strips

If the strip is straight in the sense that $k \equiv 0$, *i.e.* the curvature of the reference curve is equal to zero everywhere on $\mathbb{R}$, then the Hamiltonian coincides with the decoupled operator

$$H_0^\iota := \overline{-\Delta^{\mathbb{R}} \otimes Id + Id \otimes (-\Delta_\iota^I)} \qquad \text{on} \quad L^2(\mathbb{R}) \otimes L^2(I), \tag{3.9}$$

where Id denotes the identity operator on appropriate spaces. The operators on the transverse section, $-\Delta_\iota^I$, are the usual Laplacians on $L^2(I)$ with the Dirichlet boundary conditions if $\iota = D$, the Neumann conditions if $\iota = N$, or the Dirichlet condition at 0 and the Neumann one at d if $\iota = DN$. The eigenvalues of $-\Delta_\iota^I$ are given by

$$E_n^D := (\pi/d)^2 n^2, \quad E_n^N := (\pi/d)^2 (n-1)^2, \quad E_n^{DN} := (\pi/d)^2 (n-\tfrac{1}{2})^2, \tag{3.10}$$

where $n \in \mathbb{N} \setminus \{0\}$. The corresponding family of normalised eigenfunctions $\{\chi_n^\iota\}_{n=1}^\infty$ can be chosen in the following way:

$$\chi_n^\iota(u) := \sqrt{\tfrac{2}{d}}\,\sin\sqrt{E_n^\iota}\,u \quad \text{for} \quad \iota \in \{D, DN\}; \tag{3.11}$$

$$\chi_n^N(u) := \begin{cases} \sqrt{\tfrac{1}{d}} & \text{if} \quad n = 1,\\ \sqrt{\tfrac{2}{d}}\,\cos\sqrt{E_n^N}\,u & \text{if} \quad n \geq 2. \end{cases} \tag{3.12}$$

In view of (3.9), the straight strip has an absolutely continuous spectrum starting from the first eigenvalue of the transverse Laplacian, *i.e.*,

$$\sigma(H_0^\iota) = \sigma_{\mathrm{ess}}(H_0^\iota) = [E_1^\iota, \infty). \tag{3.13}$$

We shall use this trivial case of quantum strips as a comparative system whose spectrum is known explicitly.

4 Main Results

As we have seen, the essential spectrum of a straight waveguide, *i.e.* $k \equiv 0$, is the interval $[E_1^\iota, \infty)$. In Section 5, we prove that the same spectral result holds for any curved waveguide which is straight *asymptotically* in the sense that the curvature k vanishes at infinity, *i.e.*,

$$\langle \mathsf{d} \rangle \qquad k(s) \xrightarrow[|s|\to\infty]{} 0 \ .$$

Theorem 4.1 (Essential spectrum). *Suppose* $\langle \mathsf{H} \rangle$. *If the strip obeys* $\langle \mathsf{d} \rangle$, *then*

$$\sigma_{\text{ess}}(H^\iota) = [E_1^\iota, \infty) \qquad \textit{for} \quad \iota \in \{D, N, DN\}.$$

To the best of our knowledge, the spectrum of the Neumann Laplacian H^N, has been previously investigated just for strips which were straight and contained an obstacle, [17, 10]. Hence, our Theorem 4.1 represents a quite new result concerning the spectral theory of curved Neumann strips.

The Dirichlet-Neumann case, *i.e.* $\iota = DN$, was previously considered just in the recent letter [13]. It is mentioned there that $\inf \sigma_{\text{ess}}(H^{DN}) = E_1^{DN}$ provided k has a compact support. Here we have proved that the whole interval $[E_1^{DN}, \infty)$ is in the essential spectrum under much weaker condition $\langle \mathsf{d} \rangle$.

Although the case of Dirichlet strips, *i.e.* $\iota = D$, has already been considered in many works, our Theorem 4.1 represents a new result in this situation as well, since it is for the first time when the whole essential spectrum has been localised under a condition which does not contain derivatives of k. Some decay assumptions about the derivatives of the curvature were even required in order to localise the threshold $\inf \sigma_{\text{ess}}(H^D)$ itself in the previous works, *cf* [14, 53]. (An exception is the paper [45] where, however, only a lower bound on the threshold is given.) Let us mention that the result of Theorem 4.1 was achieved in the thesis [44] under an additional condition about vanishing of the first derivative of k.

Since H^N is non-negative, it follows immediately from Theorem 4.1 that there is no discrete spectrum in asymptotically straight Neumann strips.

Theorem 4.2 (Neumann case). *Suppose* $\langle \mathsf{H} \rangle$. *Then*

$$\inf \sigma(H^N) = E_1^N \equiv 0.$$

Consequently, if the strip is asymptotically straight, i.e. $\langle \mathsf{d} \rangle$, *then*

$$\sigma(H^N) = \sigma_{\text{ess}}(H^N) = [0, \infty),$$

i.e., $\sigma_{\text{disc}}(H^N) = \emptyset$.

Here the fact that the spectral threshold of H^N starts exactly at 0 for any strip can be easily proved by means of a suitable trial function (see Proposition 6.1).

An interesting result in the theory of quantum waveguides is that the curved geometry may produce a non-trivial spectrum below the energy E_1^ι for $\iota \in$

$\{D, DN\}$. The phenomenon is examined in this paper. Notice that any result of the type $\inf \sigma(H^\iota) < E_1^\iota$ together with the decay condition $\langle \mathsf{d} \rangle$ yield that the spectrum below E_1^ι consists of isolated eigenvalues of finite multiplicity only, *i.e.* $\sigma_{\text{disc}}(H^\iota) \neq \emptyset$. However, we do not restrict ourselves to the particular case of asymptotically straight strips, *i.e.*, the geometrically induced spectrum below E_1^ι may have a non-zero Lebesgue measure, too.

Sufficient conditions for the Hamiltonians H^ι with $\iota \in \{D, DN\}$ to have a non-empty spectrum below E_1^ι are known. In particular, any non-trivial curvature of the reference curve pushes the spectrum of H^D down the corresponding spectral threshold of the straight strip.

Theorem 4.3 (Dirichlet case). *Suppose* $\langle \mathsf{H} \rangle$.

If $k \not\equiv 0$, *then* $\inf \sigma(H^D) < E_1^D$.

Consequently, if the strip is not straight but it is straight asymptotically, i.e. $\langle \mathsf{d} \rangle$, *then* H^D *has at least one eigenvalue of finite multiplicity below its essential spectrum* $[E_1^D, \infty)$, i.e., $\sigma_{\text{disc}}(H^D) \neq \emptyset$.

This property was shown first in [31] for sufficiently thin strips with a rapidly decaying curvature and since various improvements have been achieved (see the references mentioned in Introduction, mainly [39]). Since the present paper is intended also as a survey paper, we find useful to make a proof of Theorem 4.3 in Section 6.1, simultaneously with the proof of the new result contained in condition (a) of Theorem 4.4 below.

As for the operator H^{DN}, its spectrum was studied for the first time in the recent letter [13]. It shows that the position of the infimum of spectrum essentially depends on the sign of the total bending angle

$$\alpha := \int_{\mathbb{R}} k(s)\, ds, \tag{4.1}$$

which is well defined if we assume that the curvature is integrable. In detail, the authors of [13] proved that: i) the spectrum of H^{DN} in a non-trivially curved strip starts below E_1^{DN} provided $\alpha \leq 0$ and the curvature k is non-positive out of some bounded interval. On the other hand, ii) if $k(s) \geq 0$ for all $s \in \mathbb{R}$, then the spectrum below the energy E_1^{DN} is empty. Our improvement is two-fold. Firstly, we generalise the first claim in the sense that we skip the condition on k. Secondly, we find a sufficient condition which guarantees the existence of spectrum below E_1^{DN} even for some strips with $\alpha > 0$. In addition to these substantial generalisations, we will derive the same result also for periodic waveguides. Let us summarise the spectral properties of H^{DN} into the following theorem.

Theorem 4.4 (Dirichlet-Neumann case). *Suppose* $\langle \mathsf{H} \rangle$.

(i) *If* $k \not\equiv 0$, *then any of the three conditions*

(a) $k \in L^1(\mathbb{R})$ *and* $\alpha \equiv \int_{\mathbb{R}} k(s)\, ds \leq 0$

(b) k *is periodic*

(c) $k_- \not\equiv 0$ *and* d *is small enough*

is sufficient to guarantee that $\inf \sigma(H^{DN}) < E_1^{DN}$.

(ii) *If* $k_- \equiv 0$, *then* $\inf \sigma(H^{DN}) \geq E_1^{DN}$.

Consequently, if the strip is not straight but it is straight asymptotically, i.e. $\langle \mathsf{d} \rangle$, *then any of the conditions* (a) *or* (c) *is sufficient to guarantee that* H^{DN} *has at least one eigenvalue of finite multiplicity below its essential spectrum* $[E_1^{DN}, \infty)$, i.e., $\sigma_{\mathrm{disc}}(H^{DN}) \neq \emptyset$. *On the other hand, if the strip is asymptotically straight and* $k_- \equiv 0$, *then* $\sigma(H^{DN}) = \sigma_{\mathrm{ess}}(H^{DN}) = [E_1^{DN}, \infty)$, i.e., $\sigma_{\mathrm{disc}}(H^{DN}) = \emptyset$.

