On Abelian C*-Algebras which are Independent with Respect to a Filter

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Abstract

We examine extensions of the concept of asymptotic independence, introduced in [14] and useful in the study of abelian C^*-algebras generated by the union of two of its C^*-subalgebras. We also give non-trivial exemples in terms of classes of bounded, continuous functions defined on a locally compact group.

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Introduction

Let BC be the C^*-algebra of all bounded, continuous functions on a locally compact, non-compact space X and C_0 the ideal of the functions in BC that converge to zero at infinity. The norm of the image of $\varphi \in BC$ in the quotient C^*-algebra BC/C_0 will be denoted by $\| \varphi \|_{\text{asy}}$; this defines a seminorm on BC. Following [14] we say that two unital C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of BC are *asymptotically independent* if for any $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ we have $\| \varphi\psi \|_{\text{asy}} = \| \varphi \|_{\text{asy}} \| \psi \|_{\text{asy}}$. S. Power introduced a non-abelian analogue of this concept in order to study the space of characters of a C^*-algebra that is generated by (at least) two other $\mathcal{C}^*$-algebras, cf. [13], [14] and [15]. In the abelian case, for $X = \mathbb{R}$, he showed that AP (the algebra of all almost periodic functions) and VO (the algebra of all vanishing oscillation functions) are indeed asymptotically independent.

We shall extend considerably this result. In fact we generalize the concept, by replacing the ideal C_0 by the most general ideal $BC^{\mathfrak{F}}$ of BC, composed of all the functions converging to zero along some filter $\mathfrak{F}$ on X. The seminorm $\| \varphi \|_{\text{asy}}$ will be replaced by $\| \varphi \|_{\mathfrak{F}} := \| \varphi_{\mathfrak{F}} \|$, where $\varphi_{\mathfrak{F}}$ is the canonical image of φ in the quotient $BC/BC^{\mathfrak{F}}$. The analogue of asymptotic independence in this case will be called $\mathfrak{F}$-*independence*. If each element φ of the C^*-subalgebra $\mathcal{A}$ is $\mathfrak{F}$-independent with respect to each element ψ of the C^*-subalgebra $\mathcal{B}$, then the C^*-algebra generated by the union of $\mathcal{A}/\mathcal{A} \cap BC^{\mathfrak{F}}$ and $\mathcal{B}/\mathcal{B} \cap BC^{\mathfrak{F}}$ will be naturally isomorphic to the tensor product $\left[\mathcal{A}/\mathcal{A} \cap BC^{\mathfrak{F}}\right] \otimes \left[\mathcal{B}/\mathcal{B} \cap BC^{\mathfrak{F}}\right]$ and this is the main reason for which we are interested in $\mathfrak{F}$-independence.

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Independence with respect to the filter $\mathfrak{F}$ is, a priori, a rather implicit notion even when $\mathfrak{F}$ is simple to understand. In order to have access to sufficiently large classes of examples, we shall study systematically ε-level sets. An ε-level set for φ is a set of the form $M_\varepsilon(\varphi;\lambda) := \{x \in X \mid |\varphi(x) - \lambda| < \varepsilon\}$ for some $\lambda \in \mathbb{C}$. We put into evidence classes $\mathfrak{M}$, $\mathfrak{N}$ of subsets of X (called $\mathfrak{F}$-*overlapping*) such that if for each $\varepsilon > 0$, $M_\varepsilon(\varphi; \| \varphi \|_{\mathfrak{F}})$ belongs to $\mathfrak{M}$ and $M_\varepsilon(\psi; \| \psi \|_{\mathfrak{F}})$ belongs to $\mathfrak{N}$, then, once again for each $\varepsilon > 0$, $M_\varepsilon(\varphi\psi; \| \varphi\psi \|_{\mathfrak{F}})$ will not be the complement of some element of $\mathfrak{F}$. This will imply that φ and ψ are $\mathfrak{F}$-independent.

Even so, relevant nontrivial $\mathfrak{F}$-overlapping classes as above are not so easy to find. Exept some general considerations, we shall be mainly concerned with the case in which X is a locally compact group. We shall introduce two families: $\mathfrak{G}\mathfrak{S}$ (of $\mathfrak{G}$-*syndetic* sets) and $\mathfrak{G}^{-1}\mathfrak{R}$ (of $\mathfrak{G}^{-1}$-*replete* sets) and prove that they are $\mathfrak{F}$-overlapping. ($\mathfrak{G}$ is the family composed of the complements of all the sets in $\mathfrak{F}$ and is a convenient label and $\mathfrak{G}^{-1} := \{G^{-1} \mid G \in \mathfrak{G}\}$). The corresponding classes of functions do not form in general C^*-algebras. But under suitable hypothesis, some of their subclasses do, thus we get families of couples of mutually $\mathfrak{F}$-independent C^*-algebras.

As examples, we introduce various algebras of functions that have *vanishing oscillation*, which come out to have repleteness properties. These types of functions may present their own interest. If $\mathfrak{F} = \mathfrak{F}_{\rm st}$ is the standard filter, composed of the complements of all the relatively compact subsets of X, we show that all the C^*-algebras of *minimal* functions verify $\mathfrak{G}$-syndeticity. It follows that the (standard) algebra VO of vanishing oscillation functions and any algebra of minimal functions are mutually $\mathfrak{F}_{\rm st}$-independent. This extends significantly the result of [14], since any maximal algebra of minimal functions contains AP as a small subset, also containing larger classes of functions, (distal or almost automorphic for example).

My interest in the present topic was arisen by certain problems in the spectral analysis of self-adjoint, Schrödinger-type operators. Recently, C^*-algebra techniques proved relevant in this direction; see [1], [8] and references therein. Suppose given an operator of the form $H = H_0 + V$ in the Hilbert space $\mathcal{H} = L^2(X)$. We assume, for instance, that $X = \mathbb{R}^n$, $H_0 = -\nabla^2$ (the Laplace operator may be replaced by more general convolution operators) and V is (for simplicity) the operator of multiplication by a bounded, uniformly continuous real function on X. We would like to connect some of the spectral properties of H to the fact that V belongs to a specific C^*-algebra of functions $\mathcal{C}$. In [11], [2] and [12] it is shown how to exploit informations about the Gelfand spectrum of the quotient of $\mathcal{C}$ by suitable ideals $\mathcal{J}$ to get formulae for the essential spectrum of H as well as localization results leading to certain propagation properties of the unitary group $\{e^{-itH}\}_{t\in\mathbb{R}}$. An interesting case is that in which $\mathcal{C}$ is generated by two other abelian C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ (this will cover $V = \sum U_j W_j$, $U_j \in \mathcal{A}$ and $W_j \in \mathcal{B}$). Just to find the essential spectrum would require to take $\mathcal{J} = C_0$, but more general ideals are needed to obtain the localization properties. Thus, we are placed in the framework presented above.

Section 3 of the present paper leads to improvements of certain results of [11], [2] and [12]. In fact, in Section 4 of [12] some results were announced only with a brief sketch of a proof. Improved versions are now covered in a general setting. We do not give details here; they should be straightforward to the interested reader.

In the sequel we assume that X is not compact mainly to allow important examples of filters, as the filter composed of the complements of all relatively compact sets. But the reader will remark that many assertions are correct also on compact spaces and groups.

1 Quotients of Abelian C^*-Algebras

In order to prepare the framework of the future sections, we shall describe here some general constructions related to abelian C^*-algebras and filters on a locally compact space X. Only the case in which X is not compact will be really interesting. Most of the facts we state follow straightforwardly from standard Gelfand theory. In **[1.1]** we give a convenient indexation of ideals and quotients of an abelian C^*-algebra $\mathcal{A}$ of functions on X in terms of filters and describe the corresponding spectra. In **[1.2]** objects naturally associated to quotients are discussed in terms of filters; in particular we introduce the range and the seminorm of a function with respect to a filter.

1.1 Ideals and quotients associated to a filter

Let us fix a locally compact, non-compact space X; the term "locally compact" will include the Hausdorff property, by convention.

For any unital C^*-subalgebra $\mathcal{A}$ of $BC \equiv BC(X)$ (the C^*-algebra of all bounded, continuous complex functions on X) we denote by $X(\mathcal{A})$ its spectrum. One often identifyes (by the Gelfand isomorphism) $\mathcal{A}$ with the C^*-algebra $C[X(\mathcal{A})]$ of all the continuous, complex functions on the compact space $X(\mathcal{A})$. We use the standard notation $\beta X \equiv X(BC)$ for the Stone-Čech compactification of X; one always has a canonical surjection $s_{\mathcal{A}} : \beta X \to X(\mathcal{A})$. The C^*-subalgebra $\mathcal{A}$ will be called *of the first kind* if it contains $C_0 \equiv C_0(X)$, the ideal of continuous, complex functions on X which converge to zero at infinity. If $\mathcal{A} \cap C_0 = \{0\}$, then $\mathcal{A}$ will be called *of the second kind*. If $\mathcal{A}$ is of the first kind, then $X(\mathcal{A})$ is a compactification of X. This means that X can be identified with a dense subspace of the compact space $X(\mathcal{A})$. The identification is given by the map $x \mapsto \delta_x$, $\delta_x(\varphi) = \varphi(x)$ for all $\varphi \in \mathcal{A}$, which is injective (continuous and with dense image) if $\mathcal{A}$ is of the first kind. In this case the restriction of $s_{\mathcal{A}}$ to X is the identity map.

