Geometric properties of Bernoulli-type minimizers

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Abstract

We consider a Bernoulli-type variational problem and we prove some geometric properties for minimizers, such as: gradient bounds, linear growth from the free boundary, density estimates, uniform convergence of level sets and the existence of plane-like minimizers in periodic media.

1 Introduction

In this paper we study various geometric properties of minimizers of the energy functional

$$\mathcal{J}_\Omega(u) = \int_\Omega \left[A(x,\nabla u) + Q(x)\chi_{(-1,1)}(u)\right] dx, \tag{1.1}$$

where Ω is a (possibly unbounded) domain in $\mathbf{R}^n$. We assume that $A(x,\eta)$, $\eta \in \mathbf{R}^n$, behaves as $|\eta|^p$ for some $1 < p < \infty$ and that $Q(x) > 0$ is uniformly bounded away from zero and infinity (see below for precise hypotheses.)

One can think of the set $\{-1 < u < 1\}$ as a jet of fluid in a certain media and that A and Q reflect the underlying physics and geometry. In this interpretation u is the stream function. In fact, the minimizers satisfy (in a distributional sense) the equation

$$\operatorname{div} a(x,\nabla u) = 0 \quad \text{in } \{-1 < u < 1\}, \tag{1.2}$$

where

$$a(x,\eta) := D_\eta A(x,\eta).$$

Moreover on the *free boundary*

$$\Gamma := \partial\{-1 < u < 1\} \cap \Omega$$

a Bernoulli-type condition (hence the name)

$$b(x,\nabla u) = Q(x)$$

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is satisfied in a certain weak sense, see e.g. [AC81]. Here $b(x,\eta) := \eta \cdot a(x,\eta) - A(x,\eta)$ and behaves as $|\eta|^p$ under conditions we impose on A and a.

The functional $\mathcal{J}$ and its "one-side" analogue (see Lemma 3.3)

$$\mathcal{F}_\Omega(v) = \int_\Omega A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v) \tag{1.3}$$

have been studied earlier in the cases

$$A(x,\eta) = |\eta|^2, \quad A(x,\eta) = a_{ij}(x)\eta_i\eta_j, \quad A(x,\eta) = |\eta|^p$$

respectively by [AC81], [Val01] and [DP03] and we borrow many of the ideas from those papers. We prove gradient bounds, linear growth from the free boundary, density estimates, uniform convergence of level sets to a set of minimal "perimeter" (see [Bou90]) and the existence of plane-like minimizers in periodic media. For the latter two results, we were influenced mostly by [CC95] and [CdlL01].

We now state explicitly the hypotheses we assume on our functional.

(H1) $A \in C^1(\mathbf{R}^n \times \mathbf{R}^n)$, $a \in C(\mathbf{R}^n \times \mathbf{R}^n) \cap C^1(\mathbf{R}^n \times \mathbf{R}^n - \{0\})$; $A(x,0) = 0$, $a(x,0) = 0$;

(H2) $\zeta \cdot D_\eta a(x,\eta)\,\zeta \geq \lambda\,|\zeta|^2\,|\eta|^{p-2}$, for any $\zeta \in \mathbf{R}^n$; Particularly, $A(x,\eta)$ is convex in η;

(H3) $|D_\eta a(x,\eta)| \leq \Lambda\,|\eta|^{p-2}$;

(H4) $|D_x a(x,\eta)| \leq \Lambda\,|\eta|^{p-1}$;

(H5) $|a(x,\eta)| \leq \Lambda\,|\eta|^{p-1}$;

(H6) $\eta \cdot a(x,\eta) \geq \lambda\,|\eta|^p$;

(H7) $0 < Q_{\min} \leq Q(x) \leq Q_{\max}$.

Here above and in the sequel we assume $\lambda \leq \Lambda$ and $Q_{\min} \leq Q_{\max}$ to be strictly positive constants. We will refer to λ, Λ, $Q_{\min}$, $Q_{\max}$, p and n as *structural constants.* Quantities depending only on them will be referred as *universal constants.* Hypotheses **(H1)**–**(H5)** are needed in order to apply the interior regularity results of [Tol84]. Under hypothesis **(H6)**, we are also able to employ the Harnack inequality of [Tru81].

Also, when dealing with global geometric results, we will consider periodic media, i.e. we will require the following condition:

(H8) $A(x,\eta) = A(x+k,\eta)$ and $Q(x) = Q(x+k)$ for any $k \in \mathbf{Z}^n$.

In what follows, we will take $u \in L^1_{\mathrm{loc}}$ with $\nabla u \in L^p_{\mathrm{loc}}$ and we will consider several types of minimizers for the functional $\mathcal{J}$. Namely, if $K \subset \mathbf{R}^n$ is a compact domain, we say that u is an *absolute minimizer* for $\mathcal{J}$ in K if $\mathcal{J}_K(u) \leq \mathcal{J}_K(u+\varphi)$, for any $\varphi \in C^\infty_0(K)$.

We say that a function u in $\mathbf{R}^n$ is a *class A minimizer* if $\mathcal{J}_K(u) \le \mathcal{J}_K(u+\varphi)$, for any compact domain $K \subset \mathbf{R}^n$ and for any $\varphi \in C_0^\infty(K)$; i.e., a class A minimizer is an absolute minimizer in any compact domain.

Also, given a domain Ω in $\mathbf{R}^n$, a subset $\Sigma \subset \partial\Omega$ and a function $-1 \le u_0 \le 1$ we consider the class of functions $\mathcal{X} = \mathcal{X}_{\Omega,\Sigma,u_0}$ that are equal to u_0 on Σ, see (3.1) below. We say that u is a *constrained minimizer* if $\mathcal{J}_\Omega(u) \le \mathcal{J}_\Omega(v)$, for any $v \in \mathcal{X}$.

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2 Main results

Below we state the main results of this paper.

As a first result, we prove uniform gradient bounds and linear growth from the free boundary for minimizers of (1.3).

Theorem 2.1 *Assume hypotheses **(H1)**–**(H7)**. Let v be a constrained minimizer for $\mathcal{F}$ in a domain Ω. Then v is Lipschitz continuous. Moreover, given $K \subset\subset \Omega$, the Lipschitz norm of v on K depends only on the structural constants and the distance between K and $\partial\Omega$. Also, if $B_r(x) \subset K \cap \{v > 0\}$ touches $\partial\{v > 0\}$, then*

$$c\, r \le v(x) \le C\, r,$$

for suitable positive universal constants c and C.

This result is an extension of [AC81], [Val01] and [DP03], which has considered, respectively, the cases $A(x,\eta) = |\eta|^2$, $A(x,\eta) = a_{ij}(x)\eta_i\eta_j$ and $A(x,\eta) = |\eta|^p$.

Next, we prove the following density estimates for the level sets of minimizers:

Theorem 2.2 *Assume hypotheses **(H1)**–**(H7)** hold. Let u be a constrained minimizer for $\mathcal{J}$ in a domain Ω and $x \in \{-1 < u < 1\}$. Then, for fixed $r_1 > 0$, there exist positive r_0, c and C depending only on the structural constants and on r_1 such that:*

$$c\, r^{n-1} \le \mathcal{J}_{B_r(x)}(u) \le C\, r^{n-1},$$
$$\mathcal{L}^n\big(B_r(x) \cap \{u > -1\}\big) \ge c\, r^n \quad \text{and} \quad \mathcal{L}^n\big(B_r(x) \cap \{u < 1\}\big) \ge c\, r^n,$$

for any $r \ge r_0$ provided $B_{r+r_1}(x) \subset\subset \Omega$.

Analogous density estimates for $A(x, \eta) = |\eta|^2$ and $A(x, \eta) = a_{ij}(x)\eta_i\eta_j$ have been dealt with in [CC95] and [Val01], respectively.

By Γ-convergence methods, it has been proved in [Bou90] that minimizers u_ε of

$$\mathcal{J}^\varepsilon(u) = \int A(x, \varepsilon \nabla u) + Q(x)\chi_{(-1,1)}(u) \tag{2.1}$$

converge in L^1_{loc} to a function u_0 which has a minimal "interface" with respect to some weighted area. Indeed, a consequence of the above density estimates is that level sets converge in L^∞_{loc}:

Theorem 2.3 *Assume hypotheses **(H1)**–**(H7)**. Let u_ε be an absolute minimizer of (2.1) in a bounded domain D. Assume that, as $\varepsilon \to 0$, u_ε converges in L^1_{loc} to*

$$u_0 := \chi_E - \chi_{D-E}$$

for a suitable $E \subset D$. Then, $\{|u_\varepsilon| < 1\}$ converges locally uniformly to ∂E.

The latter convergence is understood in the sense that $\operatorname{dist}(x, \partial E) \to 0$ uniformly for $x \in \{|u_\varepsilon| < 1\} \cap K$ for any $K \subset\subset D$.