Remark 4.1. The signs of $k(s)$ and the corresponding total bending angle α change after the change of arc-length parameter given by $s \mapsto -s$. It has to be stressed here that such a re-parameterisation of the reference curve Γ leads to another strip due to (3.1) and, consequently, there is no ambiguity in stating the spectral results on H^{DN} in terms of the sign of α and k, see Figure 2. $\square$

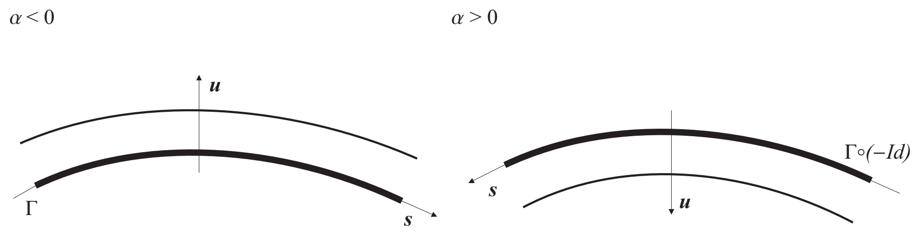

Figure 2: Inversion of orientation of the reference curve (given by the re-parameterisation $s \mapsto -s$). Thick lines denote the Dirichlet boundary condition, thin lines the Neumann one.

The sufficient conditions (a)–(c) of the first part of the theorem are proved in Section 6.1, while the result of (ii) is already known from [13] (for completeness, we sketch the main ideas of the proof). A comparison of the condition (a) with the assumptions in [13] is done in Remark 6.1.

Consider now a situation when the discrete spectrum of H^ι, $\iota \in \{D, DN\}$, below the energy E_1^ι is not empty.

Although this paper in not intended to investigate the *number* of eigenvalues of H^ι, let us point out the following remarkable property of H^{DN} which we establish at the end of Section 6.1.

Proposition 4.1 (Number of bound states in the DN case). *Suppose* $\langle \mathsf{H} \rangle$ *and* $\langle \mathsf{d} \rangle$. *If* $k_- \not\equiv 0$ *then*

$$\forall n \in \mathbb{N} \quad \exists d_n > 0 \ : \quad d < d_n \implies N(H^{DN}) \geq n \,,$$

where $N(H^{DN})$ denotes the number of discrete eigenvalues of H^{DN}, counting multiplicity.

The number of bound states in thin strips is another property, which demonstrates a significant influence of the choice of boundary conditions on the spectrum. To see it, we recall that an upper SKN-type (*cf* [54, 41, 50]) bound on the number of bound states in thin Dirichlet strips was derived in [14, Sec. 2.3] and it showed that $N(H^D)$ is bounded from above by a finite constant which *does not* depend on the strip width d. On the other hand, Proposition 4.1 shows that $N(H^{DN})$ can reach arbitrarily large value by shrinking the strip width to zero.

The last objective of this paper is to estimate the distance between the spectral threshold, *i.e.* the lowest eigenvalue of H^ι, and E_1^ι. We derive the following upper bounds, which are again qualitatively different for the Dirichlet and mixed Dirichlet-Neumann situations, respectively.

Theorem 4.5 (Estimates of the spectral threshold). *Suppose* $\langle \mathsf{H} \rangle$ *and assume that* k *has a compact support in an interval of width* $2s_0$*.*

(i) *If* $\alpha \leq 0$*, then* $\inf \sigma(H^{DN}) \leq E_1^{DN} - C^{DN}(s_0, d, \alpha)^2 \alpha^2$, *where*

$$C^{DN}(s_0, d, \alpha) := \sqrt{E_1^{DN}} \, \frac{\sqrt{3}/\pi}{1 + \sqrt{1 - \frac{3}{2}\frac{\alpha s_0}{d} + \frac{3}{4}\alpha^2\left(\frac{1}{2} + \frac{2}{\pi^2}\right)}} .$$

(ii) $\inf \sigma(H^D) \leq E_1^D - C^D(s_0, d, \alpha)^2 \alpha^4$, *where*

$$C^D(s_0, d, \alpha) := \frac{2^4}{3^3} \, \frac{\sqrt{3}/\pi^2}{d\left(\frac{s_0}{d} - \frac{\alpha}{4} + \frac{2}{3\,\pi}\right)} \, \frac{1}{1 + \sqrt{1 + \left(\frac{4\,\alpha}{3\,\pi}\right)^2 \frac{4s_0 - \alpha\, d}{4s_0 - \alpha\, d + \frac{8d}{3\,\pi}}}} .$$

These estimates are new in the theory of quantum waveguides and we derive them in Section 6. One can immediately see that for small total bending angles, the leading term in the estimate (i) is proportional to the second power of α, while it is the fourth power of α in the estimate (ii). Another essential difference in our estimates appears in the limit case of thin strips. We discuss these interesting disparities in Remarks 6.4 and 6.5. We also compare there the result (ii) with the exact eigenvalue asymptotics obtained in [14] by perturbation methods applied to mildly curved or thin strips, respectively.

5 Essential Spectrum

This section is devoted to the proof of Theorem 4.1. It is achieved in two steps. Firstly, in Lemma 5.2, we employ a Neumann bracketing argument in order to show that the threshold of the essential spectrum does not descend below the energy E_1^ι. Secondly, in Lemma 5.3, we prove that all energies above E_1^ι belong to the spectrum by means of the following general characterisation of essential spectrum, which we have adopted from [11].

Lemma 5.1. *Let H be a non-negative self-adjoint operator in a complex Hilbert space $\mathcal{H}$ and Q be the associated quadratic form. Then $\eta \in \sigma_{\mathrm{ess}}(H)$ if and only if*

$$\exists\{\psi_n\}_{n=1}^{\infty} \subset \operatorname{Dom} Q : \begin{cases} \text{(i)} & \forall n \in \mathbb{N} \setminus \{0\} : \ \|\psi_n\| = 1, \\ \text{(ii)} & \psi_n \xrightarrow[n\to\infty]{w} 0 \quad \text{in } \mathcal{H}, \\ \text{(iii)} & (H-\eta)\psi_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } (\operatorname{Dom} Q)^* . \end{cases}$$

Here $(\operatorname{Dom} Q)^*$ denotes the dual of the space $\operatorname{Dom} Q$. We note that $H+1 : \operatorname{Dom} Q \to (\operatorname{Dom} Q)^*$ is an isomorphism and

$$\|\psi\|_{-1} := \|\psi\|_{(\operatorname{Dom} Q)^*} = \sup_{\phi \in \operatorname{Dom} Q \setminus \{0\}} \frac{|(\phi,\psi)|}{\|\phi\|_1} \tag{5.1}$$

with

$$\|\phi\|_1 := \sqrt{Q[\phi] + \|\phi\|^2} \, .$$

Lemma 5.1 is proved in a quite similar fashion as the Weyl criterion, [56, Thms. 7.22–7.24]. The advantage of the present characterisation is that it requires to find a sequence from the form domain of H only, and not from $\operatorname{Dom} H$ as it is required by the Weyl criterion. Moreover, in order to check the limit from (iii), it is still sufficient to consider the operator H in the form sense, *i.e.*, we will not need to assume that (G_{ij}) is differentiable in our case.

We start by an estimate on the threshold of the essential spectrum.

Lemma 5.2. *If $\langle \mathsf{d} \rangle$ holds true, then* $\inf \sigma_{\mathrm{ess}}(H^\iota) \geq E_1^\iota$.

Proof. Since the curvature vanishes at infinity, for any fixed $\epsilon > 0$, there exists s_ϵ such that

$$\forall (s,u) \in \Omega_{\mathrm{ext}} \ : \ (1 - d\epsilon) \leq 1 - u\,k(s) \leq (1 + d\epsilon), \tag{5.2}$$

where $\Omega_{\mathrm{ext}} := \Omega_0 \setminus \overline{\Omega}_{\mathrm{int}}$ with $\Omega_{\mathrm{int}} := (-s_\epsilon, s_\epsilon) \times I$. Denote by H_N^ι the operator H^ι with a supplementary Neumann boundary condition imposed on the two segments $\{\pm s_\epsilon\} \times I$, that is, the operator associated with the form $Q_N^\iota := Q_N^{\iota,\mathrm{int}} \oplus Q_N^{\iota,\mathrm{ext}}$, where

$$\begin{aligned} Q_N^{\iota,\omega}[\psi] &:= \left(\psi_{,i}, G^{ij}\psi_{,j}\right)_{L^2(\Omega_\omega, d\Omega)} , \\ \operatorname{Dom} Q_N^{D,\omega} &:= \left\{\psi \in W^{1,2}(\Omega_\omega, d\Omega) \,|\, \psi(s,0) = \psi(s,d) = 0 \text{ for a.e. } s \in \mathbb{R} \cap \overline{\Omega}_\omega\right\} , \\ \operatorname{Dom} Q_N^{N,\omega} &:= W^{1,2}(\Omega_\omega, d\Omega) , \\ \operatorname{Dom} Q_N^{DN,\omega} &:= \left\{\psi \in W^{1,2}(\Omega_\omega, d\Omega) \,|\, \psi(s,0) = 0 \text{ for a.e. } s \in \mathbb{R} \cap \overline{\Omega}_\omega\right\} \end{aligned}$$

for $\omega \in \{\mathrm{int}, \mathrm{ext}\}$. Since $H^\iota \geq H_N^\iota$ and the spectrum of the operator associated with $Q_N^{\iota,\mathrm{int}}$ is purely discrete, *cf* [9, Chap. 7], the minimax principle gives the estimate

$$\inf \sigma_{\mathrm{ess}}(H^\iota) \geq \inf \sigma_{\mathrm{ess}}(H_N^{\iota,\mathrm{ext}}) \geq \inf \sigma(H_N^{\iota,\mathrm{ext}}),$$

where $H_N^{\iota,\mathrm{ext}}$ denotes the operator associated with $Q_N^{\iota,\mathrm{ext}}$. Neglecting the non-negative "longitudinal" part of the Laplacian in (3.8) (*i.e.*, the term where one sums over $i = j = 1$) and using the estimates (5.2), we arrive easily at the following lower bound

$$H_N^{\iota,\mathrm{ext}} \geq \frac{1-d\epsilon}{1+d\epsilon}\, E_1^\iota \qquad \text{in} \quad L^2(\Omega_{\mathrm{ext}}, d\Omega),$$

which holds in the form sense (see also proof of Theorem 4.1 in [16]). The claim then follows by the fact that ϵ can be chosen arbitrarily small. □