We are interested in quotients of $\mathcal{A}$ by ideals (always assumed to be closed). It is known that $Z \mapsto C^Z[X(\mathcal{A})] := \{\varphi \in C[X(\mathcal{A})] \mid \varphi|_Z = 0\}$ is a bijection from the family of all the closed subsets of $X(\mathcal{A})$ to the family of all ideals of $\mathcal{A} \cong C[X(\mathcal{A})]$. We shall prefer another indexation (which is not one-to-one), in terms of convergence along filters, because this will take into account only the behaviour of the functions on X. Let $\mathfrak{F}$ be a filter on X. Its image through $s_{\mathcal{A}}$ is a filter base in $X(\mathcal{A})$, which generates a filter that we call $\mathfrak{F}(\mathcal{A})$. Let us set $\mathcal{A}^{\mathfrak{F}} := \{\varphi \in \mathcal{A} \mid \lim_{\mathfrak{F}} \varphi = 0\}$. Any ideal of $\mathcal{A}$ is indeed of this type. By adjoining a unit to $\mathcal{A}^{\mathfrak{F}}$ we get a C^*-algebra which will be identified with $\tilde{\mathcal{A}}^{\mathfrak{F}} := \{\varphi \in \mathcal{A} \mid \exists \lim_{\mathfrak{F}} \varphi\}$. The quotient C^*-algebra $\mathcal{A}/\mathcal{A}^{\mathfrak{F}}$ will be denoted by $\mathcal{A}_{\mathfrak{F}}$. Denote also by $\mathrm{Cl}\,[\mathfrak{F}(\mathcal{A})] := \cap_{F \in \mathfrak{F}(\mathcal{A})} \overline{F}$ the set of all cluster points of the filter $\mathfrak{F}(\mathcal{A})$. Then if $\varphi \in \mathcal{A} \cong C[X(\mathcal{A})]$, $\lim_{\mathfrak{F}} \varphi = 0$ is equivalent to the fact that φ is null on $\mathrm{Cl}\,[\mathfrak{F}(\mathcal{A})]$. Hence all the filters $\mathfrak{F}$ on X for which the images $\mathfrak{F}(\mathcal{A})$ have the same set of cluster points (called *equivalent filters* in the sequel) define the same ideal in $\mathcal{A}$. By standard arguments one identifies the spectrum $X(\mathcal{A}_{\mathfrak{F}})$ of $\mathcal{A}_{\mathfrak{F}}$ with $\mathrm{Cl}\,[\mathfrak{F}(\mathcal{A})]$ and the spectrum $X\left(\mathcal{A}^{\mathfrak{F}}\right)$ of $\mathcal{A}^{\mathfrak{F}}$ with the complement $X(\mathcal{A}) \setminus \mathrm{Cl}\,[\mathfrak{F}(\mathcal{A})]$. We note that ultrafilters correspond to singletons of $X(\mathcal{A})$ as well as to maximal ideals of $\mathcal{A}$. Thus two different ultrafilters cannot be equivalent.

1.2 $\mathfrak{F}$-range and $\mathfrak{F}$-seminorm

We are placed in the same setting as before: $\mathcal{A}$ is a unital C^*-algebra of bounded, continuous functions on the locally compact space X (tacitly assumed to be non-compact) and $\mathfrak{F}$ is a filter on X which defines the ideal $\mathcal{A}^{\mathfrak{F}}$ and the quotient $\mathcal{A}_{\mathfrak{F}} = \mathcal{A}/\mathcal{A}^{\mathfrak{F}}$. The canonical epimorphism $\mathcal{A} \mapsto \mathcal{A}_{\mathfrak{F}}$ will be denoted by $\pi^{\mathcal{A}}_{\mathfrak{F}}$. By identifying $\mathcal{A}$ with $C\left[X(\mathcal{A})\right]$ and $\mathcal{A}_{\mathfrak{F}}$ with $C\left[X(\mathcal{A}_{\mathfrak{F}})\right] \equiv C\left(\mathrm{Cl}\left[\mathfrak{F}(\mathcal{A})\right]\right)$, $\pi^{\mathcal{A}}_{\mathfrak{F}}$ is just the restriction map $\varphi \mapsto \varphi|_{\mathrm{Cl}[\mathfrak{F}(\mathcal{A})]}$.

The range $R(f)$ of a continuous function f between two topological spaces Y and Z will be the closure in Z of the set $f(Y)$.

Definition 1.1 *(a)* The $\mathfrak{F}$-range $R_{\mathfrak{F}}(\varphi)$ *of* $\varphi \in \mathcal{A}$ *is the range of* $\pi^{\mathcal{A}}_{\mathfrak{F}}(\varphi)$ *(viewed as a function on Cl* $\left[\mathfrak{F}(\mathcal{A})\right]$*).*

(b) The $\mathfrak{F}$-seminorm $\| \varphi \|_{\mathfrak{F}}$ *of* $\varphi \in \mathcal{A}$ *is the norm of its image* $\pi^{\mathcal{A}}_{\mathfrak{F}}(\varphi)$ *in* $\mathcal{A}_{\mathfrak{F}}$.

Obviously, $R_{\mathfrak{F}}(\varphi)$ and $\| \varphi \|_{\mathfrak{F}}$ do not depend on the algebra $\mathcal{A}$. $\| \varphi \|_{\mathfrak{F}}$ is a seminorm on $\mathcal{A}$ with kernel $\mathcal{A}^{\mathfrak{F}}$. It coincides with the usual sup-norm of the restriction of φ to $\mathrm{Cl}[\mathfrak{F}(\mathcal{A})]$ and it can also be writen: $\| \varphi \|_{\mathfrak{F}} = \sup R_{\mathfrak{F}}(|\varphi|) = \limsup_{\mathfrak{F}} |\varphi| = \inf_{F\in\mathfrak{F}} \sup_{x\in\overline{F}} |\varphi(x)|$.

We shall give in Lemma 1.3 suitable descriptions of the closed subset $R_{\mathfrak{F}}(\varphi)$ of $\mathbb{C}$ (usually called *the set of cluster points of the function* φ *with respect to the filter* $\mathfrak{F}$) in terms of the behaviour of φ on X. The first one is standard, in terms of ultrafilters. For the second one, some extra notations are needed. We set $\mathfrak{G} \equiv \mathfrak{F}^{\complement} := \{\complement F | F \in \mathfrak{F}\}$ for the family of all the complements of the sets in $\mathfrak{F}$. $\mathfrak{G}$ will be *an antifilter*, i.e. it is non-void, it does not contain X and it is stable under finite unions and subsets.

Definition 1.2 *For any* $\varphi \in BC$*,* $\varepsilon > 0$ *and* $\lambda \in \mathbb{C}$ *we set* $M_\varepsilon(\varphi;\lambda) := \{x \in X \mid |\varphi(x) - \lambda| < \varepsilon\}$ *and we call it* an ε-level set.

Clearly $M_\varepsilon(\varphi;\lambda)$ is open in X and it is nonvoid for any $\varepsilon > 0$ iff λ belongs to the range of φ.

Lemma 1.3 *For* $\varphi \in BC$ *and* $\lambda \in \mathbb{C}$*, the following conditions are equivalent:*

(i) $\lambda \in R_{\mathfrak{F}}(\varphi)$.

(ii) There exists an ultrafilter $\mathfrak{U}$ *on* X*, finer than* $\mathfrak{F}$*, such that* $\lim_{\mathfrak{U}} \varphi = \lambda$.

(iii) For any $\varepsilon > 0$*,* $M_\varepsilon(\varphi;\lambda) \notin \mathfrak{G}$.

The proof is a simple exercice in topology.

Three simple examples are of distinguished interest and will be met in subsequent sections:

The trivial filter $\mathfrak{F}_{\mathrm{tr}} = \{X\}$ will be mainly considered for illustrational purposes. For any unital C^*-subalgebra $\mathcal{A}$ of BC, one has $\mathcal{A}^{\mathfrak{F}_{\mathrm{tr}}} = \{0\}$, $\tilde{\mathcal{A}}^{\mathfrak{F}_{\mathrm{tr}}} = \mathbb{C}$ and $\mathcal{A}_{\mathfrak{F}_{\mathrm{tr}}} = \mathcal{A}$. The seminorm $\| \cdot \|_{\mathfrak{F}_{tr}}$ coincides with the usual norm $\| \cdot \|$ and one has $R_{\mathfrak{F}_{\mathrm{tr}}}(\varphi) = R(\varphi)$ for any φ.

Of direct interest will be *the standard filter* $\mathfrak{F}_{\mathrm{st}}$, composed of the complements of all the relatively compact subsets of X. Then we have $\mathcal{A}^{\mathfrak{F}_{\mathrm{st}}} = \mathcal{A} \cap C_0$ and $\tilde{\mathcal{A}}^{\mathfrak{F}_{\mathrm{st}}} = \mathcal{A} \cap C(\dot{X})$ (the algebra of all functions in $\mathcal{A}$ which can be extended continuously to the one-point compactification of X). We shall call respectively the objects $\| \varphi \|_{\mathfrak{F}_{\mathrm{st}}} \equiv \| \varphi \|_{\mathrm{asy}}$ and $R_{\mathfrak{F}_{\mathrm{st}}}(\varphi) \equiv R_{\mathrm{asy}}(\varphi)$ *the asymptotic seminorm* and *the asymptotic range* of $\varphi \in BC$. The complex number λ is in the asymptotic range of φ iff for any $\varepsilon > 0$ the level set $M_\varepsilon(\varphi;\lambda)$ is not relatively compact. This example motivated our work and many of the results below are interesting and non-trivial for $\mathfrak{F} = \mathfrak{F}_{\mathrm{st}}$.

Occasionally we shall meet *the Fréchet filter*, composed of the complements of the finite subsets of X; it is contained in the standard filter. One has $\mathfrak{F}_{\rm Fr} = \mathfrak{F}_{\rm st}$ if X is discrete. In the opposite case, in which the single finite open set is $\emptyset$, we have obviously $\| \cdot \|_{\mathfrak{F}_{\rm Fr}} = \| \cdot \|$ and $R_{\mathfrak{F}_{\rm Fr}} = R$. There are also many intermediary cases, for which we have nothing special to say.

2 $\mathfrak{F}$-Independence, the General Case

As in the preceding section, $\mathfrak{F}$ will be a filter on the locally compact (non-compact) space X. We shall be concerned with the C^*-algebra $\mathcal{C}$ generated by two other unital C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of BC. Generalizing some ideas of Power ([13] [14] [15]), we introduce a concept of $\mathfrak{F}$-independence for the couple $(\mathcal{A}, \mathcal{B})$ which is equivalent to the fact that the spectum of $\mathcal{C}_{\mathfrak{F}}$ is homeomorphic with the product between the spectum of $\mathcal{A}_{\mathfrak{F}}$ and the spectum of $\mathcal{B}_{\mathfrak{F}}$. The relevant facts are overviewed in [**2.1**] and they are a convenient rearrangement and extension of the framework of Power. In [**2.2**] we start our systematic study of $\mathfrak{F}$-independence by giving a criterion in terms of ε-level sets.