Finally, we study the minimizers of $\mathcal{J}$ in periodic media. We prove the existence of class A minimizers in periodic media constrained in a strip of universal width. Also, we show that any periodic minimizer constrained in a strip wide enough is indeed unconstrained and class A. More precisely, we prove:

Theorem 2.4 *Assume hypotheses **(H1)**–**(H8)**. There exists a positive universal constant M_0, such that:*

(T1) *Given any $\omega \in \mathbf{R}^n - \{0\}$, there exists a class A minimizer $u = u_\omega$ for the functional $\mathcal{J}$ for which the set $\{|u| < 1\}$ is constrained in the strip $\{x \cdot \omega \in [0, M_0|\omega|]\}$.*

Furthermore, such u enjoys the following property of "quasi-periodicity":

- *if $\omega \in \mathbf{Q}^n - \{0\}$, then u is periodic (with respect to the identification induced by ω);*
- *if $\omega \in \mathbf{R}^n - \mathbf{Q}^n$, then u can be approximated uniformly on compact sets by periodic class A minimizers.*

(T2) *If $\omega \in \mathbf{Q}^n - \{0\}$ and v is any ω-periodic minimizer of $\mathcal{J}$ constrained in the strip $S^\omega_M := \{x \cdot \omega \in [0, M|\omega|]\}$, $M \geq M_0$, then it is also an unconstrained and class A minimizer.*

Notice that M_0 above is *independent* on the frequency ω.

The paper is organized as follows. In Section 3, we discuss existence of constrained minimizers and in Section 4 we prove optimal Lipschitz regularity (i.e., the first part of Theorem 2.1). Section 5 deals with the linear growth from the free boundary (i.e., the second part of Theorem 2.1). The density estimates and the uniform convergence of level sets will be dealt with in Section 6, where we also prove Theorems 2.2 and 2.3. In Section 7 we recall a geometric result, which will be of use in Section 8, where the proof of Theorem 2.4 will be carried out. The Appendix collects some utilities. The results in Sections 3–7 will make use only of hypotheses **(H1)**–**(H7)**, while in Section 8 will also make use of hypothesis **(H8)**.

3 Existence

In this section we show the existence and recall some properties of the constrained minimizers of the functional

$$\mathcal{J}_\Omega(u) = \int_\Omega A(x, \nabla u) + Q(x)\chi_{(-1,1)}(u)$$

over the class of admissible functions

$$\mathcal{X}_\Omega = \mathcal{X}_{\Omega,\Sigma,u_0} := \{u \in L^1_{\mathrm{loc}}(\Omega) : \nabla u \in L^p(\Omega) \ \text{ and } \ u = u_0 \ \text{ on } \ \Sigma\}. \tag{3.1}$$

Here Ω is a (possibly unbounded) domain in $\mathbf{R}^n$ with Lipschitz regular boundary $\partial\Omega$, Σ is a given measurable subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Sigma) > 0$ and u_0 is a given function with

$$-1 \le u_0 \le 1, \quad u_0 \in L^1_{\mathrm{loc}}(\Omega), \quad \nabla u_0 \in L^p(\Omega).$$

We start by discussing the known existence and regularity of such constrained minimizers.

Proposition 3.1 *(i) Suppose that $\mathcal{J}_\Omega(u_1) < \infty$ for some $u_1 \in \mathcal{X}_\Omega$. Then there exists a constrained minimizer of $\mathcal{J}_\Omega$ in class $\mathcal{X}_\Omega$. (ii) Every constrained minimizer u is Hölder continuous in the interior of Ω. Furthermore, given $K \subset\subset \Omega$, the Hölder exponent $0 < \alpha < 1$ and the C^α norm of u on K depend only on structural constants and the distance between K and $\partial\Omega$.*

Proof. (i) The existence is obtained by a direct method of Calculus of Variations. Our proof below is the same as in [AC81], Proposition 1.3 (see also [Val01], Theorem 9.1).

Let u_k be a minimizing sequence for $\mathcal{J}_\Omega$ in the sense that

$$\mathcal{J}_\Omega(u_k) \to \inf_{\mathcal{X}_\Omega} \mathcal{J}_\Omega.$$

Then, from **(H1)**, **(H2)** and **(H6)** it follows that

$$A(x, \eta) \ge \lambda |\eta|^p,$$

hence, we will have that the sequence ∇u_k is uniformly bounded in $L^p(\Omega)$. Moreover, we will also have that $u_k - u_0$ is uniformly bounded in $L^p(\Omega \cap B_R)$

(since $\mathcal{H}^{n-1}(\Sigma) > 0$) for any large R. Thus we can extract a subsequence (still denoted by u_k) such that

$$\begin{aligned} \nabla u_k &\rightharpoonup \nabla u \qquad \text{weakly in } L^p(\Omega) \\ u_k &\to u \qquad \text{a.e. in } \Omega. \end{aligned}$$

Since we assume $A(x,\eta)$ is convex in η we have (see, for instance, Theorem 1.2 in [Dac89])

$$\int_{\Omega\cap B_R} A(x,\nabla u) \le \liminf_{k\to\infty} \int_{\Omega\cap B_R} A(x,\nabla u_k).$$

Moreover, the pointwise convergence implies

$$\int_{\Omega\cap B_R} Q(x)\chi_{(-1,1)}(u) \le \liminf_{k\to\infty} \int_{\Omega\cap B_R} Q(x)\chi_{(-1,1)}(u_k),$$

since $\chi_{(-1,1)}$ is lower semicontinuous, and letting $R \to \infty$ we obtain that

$$\mathcal{J}_\Omega(u) \le \liminf_{k\to\infty} \mathcal{J}_\Omega(u_k).$$

(ii) The Hölder regularity of minimizers follows from a theorem of Giaquinta and Giusti [GG82]. □

Since functional $\mathcal{J}_\Omega$ is not convex, there could be more than one minimizer. We will denote by $\mathcal{M}_\Omega$ the class of all constrained minimizers in $\mathcal{X}_\Omega$.

Observe next, that for any $u \in \mathcal{M}_\Omega$

$$-1 \le u \le 1 \quad \text{a.e. in} \quad \Omega.$$

Indeed, otherwise we would have that $\max(\min(u,1),-1)$ has less energy.

Now we turn to the PDE properties of minimizers.

Proposition 3.2 *Let $u \in \mathcal{M}_\Omega$. Then in the week (distributional) sense*

$$\operatorname{div} a(x,\nabla u) = 0 \quad \textit{in } \{-1 < u < 1\}. \tag{3.2}$$

Moreover, on the free boundary $\Gamma = \partial\{-1 < u < 1\} \cap \Omega$ *a Bernoulli-type condition*

$$b(x,\nabla u) = Q(x) \tag{3.3}$$

is satisfied in a certain week sense, where $b(x,\eta) := a(x,\eta)\cdot\eta - A(x,\eta)$.

Observe that the structural conditions imply that $c|\eta|^p \le b(x,\eta) \le C|\eta|^p$.

Proof. We leave the standard proof of (3.2) to the reader. As for the interpretation of the condition (3.3) we refer to [AC81]. □

The free boundary Γ in the proposition above consists of two parts

$$\Gamma_i = \partial\{-1 < u < 1\} \cap \{u = i\}, \quad i = -1, 1,$$

that are well separated on every compact subset of Ω, as it follows from Hölder regularity of u. We find the following simple observation useful, and use it extensively throughout the paper.

Lemma 3.3 *Suppose that in a subdomain $D \subset\subset \Omega$ the minimizer u does not take the value 1 (respectively -1). Then the function $v = u + 1$ (respectively $v = 1 - u$) is an absolute minimizer of the functional*

$$\mathcal{F}_D(v) = \int_D A(x, \nabla v) + Q(x)\chi_{(0,\infty)}(v). \tag{3.4}$$

We conclude this section with a remark about rescalings.

We will use two kinds of rescalings. The first kind preserves the gradient and is suitable for proving the Lipschitz regularity of minimizers. It is better formulated in the terms of functional $\mathcal{F}$. Namely, let v be a minimizer of $\mathcal{F}_\Omega$ and $x_0 \in \mathbf{R}^n$. Then the function

$$v_\varepsilon(x) = \frac{1}{\varepsilon}\, v(x_0 + \varepsilon x)$$

is a minimizer of

$$\int_{\Omega_{1/\varepsilon}} A_\varepsilon(x, \nabla v) + Q_\varepsilon(x)\chi_{(0,\infty)}(v),$$

where $\Omega_{1/\varepsilon} = \Omega_{x_0,1/\varepsilon} = \{x : x_0 + \varepsilon x \in \Omega\}$ and

$$A_\varepsilon(x, \eta) = A(x_0 + \varepsilon x, \eta), \quad Q_\varepsilon(x) = Q(x_0 + \varepsilon x). \tag{3.5}$$

Note that A_ε and Q_ε satisfy the same (if not better) structural conditions as A and Q.

The second kind of rescaling preserves the value of the function. We will need it in the proof of Theorem 2.3, see the end of Section 6. We state it in the terms of the functional $\mathcal{J}$. Namely, if u_ε is a minimizer of

$$\int_\Omega A(x, \varepsilon\, \nabla u) + Q(x)\chi_{(-1,1)}(u)$$

Then the function

$$w_\varepsilon(x) = u_\varepsilon(x_0 + \varepsilon x) \tag{3.6}$$

is a minimizer of

$$\int_{\Omega_{1/\varepsilon}} A_\varepsilon(x, \nabla w) + Q_\varepsilon(x)\chi_{(-1,1)}(w),$$

with A_ε and Q_ε as in (3.5).

4 Lipschitz regularity

In this section we prove that every constrained minimizer u of $\mathcal{J}$ is in fact Lipschitz.