Remark 5.1 (Neumann case). Since $E_1^N = 0$ and H^N is a non-negative operator, the statement of Lemma 5.2 holds trivially true for the Neumann boundary conditions, *i.e.*, $\iota = N$, even without the assumption $\langle \mathsf{d} \rangle$. □

Example 5.1 (Periodic waveguides). The periodic strip (*i.e.* assumption $\langle \mathsf{d} \rangle$ is not obeyed) is the simplest example for which

$$\inf \sigma_{\mathrm{ess}}(H^\iota) < E_1^\iota\,, \qquad \iota \in \{D, DN\}.$$

Let $k \not\equiv 0$ be a periodic function of a period $L > 0$, *i.e.*, $\forall s \in \mathbb{R}:\ k(s+L) = k(s)$, and such that the hypothesis $\langle \mathsf{H} \rangle$ holds true for some $d > 0$. Then the operator H^ι is invariant w.r.t. the transformation $s \mapsto s + jL$ for every $j \in \mathbb{Z}$, which implies that there is no discrete eigenvalue in its spectrum, *i.e.* $\sigma(H^\iota) = \sigma_{\mathrm{ess}}(H^\iota)$. However, Theorem 4.3 and Theorem 4.4 state that $\inf \sigma(H^\iota) < E_1^\iota$.

According to a common belief, second order elliptic differential operators with sufficiently regular periodic coefficients should not have degenerate bands in their spectra, or, in other words, their spectra should be purely absolutely continuous (see [55] and references therein). An elegant rigorous proof of this fact for Dirichlet and Neumann periodic waveguides was given by E. Shargorodsky and A. Sobolev in [55]. □

The precedent Lemma 5.2 together with the following one establish Theorem 4.1.

Lemma 5.3. *If $\langle \mathsf{d} \rangle$ holds true, then* $\sigma_{\mathrm{ess}}(H^\iota) \supseteq [E_1^\iota, \infty)$.

Proof. Let $n \in \mathbb{N} \setminus \{0\}$. We shall construct a sequence $\{\psi_n^\iota\}$ satisfying (i)–(iii) of Lemma 5.1 with $\eta^\iota := \lambda^2 + E_1^\iota$ for all $\lambda \in \mathbb{R}$. We start with the following family of functions

$$\hat{\psi}_n^\iota(s,u) := \varphi_n(s)\, \chi_1^\iota(u)\, e^{i\lambda s}\,,$$

where χ_1^ι is the lowest transverse-mode function (3.11) if $\iota \in \{D, DN\}$, or (3.12) if $\iota = N$, respectively, and $\varphi_n(s) := \varphi(n^{-1}s - n)$ with φ being a C^∞-smooth function with a compact support in $(-1, 1)$. Note that $\operatorname{supp} \varphi_n \subset (n^2 - n, n^2 + n)$ and, consequently, the sequence $\{\varphi_n\}$ is "localised at $+\infty$" for large n. It is clear that $\hat{\psi}_n^\iota$ belongs to the form domain of H^ι. Since it is not normalised in $\mathcal{H}$, we introduce $\psi_n^\iota := \hat{\psi}_n^\iota / \|\hat{\psi}_n^\iota\|$. Hereafter we shall use the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{L^2(\Omega_0)}$, which follows by (3.3). In particular, one has

$$C_- \|\varphi_n\|_{L^2(\mathbb{R})}^2 \leq \left\|\hat{\psi}_n^\iota\right\|^2 \leq C_+ \|\varphi_n\|_{L^2(\mathbb{R})}^2 \tag{5.3}$$

due to the normalisation of χ_1^ι.

First of all, we note that one does not need to check the weak convergence. Indeed, if we show the condition (iii) for our sequence $\{\psi_n^\iota\}$ for all $\lambda \in \mathbb{R}$, we get that $[E_1^\iota, \infty)$ belongs to the spectrum of H^ι. Then it follows immediately that it must be the essential part of the spectrum because intervals have no isolated points.

Hence, it remains to check that $\|(H^\iota - \eta^\iota)\psi_n\|_{-1} \to 0$ as $n \to \infty$. Employing the diagonal form (3.2) of the metric tensor, we can split the Hamiltonian (3.8) into a sum of two parts, $H^\iota = H_1^\iota + H_2^\iota$, where H_i^ι, $i \in \{1, 2\}$, corresponds to the term with G^{ii} in (3.8). This decomposition leads to the trivial bound

$$\left\|(H^\iota - \eta^\iota)\psi_n^\iota\right\|_{-1} \le \left\|(H_1^\iota - \lambda^2)\psi_n^\iota\right\|_{-1} + \left\|(H_2^\iota - E_1^\iota)\psi_n^\iota\right\|_{-1}. \tag{5.4}$$

We will show that the norms at the r.h.s. of this inequality tends to zero as $n \to \infty$ separately. Denote $\|f\|_{\infty,n} := \sup\{|f(s,u)| \mid (s,u) \in \operatorname{supp}\varphi_n \times I\}$.

An explicit calculation using (3.3) and the fact that χ_1^ι is an eigenfunction of $-\Delta_\iota^I$ corresponding to the energy E_1^ι yields

$$\left|\left(\phi, (H_2^\iota - E_1^\iota)\hat{\psi}_n^\iota\right)\right| = \left|\left(\phi, k\,\hat{\psi}_{n,2}^\iota\right)_{L^2(\Omega_0)}\right| \le C_-^{-1}\sqrt{E_1^\iota}\;\|k\|_{\infty,n}\,\|\phi\|\,\|\hat{\psi}_n^\iota\| \tag{5.5}$$

for all $\phi \in \operatorname{Dom} Q^\iota$. Consequently, the second term at the r.h.s. of (5.4) goes to zero as $n \to \infty$ by the assumption $\langle \mathsf{d} \rangle$.

A little more toilsome but still direct calculation yields

$$\begin{aligned}
\left(\phi, (H_1^\iota - \lambda^2)\hat{\psi}_n^\iota\right) &= \lambda^2 \left(\phi, (1 - G^{\frac{1}{2}})\,\hat{\psi}_n^\iota\right)_{L^2(\Omega_0)} \\
&+ \left(\phi_{,1}, (G^{\frac{1}{2}}G^{11} - 1)\,(\dot{\varphi}_n + i\lambda\varphi_n)\,\chi_1^\iota e^{i\lambda s}\right)_{L^2(\Omega_0)} \\
&- \left(\phi, (\ddot{\varphi}_n + 2i\lambda\dot{\varphi}_n)\,\chi_1^\iota e^{i\lambda s}\right)_{L^2(\Omega_0)}
\end{aligned}$$

for all $\phi \in \operatorname{Dom} Q^\iota$. Estimating all the terms at the r.h.s. of this equality in the same way as in (5.5), it is enough to show that the following sequences

$$\left\|1 - G^{\frac{1}{2}}\right\|_{\infty,n}, \quad \left\|G^{\frac{1}{2}}G^{11} - 1\right\|_{\infty,n}, \quad \frac{\|\dot{\varphi}_n\|_{L^2(\mathbb{R})}}{\|\varphi_n\|_{L^2(\mathbb{R})}}, \quad \frac{\|\ddot{\varphi}_n\|_{L^2(\mathbb{R})}}{\|\varphi_n\|_{L^2(\mathbb{R})}},$$

has the zero limit as $n \to \infty$. However, this is evident for the first and second ones by virtue of (3.2) and $\langle \mathsf{d} \rangle$, while for the rest it follows by the definition of the sequence $\{\varphi_n\}$. □

If the strip is asymptotically straight, *i.e.* $\langle \mathsf{d} \rangle$, then $\sigma(H^N) = [0, \infty)$ by Theorem 4.1, (3.10) and non-negativity of H^N; see also Theorem 4.2. We conclude this section by proving the following result about the spectral threshold of the operators H^D and H^{DN}.

Proposition 5.1. *Suppose $\langle \mathsf{H} \rangle$. If the strip obeys $\langle \mathsf{d} \rangle$, then*

$$\inf \sigma(H^\iota) > 0 \qquad \textit{for} \quad \iota \in \{D, DN\}.$$

Proof. We have $H^\iota \geq 0$ and $E_1^\iota > 0$. By virtue of Theorem 4.1, it is enough to prove that $0 \notin \sigma_{\mathrm{p}}(H^\iota)$. Assume that there exists $\psi \in \operatorname{Dom} H^\iota$ such that $H^\iota \psi = 0$. Then $\psi \in \operatorname{Dom} Q^\iota$ and $0 = (\psi, H^\iota \psi) = Q^\iota[\psi] \equiv \int_{\Omega_0} \overline{\psi}_{,i} G^{ij} \psi_{,j} \, d\Omega$ with (G_{ij}) being a strictly positive definite matrix, hence $\psi = 0$. □

6 Curvature-Induced Spectrum

Now we will be interested in the proofs concerning the existence and properties of the spectrum of H^ι below the energy E_1^ι. Since H^N is a non-negative operator and $E_1^N = 0$, only the situations $\iota \in \{D, DN\}$ are relevant here, however, we do not exclude the Neumann case from the preliminary considerations here in order to establish a minor result contained in Proposition 6.1 below.