2.1 $\mathfrak{F}$-independence of functions and algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-subalgebras of $BC \equiv BC(X)$. We denote by $\langle \mathcal{A} \cdot \mathcal{B} \rangle$ or by $\mathcal{C}$ the C^*-subalgebra of BC generated by the union of $\mathcal{A}$ and $\mathcal{B}$. We shall be interested in the spectrum of $\mathcal{C}_{\mathfrak{F}} = \mathcal{C}/\mathcal{C}^{\mathfrak{F}}$; we recall that $\mathcal{C}^{\mathfrak{F}} \equiv \langle \mathcal{A} \cdot \mathcal{B} \rangle^{\mathfrak{F}}$ is the ideal composed of all the functions $\varphi \in \mathcal{C}$ such that $\lim_{\mathfrak{F}} \varphi = 0$. $\mathcal{C}_{\mathfrak{F}}$ may be identified canonically with the C^*-algebra $\langle \mathcal{A}_{\mathfrak{F}} \cdot \mathcal{B}_{\mathfrak{F}} \rangle$ generated by $\mathcal{A}_{\mathfrak{F}}$ and $\mathcal{B}_{\mathfrak{F}}$ in $BC_{\mathfrak{F}}$.

Definition 2.1 *An element $(\sigma, \tau) \in X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})$ is called* a permanent point with respect to $\mathfrak{F}$ *if for any positive elements $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ such that $\sigma\left[\pi_{\mathfrak{F}}^{\mathcal{A}}(\varphi)\right] = \| \varphi \|_{\mathfrak{F}}$ and $\tau\left[\pi_{\mathfrak{F}}^{\mathcal{B}}(\psi)\right] = \| \psi \|_{\mathfrak{F}}$, one has $\| \varphi \|_{\mathfrak{F}} \| \psi \|_{\mathfrak{F}} = \| \varphi\psi \|_{\mathfrak{F}}$. The set of all permanent points with respect to $\mathfrak{F}$ will be denoted by $P_{\mathfrak{F}}(\mathcal{A}, \mathcal{B})$.*

We shall compare now the topological spaces $X(\langle \mathcal{A}_{\mathfrak{F}} \cdot \mathcal{B}_{\mathfrak{F}} \rangle)$ and $X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})$; we define

$$\Theta : X(\langle \mathcal{A}_{\mathfrak{F}} \cdot \mathcal{B}_{\mathfrak{F}} \rangle) \rightarrow X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}}), \quad \Theta(\chi) := \left(\chi|_{\mathcal{A}_{\mathfrak{F}}}, \chi|_{\mathcal{B}_{\mathfrak{F}}}\right).$$

Proposition 2.2 *The spectrum of the C^*-algebra $\mathcal{C}_{\mathfrak{F}} \equiv \langle \mathcal{A}_{\mathfrak{F}} \cdot \mathcal{B}_{\mathfrak{F}} \rangle$ is naturally homeomorphic by the map Θ to the set $P_{\mathfrak{F}}(\mathcal{A}, \mathcal{B})$ endowed with the product topology.*

The result follows easily from a particular case of Theorem 1.2 of [15].

Definition 2.3 *(a) The functions $\varphi, \psi \in BC$ are called* $\mathfrak{F}$-independent *if $\| \varphi\psi \|_{\mathfrak{F}} = \| \varphi \|_{\mathfrak{F}} \| \psi \|_{\mathfrak{F}}$.*

(b) The unital C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of BC are called $\mathfrak{F}$-independent *if any $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ are $\mathfrak{F}$-independent.*

Instead of "$\mathfrak{F}_{\rm st}$-independent" we shall say "asymptotically independent". This concept has already been used in [14].

Corollary 2.4 *Two unital C^*-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of BC are $\mathfrak{F}$-independent if and only if $X\left(\langle \mathcal{A} \cdot \mathcal{B} \rangle_{\mathfrak{F}}\right)$ and $X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})$ are homeomorphic by the map Θ introduced above.*

Proof: By Proposition 2.2 it is enough to check that $P_{\mathfrak{F}}(\mathcal{A},\mathcal{B})$ coincides with $X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{F}$-independent. The "if" part is trivial.

To prove the converse we have to note that, by compactness, for any $\varphi \in \mathcal{A}$ there exists $\sigma \in X(\mathcal{A}_{\mathfrak{F}})$ such that $\sigma\left[\pi_{\mathfrak{F}}^{\mathcal{A}}(|\varphi|)\right] = \| \, |\varphi| \, \|_{\mathfrak{F}} = \| \, \varphi \, \|_{\mathfrak{F}}$ and that an analogous result is true for $\psi \in \mathcal{B}_{\mathfrak{F}}$. Then we apply the definition of a permanent point. ∎

Remark. It follows immediately from the Corollary that for an $\mathfrak{F}$-independent couple of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the algebra $\langle \mathcal{A} \cdot \mathcal{B} \rangle_{\mathfrak{F}}$ is canonically isomorphic to the tensor product $\mathcal{A}_{\mathfrak{F}} \otimes \mathcal{B}_{\mathfrak{F}}$. More generally, there is a canonical epimorphism $\theta : \mathcal{A}_{\mathfrak{F}} \otimes \mathcal{B}_{\mathcal{F}} \to \langle \mathcal{A} \cdot \mathcal{B} \rangle_{\mathfrak{F}}$ sending $\varphi_{\mathfrak{F}} \otimes \psi_{\mathfrak{F}}$ in $\varphi_{\mathfrak{F}} \psi_{\mathfrak{F}}$ for any $\varphi_{\mathfrak{F}} \in \mathcal{A}_{\mathfrak{F}}$ and $\psi_{\mathfrak{F}} \in \mathcal{B}_{\mathfrak{F}}$. If we identify $\mathcal{A}_{\mathfrak{F}} \otimes \mathcal{B}_{\mathfrak{F}}$ with $C\left[X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})\right]$, the kernel of θ will coincide with the set of elements of $C\left[X(\mathcal{A}_{\mathfrak{F}}) \times X(\mathcal{B}_{\mathfrak{F}})\right]$ that are null on $P_{\mathfrak{F}}(\mathcal{A},\mathcal{B})$.

Remark. It is a trivial matter to show that $BC^{\mathfrak{F}} = \{\varphi \in BC \mid \exists \lim_{\mathfrak{F}} \varphi\}$ and BC are $\mathfrak{F}$-independent. We note that if $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{F}$-independent, then $\mathcal{A}_{\mathfrak{F}} \cap \mathcal{B}_{\mathfrak{F}} = \mathbb{C}$. To prove this, set $\mathcal{D} = \mathcal{A} \cap \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{F}$-independent then $\mathcal{D}$ and $\mathcal{D}$ are $\mathfrak{F}$-independent. It follows that $\tau \mapsto (\tau,\tau)$ is a homeomorphism between $X(\mathcal{D}_{\mathfrak{F}})$ and $X(\mathcal{D}_{\mathfrak{F}}) \times X(\mathcal{D}_{\mathfrak{F}})$. Then $X(\mathcal{D}_{\mathfrak{F}})$ must be a singleton, which finishes the proof. However, the condition $\mathcal{A}_{\mathfrak{F}} \cap \mathcal{B}_{\mathfrak{F}} = \mathbb{C}$ by no means implies $\mathfrak{F}$-independence. Simple counterexamples can be obtained, for the trivial filter, in terms of two unital C^*-algebra generated respectively by two suitable bounded, continuous functions with disjoint supports.

2.2 $\mathfrak{F}$-overlapping families of ε-level sets

In the sequel I shall be guided by the rough intuition that "$\mathfrak{F}$ contains mainly large sets" (since it is stable under supersets) and that "$\mathfrak{G}$ (the antifilter defined by $\mathfrak{F}$, composed of the complements of the elements of $\mathfrak{F}$) contains mainly small sets" (being stable under subsets). One should not push this intuition too far.

Definition 2.5 *(a) Let $\mathfrak{M}, \mathfrak{N} \subset 2^X$ two families of subsets of X. We say that $\mathfrak{M}$ and $\mathfrak{N}$ are* $\mathfrak{F}$-overlapping *if for any $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$, $M \cap N \notin \mathfrak{G}$.*

(b) For $\mathfrak{M} \subset 2^X$ we set $\mathfrak{M}^{[\mathfrak{F}]} := \{N \subset X \mid M \cap N \notin \mathfrak{G}, \ \forall M \in \mathfrak{M}\}$.

Instead of "$\mathfrak{F}_{\mathrm{tr}}$-overlapping" we shall simply say "overlapping". For "$\mathfrak{F}_{\mathrm{st}}$-overlapping" we reserve the term "asymptotically overlapping". The condition $M \cap N \notin \mathfrak{G}$ is equivalent to $\complement M \cup \complement N \notin \mathfrak{F}$. Our rough interpretation will be that the intersection between any element of $\mathfrak{M}$ and any element of $\mathfrak{N}$ should not be a small set. In particular this intersection is never void, since $\emptyset \in \mathfrak{G}$. Note that $\mathfrak{M}^{[\mathfrak{F}]}$ is the largest family of subsets of X that is $\mathcal{F}$-overlapping with respect to $\mathfrak{M}$. The finer the filter $\mathfrak{F}$ is, the lesser $\mathfrak{F}$-overlapping couples you get. We collect for familiarity or for further use some simple properties:

(i) Subfamilies of $\mathfrak{F}$-overlapping families are $\mathfrak{F}$-overlapping.

(ii) $\mathfrak{M} \mapsto \mathfrak{M}^{[\mathfrak{F}]}$ is decreasing.

(iii) If $\emptyset \in \mathfrak{M}$, then $\mathfrak{M}^{[\mathfrak{F}]} = \emptyset$ for any $\mathfrak{F}$; we shall typically avoid this situation.

(iv) If $X \in \mathfrak{M}$, then $\complement \mathfrak{G} \equiv 2^X \setminus \mathfrak{G}$ contains $\mathfrak{M}^{[\mathfrak{F}]}$, strictly in general.

(v) $\mathfrak{M} \subset \left(\mathfrak{M}^{[\mathfrak{F}]}\right)^{[\mathfrak{F}]}$, strictly in general.

In the sequel it will be important to isolate situations in which $\mathfrak{M}^{[\mathfrak{F}]}$ can be explicitly calculated.

Definition 2.6 *The family $\mathfrak{M}$ of subsets of X will be called* $\mathfrak{F}$-admissible *if*

(i) $\mathfrak{M}$ is stable under supersets,

(ii) for any $G \in \mathfrak{G}$, if $S \notin \mathfrak{M}$ then $G \cup S \notin \mathfrak{M}$.