Theorem 4.1 *Let $u \in \mathcal{M}_\Omega$. Then u is Lipschitz continuous. Moreover, given $K \subset\subset \Omega$, the Lipschitz norm of u on K depends only on the structural constants and the distance between K and $\partial\Omega$.*

The idea is to show that u grows at most linearly away from the free boundary

$$\Gamma = \partial\{-1 < u < 1\}.$$

The Hölder regularity of u implies that the two parts of the free boundary, $\Gamma_{\pm 1}$ are well separated on K in the sense that

$$\operatorname{dist}(\Gamma_{-1} \cap K, \Gamma_1 \cap K) \geq \delta_0 > 0.$$

Let now $x_0 \in \Gamma \cap K$. Without loss of generality we may assume that $x_0 \in \Gamma_{-1} \cap K$. Then for $\varepsilon < \min(\delta_0, \operatorname{dist}(K, \partial\Omega))$, the ball $B_\varepsilon(x_0)$ is contained Ω and does not intersect Γ_1. Hence the function $v = u + 1$ will be a minimizer of $\mathcal{F}_{B_\varepsilon(x_0)}$, as defined in Lemma 3.3.

The next lemma is the main step in establishing Theorem 4.1. The proof is based on that of Lemma 3.2 in [DP03].

Lemma 4.2 *Let v be a bounded absolute minimizer of $\mathcal{F}_{B_\varepsilon(x_0)}$, with $v(x_0) = 0$. Then*

$$\sup_{B_{\varepsilon/4}(x_0)} v \leq C\,\varepsilon, \tag{4.1}$$

and

$$|\nabla v(x_0)| \leq C, \tag{4.2}$$

where $C > 0$ is a universal constant.

Proof. Observe that (4.2) is a consequence of (4.1) and the interior gradient estimates, see e.g. [Tol84]. Moreover, by the rescaling arguments in the end of the previous section, it is enough to consider the case $\varepsilon = 1$ and $x_0 = 0$ to prove (4.1).

Next we argue by contradiction. Indeed, assume that the estimate (4.1) is false for $\varepsilon = 1$. Then there exists a sequence of bounded absolute minimizers v_k of functionals $\mathcal{F}^k(v) = \int A^k(x, \nabla v) + Q^k(x)\chi_{(0,\infty)}(v)$ in B_1, $k = 1, 2, \ldots$, such that

$$\max_{\overline{B}_{1/4}} v_k(x) > k.$$

Set

$$d_k(x) = \operatorname{dist}(x, \{v_k = 0\}) \quad \text{in } B_1$$

and define

$$\mathcal{O}_k = \{x \in B_1 : d_k(x) \leq (1 - |x|)/3\}.$$

Observe that $\overline{B}_{1/4} \subset \mathcal{O}_k$. In particular

$$m_k := \sup_{\mathcal{O}_k} (1 - |x|)\, v_k(x) \geq \frac{3}{4} \max_{\overline{B}_{1/4}} v_k(x) > \frac{3}{4}\, k.$$

Since $v_k(x)$ is bounded (for fixed k), we will have $(1 - |x|)\, v_k(x) \to 0$ as $|x| \to 1$, and therefore m_k will be attained at some point $x_k \in \mathcal{O}_k$:

$$(1 - |x_k|)\, v_k(x_k) = \max_{\mathcal{O}_k} (1 - |x|)\, v_k(x). \tag{4.3}$$

Clearly,

$$v_k(x_k) = \frac{m_k}{1-|x_k|} \geq m_k > \frac{3}{4}\, k.$$

Since $x_k \in \mathcal{O}_k$, by the definition we will have

$$d_k := d_k(x_k) \leq (1-|x_k|)/3. \tag{4.4}$$

Let now $y_k \in \partial\{v_k > 0\} \cap B_1$ be such that

$$|y_k - x_k| = d_k. \tag{4.5}$$

The two inclusions

$$B_{2d_k}(y_k) \subset B_1 \qquad \text{and} \qquad B_{d_k/2}(y_k) \subset \mathcal{O}_k,$$

both follow from (4.4)–(4.5). In particular, for $z \in B_{d_k/2}(y_k)$

$$(1-|z|) \geq (1-|x_k|) - |x_k - z| \geq (1-|x_k|) - (3/2)\, d_k \geq (1-|x_k|)/2.$$

This, in conjunction with (4.3), implies that

$$\max_{\overline{B}_{d_k/2}(y_k)} v_k \leq 2\, v_k(x_k).$$

Next, since $B_{d_k}(x_k) \subset \{v_k > 0\}$, v_k satisfies

$$\operatorname{div} a^k(x, \nabla v_k) = 0 \quad \text{in } B_{d_k}(x_k)$$

with $a^k = D_\eta A^k$. By the Harnack inequality, see Theorem 4.1–4.3 in [Tru81], there is a constant $c > 0$ such that

$$\min_{\overline{B}_{3d_k/4}(x_k)} v_k \geq c\, v_k(x_k).$$

In particular,

$$\max_{\overline{B}_{d_k/4}(y_k)} v_k \geq c\, v_k(x_k).$$

Consider now the rescalings

$$w_k(x) = \frac{v_k(y_k + (d_k/2)\, x)}{v_k(x_k)}.$$

¿From the properties of v_k above, we obtain

$$\max_{\overline{B}_1} w_k \leq 2, \quad \max_{\overline{B}_{1/2}} w_k \geq c > 0, \quad w_k(0) = 0. \tag{4.6}$$

We will also have that w_k is an absolute minimizer of

$$\tilde{\mathcal{F}}^k(w) = \int \tilde{A}^k(x, \nabla w) + \tilde{Q}^k(x)\chi_{(0,\infty)}(w),$$

in B_1 with

$$\begin{aligned}\tilde{A}^k(x,\eta) &= \varepsilon_k^p A^k(y_k + (d_k/2)\,x, \eta/\varepsilon_k)\\ \tilde{Q}^k(x,\mu) &= \varepsilon_k^p Q^k(y_k + (d_k/2)\,x), \qquad \varepsilon_k = d_k/(2v_k(x_k))\end{aligned}$$

Then over a subsequence

$$\tilde{a}^k(x,\eta) \to a^0(\eta), \quad x \in B_1$$

uniformly for η in compact subsets of $\mathbf{R}^n$, where $a^k = D_\eta A^k$ and a^0 satisfies the same structural conditions as a^k. Besides, we have

$$0 \le Q^k(x) \le Q_{\max}\,\varepsilon_k^p \to 0.$$

Let now v_k be such that $v_k - w_k \in W_0^{1,p}(B_{3/4})$ and $\operatorname{div} a^k(x, \nabla v_k) = 0$ in $B_{3/4}$, where $a^k(x,\eta) = D_h A^k(x,\eta)$. Then from Lemma A.1 in Appendix we have

$$\|\nabla(w_k - v_k)\|_{L^p(B_{3/4})} \le C(\varepsilon_k) \to 0. \tag{4.7}$$

Besides, from a theorem of Giaquinta and Giusti [GG82], w_k and v_k are uniformly C^α in $B_{5/8}$. Thus we can extract subsequences (still denoted by w_k and v_k) such that $w_k \to w_0$ and $v_k \to v_0$ in uniformly on $B_{5/8}$. On the other hand v_k satisfy uniform $C^{1,\alpha}$ estimates on $B_{5/8}$, see [Tol84]. Hence we can assume that the convergence $v_k \to v_0$ is actually in $C^{1,\beta}$ on $B_{5/8}$ and v_0 solves

$$\operatorname{div} a^0(\nabla v_0) = 0 \quad \text{in } B_{5/8}$$

We next observe that (4.7) implies that $w_0 = v_0 + c$ in $B_{5/8}$. Hence also

$$\operatorname{div} a^0(\nabla w_0) = 0 \quad \text{in } B_{5/8}.$$

Since $w_0 \ge 0$ and $w_0(0) = 0$, from Harnack inequality, see [Tru81], we obtain that $w_0 = 0$ in $B_{5/8}$. On the other hand, (4.6) implies

$$\max_{\overline{B}_{1/2}} w_0 \ge c > 0,$$

which is a contradiction.