All the proofs of the following subsections are based on the variational strategy of finding a trial function ψ^ι from the form domain of H^ι such that

$$Q_1^\iota[\psi^\iota] := Q^\iota[\psi^\iota] - E_1^\iota \, \|\psi^\iota\|^2 < 0. \tag{6.1}$$

We construct such a trial function by modifying the generalised eigenfunction (3.11) of energy E_1^ι for the straight strip. This idea goes back to J. Goldstone and R. L. Jaffe, [39]; see also [14, 16, 13, 45].

As a preliminary, let us express the form (6.1) in the situation when the variables are separated in the following way:

$$\psi^\iota(s,u) := \varphi(s) \, \chi_1^\iota(u), \tag{6.2}$$

where χ_1^ι is the first transverse mode (3.11) or (3.12) and φ is a suitable function from $W^{1,2}(\mathbb{R})$. In view of (3.3), it is clear that ψ^ι belongs to $\operatorname{Dom} Q^\iota$, given by (3.5), (3.6) or (3.7), respectively. An explicit calculation yields

$$Q_1^\iota[\psi^\iota] = \left(\dot{\varphi}, \langle G^{-\frac{1}{2}} \rangle_\iota \, \dot{\varphi}\right)_{L^2(\mathbb{R})} + \tfrac{1}{2} \left[\chi_1^\iota(d)^2 - \chi_1^\iota(0)^2\right] (\varphi, k \, \varphi)_{L^2(\mathbb{R})} \,, \tag{6.3}$$

where $\langle \cdot \rangle_\iota$ denotes the expectation w.r.t. χ_1^ι, *i.e.* $\langle f \rangle_\iota := \int_I f(\cdot, u) \chi_1^\iota(u)^2 du$ with $f \in L^\infty(\Omega_0)$. It is clear from (3.11) and (3.12) that the second term at the r.h.s. of (6.3) is absent for $\iota \in \{D, N\}$, while $\chi_1^{DN}(d) = \sqrt{2/d}$ and $\chi_1^{DN}(0) = 0$.

6.1 The Existence

***Proof of* Theorem 4.3 and Theorem 4.4, *condition* (a).** We set

$$\psi_n^\iota(s,u) := \varphi(s;n) \, \chi_1^\iota(u), \tag{6.4}$$

where $\varphi : \mathbb{R} \times (0, \infty) \to [0,1]$ is supposed to satisfy:

(i) $\forall n \in (0, \infty): \ \varphi(\cdot; n) \in W^{1,2}(\mathbb{R})$,

(ii) $\varphi(s;n) \xrightarrow[n \to \infty]{} 1 \quad$ for a.e. $s \in \mathbb{R}$,

(iii) $\|\varphi_{,1}(\cdot;n)\|_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0$,

that is, φ is a suitable mollifier of 1 (for an example of such a function, see (6.13) below). Substituting this trial function to (6.3), we get

$$Q_1^D[\psi_n] \xrightarrow[n\to\infty]{} 0\,, \qquad Q_1^N[\psi_n] \xrightarrow[n\to\infty]{} 0\,, \qquad Q_1^{DN}[\psi_n] \xrightarrow[n\to\infty]{} \frac{\alpha}{d}\,, \tag{6.5}$$

where α is the total bending angle (4.1). The limits hold true by virtue of the required properties of φ, the fact that $\langle G^{-\frac{1}{2}}\rangle_\iota$ are bounded functions and, in the case $\iota = DN$, also by the dominated convergence theorem. That is why we need to assume in addition that k is integrable for $\iota = DN$. Consequently, if α is strictly negative, then there exists a finite $n_0 > 0$ such that $Q_1^{DN}[\psi_{n_0}] < 0$ and the proof for $\iota = DN$ is finished in this case.

To obtain the result for $\iota = DN$ in the limit case $\alpha = 0$, and for any Dirichlet strip, we modify the function ψ_n^ι, in a curved part of the waveguide. We define

$$\psi_{n,\varepsilon}^\iota(s,u) := \psi_n^\iota(s,u) + \varepsilon\,\phi(s)\,\upsilon^\iota(u)\,\chi_1^\iota(u), \qquad \iota \in \{D, DN\} \tag{6.6}$$

where $\varepsilon \in \mathbb{R}$, $\phi \in W^{1,2}(\mathbb{R})$ is a real, non-negative function with compact support contained in a bounded interval in $\mathbb{R}$ where k is not zero and does not change sign (such an interval surely exists because $k \not\equiv 0$ and is continuous), and $\upsilon^D(u) := -2u/d$ and $\upsilon^{DN}(u) := 1$. The family $\{\psi_{n,\varepsilon}^\iota\}$ is a subset of $\operatorname{Dom} Q^\iota$ and we can write

$$Q_1^\iota[\psi_{n,\varepsilon}^\iota] = Q_1^\iota[\psi_n^\iota] + 2\varepsilon\, Q_1^\iota(\phi\upsilon^\iota\chi_1^\iota, \psi_n^\iota) + \varepsilon^2\, Q_1^\iota[\phi\upsilon^\iota\chi_1^\iota]. \tag{6.7}$$

The last term at the r.h.s. of (6.7) does not depend on n, while the first one tends to zero as $n \to \infty$ by (6.5). An explicit calculation of the central term gives (*cf* (6.3) for $\iota = DN$)

$$Q_1^\iota(\phi\upsilon^\iota\chi_1^\iota, \psi_n) = \big(\dot\phi, \langle \upsilon^\iota G^{-\frac{1}{2}}\rangle_\iota\,, \dot\varphi_n\big)_{L^2(\mathbb{R})} + \tfrac{1}{d}\,(\phi, k\,\varphi_n)_{L^2(\mathbb{R})}\,,$$

where we have denoted $\varphi_n := \varphi(\cdot;n)$ and $\dot\varphi_n := \varphi_{,1}(\cdot;n)$. Using then the properties of the function φ together with the dominated convergence theorem (notice that $\phi\,k \in L^1(\mathbb{R})$), we have

$$Q_1^\iota[\psi_{n,\varepsilon}^\iota] \xrightarrow[n\to\infty]{} \tfrac{2}{d}\,\varepsilon\,(\phi,k)_{L^2(\mathbb{R})} + \varepsilon^2\, Q_1^\iota[\phi\upsilon^\iota\chi_1^\iota]. \tag{6.8}$$

Since the integral $(\phi,k)_{L^2(\mathbb{R})} \neq 0$ by the construction of ϕ, we can take ε sufficiently small and of an appropriate sign so that the sum of the last two terms at the r.h.s. of (6.8) is negative, and then choose n sufficiently large so that $Q_1^\iota[\psi_{n,\varepsilon}^\iota] < 0$. □

The intermediate results (6.5) of the precedent proof give the following upper bounds on the spectral threshold of H^ι:

Proposition 6.1. *Suppose* $\langle \mathsf{H} \rangle$. *One has*

(i) $\inf \sigma(H^\iota) \leq E_1^\iota$ *for* $\iota \in \{D, N\}$;

(ii) $\inf \sigma\left(H^{DN} - \frac{k(s)}{d(1-uk(s))}\right) \leq E_1^{DN}$ *provided* $k \in L^1(\mathbb{R})$.

Actually, in view of Theorem 4.3, a stronger result than (i) holds for any Dirichlet strip. The assertion (i) for the Neumann case, together with the fact that H^N is non-negative, establishes the first claim of Theorem 4.2.

Remark 6.1 (Condition (a) of Theorem 4.4 *vs* the assumptions in [13]). The non-positivity of the total bending angle, *i.e.* $\alpha \leq 0$, is a nice sufficient condition which guarantees the existence of geometrically induced spectrum for H^{DN}. This was established already in [13] under the additional hypothesis that "k is non-positive everywhere outside of some bounded interval". Since the latter is not assumed in this paper, we extend significantly the class of admissible geometries. Nevertheless, in order to justify the use of the dominated convergence theorem, we need to assume that "k is integrable" instead; *cf* the condition (a) of Theorem 4.4. Hence, a natural question is to ask whether the assumptions in [13] may after all present an alternative criterion which is not contained in our condition (a). The answer is negative due to the following (purely geometrical) result, which can be easily shown using the so-called "Umlaufsatz", [43, Thm. 2.2.1].