It is natural to assume (i). Condition (ii) is once again connected to our image that "the elements of $\mathfrak{G}$ should be small". It is satisfied for any $\mathfrak{M}$ if $\mathfrak{F} = \mathfrak{F}_{\mathrm{tr}}$. But (ii) is by no means automatical; a simple counterexample is given by $\mathfrak{F} = \mathfrak{F}_{\mathrm{st}}$ and $\mathfrak{M} = \{X\}$.

Lemma 2.7 *Let $\mathfrak{M} \subset 2^X$ an $\mathfrak{F}$-admissible family and set $\mathfrak{M}^{\complement} := \{\complement M \mid M \in \mathfrak{M}\}$. Then $\mathfrak{M}^{[\mathfrak{F}]} = \complement\left(\mathfrak{M}^{\complement}\right)$.*

Proof: (a) Let $N \subset X$ be such that $M \cap N \notin \mathfrak{G}$ for all $M \in \mathfrak{M}$. Then $N \notin \complement\left(\mathfrak{M}^{\complement}\right)$ would mean that $\complement N \in \mathfrak{M}$, implying that $\emptyset \notin \mathfrak{G}$, which is impossible. Thus $\mathfrak{M}^{[\mathfrak{F}]} \subset \complement\left(\mathfrak{M}^{\complement}\right)$ without any assumption on $\mathfrak{M}$ and $\mathfrak{F}$.

(b) Assume now that $\mathfrak{M}$ is $\mathfrak{F}$-admissible, let $M \in \mathfrak{M}$ and $N \in \complement\left(\mathfrak{M}^{\complement}\right)$, i.e. $\complement N \notin \mathfrak{M}$. One has $M \subset (M \cap N) \cup (\complement N)$ and thus $M \cap N \in \mathfrak{G}$ would imply $M \notin \mathfrak{M}$, which is a contradiction. ■

Note that $\mathfrak{M}^{[\mathfrak{F}]}$ is the same for all the filters $\mathfrak{F}$ with respect to which $\mathfrak{M}$ is admissible.

Definition 2.8 *Let $\varphi \in BC$, $\lambda \in \mathbb{C}$ and $\mathfrak{M} \subset 2^X$. One says that φ* is of class $\mathfrak{M}$ *at λ if $M_\varepsilon(\varphi, \lambda) \in \mathfrak{M}$ for all $\varepsilon > 0$.*

As a simple example, $\lambda \in R(\varphi)$ iff φ is of class $\mathfrak{M} \equiv 2^X \setminus \{\emptyset\}$ at λ. More generally, by Lemma 1.3, $\lambda \in R_{\mathfrak{F}}(\varphi)$ iff φ is of class $\complement\mathfrak{G}$ at λ. On the other hand, φ is of class $\mathfrak{F}$ at λ iff $\lim_{\mathfrak{F}} \varphi = \lambda$, so that φ is of class $\mathfrak{F}_{\mathrm{tr}}$ at λ iff $\varphi = \lambda$ and φ is of class $\mathfrak{F}_{\mathrm{st}}$ at λ iff φ can be extended continuously on the one-point compactification $\dot{X} \equiv X \cup \{\infty\}$ and $\varphi(\infty) = \lambda$.

The notion of $\mathfrak{F}$-overlapping is relevant in our context because of the following result:

Proposition 2.9 *Let $\varphi, \psi \in BC$, $\lambda, \mu \in \mathbb{C}$ and $\mathfrak{M}, \mathfrak{N}$ two $\mathfrak{F}$-overlapping families of subsets of X.*

(a) If φ is of class $\mathfrak{M}$ at λ and ψ is of class $\mathfrak{N}$ at μ, then $\lambda\mu \in R_{\mathfrak{F}}(\varphi\psi)$.

(b) If $|\varphi|$ is of class $\mathfrak{M}$ in $\| \varphi \|_{\mathfrak{F}}$ and $|\psi|$ is of class $\mathfrak{N}$ in $\| \psi \|_{\mathfrak{F}}$, then φ and ψ are $\mathfrak{F}$-independent.

Proof: (a) For any $x \in X$ one has

$$|(\varphi\psi)(x) - \lambda\mu| \le |\varphi(x) - \lambda||\psi(x)| + |\lambda||\psi(x) - \mu| \le C\left(|\varphi(x) - \lambda| + |\psi(x) - \mu|\right),$$

where $C \equiv \max(|\lambda|, \| \psi \|)$. It follows that

$$M_{\varepsilon_1}(\varphi; \lambda) \cap M_{\varepsilon_2}(\psi; \mu) \subset M_\varepsilon(\varphi\psi; \lambda\mu)$$

for all $\varepsilon_1, \varepsilon_2 > 0$, with $C(\varepsilon_1 + \varepsilon_2) \le \varepsilon$. Therefore $M_{\varepsilon_1}(\varphi; \lambda) \in \mathfrak{M}$ and $M_{\varepsilon_2}(\psi; \mu) \in \mathfrak{N}$ will imply $M_\varepsilon(\varphi\psi; \lambda\mu) \notin \mathfrak{G}$, so $\lambda\mu \in R_{\mathfrak{F}}(\varphi\psi)$.

(b) It follows from (a) that $\| \varphi \|_{\mathfrak{F}} \| \psi \|_{\mathfrak{F}} \in R_{\mathfrak{F}}(|\varphi||\psi|)$, thus $\| \varphi\psi \|_{\mathfrak{F}} = \| \varphi \|_{\mathfrak{F}} \| \psi \|_{\mathfrak{F}}$, i.e. φ and ψ are $\mathfrak{F}$-independent.

$\blacksquare$

There is a result similar to (a) for sums of continuous functions, but we shall not need it.

By the results of this Section, it is now easy to outline a strategy to generate couples (φ, ψ) of $\mathfrak{F}$-independent bounded, continuous functions on X: Choose an $\mathfrak{F}$-admissible family $\mathfrak{M}$ of subsets of X and evaluate $\complement\left(\mathfrak{M}^{\complement}\right)$; if $|\varphi|$ is of class $\mathfrak{M}$ in $\| \varphi \|_{\mathfrak{F}}$ and $|\psi|$ is of class $\complement\left(\mathfrak{M}^{\complement}\right)$ in $\| \psi \|_{\mathfrak{F}}$, then φ and ψ are $\mathfrak{F}$-independent. In the sequel we shall study situations in which this strategy leads to interesting classes of functions on locally compact groups.

3 $\mathfrak{F}$-Independence on Locally Compact Groups

From now on we shall fix a non-compact, locally compact group X with unit e. As before, $\mathfrak{F}$ will be a filter on X defining the antifilter $\mathfrak{G} = \{G = \complement F \mid F \in \mathfrak{F}\}$. For $A, B \in X$ we set $AB := \{xy \mid x \in A \text{ and } y \in B\}$ and $A^{-1} := \{x^{-1} \mid x \in X\}$. We also use the notations $\mathfrak{F}^{-1} := \{F^{-1} \mid F \in \mathfrak{F}\}$ and $\mathfrak{G}^{-1} := \{G^{-1} \mid G \in \mathfrak{G}\}$. Plainly, $\mathfrak{F}^{-1}$ also is a filter and its associated antifilter is $\mathfrak{G}^{-1}$.

We shall introduce in [**3.1**] two types of subsets of X ($\mathfrak{G}$-syndetic and $\mathfrak{G}^{-1}$-replete) which, under suitable conditions, will be $\mathfrak{F}$-overlapping. We also consider the corresponding classes of functions. As a consequence of the results in [**2.2**], any $\mathfrak{G}$-syndetic function will be $\mathfrak{F}$-independent with respect to any $\mathfrak{G}^{-1}$-replete one. Unfortunately, neither of these two classes of functions form a C^*-algebra. In [**3.2**] and [**3.3**] we turn to conditions stronger than $\mathfrak{G}$-syndeticity and $\mathfrak{G}$-repleteness, which do define C^*-algebras. We extend and study for large classes of filters $\mathfrak{F}$ the standard notion of vanishing oscillation function. The relationship of $\mathfrak{G}_{\mathrm{st}}$-syndeticity with classical notions as minimality and almost periodicity is also examined.

Note that, due to the possible noncommutativity of the group X, we have to make choices in our definitions. For instance, my syndeticity may be called "left syndeticity" The reader will construct easily the reflected versions. In certain situations (only), the two versions define in fact the same object, but I shall not insist on this point.

3.1 $\mathfrak{G}$-syndetic and $\mathfrak{G}$-replete

Definition 3.1 *(a) A subset A of X is called* $\mathfrak{G}$-syndetic *if there is some $G \in \mathfrak{G}$ such that $AG = X$. A function $\varphi \in BC$ is called* $\mathfrak{G}$-syndetic at $\lambda \in \mathbb{C}$ *if $M_\varepsilon(\varphi; \lambda)$ is $\mathfrak{G}$-syndetic for all $\varepsilon > 0$.*

(b) A subset B of X is called $\mathfrak{G}$-replete *if for any $G \in \mathfrak{G}$ there exists an element x_G of X such that $x_G G \subset B$. A function $\psi \in BC$ is called* $\mathfrak{G}$-replete at $\lambda \in \mathbb{C}$ *if $M_\varepsilon(\psi; \lambda)$ is $\mathfrak{G}$-replete for all $\varepsilon > 0$.*

Since $\mathfrak{F}$ and $\mathfrak{G}$ determine each other, we could have said "$\mathfrak{F}$-syndetic" and "$\mathfrak{F}$-replete" and this would have led to a more uniform system of notations. I have chosen the present terminology because it seems to me closer to the substance of the two properties. Anyhow, we list here for further use equivalent conditions given in terms of $\mathfrak{F}$:

(a') A is $\mathfrak{G}$-syndetic iff there exists $F \in \mathfrak{F}$ such that $\cap_{y \in A}(yF) = \emptyset$.

(b') B is $\mathfrak{G}$-replete iff for any $F \in \mathfrak{F}$ there is some $x_F \in X$ such that $\complement B \subset x_F F$.

Obviously, the two properties above are preseved when going to supersets but, in general, are not stable under intersections. By replacing $\mathfrak{F}$ with a finer filter, one also gets a richer antifilter; therefore there will be more syndetic and less replete sets. If, for instance, $\mathfrak{F}$ is the trivial filter, then no set will be $\mathfrak{G}$-syndetic and all the sets will be $\mathfrak{G}$-replete .