The lemma is proved. □

Proof of Theorem 4.1. Lemma 4.2 implies that $|\nabla u|$ is bounded on $\Gamma \cap K$ and we need to prove the boundedness also in $\{-1 < u < 1\} \cap K$. So, fix a point $x_0 \in K$ with $-1 < u(x_0) < 1$ and consider the following two cases:

1. $\operatorname{dist}(x_0, \Gamma) > \delta/2$,
2. $\operatorname{dist}(x_0, \Gamma) \le \delta/2$,

where

$$\delta = \operatorname{dist}(K, \partial\Omega) > 0.$$

In the former case we have $|\nabla u(x_0)| \le C/\delta$ by the interior gradient estimates (see [Tol84]) and in the latter case we argue as follows. Consider the ball $B_\varepsilon(x_0)$ with $\varepsilon = \operatorname{dist}(x_0, \Gamma)$ and let $y_0 \in \Gamma \cap \partial B_\varepsilon(x_0)$. Then $\operatorname{dist}(y_0, \partial\Omega) \ge \delta/2$ and by Lemma 4.2 we have $|\nabla u(y_0)| \le C$. Without loss of generality we may assume also that $y_0 \in \Gamma_{-1}$. We claim then

$$c_0 := \inf_{B_{\kappa\varepsilon}(x_0)} (u+1) \le C\,\varepsilon$$

where the constant $0 < \kappa < 0$ will be specified below and will depend only on the structural constants of A. Define

$$\psi_\kappa(s) = s + C_\kappa s^2, \quad C_\kappa = \kappa/(1-\kappa)^2.$$

Then ψ_κ has the following properties:

$$\psi_\kappa(0) = 0, \quad \psi_\kappa(1-\kappa) = 1, \quad \psi_\kappa(0) = 1, \quad \psi_k''(s) = 2C_\kappa.$$

Consider now the function

$$\phi(x) = c_0\, \psi_\kappa(1 - |x - x_0|/\varepsilon) \quad \text{in } U_\kappa := B_\varepsilon(x_0) \setminus B_{\kappa\varepsilon}(x_0).$$

Then, using the strict convexity of ψ_κ, **(H2)** and **(H4)**, we will have

$$\operatorname{div} a(x, \nabla\phi) \ge 0 \quad \text{in } U_\kappa$$

for $\kappa_0 < \kappa < 1$ depending only on the structural constants. Besides, $\phi \le u + 1$ of ∂U_κ, hence by the standard comparison principle $\phi \le u+1$ in U_κ. Since also $\phi(y_0) = u(y_0) + 1 = 0$ we obtain that

$$|\nabla\phi(x_0)| \le |\nabla u(y_0)|,$$

hence

$$c_0\, \psi_\kappa'(0)/\varepsilon \le C$$

and consequently

$$c_0 \le C\,\varepsilon.$$

Finally, we observe that by the Harnack inequality (see [Tru81]) we also have

$$\sup_{B_{\kappa\varepsilon}(x_0)} (u+1) \le C \inf_{B_{\kappa\varepsilon}(x_0)} (u+1) \le C\,\varepsilon.$$

Then applying the interior gradient estimate, we conclude the proof of the theorem. □

5 Nondegeneracy

As we have seen in the previous section, a minimizer $u \in \mathcal{M}_\Omega$ grows at most linearly near the free boundary $\Gamma = \partial\{-1 < u < 1\}$. In this section we show that the linear growth is the optimal one.

In order to study the growth of u near Γ_{-1} (or Γ_1) we fix a domain $D \subset\subset \Omega$ where u does not take value 1 (respectively -1) in D. Then the function $v = u+1$ (respectively $v = 1 - u$) is an absolute minimizer of the functional $\mathcal{F}_D$.

As a simple corollary from Theorem 4.1 we obtain the following statement.

Lemma 5.1 *There exists a constant C depending only on structural constants and* $\operatorname{dist}(D, \partial\Omega)$ *such that for any absolute minimizer v of $\mathcal{F}_D$*

$$\frac{1}{r} \fint_{\partial B_r} v > C \quad \textit{implies} \quad v > 0 \quad \textit{in } B_r$$

for any ball $B_r \subset D$.

Proof. Indeed, otherwise we will have that $B_r \subset D$ contains a free boundary point and therefore, by Theorem 4.1, $u \le Cr$ on ∂B_r, a contradiction. □

Next we prove a key nondegeneracy lemma.

Lemma 5.2 *For any $\gamma > p-1$ and $0 < \kappa < 1$ there exists $c > 0$ depending only on κ, γ and the structural constants such that for every absolute minimizer v of $\mathcal{F}_D$ and for any ball $B_r \subset D$*

$$\frac{1}{r} \left(\fint_{B_r} v^\gamma \right)^{1/\gamma} < c \quad \textit{implies} \quad v = 0 \quad \textit{in } B_{\kappa r}.$$

In particular, if for $B_r \subset D$ we have $B_{\kappa r} \cap \partial\{v > 0\} \neq \emptyset$, then

$$\sup_{\partial B_r} v \ge cr.$$

Proof. Without loss of generality we may consider the case $r = 1$. Set

$$\varepsilon = \sup_{B_{\sqrt{\kappa}}} v.$$

By the Harnack inequality (see [Tru81], Theorem 4.1–4.3)

$$\varepsilon \le C \left(\fint_{B_1} v^\gamma \right)^{1/\gamma}.$$

Hence, if c is small, so is ε. Let $\phi(x) = \phi_\kappa(x)$ be the solution of

$$\operatorname{div} a(x, \nabla\phi) = 0 \ \text{ in } B_{\sqrt{\kappa}} - B_\kappa, \quad \phi = 0 \ \text{ on } \partial B_\kappa, \quad \phi = 1 \ \text{ on } \partial B_{\sqrt{\kappa}}$$

and put $\phi = 0$ on B_κ. Set $h = \varepsilon\,\phi$ in $B_{\sqrt{\kappa}}$. Then $h \geq v$ on $\partial B_{\sqrt{\kappa}}$ and $v^* = \min(v,h)$ is an admissible function. Therefore $\mathcal{F}(v) \leq \mathcal{F}(v^*)$, or equivalently

$$\int_{B_{\sqrt{\kappa}}} A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v) \leq \int_{B_{\sqrt{\kappa}}-B_\kappa} A(x,\nabla v^*) + Q(x)\chi_{(0,\infty)}(v^*)$$

Hence

$$\begin{aligned}
\int_{B_\kappa} A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v) &\leq \int_{B_{\sqrt{\kappa}}-B_\kappa} A(x,\nabla v^*) - A(x,\nabla v)\\
&\leq \int_{B_{\sqrt{\kappa}}-B_\kappa} a(x,\nabla v^*)\cdot\nabla(v^*-v)\\
&= \int_{(B_{\sqrt{\kappa}}-B_\kappa)\cap\{v>v^*\}} a(x,\nabla v^*)\cdot\nabla(v^*-v)\\
&= -\int_{\partial B_k} (v^*-v)\,a(x,\nabla v^*)\cdot\nu\\
&= \int_{\partial B_\kappa} v\,a(x,\nabla h)\cdot\nu.
\end{aligned}$$

Here in the second step we have used the convexity of $A(x,\eta)$ in η, see **(H2)**. Since also $|\nabla h| \leq C\varepsilon$ on ∂B_κ we find that

$$\int_{B_\kappa} A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v) \leq C\varepsilon^{p-1}\int_{\partial B_\kappa} v.$$

On the other hand, using the Fundamental Theorem of Calculus, polar coordinates and then Young inequality,

$$\begin{aligned}
\int_{\partial B_\kappa} v &\leq C\left(\int_{B_\kappa} v + \int_{B_\kappa}|\nabla v|\right)\\
&\leq C\left(\int_{B_\kappa} \varepsilon\,Q(x)\chi_{(0,\infty)}(v) + \int_{B_\kappa} A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v)\right)\\
&\leq C(1+\varepsilon)\int_{B_\kappa} A(x,\nabla v) + Q(x)\chi_{(0,\infty)}(v).
\end{aligned}$$

Therefore, if ε is small enough, we obtain $v = 0$ in B_κ. □

In light of the result above and Theorem 4.1, we infer the following:

Corollary 5.3 *There exist constants c, $C > 0$ such that if $B_r(x) \subset \{v>0\}\cap D$ touches $\partial\{v>0\}\cap D$ then*

$$cr \leq v(x) \leq Cr.$$

□

Theorem 2.1 follows now from this and Theorem 4.1.

6 Density estimates

The purpose of this section is to show that the jet $\{-1 < u < 1\}$ has a bounded width in the sense that $B_r \cap \{-1 < u < 1\}$ has a Lebesgue measure proportional to r^{n-1} for large r. On the other hand, the sets $B_r \cap \{u = \pm 1\}$ are thick and have Lebesgue measure proportional to r^n.

Lemma 6.1 *Let $u \in \mathcal{M}_\Omega$. Then, for fixed $r_1 > 0$, there exist constants r_0 and C depending only on structural constants and r_1 such that*

$$\mathcal{J}_{B_r}(u) \leq C\, r^{n-1},$$

for any $r \geq r_0$ provided $B_{r+r_1} \subset \Omega$.

Proof. Let h be a smooth function so that $h\big|_{B_{r-1}} = -1$ and $h\big|_{\partial B_r} = 1$. Let $u^* = \min\{u, h\}$. Then,

$$\begin{aligned} \mathcal{J}_{B_r}(u) &\leq \mathcal{J}_{B_r}(u^*) \leq \\ &\leq C \int_{B_r - B_{r-1}} (|\nabla u|^p + |\nabla h|^p + r^{n-1}) \leq \\ &\leq C \int_{B_r - B_{r-1}} (|\nabla u|^p + r^{n-1}). \end{aligned}$$

Let us cover $B_r - B_{r-1}$ with balls of radius $1 + r_1/2$, $\mathcal{B}_1, \ldots, \mathcal{B}_N$, with $N \leq C\, r^{n-1}$. Then, by a Caccioppoli type inequality, see [HKM93], we infer that

$$\begin{aligned} \int_{B_r - B_{r-1}} |\nabla u|^p &\leq \sum_{i=1}^{N} \int_{\mathcal{B}_i} |\nabla u|^p \leq \\ & C \sum_{i=1}^{N} \int_{2\mathcal{B}_i} |u|^p \leq C\, r^{n-1}. \end{aligned}$$

□

Lemma 6.2 *Let $u \in \mathcal{M}_\Omega$. Fix $\theta_0 \in [0, 1)$ and let $\theta \in [-\theta_0, \theta_0]$. Assume that*

$$\mathcal{L}^n\big(\{u > \theta\} \cap B_\rho\big) \geq \mu_0 > 0.$$

Then, for fixed $r_1 > 0$, there exist positive constants c and r_0, depend only on r_1, θ_0, ρ and μ_0 and the structural constants such that

$$\mathcal{L}^n\big(\{u > \theta\} \cap B_r\big) \geq c\, r^n,$$

for any $r \geq r_0$ provided $B_{r+r_1} \subset \Omega$.