Lemma 6.1. *Let Γ be an infinite plane C^2-smooth curve of bounded curvature k. If there exists a compact $\Gamma_c \subset \Gamma$ such that $\left|\int_{\Gamma_a} k\right| > 2\pi$ for any compact Γ_a obeying $\Gamma_c \subseteq \Gamma_a \subset \Gamma$, then any tubular neighbourhood of Γ overlaps.*

That is, any reference curve satisfying the assumptions of [13] but having a non-integrable curvature leads to a violation of the basic hypothesis $\langle \mathsf{H} \rangle$ (which is assumed in [13] as well). $\square$

***Proof of* Theorem 4.4, *condition* (b).** Let $L > 0$ be the period of k, *i.e.*, $\forall s \in \mathbb{R}: \ k(s+L) = k(s)$. We take the trial function of the form

$$\psi_{n,\varepsilon}^{DN}(s,u) := \varphi_n(s)\ (1 + \varepsilon\, \phi(s))\ \chi_1^{DN}(u)\,,$$

cf (6.6), where the functions φ_n and ϕ are defined as follows. Let $\varphi_1 \in C_0^\infty((-L, 2L))$ be a real function which is equal to 1 on the period cell $(0, L)$. We set, for any $n \in \mathbb{N} \setminus \{0\}$,

$$\varphi_n(s) := \begin{cases} \varphi_1(s) & \text{if} \quad s \in (-\infty, L), \\ 1 & \text{if} \quad s \in [L, nL], \\ \varphi_1\left(s - (n-1)L\right) & \text{if} \quad s \in (nL, (n+1)L]\,, \\ 0 & \text{if} \quad s \in ((n+1)L, +\infty)\,. \end{cases}$$

Let $\phi \in C^\infty(\mathbb{R})$ be non-negative, L-periodic, and such that $\operatorname{supp} \phi \restriction (0, L)$ is contained in an interval where k is not zero and does not change sign. Then

$(\phi, k)_{L^2((0,L))} \neq 0$. Finally, let $\varepsilon \in \mathbb{R}$ be chosen in such a way that (*cf* (6.8))

$$\begin{aligned} A \;:=\; & \big(\psi_{1,\varepsilon}^{DN}, \big(H^{DN} - E_1^{DN}\big)\psi_{1,\varepsilon}^{DN}\big)_{L^2((0,L)\times I, d\Omega)} \\ =\; & \tfrac{2}{d}\,\varepsilon\,(\phi, k)_{L^2((0,L))} + \varepsilon^2\,\big(\phi\chi_1^{DN}, \big(H^{DN} - E_1^{DN}\big)\phi\chi_1^{DN}\big)_{L^2((0,L)\times I, d\Omega)} \end{aligned} \tag{6.9}$$

is negative. By virtue of the definition of φ_1 and the fact that $\int_0^L k(s)ds = 0$ (*cf* Lemma 6.1), it is clear that

$$Q_1^{DN}[\psi_{1,\varepsilon}^{DN}] = A + B,$$

where B is defined as the integral at the first line of (6.9), however, with the range of integration being the set $((-L, 0) \cup (L, 2L)) \times I$. Using the periodicity of the coefficients of H^{DN} together with the definition of φ_n, we continue by induction and arrive at the identity

$$\forall n \in \mathbb{N} \setminus \{0\}: \quad Q_1^{DN}[\psi_{n,\varepsilon}^{DN}] = nA + B,$$

which becomes negative for n sufficiently large. □

Remark 6.2 (Integrability of k). If $k \not\equiv 0$ is periodic, then the curvature is not integrable. However, one has for every $n \in \mathbb{N}$, $\int_{-nL}^{nL} k(s)\,ds = 0$ due to the periodicity (*cf* Lemma 6.1). This indicates that the requirement $k \in L^1(\mathbb{R})$ in the condition (a) of Theorem 4.4 may be rather a technical hypothesis. □

***Proof of* Theorem 4.4, *condition* (c).** We take the trial function ψ^{DN} of the form (6.2). Since k is continuous and $k_- \not\equiv 0$, there exists an interval $J \subset \mathbb{R}$, such that $k(s) < 0$ for all $s \in J$. Choosing $\varphi \in W^{1,2}(\mathbb{R})$ such that $\operatorname{supp}\varphi \subseteq \overline{J}$ and substituting it to (6.3), obvious estimates yield

$$Q_1^{DN}[\psi^{DN}] \;\leq\; \|\dot{\varphi}\|_{L^2(J)}^2 \;+\; \frac{1}{d}\int_J |\varphi(s)|^2\, k(s)\, ds. \tag{6.10}$$

The second term at the r.h.s. of the last inequality is obviously negative, while the first one does not depend on d. Hence for all d sufficiently small their sum is negative. □

Remark 6.3 (An estimate of the critical width). The claim of Theorem 4.4 with the condition (c) can be rewritten as follows. Suppose $\langle \mathsf{H} \rangle$. There exists a positive d_0 such that for all $d < d_0$ the condition $k_- \not\equiv 0$ implies that $\inf\sigma(H^{DN}) < E_1^{DN}$. Let us estimate the critical width d_0 from below here. First of all, notice that the assumption $\langle \mathsf{H} \rangle$ estimates d from above by $\|k_+\|_\infty^{-1}$, hence d_0 can be infinite in the case $k_+ \equiv 0$ (for instance, in view of the condition (a) of Theorem 4.4, this happens if in addition $k \in L^1(\mathbb{R})$).

We define for $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$,

$$\alpha(s_1, s_2) := \int_{s_1}^{s_2} k(s)\, ds\,, \qquad k_m(s_1, s_2) := \sup_{s \in \mathbb{R}\setminus(s_1, s_2)} k_+(s)$$

and

$$d_1 := \sup\left\{ \frac{1}{k_m(s_1,s_2)}\,\frac{\alpha(s_1,s_2)^2}{4\,+\,\alpha(s_1,s_2)^2} \;\middle|\; s_1,s_2\in\mathbb{R}, s_1<s_2,\ \alpha(s_1,s_2)<0 \right\}.$$

Notice that $d_1>0$ because it is supposed that $k_-\not\equiv 0$, and $k\in L^\infty(\mathbb{R})$ by $\langle\mathsf{H}\rangle$ By the definition of the supremum, for any positive $d<d_1$, there exist $s_{1,d},s_{2,d}\in\mathbb{R}$ such that $s_{1,d}<s_{2,d}$, $\alpha_d:=\alpha(s_{1,d},s_{2,d})<0$ and

$$d<\frac{1}{k_d}\,\frac{\alpha_d^2}{4\,+\,\alpha_d^2}, \tag{6.11}$$

where $k_d:=k_m(s_{1,d},s_{2,d})$. If $k_d=0$, then necessarily $k\in L^1(\mathbb{R})$ by virtue of Lemma 6.1, thus the condition (a) of Theorem 4.4 is satisfied and the only restriction on d is given by the natural condition $\langle\mathsf{H}\rangle$. Let us assume therefore $k_d>0$ so that the r.h.s. of (6.11) is well defined. We set

$$\varphi_d(s):=\begin{cases}\exp\left(\sqrt{k_d\,(1-d\,k_d)/d}\,(s-s_{1,d})\right) & \text{if}\quad s\in(-\infty,s_{1,d})\,,\\ 1 & \text{if}\quad s\in[s_{1,d},s_{2,d}]\,,\\ \exp\left(-\sqrt{k_d\,(1-d\,k_d)/d}\,(s-s_{2,d})\right) & \text{if}\quad s\in(s_{2,d},+\infty)\,.\end{cases}$$

Again, $\psi_d(s,u):=\varphi_d(s)\,\chi_1^{DN}(u)$ belongs to the domain of Q^{DN} and we have, *cf* (6.3),

$$\begin{aligned}Q_1^{DN}[\psi_d] &\le \frac{1}{1-d\,k_d}\;\|\dot\varphi_d\|^2_{L^2(\mathbb{R}\setminus(s_{1,d},s_{2,d}))}+\frac{k_d}{d}\;\|\varphi_d\|^2_{L^2(\mathbb{R}\setminus(s_{1,d},s_{2,d}))}+\frac{\alpha_d}{d}\\ &=\frac{2}{d}\sqrt{\frac{d\,k_d}{1-d\,k_d}}+\frac{\alpha_d}{d}.\end{aligned}$$

The inequality uses the definition of k_d and the equality is a result of an explicit calculation of the norms. However, using (6.11) and the fact that $\alpha_d<0$, we arrive at $Q_1^{DN}[\psi_d]<0$. Hence, together with the assumption $\langle\mathsf{H}\rangle$, we get $d_0\ge\min\left\{\|k_+\|_\infty^{-1},d_1\right\}$ (with the convention $d_0:=+\infty$ if $k_+\equiv 0$). ☐

***Proof of* Theorem 4.4, *part* (ii).** Let us only sketch here the main ideas of the proof, for more details see [13, Prop. 3]. Let $k_-\equiv 0$. We show that then the functional $Q_1^{DN}[\Phi]$ is non-negative for every $\Phi\in\operatorname{Dom}Q^{DN}$. To do so, we decompose the function Φ to a transverse orthonormal basis, *i.e.*

$$\Phi(s,u)=\sum_{n=1}^{\infty}\phi_n(s)\,\chi_n(s;u),$$

where, for almost every value of the parameter $s\in\mathbb{R}$,

$$\phi_n(s):=\left(\Phi(s,\cdot)\,,\,\chi_n(s;\cdot)\right)_{L^2(I,(1-uk(s))du)}$$

and $\chi_n(s;\cdot)$ are normalised eigenfunctions of the self-adjoint operator $h(s)$ acting in $L^2\left(I,(1-uk(s))du\right)$ associated with the quadratic form

$$\begin{aligned} q(s)[\psi] &:= \Big(\psi_{,2}(s;\cdot),\psi_{,2}(s;\cdot)\Big)_{L^2(I,(1-uk(s))du)}\,, \\ \operatorname{Dom} q(s) &:= \left\{\psi(s;\cdot)\in W^{1,2}(I)\mid \psi(s;0)=0\right\}\,. \end{aligned}$$

Let us denote by $\lambda_n(s)$ the eigenvalues of this operator corresponding to eigenfunctions $\chi_n(s;\cdot)$. Neglecting the non-negative longitudinal term (*i.e.* the integral (3.4) for $i,j=1$), we obtain

$$Q_1^{DN}[\Phi] \geq \sum_{n=1}^{\infty}\int_{\mathbb{R}} |\phi_n(s)|^2\left(\lambda_n(s)-E_1^{DN}\right)ds\,.$$

Hence it is enough to show that for almost every $s\in\mathbb{R}$, the lowest eigenvalue $\lambda_1(s)$ is greater than or equal to E_1^{DN}. This can be shown by a spectral analysis of the ordinary differential operator h associated with q, *i.e.*, for a.e. $s\in\mathbb{R}$,

$$\begin{aligned} h(s) &= -\partial_u^2+\frac{k(s)}{1-uk(s)}\,\partial_u\,, \\ \operatorname{Dom} h(s) &= \left\{\psi(s;\cdot)\in W^{2,2}(I)\mid \psi(s;0)=\psi_{,2}(s;d)=0\right\}. \end{aligned}$$