Due to the results in [**2.2**], a basic fact will be the following "duality property" (see also [7] for a somewhat related setting):

Lemma 3.2 *(a) A is $\mathfrak{G}$-syndetic iff $\complement A$ is not $\mathfrak{G}^{-1}$-replete .*
(b) B is $\mathfrak{G}^{-1}$-replete iff $\complement B$ is not $\mathfrak{G}$-syndetic .

Proof: The two assertions are equivalent, so we shall only prove the second one:
$\complement B$ is not $\mathfrak{G}$-syndetic $\Leftrightarrow$ for any $G \in \mathfrak{G}$, $\exists x_G \notin (\complement B)G \Leftrightarrow$
$\Leftrightarrow$ for any $G \in \mathfrak{G}$, $\exists x_G$ such that $x_G G^{-1} \subset B \Leftrightarrow B$ is $\mathfrak{G}^{-1}$-replete .

■

Let us denote by $\mathfrak{G}\mathfrak{S}$ (resp. $\mathfrak{G}^{-1}\mathfrak{R}$) the family of $\mathfrak{G}$-syndetic (resp. $\mathfrak{G}^{-1}$-replete) subsets of X. By using Lemma 3.2 and the notations of [**2.2**] we can write $\mathfrak{G}\mathfrak{S}^{[\mathfrak{F}_{\mathrm{tr}}]} = \complement\left(\mathfrak{G}\mathfrak{S}^{\complement}\right) = \mathfrak{G}^{-1}\mathfrak{R}$, since $\mathfrak{G}\mathfrak{S}$ is obviously $\mathfrak{F}_{\mathrm{tr}}$-admissible for any $\mathfrak{G}$. Remarkably, the intersection between a $\mathfrak{G}$-syndetic and a $\mathfrak{G}^{-1}$-replete set is never void. However, in general, this intersection may belong to $\mathfrak{G}$. Let for example $X = \mathbb{R}$ and $\mathfrak{F} = \mathfrak{F}^{-1}$ the filter of all the subsets of $\mathbb{R}$ containing $\mathbb{Z}$. Then $(-\varepsilon, \varepsilon)$ is clearly $\mathfrak{G}$-syndetic for any $\varepsilon > 0$ and $\complement\mathbb{Z}$ is $\mathfrak{G}^{-1}$-replete but their (non-void) intersection is in $\mathfrak{G}$.

To eliminate examples of this type, we make use of the results of [**2.2**]. It will be convenient to impose explicit assumptions:

Hypothesis

(H1) $\mathfrak{F}$ is invariant under left translations: $x \in X,\ F \in \mathfrak{F} \ \Rightarrow\ xF \in \mathfrak{F}$.

(H2) $G_1, G_2 \in \mathfrak{G} \ \Rightarrow\ G_1G_2 \in \mathfrak{G}$.

We note that $\mathfrak{F}$ is finer than the Fréchet filter iff for every $x \in X$ there exists some $G \in \mathfrak{G}$ such that $x \in G$. In this case (H1) follows from (H2).

Theorem 3.3 *Let $\mathfrak{F}$ be a filter on the locally compact, non-compact group X satisfying (H1) and (H2). Then:*

(a) The families $\mathfrak{G}\mathfrak{S}$ and $\mathfrak{G}^{-1}\mathfrak{R}$ are $\mathfrak{F}$-overlapping.

(b) Let $\varphi, \psi \in BC$. If $|\varphi|$ is $\mathfrak{G}$-syndetic at $\parallel \varphi \parallel_{\mathfrak{F}}$ and $|\psi|$ is $\mathfrak{G}^{-1}$-replete at $\parallel \psi \parallel_{\mathfrak{F}}$, then φ, ψ are $\mathfrak{F}$-independent.

Proof: We begin by showing that the class $\mathfrak{G}\mathfrak{S}$ is $\mathfrak{F}$-admissible. As before, $\mathfrak{G}\mathfrak{S}$ is indeed stable under forming supersets so we only need to show that if $G \in \mathfrak{G}$ and $S \cup G$ is $\mathfrak{G}$-syndetic, S will also be $\mathfrak{G}$-syndetic. We know that $(S \cup G)G_0 = (SG_0) \cup (GG_0) = X$ for some $G_0 \in \mathfrak{G}$. The set S cannot be void, by (H2). But for any $x \in S$ one has $GG_0 \subset S[x^{-1}(GG_0)]$, thus $SG' = X$, where $G' = G_0 \cup [x^{-1}(GG_0)] \in \mathfrak{G}$ by (H1) and (H2).

Now, by combining Lemma 2.7 with Lemma 3.2 one gets $(\mathfrak{G}\mathfrak{S})^{[\mathfrak{F}]} = \complement\left(\mathfrak{G}\mathfrak{S}^{\complement}\right) = \mathfrak{G}^{-1}\mathfrak{R}$, that is a strengthened form of the point (a). Then (b) follows from Proposition 2.9. ■

Example. As it was already remarked before, no set is syndetic with respect to the trivial filter and all the sets are replete. So the conclusions of Theorem 3.3 hold (in a very formal way), as well as its hypothesis.

Example. Our main example is the standard filter. Instead of $\mathfrak{G}_{\mathrm{st}}$-syndetic and $\mathfrak{G}_{\mathrm{st}}$-replete we shall say, respectively, *asymptotically syndetic* and *asymptotically replete* (both for sets and for functions), although this is not standard terminology. In the vector group

$X=\mathbb{R}^n$, $\mathbb{Z}^n$ is asymptotically syndetic without being asymptotically replete and a non-void open cone is asymptotically replete without being asymptotically syndetic (except the case when it coincides with X). Both these sets are neither relatively compact (i.e. in $\mathfrak{G}_{\rm st}$) nor complements of relatively compact sets (i.e. in $\mathfrak{F}_{\rm st}$). In fact in $\mathbb{R}^n$ a subset B is asymptotically replete iff it contains balls of arbitrary radious and a subset A is asymptotically syndetic iff it meets every ball of a fixed, sufficiently large radious.

It is also easy to prove that if there exists a closed, co-compact subgroup Y of $\mathbb{R}^n$ (i.e. $\mathbb{R}^n/Y$ is compact) such that $y+Y\subset A$ for some $y\in\mathbb{R}^n$, then A is asymptotically syndetic. The converse is not true, as shown by the counter-example $A=\{n+1/n\mid n\in\mathbb{Z}\}$. In fact, sets with the property above do form an $\mathfrak{F}_{\rm st}$-admissible family, to which we can apply the general strategy outlined in **[2.2]**. We shall not pursue this here.

Obviously, the standard filter satisfies (H1) and (H2). We consider this example important enough to state apart

Corollary 3.4 *(a) The intersection between an asymptotically syndetic set and an asymptotically replete one cannot be relatively compact.*

(b) If $|\varphi|$ is asymptotically syndetic at $\|\varphi\|_{asy}$ and $|\psi|$ is asymptotically replete at $\|\psi\|_{asy}$, then φ and ψ are asymptotically independent.

Example. Traditionally, a $\mathfrak{G}_{\rm Fr}$-syndetic set is called *relatively dense*. On the real line $\cup_{n\in\mathbb{Z}}(n-\varepsilon,n+\varepsilon)$ is relatively dense for any $\varepsilon>0$. But $\mathbb{Z}$ or even $\cup_{n\in\mathbb{Z}^*}(n-1/|n|,n+1/|n|)$ are asymptotically syndetic without being relatively dense. The Fréchet filter $\mathfrak{F}_{\rm Fr}$ satisfies the assumptions of Theorem 3.3.

Example. We take $X=\mathbb{R}$. A subset F of $\mathbb{R}$ is called *large at* $+\infty$ if it contains an interval of the form $(a,+\infty)$. The family of all these sets forms a filter that we denote by $\mathfrak{F}_+$. Its associated antifilter $\mathfrak{G}_+$ is composed of all the subsets of $\mathbb{R}$ that are upper bounded, i.e. contained in an interval of the form $(-\infty,b]$, and its inverse $\mathfrak{G}_+^{-1}\equiv\mathfrak{G}_-$ consists of all the lower bounded subsets of $\mathbb{R}$. A is $\mathfrak{G}_+$-syndetic iff it is not upper bounded and B is $\mathfrak{G}_+^{-1}$-replete iff it belongs to $\mathfrak{F}_+$. In fact $\mathfrak{F}_+$-admissibility follows immediatly, implying that the conclusions of Theorem 3.3 are true in this case.

Example. In $X=\mathbb{R}^2$ we consider the antifilter $\mathfrak{G}$ of all the subsets of $\mathbb{R}_1:=\mathbb{R}\times\{0\}$; obviously (H2) is satisfied, but not (H1). Then $\mathbb{R}_1$ is $\mathfrak{G}^{-1}$-replete and any infinite vertical line is $\mathfrak{G}$-syndetic , but their intersection is a singleton belonging to $\mathfrak{G}$. It is instructive to see in which way $\mathfrak{F}$-admissibility fail.

Remark. We have now rather general results on $\mathfrak{F}$-independent functions on locally compact groups. In order to get also $\mathfrak{F}$-independent C^*-algebras, there is still some work to do. Unfortunately, the property of being of a certain class $\mathfrak{M}$ (even at *all* the resonable values λ) does not have the tendancy to force functions to form C^*-algebras. As a typical example, we sketch briefly a case in which two functions φ_1 and φ_2 are both asymptotically syndetic at all the points of their ranges (respectively) but their product is no longer asymptotically syndetic.

So let φ_1 and φ_2' be continuous functions on $\mathbb{R}$, both periodic of period 2. Plainly, they will be asymptotically syndetic at all the values λ belonging to their ranges, respectively. We assume that their product is null; for definiteness, suppose that ${\rm supp}\varphi_1=\cup_{n\in\mathbb{Z}}[2n-1,2n]$ and ${\rm supp}\varphi_2'=\cup_{n\in\mathbb{Z}}[2n,2n+1]$. Now modify φ_2' slightly by prescribing non-zero values also on $I:=\cup_{n\in\mathbb{Z}}[2n^2-1/4,2n^2]$, taking care not to enlarge the range. The new function φ_2, while no longer periodic, will still be asymptotically syndetic at all $\lambda \in R(\varphi_2)$, but now the product $\varphi_1\varphi_2$ has support $I \neq \emptyset$ and obviously it will be asymptotically syndetic only at the value $\lambda = 0$.

The same phenomenon can occur for sums of functions and, with slightly more effort, one can even take for φ_2 a translation of φ_1 (so asymptotic syndeticity does not transfer, in general, from a function to the smallest translational invariant C^*-algebra that contains it). And in **[3.3]** one sees that asymptotic repleteness is no better than that.