An analogous statement holds for the sub-level set $\{u < \theta\}$.

Proof. Let h be a smooth function so that $h\big|_{B_{r-1}} = -1$ and $h\big|_{\partial B_r} = 1$. Let $u^* = \min\{u, h\}$ and $\beta = \min\{u - u^*, 1 + \theta\}$. Define also

$$\mathcal{V}(r) = \mathcal{L}^n\big(\{u > \theta\} \cap B_r\big), \quad \text{and} \quad \mathcal{A}(r) = \mathcal{L}^n\left(\{-1 < u \leq \theta\} \cap B_r\right).$$

We would like to proof that

$$[\mathcal{V}(r-1)]^{(n-1)/n} + \mathcal{A}(r-1) \le C\big(\mathcal{V}(r) - \mathcal{V}(r-1) + \mathcal{A}(r) - \mathcal{A}(r-1)\big). \quad (6.1)$$

Indeed, from (6.1) and Lemma A.2 $\mathcal{V}(\rho+k) + \mathcal{A}(\rho+k) \ge c\,(k+1)^n$, for any natural number k. This and Lemma 6.1 would imply Lemma 6.2.

Let us now address the proof of (6.1). First of all, by definition of β,

$$\int_{B_r} |\beta|^{2n/(n-1)} \ge \int_{B_{r-1}\cap\{u>\theta\}} |\beta|^{2n/(n-1)} = (1+\theta)^{2n/(n-1)}\,\mathcal{L}^n\big(B_{r-1}\cap\{u>\theta\}\big).$$

Therefore, by the Sobolev inequality applied to β^2 and then by the Young inequality with $1/p+1/q=1$ we obtain

$$[\mathcal{V}(r-1)]^{(n-1)/n} \le C\int_{B_r\cap\{u-u^*\le 1+\theta\}} |\beta|\,|\nabla\beta| \le C\left(K\int_{B_r\cap\{u-u^*\le 1+\theta\}} |\nabla(u-u^*)|^p + \frac{1}{K}\int_{B_r\cap\{u-u^*\le 1+\theta\}} |u-u^*|^q\right).$$

Here above, $K>0$ is a free parameter that will be conveniently chosen in what follows. By our hypotheses on the functional and the minimality property of u, we deduce from the above estimate that

$$\begin{aligned}
[\mathcal{V}(r-1)]^{(n-1)/n} &\le C\left(K\int_{B_r} A(x,\nabla(u-u^*)) + \frac{1}{K}\int_{B_r\cap\{u-u^*\le 1+\theta\}} |u-u^*|^q\right)\\
&\le C\left(K\int_{(B_r-B_{r-1})\cap\{u\ne -1\}} [A(x,\nabla(u-u^*)) - A(x,\nabla u) + A(x,\nabla u^*)]\right.\\
&\quad\left. + \int_{B_r} Q(x)\left[\chi_{(-1,1)}(u^*) - \chi_{(-1,1)}(u)\right] + \frac{1}{K}\int_{B_r\cap\{u-u^*\le 1+\theta\}} |u-u^*|^q\right).
\end{aligned}$$

By the our assumptions on the functional and the gradient estimates in Theorem 4.2, we have that

$$\int_{(B_r-B_{r-1})\cap\{u\ne -1\}} [A(x,\nabla(u-u^*)) + A(x,\nabla u^*)] \le C\,\mathcal{L}^n\big((B_r-B_{r-1})\cap\{u\ne -1\}\big).$$

Thus

$$\begin{aligned}
&[\mathcal{V}(r-1)]^{(n-1)/n} \le\\
&\quad C\Big(K\int_{B_r} Q(x)\left[\chi_{(-1,1)}(u^*) - \chi_{(-1,1)}(u)\right] + \frac{1}{K}\int_{B_r\cap\{u-u^*\le 1+\theta\}} |u-u^*|^q\Big)\\
&\quad + C\,\mathcal{L}^n\big((B_r-B_{r-1})\cap\{u\ne -1\}\big).
\end{aligned}$$

To estimate the two remaining terms above, we split the domain of integration. Namely, in B_{r-1}, we have $u^* = -1$ and therefore $\chi_{(-1,1)}(u^*) = 0$. Thus, in B_{r-1} we are left with

$$-K\int_{B_{r-1}} Q(x)\chi_{(-1,1)}(u) + \frac{1}{K}\int_{B_{r-1}\cap\{-1<u\le\theta\}} |u+1|^q \le -\mathcal{A}(r-1),$$

by choosing K suitably big. On the other hand, in $B_r - B_{r-1}$, we are left with the term

$$C\int_{(B_r-B_{r-1})\cap\{u\neq-1\}} \big(Q(x)\left[\chi_{(-1,1)}(u^*) - \chi_{(-1,1)}(u)\right] + |u-u^*|^q\big) \le C\,\mathcal{L}^n\big((B_r - B_{r-1})\cap\{u\neq -1\}\big).$$

By means of these estimates, (6.1) readily follows. □

Lemma 6.3 *Let $u \in \mathcal{M}_\Omega$. Assume that $x \in \{-1 < u < 1\}$. Then, for fixed $r_1 > 0$, there exist positive r_0 depending only on structural constants and r_1 such that u takes both the values 1 and -1 in $B_{r_0}(x)$, provided $B_{r_0+r_1}(x) \subset \Omega$.*

Proof. By Lemma 6.1, u takes either the value -1 or $+1$ in $B_{r_0/2}(x)$; say it takes the value -1. If u does not assume the value 1 in $B_{r_0}(x)$, using the linear growth from the free boundary (see Lemma 5.2), we will have

$$2 \ge \sup_{B_{r_0}(x)} (u+1) \ge c\,r_0 \ge 3,$$

by choosing r_0 big enough, which is a contradiction. □

We are now in position to improve the previous density estimates:

Lemma 6.4 *Let $u \in \mathcal{M}_\Omega$. Assume that $x \in \{-1 < u < 1\}$. Then, for fixed $r_1 > 0$, there exist positive r_0 and c depending only on structural constants and r_1 such that*

$$\mathcal{J}_{B_r(x)}(u) \ge c\,r^{n-1},$$

for any $r \ge r_0$ provided $B_{r+r_1}(x) \subset \Omega$.

Proof. By Lemma 6.3, the uniform continuity of u and Lemma 6.1, we have that

$$\mathcal{L}^n\big(B_r(x)\cap\{u > 9/10\}\big) \ge c\,r^n$$

and

$$\mathcal{L}^n\big(B_r(x)\cap\{u < -9/10\}\big) \ge c\,r^n.$$

Let a be the average of u in $B_r(x)$. If $a \le 0$,

$$\begin{aligned}\int_{B_r(x)} |u-a| \quad &\ge \quad \int_{B_r(x)\cap\{u>9/10\}} (u-a) \ge \frac{9}{10}\,\mathcal{L}^n\big(B_r(x)\cap\{u>9/10\}\big) \ge \\ &\ge \quad c\,r^n.\end{aligned}$$

Analogously, if $a \geq 0$,

$$\int_{B_r(x)} |u-a| \geq \frac{9}{10}\,\mathcal{L}^n\big(B_r(x)\cap\{u<-9/10\}\big) \geq c\,r^n.$$

So, in any case, using Poincaré and Young inequalities:

$$c\,r^{n-1} \quad\leq\quad \frac{1}{r}\int_{B_r(x)} |u-a| \leq C\int_{B_r(x)} |\nabla u| \leq C\,\mathcal{J}_{B_r(x)}(u).$$

□

Let us summarize some of the density estimates obtained in this section:

Theorem 6.5 *Let $u \in \mathcal{M}_\Omega$. Assume that $x \in \{-1<u<1\}$. Then, for fixed $r_1>0$, there exist positive r_0, c and C depending only on structural constants and r_1 such that:*

$$c\,r^{n-1} \leq \mathcal{J}_{B_r(x)}(u) \leq C\,r^{n-1},$$
$$\mathcal{L}^n\big(B_r(x)\cap\{u=-1\}\big) \geq c\,r^n \quad \text{and} \quad \mathcal{L}^n\big(B_r(x)\cap\{u=1\}\big) \geq c\,r^n,$$

for any $r \geq r_0$ provided $B_{r+r_1}(x) \subset \Omega$.