More specifically, using an expansion of the explicit solutions of the corresponding eigenvalue problem and the minimax principle together with a unitary transformation, the authors of [13] prove that for a.e. $s\in\mathbb{R}$, $h(s)$ has no spectrum in $[0,E_1^{DN})$ provided $k_-\equiv 0$. $\square$

Proof of **Proposition 4.1.** The claim is trivial for $n=0$. Let us fix an integer $n\in\mathbb{N}\setminus\{0\}$. We shall find a critical width d_n such that for all $d<d_n$, there are at least n discrete eigenvalues in the spectrum of H^{DN}, counting multiplicity. As in the proof of Theorem 4.4, condition (c), let $J\subset\mathbb{R}$ be a bounded interval such that $k(s)<0$ for all $s\in J$. We set $s_0:=\inf J$ and $s_j:=s_0+j\,|J|/n$ for every $j\in\{1,\dots,n\}$. Let φ_0 be a non-zero function from $W^{1,2}(\mathbb{R})$ such that $\operatorname{supp}\varphi_0\subset(s_0,s_1)$. We define for every $j\in\{1,\dots,n\}$ and $s\in\mathbb{R}$,

$$\begin{aligned} N_j^{-2} &:= \int_{s_{j-1}}^{s_j} |\varphi_0(s_0+s-s_{j-1})|^2\,\langle G^{\frac12}\rangle_{DN}(s)\,ds\,, \\ \varphi_j(s) &:= N_j\,\varphi_0(s_0+s-s_{j-1})\,. \end{aligned}$$

Putting $\psi_j^{DN}(s,u):=\varphi_j(s)\chi_1^{DN}(u)$ for every $j\in\{1,\dots,n\}$, *cf* (6.2), we obtain an orthonormal basis of a subset of $\operatorname{Dom} Q^{DN}$. Moreover, $Q^{DN}(\psi_j^{DN},\psi_\ell^{DN})=0$ whenever $j\neq\ell$ because φ_j and φ_ℓ have disjoint supports. Therefore, it follows by [9, Lemma 4.5.4] and Theorem 4.1 that a sufficient condition for H^{DN} to have at least n discrete eigenvalue is $Q^{DN}[\psi_j^{DN}] < E_1^{DN}$, *i.e.* $Q_1^{DN}[\psi_j^{DN}] < 0$, for every $j \in \{1, \dots n\}$. However, according to (6.10),

$$Q_1^{DN}\left[\psi_j^{DN}\right] \;\leq\; N_j^2\,\|\dot{\varphi}_0\|^2_{L^2(\mathbb{R})} + \frac{N_j^2}{d}\int_{s_{j-1}}^{s_j} |\varphi_0(s_0+s-s_{j-1})|^2\, k(s)\, ds\,.$$

The r.h.s. of the last inequality is obviously negative for all $j \in \{1, \dots, n\}$ provided that $d < d_n$ with

$$d_n := \min_{j\in\{1,\dots,n\}} \frac{1}{\|\dot{\varphi}_0\|^2_{L^2(\mathbb{R})}} \int_{s_0}^{s_1} |\varphi_0(s)|^2\, |k(s-s_0+s_{j-1})|\, ds\,.$$

□

6.2 The estimates on the spectral threshold

Throughout this subsection, we consider only $\iota \in \{D, DN\}$. Obviously,

$$\inf\sigma(H^\iota) - E_1^\iota = \inf_{\psi\in\operatorname{Dom} Q^\iota} \frac{Q_1^\iota[\psi]}{\|\psi\|^2} \leq \inf_{\psi\in T^\iota} \frac{Q_1^\iota[\psi]}{\|\psi\|^2}\,, \tag{6.12}$$

where T^ι is an arbitrary subset of $\operatorname{Dom} Q^\iota$. Our strategy will be to choose a suitable T^ι and then explicitly find the infimum of the quotient at the r.h.s. of (6.12).

In Theorem 4.5, the curvature is supposed to have a compact support contained in an interval of width $2s_0$; without loss of generality we may assume that the reference curve is parameterised in such a way that $\operatorname{supp} k \subseteq [-s_0, s_0]$.

***Proof of* Theorem 4.5, *part* (i).** Let $\psi_{n,c}(s,u) := \varphi_c(s;n)\,\chi_1^{DN}(u)$ be the trial function from the beginning of the proof of the condition (a) of Theorem 4.4 in Section 6.1 with the mollifier $\varphi_c(\cdot;n)$ given explicitly by

$$\varphi_c(s;n) := \begin{cases} 1 & \text{if} \quad |s| \in [0,n), \\ (c\,n-|s|)/\left((c-1)\,n\right) & \text{if} \quad |s| \in [n,cn), \\ 0 & \text{if} \quad |s| \in [cn,\infty), \end{cases} \qquad c > 1. \tag{6.13}$$

We set $T^{DN} := \{\psi_{n,c} \,|\, n \geq s_0\,,\ c > 1\}$. An easy calculation yields

$$Q_1^{DN}[\psi_{n,c}] = \frac{2}{(c-1)\,n} + \frac{\alpha}{d}\,, \qquad \|\psi_{n,c}\|^2 = \frac{2}{3}\,(c+2)\,n - \alpha\,\langle u\rangle\,,$$

where

$$\langle u\rangle := \int_I u\,\chi_1^{DN}(u)^2\, du = d\left(\frac{1}{2}+\frac{2}{\pi^2}\right).$$

Hence, denoting by $f(n,c)$ the quotient at the r.h.s. of (6.12), we have

$$f(n,c) = \frac{\frac{2}{c-1}+\frac{\alpha}{d}n}{\frac{2}{3}(c+2)\,n^2 - \alpha\,\langle u\rangle\, n}\,. \tag{6.14}$$

Now we shall seek the infimum of the continuous function f in the region $[s_0, \infty) \times (1, \infty)$; the result establishes the bound from Theorem 4.5.

Let us solve the equation $f_{,2}(n,c) = 0$. Calculating the derivative

$$f_{,2}(n,c) = \frac{2}{3}\, \frac{-\frac{\alpha\, n}{d}\,(c-1)^2 - 2(2c+1) + \frac{3\alpha\langle u\rangle}{n}}{(c-1)^2\,\left(\frac{2}{3}(c+2)\,n - \alpha\,\langle u\rangle\right)^2}\,,$$

we see that its numerator is a quadratic polynom in c which has two roots

$$c_{\pm}(n) := -\frac{2d}{\alpha\, n} + 1 \mp \frac{d}{\alpha\, n}\sqrt{-6\frac{\alpha\, n}{d} + 4 + 3\frac{\alpha^2\,\langle u\rangle}{d}}\,. \tag{6.15}$$

Since $\alpha < 0$, it is evident that $c_+ > 1$, while $c_- < 1$ is not allowed. A straightforward calculation leads to the following (negative) value

$$f\,(n, c_+(n)) = \frac{-3\,\alpha^2/d^2}{4\,\left(1 + \sqrt{1 - \frac{3}{2}\alpha\, n/d + \frac{3}{4}\alpha^2\langle u\rangle/d}\right)^2}\,, \tag{6.16}$$

which is nothing else than the global minimum of the function $c \mapsto f(n,c)$ for any fixed $n \in [s_0, \infty)$. Obviously, $g(n) := f(n, c_+(n))$ is a strictly increasing function of the variable n in the interval $[s_0, \infty)$, *i.e.* $\forall n \in [s_0,\infty) :\ \dot g(n) > 0$. However,

$$0 < \dot g(n) = f_{,1}(n, c_+) + f_{,2}(n, c_+)\,\dot c_+(n) = f_{,1}(n, c_+)\,,$$

and that is why there is no candidate for a local minimum of the function f in the interior of its domain, *i.e.* for every point $(n,c) \in (s_0,\infty)\times(1,\infty)$, $f_{,1}(n,c) \neq 0$ or $f_{,2}(n,c) \neq 0$.

Thus the problem reduces to the study of the behaviour of f on the boundary set $\{s_0\}\times(1,\infty)$ and its limits as $n \to \infty$, $c \to 1$ and $c \to \infty$, respectively. Using the estimate $f(n,c) \geq \alpha/(2dn)$, we obtain

$$\liminf_{n\to\infty} f(n,c) \geq 0$$

uniformly in c. Hence, there exists a (finite) $n_0 > s_0$ such that for every $n > n_0$ holds true $f(n,c) \geq f(s_0, c_+(s_0))$ (recall that $f(s_0, c_+(s_0)) < 0$, *cf* (6.16)) uniformly in c. Therefore since we seek the infimum of f we can consider only $n \in [s_0, n_0]$ in the rest of the proof. However, for those values of n we have

$$f(n,c) \geq \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} + \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle} \tag{6.17}$$

and since

$$\lim_{c\to1} \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} = \infty\,, \qquad \left|\lim_{c\to1} \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle}\right| < \frac{1}{d\langle u\rangle}\,,$$

we obtain

$$\lim_{c\to 1} f(n,c) = \infty$$

uniformly in n. Finally,

$$\lim_{c\to\infty} \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} = 0\,,$$

$$\lim_{c\to\infty} \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle} \geq \frac{\alpha}{dn} \lim_{c\to\infty} \frac{3}{2(c+2)} = 0$$

because $n \mapsto \alpha/(dn)$ is bounded on $[s_0, n_0]$; hence, in view of (6.17),

$$\liminf_{c\to\infty} f(n,c) \geq 0$$

uniformly in n. Since the infimum of f should be negative, we infer from the above results that

$$\inf_{(n,c)\in[s_0,\infty)\times(1,\infty)} f(n,c) = \inf_{c\in(1,\infty)} f(s_0,c) = f\left(s_0, c_+(s_0)\right),$$

where $f(s_0, c_+(s_0)) < 0$ given by (6.16) provides an upper bound on the r.h.s. of (6.12) for the case $\iota = DN$. $\square$