For the moment we "solve" the problem above just by giving it a name:

Definition 3.5 *(a) The function $\varphi \in BC$ is called* $\mathfrak{G}$-syndetic *if it is $\mathfrak{G}$-syndetic at any value $\lambda \in R_{\mathfrak{F}}(\varphi)$. The unital C^*-subalgebra $\mathcal{A}$ of BC is called* $\mathfrak{G}$-syndetic *if every $\varphi \in \mathcal{A}$ is $\mathfrak{G}$-syndetic.*

(b) The function $\psi \in BC$ is called $\mathfrak{G}$-replete *if it is $\mathfrak{G}$-replete at any value $\lambda \in R_{\mathfrak{F}}(\varphi)$. The unital C^*-subalgebra $\mathcal{B}$ of BC is called* $\mathfrak{G}$-replete *if every $\psi \in \mathcal{B}$ is $\mathfrak{G}$-replete.*

Of course, there is some arbitrariness in choosing to verify the properties at the values of the $\mathfrak{F}$-range. The main reason for this choice is the fact that for a large class of filters $\mathfrak{F}$ (for the standard filter for instance), the elements of $\mathfrak{G}$ are neither $\mathfrak{G}$-syndetic nor $\mathfrak{G}^{-1}$-replete . Then functions cannot be $\mathfrak{G}$-syndetic or $\mathfrak{G}^{-1}$-replete at values not belonging to their $\mathfrak{F}$-range, thus the framework of Definition 3.5 is maximal in these cases.

Remark. Even for $\mathfrak{F} = \mathfrak{F}_{\rm st}$ there are not enough constraints to assure uniform continuity. The function $\varphi(x) = \cos x^2$ is asymptotically syndetic without being uniformly continuous on $\mathbb{R}$. And one can get asymptotic repleteness just because of the behaviour at $+\infty$, so that the behaviour at $-\infty$ may lead easily to the failure of uniform continuity. Take, for example, $\psi(x) = \cos x^{1/2}$ on $\mathbb{R}_+$ and $\psi(x) = \cos x^2$ on $\mathbb{R}_-$.

In the following two subsections it will be shown that Definition 3.5 often leads to large and interesting C^*-algebras, so it is worth stating

Corollary 3.6 *Let $\mathfrak{F}$ be a filter on the locally compact, non-compact group X, satisfying (H1) and (H2). Let $\mathcal{A}$ and $\mathcal{B}$ unital C^*-subalgebras of BC. Assume that $\mathcal{A}$ is $\mathfrak{G}$-syndetic and $\mathcal{B}$ is $\mathfrak{G}^{-1}$-replete . Then $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{F}$-independent, thus $\langle \mathcal{A} \cdot \mathcal{B} \rangle_{\mathfrak{F}}$ is canonically isomorphic to $\mathcal{A}_{\mathfrak{F}} \otimes \mathcal{B}_{\mathfrak{F}}$ and $X\left(\langle \mathcal{A} \cdot \mathcal{B} \rangle_{\mathfrak{F}}\right)$ is canonically homeomorphic to $X\left(A_{\mathfrak{F}}\right) \times X\left(B_{\mathfrak{F}}\right)$.*

3.2 Syndeticity in the sense of translations and minimality

If a family $\mathfrak{M}$ of subsets of X is invariant under right translations, the fact that φ is of class $\mathfrak{M}$ at a value λ can be expressed in translational terms. We present here only the case $\mathfrak{M} = \mathfrak{G}\mathfrak{S}$, since this is connected (at least in the case $\mathfrak{F} = \mathfrak{F}_{\rm st}$) with classical types of functions generalizing the usual almost periodic functions.

Definition 3.7 *Let $\varphi \in BC$, $\varepsilon > 0$ and $S \subset X$. An element y of X is called* an ε-translation element for φ with respect to S *if $|\varphi(yx) - \varphi(x)| < \varepsilon$ for all $x \in S$. The set of all these elements will be denoted by $T_\varepsilon(\varphi; S)$.*

The set $T_\varepsilon(\varphi; S)$ is increasing with ε and decreasing with S. When $S = X$, ε-translation elements are called *ε-quasiperiods.* An element of $\cap_{\varepsilon>0} T_\varepsilon(\varphi; X)$ is *a period* and this was the primitive notion that led to these generalizations. The set of all periods of a function φ is a closed subgroup of X. Note that $T_\varepsilon(\varphi; \cup_j S_j) = \cap_j T_\varepsilon(\varphi; S_j)$, so that $T_\varepsilon(\varphi; S) = \cap_{x \in S} T_\varepsilon(\varphi; \{x\})$.

Definition 3.8 *Let $\varphi \in BC$. We shall say that φ* is $\mathfrak{G}$-syndetic in the sense of translations in $x \in X$ *if the set $T_\varepsilon(\varphi; \{x\})$ is $\mathfrak{G}$-syndetic for any $\varepsilon > 0$.*

Proposition 3.9 *Assume that the filter $\mathfrak{F}$ is invariant under left translations. For the bounded, continuous function φ on X and the complex number λ the following assertions are equivalent:*

(i) φ is $\mathfrak{G}$-syndetic at λ.

(ii) φ is $\mathfrak{G}$-syndetic in the sense of translations in some point x such that $\varphi(x) = \lambda$.

(iii) φ is $\mathfrak{G}$-syndetic in the sense of translations in any point x such that $\varphi(x) = \lambda$.

As a particular case, φ is $\mathfrak{G}$-syndetic iff it is $\mathfrak{G}$-syndetic in the sense of translations in every point of $\varphi^{-1}[R_{\mathfrak{F}}(\varphi)]$.

Proof: It is easily shown that $M_\varepsilon(\varphi; \varphi(x)) = T_\varepsilon(\varphi, x)x$ for any $x \in X$. Since $\mathfrak{F}$ is invariant under left translations, the family of $\mathfrak{G}$-syndetic sets will be invariant under right translations, so $M_\varepsilon(\varphi; \varphi(x))$ and $T_\varepsilon(\varphi, x)$ will be simultaneously $\mathfrak{G}$-syndetic . ■

Our rephrasing of $\mathfrak{G}$-syndeticity in translational terms leads to connections with standard classes of functions, and some of these do form C^*-algebras. Some preparation is needed.

For each $y \in X$ one set $r_y, l_y : X \to X$, $r_y(x) := xy$ and $l_y(x) := yx$ for all $x \in X$. By duality, one gets automorphisms of BC: $R_y(\varphi) := \varphi \circ r_y$ and $L_y(\varphi) := \varphi \circ l_y$ for all $\varphi \in BC$. Let us denote by RUC the C^*-algebra of all bounded, continuous functions φ on X satisfying $\lim_{y \to e} \parallel R_y\varphi - \varphi \parallel = 0$. Equivalently, φ is uniformly continuous with respect to the left uniform structure on X.

A C^*-subalgebra $\mathcal{C}$ of RUC will be called *an X-algebra* if it is unital and invariant under the left translations: $L_y\mathcal{C} \subset \mathcal{C}$ for all $y \in X$. RUC itself is an X-algebra. Once again by duality, the spectrum $X(\mathcal{C})$ of $\mathcal{C}$ becomes the compact space of a (topological) dynamical system with group X. If φ is an element of RUC, we denote by $\mathcal{C}_\varphi$ the smallest X-algebra containing φ.

We say that the X-algebra $\mathcal{C}$ is *X-simple* if it contains no proper left invariant ideal. Plainly, this is equivalent with the fact that $X(\mathcal{C})$ is minimal, i.e. it containes no proper closed, invariant subset (equivalently, all the orbits are dense).

Now we are in a position to define and use minimal functions. For proofs and a detailed discussion we refer to [4], [6] and [10].

Definition 3.10 *A function $\varphi \in RUC$ is called* minimal *if the X-algebra $\mathcal{C}_\varphi$ it generates is X-simple.*

Our interest in minimal functions is due to the next result:

Lemma 3.11 *Let $\varphi \in RUC$. Then φ is minimal iff for any finite set G and any $\varepsilon > 0$ the set $T_\varepsilon(\varphi; G)$ is relatively dense.*

Thus, by applying Proposition 3.9, we get readily:

Proposition 3.12 *Any minimal function is $\mathfrak{G}_{Fr}$-syndetic, thus also asymptotically syndetic.*

In fact, a minimal function will also be $\mathfrak{G}$-syndetic for any filter $\mathfrak{F}$ finer than the Fréchet filter.

It was noticed before that asymptotically syndetic functions may fail to be (both right and left) uniformly continuous, thus they may also fail to be minimal. Even within the class of uniformly continuous functions there is discrepancy, since any non-null C_0-function is asymptotically syndetic, without being minimal. Furthermore, for a minimal function the range and the asymptotic range are the same, while this can be evidently false for many asymptotically syndetic functions. By one of the examples of **[3.1]** one constructs easily functions that are asymptotically syndetic without being $\mathfrak{G}_{\mathrm{Fr}}$-syndetic.

It is known that minimal functions form a closed, left invariant subset of BC which is not a vector space in general. But, fortunately, there is a way to put into evidence large C^*-algebras composed of minimal functions; they will be $\mathfrak{G}_{\mathrm{Fr}}$-syndetic, thus asymptotically syndetic C^*-algebras. First, a C^*-algebra contains only minimal functions iff it is X-simple, i.e. its spectrum is a minimal dynamical system. It is interesting to consider maximal (under inclusion) C^*-algebras of this kind. They are in one-to-one correspondence with minimal left invariant ideals of RUC. Even the intersection of all the maximal X-simple C^*-algebras contains many famous types of functions (distal, almost automorphic) and in particular it contains (often as a very small subset) the C^*-algebra AP of all the almost periodic functions on X (for which we refer to [4], [5] or [9]). Let us recall that φ is called *almost periodic* if the set $R_X(\varphi) := \{R_y\varphi \mid y \in X\}$ is norm relatively compact in BC or, equivalently, for any $\varepsilon > 0$ the set $T_\varepsilon(\varphi; X)$ of all the ε-quasiperiods of φ is relatively dense. In fact this second characterization (a kind of global version of $\mathfrak{G}_{\mathrm{Fr}}$-syndeticity) makes clear the relationship between almost periodic, minimal and $\mathfrak{G}_{\mathrm{Fr}}-$(or asymptotically) syndetic functions.