Notice that, from these density estimates, Theorem 2.3 easily follows. Indeed, in the notation of Theorem 2.3, if one assumes the claim to be false, there would be $\delta>0$ and an infinitesimal sequence ε_n for which, say, $B_\delta(x_n) \subset E$, for suitable $x_n \in \{-1<u_{\varepsilon_n}<1\}$. But then, considering $w_\varepsilon(x) := u_\varepsilon(\varepsilon x)$ (see the end of Section 3 for the rescaling argument) and applying the density estimates we would get, after scaling back to u_ε, that

$$\mathcal{L}^n(B_{\delta/2}(x_n)\cap\{u_{\varepsilon_n}=-1\}) \geq c\,\delta^n,$$

for some $c>0$. But then,

$$\int_{B_{\delta/2}(x_n)} |u_{\varepsilon_n}-u_0| \geq 2c\,\delta^n,$$

in contradiction with the hypothesis. This proves Theorem 2.3.

7 Geometric Lemma

We show in this section that for a minimizer u of $\mathcal{J}_\Omega$ the sets $B\cap\{u=\pm1\}$, where B is a large ball centered in the jet $\{-1<u<1\}$, contain balls proportional to B. It is a consequence of the density estimates in Theorem 6.5 and the following result, know as the Geometric Lemma.

Lemma 7.1 (Geometric Lemma) *Let $\beta \subset E \subset \mathbf{R}^n$, μ and ν be Radon measures on $\mathbf{R}^n$. Take $x_0 \in \beta$. Fix $r>0$. Assume that there exist $\delta' \in (0,1/10]$ such that:*

(i) for all $x \in B_r(x_0)$ *and* $\varrho \in [\delta' r, r/10]$,

$$\mu(B_\varrho(x)) \le C\, \varrho^n;$$

(ii) for all $x \in \beta \cap B_r(x_0)$ *and* $\varrho \in [\delta' r, r/10]$

$$\nu(B_\varrho(x)) \ge c\, \varrho^{n-1};$$

(iii) $\mu(E \cap B_r(x_0)) \ge c\, r^n$;

(iv) $\nu(B_{2r}(x_0)) \le C\, r^{n-1}$.

Then, there exists δ*, depending only on* n *and on the constants involved in (i)–(iv) above, so that, if* $\delta' \le \delta$*, we can find* $\bar{x} \in B_r(x_0)$ *such that*

$$B_{\delta r}(\bar{x}) \subset (E - \beta) \cap B_{2r}(x_0). \tag{7.1}$$

Proof. For a complete proof see [Val01], Lemma 6.1. Here we just outline the rough idea of the proof. We first cover β with balls of radius δr: since β behaves like an $(n-1)$-dimensional set, we need $O(1/\delta^{n-1})$ of such balls to cover β. We then extend this cover to the whole set E: since E behaves like an n-dimensional set, we need $O(1/\delta^n)$ of such balls to cover E. Therefore, if δ is small enough, there is at least one ball in the cover of E that does not touch δ. ☐

Following is the application of the Geometric Lemma to our case:

Proposition 7.2 *Let* $u \in \mathcal{M}_\Omega$*. Then, for any fixed* $r_1 > 0$*, there exist positive constants* κ *and* r_0*, depending only on the structural constants and* r_1*, such that for any* $x \in \{-1 < u < 1\}$ *and any* $r \ge r_0$*, there exist* z_1 *and* z_2 *with*

$$B_{\kappa r}(z_1) \subset \{u = 1\} \cap B_r(x), \quad \text{and} \quad B_{\kappa r}(z_2) \subset \{u = -1\} \cap B_r(x),$$

provided $B_{2r+r_1}(x) \subset \Omega$.

The proof of the result above easily follows by applying the Geometric Lemma to the sets $E := \{u > -1\}$, or $E := \{u < 1\}$, and $\beta := \{|u| < 1\}$, with $\mu := \mathcal{L}^n$ and $\nu(B) := \mathcal{J}_B(u)$. Notice that the hypotheses of the Geometric Lemma are fulfilled in light of the density estimates (see Theorem 6.5).

8 Plane-like minimizers

We will now deal with the proof of Theorem 2.4. We consider $\omega \in \mathbf{Q}^n - \{0\}$ (the case $\omega \in \mathbf{R}^n - \mathbf{Q}^n$ will then follow by uniform approximation). We will denote by $\mathcal{M}^\omega_M$ the set of all periodic constrained minimizers in the strip

$$S^\omega_M := \{x \in \mathbf{R}^n : x \cdot \omega \in [0, M|\omega|]\}.$$

When $\omega \in \mathbf{Q}^n$, we can recover compactness by an identification of the space. Namely, we consider the following relation $\sim$ induced by ω:

$$x \sim y \qquad \text{iff} \qquad x - y = k \in \mathbf{Z}^n \quad \text{with} \quad k \cdot \omega = 0 \tag{8.1}$$

We will denote by $\tilde{S}_M^\omega = S_M^\omega / \sim$ the quotient space induced by such relation (which is topologically equivalent to the product of an $(n-1)$-dimensional torus and a real interval).

In what follows we consider the functional $\mathcal{J}$ on functions u periodic with respect to the identification $\sim$ in S_M^ω, and by $\mathcal{J}_{\tilde{S}_M^\omega}$ we understand the integral over a fundamental region with respect to the relation $\sim$.

We also set

$$\begin{aligned}\mathcal{X}_M^\omega := \{&u \in L^1_{\rm loc}(\tilde{S}_M^\omega) : \nabla u \in L^p(\tilde{S}_M^\omega), \\ &u = 1 \text{ on } x \cdot \omega = 0 \quad \text{and} \quad u(x) = -1 \text{ on } x \cdot \omega = M|\omega|\}.\end{aligned} \tag{8.2}$$

In the sequel, we will assume $M \geq 10$, in order to avoid a degeneracy caused by "too thin" strips.

First of all, we notice the following elementary fact: for any u, $v \in L^1_{\rm loc}(\tilde{S}_M^\omega)$ with ∇u, $\nabla v \in L^2(\tilde{S}_M^\omega)$, one has that

$$\mathcal{J}_{S_M^\omega}(u) + \mathcal{J}_{S_M^\omega}(v) = \mathcal{J}_{S_M^\omega}(\min\{u,v\}) + \mathcal{J}_{S_M^\omega}(\max\{u,v\}). \tag{8.3}$$

Therefore, if u, $v \in \mathcal{M}_M^\omega$, then $\min\{u,v\}$, $\max\{u,v\} \in \mathcal{M}_M^\omega$.

Now, we prove a geometric property about the global behavior of the minimizers. If $k \in \mathbf{R}^n$, we denote by $\mathcal{T}_k$ the translation by the vector k. Also, we set $\mathcal{T}_k u(x) := u(x-k)$.

Definition 8.1 *Let $E \subset \mathbf{R}^n$, $\varpi \in \mathbf{R}^n$. We say that E satisfies the Birkhoff property with respect to the vector ϖ if:*

- *for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \leq 0$, $\mathcal{T}_k E \subset E$;*
- *for any $k \in \mathbf{Z}^n$ so that $k \cdot \varpi \geq 0$, $\mathcal{T}_k E \supset E$.*

In light of the above definition, we have that, if $E \subset \mathbf{R}^n$ satisfies the Birkhoff property with respect to ω, then, there exists a constant ϱ, depending only on n, such that, if $\mathcal{C}(E)$ contains a ball of radius ϱ, then $\mathcal{C}(E)$ contains a strip of width 1 that intersects the ball.

It is convenient to introduce the notion of *minimal minimizer*. Following [CdlL01] and [Val01], we now define

$$u_M^\omega(x) := \min_{u \in \mathcal{M}_M^\omega} u(x).$$

By a standard compactness argument, the reader may convince herself that that such definition is well posed. Let us now show that level sets of the minimal minimizer satisfy the Birkhoff property. We set $\mathcal{T}_k u(x) := u(x-k)$.

Proposition 8.2 *Let $\vartheta \in \mathbf{R}$. Then, the set $\{u^\omega_M > \vartheta\}$ satisfies the Birkhoff property with respect to ω. Namely:*

- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \le 0$, then $\{u^\omega_M > \vartheta\} \supset \{\mathcal{T}_k u^\omega_M > \vartheta\}$;*
- *if $k \in \mathbf{Z}^n$ and $k \cdot \omega \ge 0$, then $\{u^\omega_M > \vartheta\} \subset \{\mathcal{T}_k u^\omega_M > \vartheta\}$.*

The proof easily follows from (8.3) and the definition of minimal minimizer, by looking at $\min\{u^\omega_M,\, \mathcal{T}_k u^\omega_M\}$ and $\max\{u^\omega_M,\, \mathcal{T}_k u^\omega_M\}$.

We now show that the minimal minimizer enjoys the *doubling property* (also known in the literature as *no-symmetry-breaking-property*), i.e. it will have the period induced by the functional even if considered as defined in sets whose period is a multiple of the original one.

Proposition 8.3 *Let $\nu_1, \dots, \nu_{n-1} \in \mathbf{N} - \{0\}$. Let $\sim$ be the equivalence relation defined in (8.1). Let us assume that the $(n-1)$-dimensional lattice induced by $\sim$ is spanned by $k^*_1, \dots, k^*_{n-1} \in \mathbf{Z}^n$, i.e. that any $k \in \mathbf{Z}^n$ such that $\omega \cdot k = 0$ can be written as*

$$k = \sum_{i=1}^{n-1} m_i\, k^*_i,$$

for suitable $m_1, \dots, m_{n-1} \in \mathbf{Z}$.