***Proof of* Theorem 4.5, *part* (ii).** In the Dirichlet case, we use the mollifier (6.13) with the fixed $n = s_0$ for the construction of the functions from T^D. We set for any c_1, $c_2 > 1$ and $\varepsilon \in \mathbb{R}$,

$$\psi_{c_1,c_2,\varepsilon}(s,u) := \varphi_{c_1}(s;s_0)\,\chi_1^D(u) + \varepsilon\,\varphi_{c_2}(s;s_0)\,\chi_2^D(u) \tag{6.18}$$

and $T^D := \{\psi_{c_1,c_2,\varepsilon}\,|\,c_1,\,c_2>1\,,\,\varepsilon\in\mathbb{R}\}$. Easy explicit calculations give

$$Q_1^D[\psi_{c_1,c_2,\varepsilon}] = \frac{\pi^2}{d}\left(h(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2(2g(c_2)+h(c_2))\right),$$

$$\|\psi_{c_1,c_2,\varepsilon}\|^2 = \frac{2d}{3}\left(g(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2 g(c_2)\right),$$

where

$$h(c) := \frac{2}{\pi^2}\,\frac{d}{s_0}\,\frac{1}{c-1}\,, \qquad g(c) := \frac{s_0}{d}(c+2) - \frac{3}{4}\alpha\,.$$

Thus, the quotient at the r.h.s. of (6.12) can be written as

$$\tilde f(c_1,c_2,\varepsilon) := \frac{3\pi^2}{2d^2}\,\frac{h(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2(2g(c_2)+h(c_2))}{g(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2 g(c_2)}\,. \tag{6.19}$$

Clearly, $\tilde f$ is a continuous function of the three variables defined in the region $(1,\infty)^2\times\mathbb{R}$ (the denominator is positive since it is the squared norm of a nonzero function) and one could look for its infimum. However, from the technical point of view, it seems to be a rather complicated task and that is why we make first the following simplification.

We start by verifying that the infimum of $\tilde{f}$ is negative, *i.e.*, $\psi_{c_1,c_2,\varepsilon}$ is an admissible trial function to estimate $\inf(H^D) - E_1^D < 0$, *cf* Theorem 4.3. Obviously, $h(c) > 0$ for any $c \in (1, \infty)$. Using the definition of α, the assumption $\langle \mathsf{H} \rangle$ and obvious estimates, we check that the same holds true for g:

$$g(c) > 3\left(\frac{s_0}{d} - \frac{\alpha}{4}\right) > 3\left(\frac{s_0}{d} - \frac{1}{2}s_0\|k_+\|_\infty\right) > \frac{3}{2}\,\frac{s_0}{d}\,. \tag{6.20}$$

Hence, the only term in the numerator of $\tilde{f}$ which can attain negative values is the term linear in ε. However, for any given $c_2 > 0$, there exists $\varepsilon \in \mathbb{R}$ of such a sign that $\alpha\,\varepsilon < 0$ and with a sufficiently small absolute value so that the negative term linear in ε dominates over the quadratic one. Then we can find c_1 large enough to make the numerator of the r.h.s. of (6.19) negative. Recalling that the denominator is positive, we can restrict ourselves to those values of the triple (c_1, c_2, ε), for which $\tilde{f}(c_1, c_2, \varepsilon) < 0$; let us denote $\mathcal{N} := \{(c_1, c_2, \varepsilon) \in (1, \infty)^2 \times \mathbb{R} \,|\, \tilde{f}(c_1, c_2, \varepsilon) < 0\}$. Setting for any $(c_1, c_2, \varepsilon) \in \mathcal{N}$,

$$f(c_1, c_2, \varepsilon) := \frac{3\pi^2}{2d^2}\,\frac{h(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2(2g(c_2) + h(c_2))}{g(c_1)}\,, \tag{6.21}$$

we arrive easily at the inequality $\tilde{f}(c_1, c_2, \varepsilon) \leq f(c_1, c_2, \varepsilon)$, because the (positive) denominator in (6.19) is bounded from above by $g(c_1)$ due to the above considerations. Consequently,

$$\inf\sigma(H^D) - E_1^D \leq \inf_{(c_1,c_2,\varepsilon)\in\mathcal{N}} f(c_1, c_2, \varepsilon). \tag{6.22}$$

Calculating the partial derivatives of f, it is straightforward to see that the system of equations $f_{,i} = 0$, $i = 1, 2, 3$, can be cast into the following form:

$$\begin{aligned}
\frac{s_0}{d}\,A(c_2,\varepsilon)\,(c_1-1)^2 + \frac{4}{\pi^2}\,(c_1-1) + \frac{6}{\pi^2}\,\frac{d}{s_0}\,\left(\frac{s_0}{d} - \frac{\alpha}{4}\right) &= 0\,,\\
(c_2-1)^2 - \left(\frac{d}{\pi s_0}\right)^2 &= 0\,,\\
\varepsilon + \frac{8\alpha}{3\pi^2}\,\frac{1}{h(c_2) + 2g(c_2)} &= 0\,,
\end{aligned}$$

respectively, where, for any $(c_1, c_2, \varepsilon) \in \mathcal{N}$,

$$A(c_2, \varepsilon) := \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2(2g(c_2) + h(c_2)) < 0.$$

¿From the second equation we can immediately express c_2; of course, we choose that root c_{2+} which is greater than 1. Substituting c_{2+} to the third equation of our system, we obtain the root ε_0 (notice that really $\alpha\,\varepsilon_0 < 0$). Finally, putting

c_{2+} and ε_0 to the first equation, we choose that root c_{1+} which is greater than 1. A tedious but straightforward calculation yields

$$f(c_{1+}, c_{2+}, \varepsilon_0) = -\frac{3\pi^4}{4d^2} \frac{A(c_{2+}, \varepsilon_0)^2}{\left(1 + \sqrt{1 - \frac{3}{2} A(c_{2+}, \varepsilon_0) \pi^2 \left(\frac{s_0}{d} - \frac{\alpha}{4}\right)}\right)^2}$$

with

$$A(c_{2+}, \varepsilon_0) = -\frac{32\alpha^2}{9\pi^4} \frac{1}{\frac{2}{\pi} + 3(\frac{s_0}{d} - \frac{\alpha}{4})} .$$

(Recall that $\frac{s_0}{d} - \frac{\alpha}{4} > 0$, so the square root in the first formula is well defined in $\mathbb{R}$.) Hence really $(c_{1+}, c_{2+}, \varepsilon_0) \in \mathcal{N}$. Moreover, one can check that the matrix of second derivatives of f is in the point $(c_{1+}, c_{2+}, \varepsilon_0)$ diagonal with all positive elements, that is, the function f reaches its local minimum in that point.

To see that it is the global minimum too, we study the behaviour of the limits of f as $c_i \to 1, \infty$, $i \in \{1, 2\}$ and $\varepsilon \to \pm\infty$. We restrict ourselves to that cases, where the limit is reached by negative values of f; the rest of the "boundary" of the set $\mathcal{N}$ consists of those triples $(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon})$, for which $f(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon}) = 0$, that is, $f(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon}) > f(c_{1+}, c_{2+}, \varepsilon_0)$. Since (6.20) gives

$$g(c) > \frac{3\,|\alpha|}{4} , \tag{6.23}$$

we obtain

$$f(c_1, c_2, \varepsilon) > \frac{3\,\pi^2}{2\,d^2} \frac{\frac{16}{3\,\pi^2} \alpha\,\varepsilon + \frac{3\,|\alpha|}{2} \varepsilon^2}{g(c_1)}$$

and the condition $f(c_1, c_2, \varepsilon) < 0$ yields

$$|\varepsilon| < \frac{32}{9\,\pi^2} . \tag{6.24}$$

Hence we do not study the limits as $\varepsilon \to \pm\infty$ and we may assume in the following that ε is bounded. Using (6.23) in the denominator of (6.21), neglecting $h(c_1)$ and minimising the remaining polynom in ε in the numerator of (6.21), we arrive at the lower bound

$$f(c_1, c_2, \varepsilon) > -\frac{128}{9\,\pi^2\,d^2} \frac{|\alpha|}{h(c_2) + 2\,g(c_2)}$$

for any $c_2 \in (1, \infty)$. Thus

$$\liminf_{c_2 \to \infty} f(c_1, c_2, \varepsilon) \geq 0\,, \qquad \liminf_{c_2 \to 1} f(c_1, c_2, \varepsilon) \geq 0$$

uniformly in c_1 and ε. Finally, using (6.24) we can see that

$$f(c_1, c_2, \varepsilon) > \frac{3\,\pi^2}{2\,d^2} \frac{h(c_1)}{g(c_1)} - \frac{256\,|\alpha|}{9\,\pi^2\,d^2} \frac{1}{g(c_1)}$$

and therefore

$$\liminf_{c_1\to\infty} f(c_1,c_2,\varepsilon) \geq 0\,, \qquad \lim_{c_1\to 1} f(c_1,c_2,\varepsilon) = \infty$$

uniformly in c_2 and ε. Summing up the considerations, we conclude that $f(c_{1_+},c_{2_+},\varepsilon_0)$ is the global minimum and the claim (ii) of Theorem 4.5 then follows from (6.22). □