Remark: The class WAP of all weakly almost periodic functions on X (cf. [4]) is a C^*-algebra containing $C_0 + AP$. There exist locally compact groups X (called *minimally weakly almost periodic*) for which $C_0 + AP = WAP$, but in most cases (ex: X abelian) the inclusion is strict. Unfortunately, weakly almost periodic functions may fail to have good syndeticity properties. Let, for example, $S = \{x_n\}_{n\in\mathbb{N}} \subset X \equiv \mathbb{Z}$ such that $x_{n+1} - x_n \to \infty$. Then the characteristic function χ_S of S is weakly almost periodic, by a result of [3], but $S = M_\varepsilon(\chi_S; 1)$ (ε small) is not asymptotically syndetic.

3.3 Vanishing oscillation

First we shall introduce a subset of BC that is large enough to contain all the possible $\mathfrak{G}^{-1}$-replete functions on X. This will give us an ideea on the behaviour of these functions.

Definition 3.13 *(a) For any $\psi \in BC$ and any subset M of X we set $\widehat{\mathrm{osc}}_M(\psi) : X \to \mathbb{R}_+$, $[\widehat{\mathrm{osc}}_M(\psi)](x) := \sup_{y,y'\in M} |\psi(xy) - \psi(xy')|$, $\forall x \in X$.*

(b) The function $\psi \in BC$ is in $\widehat{VO}_{\mathfrak{G}^{-1}}$ if for any $G \in \mathfrak{G}$ there exists a filter $\mathfrak{F}^G$ such that $\lim_{\mathfrak{F}^G} [\widehat{\mathrm{osc}}_{G^{-1}}(\psi)] = 0$.

The letters VO stand for "vanishing oscillation". It is easy to check that $\widehat{VO}_{\mathfrak{G}^{-1}}$ is a C^*-algebra (we give below the proof of the similar assertion for $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$). Our main interest in $\widehat{VO}_{\mathfrak{G}^{-1}}$ is due to

Proposition 3.14 *If ψ is $\mathfrak{G}^{-1}$-replete at some $\lambda \in \mathbb{C}$, then $\psi \in \widehat{VO}_{\mathfrak{G}^{-1}}$.*

Proof: One has $[\widehat{\mathrm{osc}}_{G^{-1}}(\psi)](x) \le 2\sup_{y\in G} |\psi(xy^{-1}) - \lambda|$. But ψ is $\mathfrak{G}^{-1}$-replete at λ means that for all $G \in \mathfrak{G}$ and all $\varepsilon > 0$ there exists $x_{G,\varepsilon}$ such that $|\psi(x_{G,\varepsilon}y^{-1}) - \lambda| < \varepsilon/2$, $\forall y \in G$. Thus, for all $G \in \mathfrak{G}$ and all $\varepsilon > 0$ there exists $x_{G,\varepsilon} \in X$ such that $[\widehat{\mathrm{osc}}_{G^{-1}}(\psi)](x_{G,\varepsilon}) < \varepsilon$.

So, for each element $G \in \mathfrak{G}$ there is some net $\bar{x}_G = (x_{G,\varepsilon})_{\varepsilon>0}$ in X such that one has $\lim_{\varepsilon\to 0} [\widehat{\mathrm{osc}}_{G^{-1}}(\psi)]\,(x_{G,\varepsilon}) = 0$. We denote by $\mathfrak{F}^G$ the filter canonically assigned to the net $\bar{x}_G$. Then $\lim_{\mathfrak{F}^G} [\widehat{\mathrm{osc}}_{G^{-1}}(\psi)] = 0$. ■

We introduce now another (more restricted) type of vanishing oscillation, that will lead to C^*-algebras of $\mathfrak{G}^{-1}$-replete functions.

Definition 3.15 *(a) Let $\psi \in BC$ and $M \subset X$. We set* $\mathrm{osc}_M(\psi) : X \to \mathbb{R}_+$, $[\mathrm{osc}_M(\psi)]\,(x) := \sup_{y\in M} |\psi(xy) - \psi(x)|$*; we call it* the oscillation of the function ψ on the set M*.*

(b) Let $\mathfrak{F}$ and $\mathfrak{F}'$ two filters on X. We say that the function $\psi \in BC$ has vanishing $\mathfrak{G}^{-1}$-oscillation along the filter $\mathfrak{F}'$ *if for all $G \in \mathfrak{G}$ one has* $\lim_{\mathfrak{F}'} [\mathrm{osc}_{G^{-1}}(\psi)] = 0$*. The family of all these functions will be denoted by $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$.*

The behaviour required by the property of belonging to $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is quite transparent in certain particular cases. Let us suppose, for instance, that $X = \mathbb{R}^n$ and that the bounded, continuous function ψ is of class C^1. One can write $\psi(x+y)-\psi(x) = \int_0^1 ds\, y\cdot(\nabla\psi)(x+sy)$, thus $[\mathrm{osc}_M(\psi)]\,(x) \le \sup_{y\in M} [|y| \sup_s |(\nabla\psi)(x+sy)|]$, which reduces our checking to estimates on the derivatives of ψ. In the particular case $\mathfrak{F} = \mathfrak{F}' = \mathfrak{F}_{\mathrm{st}}$ one has just to insure that $\partial_j\psi \in C_0$, $\forall j = 1, \dots, n$; in fact this type of functions forms a dense subset of $VO(\mathbb{R}^n) \equiv VO_{\mathfrak{G}_{\mathrm{st}}^{-1}}(\mathfrak{F}_{\mathrm{st}})$. Anyhow, the sum between a C_0-function and a continuous function that is homogeneous of degree zero ouside a ball is a simple example of function with vanishing oscillation on $\mathbb{R}^n$.

Proposition 3.16 *(a) $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is a unital C^*-subalgebra of BC.*

(b) If $\psi \in VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$, then ψ is $\mathfrak{G}^{-1}$-replete at any value $\lambda \in R_{\mathfrak{F}'}(\psi)$.

Proof: (a) Obviously, the constant functions have vanishing oscillation. The fact that $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is a *-algebra follows easily from the properties of "$\lim_{\mathfrak{F}'}$" and the relations

$$\begin{aligned}
&\mathrm{osc}_M(\psi + \psi') \le \mathrm{osc}_M(\psi) + \mathrm{osc}_M(\psi'),\\
&\mathrm{osc}_M(\beta\psi) = |\beta| \mathrm{osc}_M(\psi),\\
&\mathrm{osc}_M(\psi\psi') \le \parallel \psi' \parallel \mathrm{osc}_M(\psi) + \parallel \psi \parallel \mathrm{osc}_M(\psi'),\\
&\mathrm{osc}_M(\overline{\psi}) = \mathrm{osc}_M(\psi),
\end{aligned}$$

easy to prove for all $\psi, \psi' \in BC, M \subset X$ and $\beta \in \mathbb{C}$.

To show that $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is closed, let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of elements of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ converging to some function $\psi \in BC$. For any $G \in \mathfrak{G}$ one has

$$\begin{aligned}
\mathrm{osc}_{G^{-1}}(\psi) &\le \mathrm{osc}_{G^{-1}}(\psi_n) + |\mathrm{osc}_{G^{-1}}(\psi) - \mathrm{osc}_{G^{-1}}(\psi_n)|\\
&\le \mathrm{osc}_{G^{-1}}(\psi_n) + \mathrm{osc}_{G^{-1}}(\psi - \psi_n) \le \mathrm{osc}_{G^{-1}}(\psi_n) + 2 \parallel \psi - \psi_n \parallel,
\end{aligned}$$

which leads immediately to the conclusion.

(b) Let us fix $\varepsilon > 0$. The condition $\lim_{\mathfrak{F}'} [\mathrm{osc}_{G^{-1}}(\psi)] = 0$ implies that there is some $F_0' \in \mathfrak{F}'$ such that $|\psi(xy^{-1}) - \psi(x)| < \varepsilon/2$ for all $x \in F_0'$ and all $y \in G$.

Let $\lambda \in R_{\mathfrak{F}'}(\psi)$; then for any $F' \in \mathfrak{F}'$ there exists $x_{F'} \in F'$ such that $|\psi(x_{F'}) - \lambda| < \varepsilon/2$. Let us take $F' = F_0'$; for any $y \in G$ one has

$$|\psi(x_{F_0'} y^{-1}) - \lambda| \le |\psi(x_{F_0'} y^{-1}) - \psi(x_{F_0'})| + |\psi(x_{F_0'}) - \lambda| < \varepsilon.$$

Thus for every $\varepsilon > 0$ and $G \in \mathfrak{G}$ there is some $x \in X$ such that $xG^{-1} \subset M_\varepsilon(\psi; \lambda)$.

$\blacksquare$

Remark The C^*-algebra $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is increasing with $\mathfrak{F}'$ and decreasing with $\mathfrak{F}$ (so, when $\mathfrak{F} = \mathfrak{F}'$, in general there is no monotony). Admitting that $\mathfrak{F}$ is fixed, one would like to take $\mathfrak{F}'$ as big as possible (an ultrafilter) in order to increase $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$. But this may lead to small sets $R_{\mathfrak{F}'}(\psi)$. To be sure to cover $R_{\mathfrak{F}}(\psi)$ entirely, one should consider the case $\mathfrak{F}' = \mathfrak{F}$: *all the functions in $VO_{\mathfrak{G}^{-1}}(\mathfrak{F})$ are $\mathfrak{G}^{-1}$-replete* . However, it might happen that the important value $\lambda = \parallel \psi \parallel_{\mathfrak{F}}$ still belongs to $R_{\mathfrak{F}'}(\psi)$ even if $\mathfrak{F} \subsetneqq \mathfrak{F}'$.

Example. Proposition 3.16 (b) gives us a hint about the way the class of all the $\mathfrak{G}^{-1}$-replete functions fails to form an algebra or at least a vector space; the reason is the possible dependence of the filter $\mathfrak{F}'$ on ψ. Let, for example, $X = \mathbb{R}$, $\psi(x) = \cos x$ if $x < 0$, $\psi(x) = \cos(|x|^{1/2})$ if $x \geq 0$ and $\psi'(x) = \psi(-x)$. The oscillation of ψ on any compact set converges to zero along the filter $\mathfrak{F}_+$ composed of all the sets that are large at $+\infty$ (cf. **[3.1]**), thus ψ is asymptotically replete at any value $\lambda \in R_{\mathfrak{F}_+}(\psi) = [-1, 1]$ (and it would be impossible, of course, to be $\mathfrak{G}^{-1}$-replete outside this set). The same is true for ψ', just by taking into account the filter $\mathfrak{F}_- := -\mathfrak{F}_+$. But the sum and the product of the two functions ψ and ψ' are no longer asymptotically replete. In $X = \mathbb{R}^n$, $n \geq 2$ one has even a wider freedom to construct counterexamples.