Let us consider the equivalence relation:

$$x \simeq y \quad \textit{iff} \quad x - y = \sum_{i=1}^{n-1} m_i\, \nu_i\, k^*_i,$$

for suitable $m_1, \dots, m_{n-1} \in \mathbf{Z}$. Let u and v be the minimal minimizers in $S^\omega_M/\!\sim$ and in $S^\omega_M/\simeq$, respectively (constrained to be 1 when $\omega \cdot x = 0$ and to be -1 when $\omega \cdot x = M|\omega|$). Then, v is also periodic with respect to the quotient induced by $\sim$ in (8.1), i.e.

$$v(x + k) = v(x), \tag{8.4}$$

for any $k \in \mathbf{Z}^n$ so that $\omega \cdot k = 0$. Also,

$$v = u. \tag{8.5}$$

Proof. Since (8.4) and the definition of minimal minimizer directly imply (8.5), we just consider the proof of (8.4). The periodicity induced by $\simeq$ on v reads

$$v\Big(x + \sum_{i=1}^{n-1} m_i\, \nu_i\, k^*_i\Big) = v(x),$$

for any $m_1, \dots, m_{n-1} \in \mathbf{Z}$, for any $x \in \mathbf{R}^n$, or, equivalently,

$$v(x + m_i\, \nu_i\, k^*_i) = v(x),$$

for any $m_i \in \mathbf{Z}$, for any $x \in \mathbf{R}^n$. We want to prove that

$$v\Big(x + \sum_{i=1}^{n-1} m_i\, k_i^*\Big) = v(x),$$

for any $m_1, \ldots, m_{n-1} \in \mathbf{Z}$, for any $x \in \mathbf{R}^n$, or, equivalently, that

$$v(x + k_i^*) = v(x), \tag{8.6}$$

for any $x \in \mathbf{R}^n$. Let us do the case $i = 1$, the others being analogous. Let

$$\begin{aligned} v_*(x) &:= \min\{v(x), v(x + k_1^*), v(x + 2\,k_1^*), \ldots, v(x + (\nu_1 - 1)\,k_1^*)\} \\ &= \min\{v(x), \min\{v(x + k_1^*), \min\{v(x + k_2^*), \ldots \\ &\quad \ldots, \min\{v(x + (\nu_1 - 2)\,k_1^*), v(x + (\nu_1 - 1)\,k_1^*)\}\}\}\}\}. \end{aligned}$$

Since v_* satisfies the same constrains and periodicity as v, it is also a minimizer, thanks to (8.3). From the fact that v is the minimal minimizer we thus infer $v \le v_*$; Hence, since $v_* \le v$ by construction, (8.6) is proved. □

Such doubling property will play a role in Proposition 8.5 here below.

We now show that constrained minimizers are indeed unconstrained and class A, provided that the strip is wide enough. In order to clearly state the next result, we introduce the following notation. Let $m \le M$ and define

$$S^\omega_{m,M} := \{x \in \mathbf{R}^n : x \cdot \omega \in [m|\omega|, M|\omega|]\}.$$

Set also

$$\begin{aligned} \mathcal{X}^\omega_{m,M} := \{&u \in L^1_{\rm loc}(\tilde S^\omega_{m,M}) : \nabla u \in L^p(\tilde S^\omega_{m,M}), \\ &u = 1 \text{ on } x \cdot \omega = m|\omega| \quad \text{and} \quad u(x) = -1 \text{ on } x \cdot \omega = M|\omega|\}. \end{aligned}$$

We also denote by $\mathcal{M}^\omega_{m,M}$ the set of all periodic constrained minimizers in the strip $S^\omega_{m,M}$. Of course, $S^\omega_M = S^\omega_{0,M}$, $\mathcal{X}^\omega_M = \mathcal{X}^\omega_{0,M}$ and $\mathcal{M}^\omega_M = \mathcal{M}^\omega_{0,M}$. Also, we define the minimal minimizer

$$u^\omega_{m,M}(x) := \min_{u \in \mathcal{M}^\omega_{m,M}} u(x).$$

For keeping the notation, we also set $u^\omega_M := u^\omega_{0,M}$.

We now prove that the minimal minimizer constrained in a strip large enough is indeed *unconstrained*.

Proposition 8.4 *There exists $M_0 > 0$, depending only on the structural constants, such that if $M \ge M_0$, then $u^\omega_M = u^\omega_{-a,M+b}$, for any $a, b \ge 0$.*

Proof. Let $u = u^\omega_{-a,M+b}$. Let S be a strip of width 3 around the hyperplane $L := \{x \cdot \omega = M/2\}$. We now prove that there exists a strip H of width 1 parallel to the constraints and at distance universally bounded from L, on which

$$\text{either } \ u = -1 \ \text{ or } \ u = 1\,. \tag{8.7}$$

For that, we may assume that there exists a point $x^* \in S$ so that $u(x^*) \in (-1,1)$, otherwise (8.7) is obviously true. Then, from Proposition 7.2 there exists a ball (whose radius is big if M is big), contained in $\{u=-1\} = \mathcal{C}\{u>-1\}$. Hence, we obtain a strip of width 1 intersecting such ball and contained in $\{u=-1\}$. This finishes the proof of (8.7).

Since u is the minimal minimizer, it cannot be that $u=1$ in S, otherwise we could lower it. Hence $u=-1$ in S and thus, for any $a,b\geq 0$

$$u^\omega_{-a,M+b} \in \mathcal{M}^\omega_{-a,M}. \tag{8.8}$$

Roughly speaking, this says that "the upper constrained has become irrelevant". We now show that the lower constrained is not necessary either. Indeed, given any $v \in \mathcal{X}^\omega_{-a,M+b}$, we can translate it up by an integer vector to a $w \in \mathcal{X}^\omega_{0,M+\hat{b}}$ for a suitable $\hat{b}\geq b$. By means of (8.8), the energy of w (which agrees with the one of v and is not less than the one of $u^\omega_{M+\hat{b}}$) is not less than the one of $u^\omega_{0,M}$. Therefore a minimizer in $\mathcal{X}^\omega_{0,M}$ a minimizer in $\mathcal{X}^\omega_{-a,M+b}$. □

We now show that the minimal minimizer in a wide enough strip is indeed a class A minimizers.

Proposition 8.5 *There exists a universal constant M_0, so that if $M\geq M_0$, then u^ω_M is a class A minimizer.*

Proof. Given a compact perturbation, by the doubling property (see Proposition 8.3), we can always consider u^ω_M as periodic with a period larger than the diameter of the perturbation and (in light of Proposition 8.4) as a minimizer in a strip of width bigger than the size of the perturbation. □

This completes the proof of **(T1)** in Theorem 2.4: indeed, the case of irrational frequency can be easily inferred from the rational frequency case by a limiting argument, noticing that M_0 does not depend on the frequency.

We now prove **(T2)** in Theorem 2.4. We will keep on assuming here $\omega \in \mathbf{Q}^n - \{0\}$.

Proposition 8.6 *There exists a universal constant M_0, so that if $M\geq M_0$, then $\mathcal{M}^\omega_M \subset \mathcal{M}^\omega_{-a,M+b}$, for any $a,b\geq 0$.*

Proof. Take $u\in\mathcal{M}^\omega_M$ and $v\in\mathcal{X}^\omega_{-a,M+b}$ and consider the minimal minimizer u^ω_M in $\tilde{S}^\omega_M$. Since $u, u^\omega_M \in \mathcal{M}^\omega_M$,

$$\mathcal{J}_{\tilde{S}^\omega_M}(u^\omega_M) = \mathcal{J}_{\tilde{S}^\omega_M}(u).$$

Since $|u| = 1 = |u^\omega_M|$ when $x\cdot\omega\leq 0$ or $x\cdot\omega\geq M$,

$$\mathcal{J}_{\tilde{S}^\omega_M}(u) = \mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(u) \qquad \text{and} \qquad \mathcal{J}_{\tilde{S}^\omega_M}(u^\omega_M) = \mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(u^\omega_M).$$

Also, by Proposition 8.4, u^ω_M still minimizes $\mathcal{J}$ in $\tilde{S}^\omega_{-a,M+b}$, and so

$$\mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(v) \geq \mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(u^\omega_M).$$

Collecting the relations above, it follows that $\mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(v) \geq \mathcal{J}_{\tilde{S}^\omega_{-a,M+b}}(u)$. □

Proposition 8.7 *There exists a universal constant M_0, so that if $M \geq M_0$, then any constrained minimizer $u \in \mathcal{M}_M^\omega$ is a class A minimizer.*