Remark 6.4 (Mildly curved strips). Let us compare our estimate (ii) of Theorem 4.5 with the exact ground-state eigenvalue asymptotics derived in [14, Thm. 4.1] for *mildly* curved Dirichlet strips by the Birman-Schwinger perturbation technique. We consider families of generating curves Γ_β characterised by the curvature $k_\beta(s) := \beta\, k(s)$, where k is a fixed curvature function and $\beta > 0$ is a small parameter. Since $\alpha_\beta := \int_{\mathbb{R}} k_\beta(s)ds = \beta\,\alpha$, we see that β controls the total bending of the strip, too. The result of [14] can be written as

$$\inf(H^D) = E_1^D \,-\, C(d,k)^2\,\beta^4 + \mathcal{O}(\beta^5)\,,$$

where $C(d,k)$ is a positive constant depending only on the fixed width d and (integrals of) k, while our estimate (ii) yields

$$\inf(H^D) \leq E_1^D \,-\, C^D(s_0,d,0)^2\,\alpha^4\,\beta^4 + \mathcal{O}(\beta^5)\,.$$

Hence we observe the same dependence of the leading terms on the perturbation parameter β. Let us quantitatively compare the actual gap-width asymptotic given by $C(d,k)^2$ with our estimate $C^D(s_0,d,0)^2\alpha^4$. Since $C(d,k)$ has rather a complicated structure, we restrict ourselves to small values of the width d when

$$C(d,k) = \frac{1}{8}\,\|k\|^2_{L^2(\mathbb{R})} + \mathcal{O}(d^2)\,. \tag{6.25}$$

We have $C^D(s_0,d,0) = 8/(9\sqrt{3}\pi^2 s_0) + \mathcal{O}(d)$. Since $\alpha^2 \leq 2s_0\|k\|^2_{L^2(\mathbb{R})}$ by the Schwarz inequality, we see that

$$\frac{C^D(s_0,0,0)\,\alpha^2}{C(0,k)} \leq \frac{128}{9\sqrt{3}\,\pi^2} \approx 0.83\,.$$

As for the mixed Dirichlet-Neumann case, our estimate (i) of Theorem 4.5 leads to

$$\inf(H^{DN}) \leq E_1^{DN} - \frac{3\,\alpha^2}{8\,d^2}\,\beta^2 + \mathcal{O}(\beta^3)$$

and we observe that the leading term is proportional to the second power of β now. In particular, it is much greater than the leading term in the identical mildly curved strip with the pure Dirichlet boundary condition. Unfortunately, no exact asymptotics are known for $\inf\sigma(H^{DN})$, so we cannot perform any comparison in this case. □

Remark 6.5 (Thin strips). Another natural perturbation parameter is the strip width d. Calculating the expansions w.r.t. d of the constants $C^{\iota}(s_0,d,\alpha)$ from our Theorem 4.5, we arrive at

$$\begin{aligned} E_1^{DN} - \inf(H^{DN}) &\geq -\frac{\alpha}{2\,s_0\,d} + \mathcal{O}(d^{-\frac{1}{2}})\,, \\ E_1^{D} - \inf(H^{D}) &\geq \frac{2^8\,\alpha^4}{3^5\,\pi^4\,s_0^2\left(1+\sqrt{1+\left(\frac{4\alpha}{3\pi}\right)^2}\right)^2} + \mathcal{O}(d). \end{aligned}$$

Again, we observe qualitatively different behaviour of our estimates w.r.t. the perturbation parameter.

In particular, the leading term in our lower estimate of the gap between the essential spectrum threshold and the lowest Dirichlet eigenvalue is independent of the strip width. This is in accordance with the perturbation expansion of the ground-state eigenvalue derived in [14, Thm. 5.1]:

$$E_1^{D} - \inf(H^{D}) = -\lambda(k) + \mathcal{O}(d)\,.$$

Here $\lambda(k)$ denotes the first (negative) eigenvalue of the one-dimensional Schrödinger operator $l := -\Delta - \frac{1}{4}\,k^2$ on $L^2(\mathbb{R})$ with $\operatorname{Dom} l := W^{2,2}(\mathbb{R})$, which is naturally associated with the problem and reflects the geometry of Γ only. (We remark that, under our assumptions, the operator l has always a negative eigenvalue, *cf* [51, Thm. XIII.11].)

The leading term in the Dirichlet-Neumann estimate tends to $+\infty$ as $d \to 0$ (notice, however, that this fact does not conflict with anything because $E_1^{DN} = \mathcal{O}(d^{-2})$). That is, we again observe the effect of stronger binding of the particle in the case when a Dirichlet boundary curve of the strip is replaced by the Neumann one. A similar asymptotic estimate can be also deduced directly from the crude bound (ii) of Proposition 6.1. Since no perturbation expansion w.r.t. d for the lowest eigenvalue in the Dirichlet-Neumann case is known yet, we cannot compare our estimate with exact asymptotics. □

7 Conclusions

Motivated by the theory of curved quantum waveguides, we were interested in spectral properties of the Laplace operator in a strip built over an infinite planar curve, see Figure 1, subject to three different types of boundary conditions (Dirichlet, Neumann or a combination of these ones, respectively). We localised the essential spectrum as a set under a very natural and weak condition about vanishing of curvature at infinity only, *cf* Theorem 4.1. We stress that no condition about the decay of derivatives of the curvature was required throughout this paper (the derivatives may not even exist because the reference curve is supposed to be C^2-smooth only). Then we were interested in the geometrically induced spectrum, *i.e.*, the spectrum below the spectral threshold of the corresponding straight strip; we made a survey of known results and established new ones, *cf* Theorems 4.2–4.4. Here the most important progress was achieved in the case of combined Dirichlet-Neumann boundary conditions where we generalised the only one known result of [13] and established two new sufficient conditions which guaranteed the existence of geometrically induced spectrum, *cf* Theorem 4.4. We recall that the geometrically induced spectrum consists of discrete eigenvalues only, whenever the above asymptotic behaviour of curvature holds true. Finally, we established two upper bounds on the infimum of the spectral threshold in a situation when the geometrically induced spectrum is present, *cf* Theorem 4.5. These estimates are new in the theory of curved quantum waveguides and their remarkable behaviour in the limit of mild curvature or small width of the strip was discussed, *cf* Remarks 6.4 and 6.5. Summing up briefly the main contribution of the paper, we gave answers to the two questions formulated in Section 2.

Let us now mention some directions in which the above mentioned results could be strengthened or extended.

In Theorem 4.1, we succeeded to localise the essential spectrum as a set, however, an open problem is to examine its nature. Here a particularly interesting question is whether the curved geometry may produce a singular continuous spectrum.

Theorem 4.3 concerning the existence of geometrically induced spectrum in Dirichlet strips is optimal in the sense that no better result can be achieved without violating the basic hypothesis $\langle \mathsf{H} \rangle$. One is of course tempted to ask which more general regions (than the curved asymptotically straight strips) still possess a non-trivial discrete spectrum. For instance, it is easy to see that the existence result does not change if the boundary of the strip is deformed locally and in such a way that the resulting deformed region lies in the exterior of the strip, *cf* [53], however, more complicated deformations of the boundary represent a difficult problem even in the straight case, [5, 4]. In this context, it is worth to recall that the existence of discrete spectrum in V-shaped waveguides was demonstrated in [32, 2, 7] (the computed bound-state energy has been verified experimentally in a flat electromagnetic waveguide in [6]).

The Neumann case is trivial from the point of view of the existence of discrete spectrum in asymptotically straight strips, *cf* Theorem 4.2. As for the Dirichlet-Neumann strip, while our Theorem 4.4 covers various wide classes of geometries for which the geometrically induced spectrum exists, it does not represent an ultimate result. For instance, it remains to be clarified whether one can include also some thick strips with a positive total bending angle. Another open question concerning the strips with combined boundary condition is the study of the behaviour of eigenvalues in mildly curved, respectively thin, strips, *cf* Remarks 6.4 and 6.5.

The upper bounds on the spectral threshold we presented in Theorem 4.5 can be surely improved. First of all, one should include the situations when the total bending angle is equal to zero and/or the strip is curved globally.

As we have already mentioned in Introduction, the Dirichlet Laplacian in the curved strip represents a reasonable model for a quantum Hamiltonian of a particle restricted to move in a strip-like nanostructure. Assuming that the boundary is sufficiently regular, to impose the Dirichlet boundary conditions means to require the vanishing of wavefunctions, however, as pointed out in [38], this may be in general too restrictive and one should rather require the vanishing of the probability current only. The latter leads in our case to a general boundary condition of the type

$$a_0\psi(\cdot,0)+b_0\psi_{,2}(\cdot,0)=0\,,\qquad a_d\psi(\cdot,d)+b_d\psi_{,2}(\cdot,d)=0\,,$$

where $\psi\in\mathcal{H}$ denotes the wavefunction and $(a_0,a_d),(b_0,b_d)\in\mathbb{R}^2\setminus\{(0,0)\}$. However, at least from the mathematical point of view, it would be interesting to examine the influence of the choice of particular boundary conditions on the spectral properties of the Hamiltonian. Finally, it would be also possible to let the coefficients a_0,a_d,b_0,b_d depend on the longitudinal variable s.

Other obvious extensions are to consider the Laplacian in tubular neighbourhoods of non-compact submanifolds of general Riemannian manifolds. Here the spectral problem has been studied only for Dirichlet tubes, [39, 14], and layers, [16, 8], in $\mathbb{R}^3$, or strips in two-dimensional manifolds, [45]; more general boundary conditions and/or higher-dimensional generalisations are completely missing.

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