Proof. Let $B \subset \mathbf{R}^n$ be a ball and take $\varphi \in C_0^\infty(B)$. Let S be a strip obtained by $\tilde{S}_M^\omega$ by enlarging its width and by repeating its periodicity ν times in such a way that it well contains the ball B (of course, if we repeat the periodicity ν_i times in the ith direction, we have $\nu = \nu_1 \dots \nu_{n-1} \in \mathbf{N} - \{0\}$). Hence, if v enjoys the periodicity induced by $\sim$, i.e. if $v(x) = v(x+k)$ for any $k \cdot \omega = 0$, we have that

$$\mathcal{J}_S(v) = \nu \mathcal{J}_{\tilde{S}_M^\omega}(v).$$

Let now $u \in \mathcal{M}_M^\omega$ and consider the minimal minimizer u_M^ω. By the doubling property (see Proposition 8.3) and Proposition 8.4, u_M^ω still minimizes $\mathcal{J}$ in S, whence

$$\mathcal{J}_S(u_M^\omega) \leq \mathcal{J}_S(u+\varphi).$$

Also, since both u and u_M^ω are minimizers in $\tilde{S}_M^\omega$,

$$\mathcal{J}_{\tilde{S}_M^\omega}(u) = \mathcal{J}_{\tilde{S}_M^\omega}(u_M^\omega).$$

Moreover, $u = u + \varphi$ in $S - B$. Therefore, in light of the relations above,

$$\begin{aligned}
\mathcal{J}_B(u+\varphi) - \mathcal{J}_B(u) &= \mathcal{J}_S(u+\varphi) - \mathcal{J}_S(u) \\
&\geq \mathcal{J}_S(u_M^\omega) - \mathcal{J}_S(u) \\
&= \nu \mathcal{J}_{\tilde{S}_M^\omega}(u_M^\omega) - \nu \mathcal{J}_{\tilde{S}_M^\omega}(u) = 0.
\end{aligned}$$

□

Thus, **(T2)** in Theorem 2.4 follows from Propositions 8.6 and 8.7. This completes the proof of Theorem 2.4.

A Appendix

In the proof of Lemma 4.2 we have used an approximation argument of minimizers of $\mathcal{F}$ with solutions of the homogeneous equation $\operatorname{div} a(x, \nabla u) = 0$. Namely, let

$$\mathcal{F}_\Omega(w) = \int_\Omega A(x, \nabla w) + Q(x)\chi_{(0,\infty)}(w),$$

where $A(x,\eta)$ and $a(x,\eta) = D_\eta A(x,\eta)$ satisfy **(H1)**–**(H6)** and with

$$0 \leq Q(x) \leq \varepsilon.$$

A consequence of the conditions **(H1)**–**(H3)** is the following useful inequality

$$(a(x,\eta) - a(x,\eta')) \cdot (\eta - \eta') \geq \lambda \begin{cases} |\eta - \eta'|^2 (|\eta| + |\eta'|)^{p-2}, & 1 < p \leq 2 \\ |\eta - \eta'|^p & p \geq 2, \end{cases} \tag{A.1}$$

see, e.g., [Tol84].

Lemma A.1 *Let $w \in W^{1,p}(B_1)$ be an absolute minimizer of $\mathcal{F}_{B_1}$. Let also $w_h \in W^{1,p}(B_{3/4})$ satisfy*

$$\operatorname{div} a(x, \nabla w_h) = 0, \quad w_h - w \in W_0^{1,p}(B_{3/4}).$$

Then

$$\|w - w_h\|_{W^{1,p}(B_{3/4})} \leq C\,\varepsilon^{\alpha/p} \|w\|_{L^p(B_1)}^{1-\alpha},$$

with $\alpha = \min(1, p/2)$, where C_0 depends only on the structural constants of A.

Proof. ¿From the minimality of w in $B = B_{3/4}$ we have

$$\int_B A(x, \nabla w) + Q(x)\,\chi_{(0,\infty)}(w) \leq \int_B A(x, \nabla w_h) + Q(x),$$

or

$$\int_B A(x, \nabla w) - A(x, \nabla w_h) \leq \varepsilon \int_B \chi_{\{u=0\}}.$$

Set now

$$w^s(x) = s\,w(x) + (1-s) w_h(x), \quad 0 \leq s \leq 1.$$

Clearly $w^0 = w_h$ and $w^1 = w$. We thus obtain

$$\begin{aligned}
&\int_B A(x, \nabla w) - A(x, \nabla w_h) = \\
= &\int_0^1 ds \int_B a(x, \nabla w^s) \cdot \nabla(w - w_h) \\
= &\int_0^1 ds \int_B (a(x, \nabla w^s) - a(x, \nabla w_h)) \cdot \nabla(w - w_h) \\
= &\int_0^1 \frac{ds}{s} \int_B (a(x, \nabla w^s) - a(x, \nabla w_h)) \cdot \nabla(w^s - w_h),
\end{aligned}$$

where in the second step we used that $\int_B a(x, \nabla w_h) \cdot \nabla(w - w_h) = 0$.

Next, we apply the inequality (A.1).

1. For $p \geq 2$ we obtain

$$\begin{aligned}
\int_B A(x, \nabla w) - A(x, \nabla w_h) \quad &\geq \quad \lambda \int_0^1 \frac{ds}{s} \int_B |\nabla(w^s - w_h)|^p = \\
&= \quad \lambda \int_0^1 s^{p-1} ds \int_B |\nabla(w - w_h)|^p
\end{aligned}$$

and consequently

$$\int_B |\nabla(w - w_h)|^p \leq C\,\varepsilon \int_B \chi_{\{u=0\}}. \tag{A.2}$$

2. In the case $1 < p \le 2$ we have

$$\begin{aligned}\int_B A(x,\nabla w) - A(x,\nabla w_h) &\ge \lambda \int_0^1 \frac{ds}{s}\int_B |\nabla(w^s - w_h)|^2 (|\nabla w^s| + |\nabla w_h|)^{p-2}\\ &\ge c\int_0^1 s\,ds \int_B |\nabla(w-w_h)|^2(|\nabla w| + |\nabla w_h|)^{p-2}\\ &\ge c\int_B |\nabla(w-w_h)|^2(|\nabla w|+|\nabla w_h|)^{p-2}.\end{aligned}$$

On the other hand, using the Hölder inequality, we have

$$\begin{aligned}&\int_B |\nabla(w-w_h)|^p \le\\ \le\ &\left(\int_B |\nabla(w-w_h)|^2(|\nabla w|+|\nabla w_h|)^{p-2}\right)^{p/2}\left(\int_B (|\nabla w|+|\nabla w_h|)^p\right)^{1-p/2},\end{aligned}$$

hence

$$\int_B |\nabla(w-w_h)|^p \le C\varepsilon^{p/2}\left(\int_B \chi_{\{w=0\}}\right)^{p/2}\left(\int_{B_1} w^p\right)^{1-p/2} \tag{A.3}$$

where we have used that $\int_B |\nabla w_h|^p \le C\int_B |\nabla w|^p$ and the Cacciopolli type inequality

$$\int_B |\nabla w|^p \le C\int_{B_1} w^p,$$

see [HKM93], Lemma 3.27. The lemma follows now from (A.2)–(A.3). □

The following estimate was used in the proof Lemma 6.2 by using the recurrence relationship (6.1).

Lemma A.2 *Let $v_k,\ a_k \in \mathbf{R}$ verify: $v_k \ge 0$, $a_k \ge 0$, $v_{k-1} + a_{k-1} \le v_k + a_k$, $v_0 + a_0 \ge c_0 > 0$ and*

$$v_{k-1}^{(n-1)/n} + a_{k-1} \le C\Big(v_k - v_{k-1} + a_k - a_{k-1}\Big),$$

for any $k \in \mathbf{N}$, $k \ge 1$, for some positive c_0 and C. Then, there exists a positive c, depending only on C and c_0, such that

$$v_k + a_k \ge c\,(k+1)^n, \tag{A.4}$$

for any $k \in \mathbf{N}$.

Proof. First of all, notice that, if C_* is big enough,

$$C_* x + 1 \ge (1+x)^n, \qquad \forall x \in [0,1]. \tag{A.5}$$

Also, if $k \le 2CC_*$,

$$v_k + a_k \ge v_0 + a_0 \ge c_0 \ge c\,(k+1)^n,$$

provided $c \le c_0/(2CC_* + 1)^n$. Hence we can assume $k \ge 2CC_*$. We make an iterative argument: we assume

$$v_{k-1} + a_{k-1} \ge c\, k^n, \tag{A.6}$$

and we prove (A.4).

In fact, in order to prove (A.4), by (A.5), it is enough to prove

$$v_k + a_k \ge c\, k^n \Big(\frac{C_*}{k} + 1\Big), \tag{A.7}$$

since, then, (A.4) follows from (A.7) applied with $x := 1/k$.

Hence, we now prove (A.7). From the hypothesis and (A.6), we have:

$$v_k + a_k \ge \frac{1}{C} \left(v_{k-1}^{(n-1)/n} + a_{k-1} \right) + c\, k^n. \tag{A.8}$$

Also, using again (A.6), we infer that either $v_{k-1} \ge c\, k^n/2$ or $a_{k-1} \ge c\, k^n/2$.

If $v_{k-1} \ge c\, k^n/2$, from (A.8),

$$v_k + a_k \ge c\,(C_* k^{n-1} + k^n),$$

provided $C_* \le 1/(C\, 2^{(n-1)/n} c^{1/n})$, which is fulfilled if c is small enough. This proves (A.7) in this case.

On the other hand, if $a_{k-1} \ge c\, k^n/2$, from (A.8) we get

$$v_k + a_k \ge c\, k^n \Big(\frac{1}{2C} + 1\Big),$$

which implies (A.7) since $k \ge 2CC_*$. □

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