Let us close this section by a discussion of the spectrum of the C^*-algebra $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ that is useful for applications to spectral analysis, as hinted in the Introduction. We prove first

Proposition 3.17 *Suppose that for all $G' \in \mathfrak{G}'$ and $G \in \mathfrak{G}$, we have $G'G \in \mathfrak{G}'$ (we write briefly $\mathfrak{G}'\mathfrak{G} \subset \mathfrak{G}'$). Then $\widetilde{BC}^{\mathfrak{F}'} \subset VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$. In particular, if $\mathfrak{F}'$ is in addition finer than the standard filter, $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is a C^*-algebra of the first kind.*

Proof: Let $\psi \in \widetilde{BC}^{\mathfrak{F}'}$ and $c = \lim_{\mathfrak{F}'} \psi$. Then for all $\varepsilon > 0$, $M_{\varepsilon/2}(\psi; c) \in \mathfrak{F}'$. For any $G \in \mathfrak{G}$ one has

$$[\mathrm{osc}_{G^{-1}}(\psi)]\,(x) \leq \sup_{y \in G} |\psi(xy^{-1}) - c| + |\psi(x) - c|.$$

But $\sup_{y \in G} |\psi(xy^{-1}) - c| < \varepsilon/2$ if xy^{-1} is in $M_{\varepsilon/2}(\psi; c)$ for all $y \in G^{-1}$, i.e. if x belongs to $\cap_{y \in G} M_{\varepsilon/2}(\psi; c)y$. If $F' \in \mathfrak{F}'$ then $\cap_{z \in G} F'z \in \mathfrak{F}'$, by using the hypothesis of the Proposition:

$$\complement \left[\cap_{y \in G} F'y\right] = \cup_{y \in G}(\complement F')y = (\complement F')G \in \mathfrak{G}'.$$

Thus, we have shown that for every $\varepsilon > 0$

$$M_\varepsilon\left(\mathrm{osc}_{G^{-1}}(\psi); 0\right) \supset M_{\varepsilon/2}(\psi; c) \cap \left[\cap_{y \in G} M_{\varepsilon/2}(\psi; c)y\right] \in \mathfrak{F}',$$

i.e. $\lim_{\mathfrak{F}'} [\mathrm{osc}_{G^{-1}}(\psi)] = 0$; the proof of the first assertion is finished.

Assume now that $\mathfrak{F}_{\mathrm{st}} \subset \mathfrak{F}$. Then $C_0 \subset C(\dot{X}) = \widetilde{BC}^{\mathfrak{F}_{\mathrm{st}}} \subset \widetilde{BC}^{\mathfrak{F}'} \subset VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$. $\blacksquare$

Example. Choose once again $X = \mathbb{R}$. Obviously, $\widetilde{BC}^{\mathfrak{F}_+}$ consists of all the bounded, continuous functions on $\mathbb{R}$ that have a limit at $+\infty$. By Proposition 3.17, it is contained in the C^*-algebra $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}_+)$ for any filter $\mathfrak{F}$ such that $\mathfrak{G}_+ + \mathfrak{G} \subset \mathfrak{G}_+$. Thus, for such a filter $\mathfrak{F}$, if $\lim_{x \to +\infty} \psi$ exists, then ψ is $\mathfrak{G}^{-1}$-replete . Simple examples are $\mathfrak{F} = \mathfrak{F}_{\mathrm{st}}$ and $\mathfrak{F} = \mathfrak{F}_+$. But $\mathfrak{F} = \mathfrak{F}_-$ is not allowed and C_0 is not contained into $VO_{\mathfrak{G}_-^{-1}}(\mathfrak{F}_+) = VO_{\mathfrak{G}_+^{-1}}(\mathfrak{F}_+)$.

The characterization of the spectrum of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is given in

Proposition 3.18 *Assume that (i) $\mathfrak{F}_{Fr} \subset \mathfrak{F}$, (ii) $\mathfrak{G}'\mathfrak{G} \subset \mathfrak{G}'$ and (iii) $\mathfrak{G}\mathfrak{G}' \subset \mathfrak{G}'$.*

Then the spectrum of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is the disjoint union of the spectrum of $BC^{\mathfrak{F}'}$ (which can be a very complicated space) and a compact space formed of points that are fixed under the left translations by elements of the group X.

Proof: By a simple calculation one gets $\mathrm{osc}_M(L_x\psi) = L_x[\mathrm{osc}_M(\psi)]$ for all $\psi \in BC$, $x \in X$ and $M \subset X$. Consequently, $L_x[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')] = VO_{\mathfrak{G}^{-1}}[L_{x^{-1}}(\mathfrak{F}')]$, where $L_{x^{-1}}(\mathfrak{F}') := \{L_{x^{-1}}(F') \mid F' \in \mathfrak{F}'\}$. But (i) and (iii) imply that $\mathfrak{F}'$ is invariant under left translations, thus $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ has the same property. By duality, the group X will act upon the spectrum of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ by $[l_x(\tau)](\psi) = \tau(L_x\psi)$, $x \in X$, $\psi \in VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ and $\tau \in X[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')]$.

Because of the assumption (ii), Proposition 3.17 tells us that $BC^{\mathfrak{F}'}$ is an ideal of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$; this ideal is also invariant under left translations. Consequently, $X[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')]$ will be the disjoint union of $X\left[BC^{\mathfrak{F}'}\right]$ and $X[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')_{\mathfrak{F}'}]$ (the spectrum of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')/BC^{\mathfrak{F}'}$), both these spaces being invariant. The space $X[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')_{\mathfrak{F}'}]$ can be identified with the set of characters of $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ that are null on $BC^{\mathfrak{F}'}$.

Let $\psi \in VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$, $x \in X$ and $G \in \mathfrak{G}$ such that $x \in G$. Then $0 \le \mathrm{osc}_{\{x\}^{-1}}(\psi) \le \mathrm{osc}_{G^{-1}}(\psi) \in BC^{\mathfrak{F}'}$, thus $\mathrm{osc}_{\{x\}^{-1}}(\psi) \in BC^{\mathfrak{F}'}$, i.e. $\lim_{\mathfrak{F}'}[L_{x^{-1}}(\psi) - \psi] = 0$. Then if $\tau \in X[VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')]$ restricted to $BC^{\mathfrak{F}'}$ is null, we shall have

$$[l_{x^{-1}}(\tau) - \tau](\psi) = \tau[L_{x^{-1}}(\psi) - \psi] = 0.$$

The proof is finished since x and ψ are arbitrary.

■

Remark In general $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is not an X-algebra; right uniform continuity is not granted without further hypothesis. The reader will prove easily that if $\mathfrak{F}' \subset \mathfrak{F}_{\mathrm{st}} \subset \mathfrak{F}$ then $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}') \subset RUC$. Thus, in this case, $VO_{\mathfrak{G}^{-1}}(\mathfrak{F}')$ is indeed an X-algebra and X acts *continuously* on its spectrum. Note that in the case $\mathfrak{F}' = \mathfrak{F}_{\mathrm{st}} = \mathfrak{F}$ all the assumptions above are verified. $VO(X) \equiv VO_{\mathfrak{G}_{\mathrm{st}}^{-1}}(\mathfrak{F}_{\mathrm{st}})$ is an X-algebra of the first kind and the spectrum of $VO(X)/C_0$ is composed of points that are fixed by the action of X.

3.4 Some extensions

Let us mention briefly an enlargement of the concepts of $\mathfrak{G}$-syndeticity and $\mathfrak{G}$-repleteness.

Definition 3.19 *Let $\mathfrak{F}$ and $\mathfrak{F}'$ two filters on X. One says that the subset A of X is $(\mathfrak{G}, \mathfrak{F}')$-*syndetic *if there exists some $G \in \mathfrak{G}$ such that $AG \in \mathfrak{F}'$. And $B \in X$ is called $(\mathfrak{G}, \mathfrak{F}')$-*replete *if for any $G \in \mathfrak{G}$ and $F' \in \mathfrak{F}'$ there exists $x \in F'$ such that $xG \subset B$. (Another terminology could be " $\mathfrak{G}$-syndetic (resp. $\mathfrak{G}$-replete) in the direction of $\mathfrak{F}'$ ".)*

Similar terminology will be used for functions.

Obviously, $(\mathfrak{G}, \mathfrak{F}_{\mathrm{tr}})$-syndetic means $\mathfrak{G}$-syndetic and $(\mathfrak{G}, \mathfrak{F}_{\mathrm{tr}})$-replete means $\mathfrak{G}$-replete. But by choosing finer filters $\mathfrak{F}'$, one gets more $(\mathfrak{G}, \mathfrak{F}')$-syndetic and less $(\mathfrak{G}, \mathfrak{F}')$-replete subsets. Let us take, for example, $X = \mathbb{R}$, $\mathfrak{F} = \mathfrak{F}_{\mathrm{st}}$ and $\mathfrak{F}' = \mathfrak{F}_+$, the filter appearing in a previous example. Then $\mathbb{N}$ is $(\mathfrak{G}_{\mathrm{st}}, \mathfrak{F}_+)$-syndetic without being $\mathfrak{G}_{\mathrm{st}}$-syndetic and $(-\infty, 0]$ is $\mathfrak{G}_{\mathrm{st}}$-replete without being $(\mathfrak{G}_{\mathrm{st}}, \mathfrak{F}_+)$-replete. But note that the set $[0, \infty)$ is $(\mathfrak{G}_{\mathrm{st}}, \mathfrak{F}_+)$-replete.

However, there are situations in which nothing new appears:

Proposition 3.20 *If $\mathfrak{F}' \subset \mathfrak{F}$ and $\mathfrak{F}$ is invariant under left translations, then A is $(\mathfrak{G}, \mathfrak{F}')$-syndetic iff A is $\mathfrak{G}$-syndetic.*

Proof: It is enough to prove that, under the assumptions of the Proposition, if A is $(\mathfrak{G}, \mathfrak{F}')$-syndetic then it is $\mathfrak{G}$-syndetic. So suppose that there exists $G \in \mathfrak{G}$ such that $AG \equiv F_0 \in \mathfrak{F}$. Set $G_0 = \complement F_0 \in \mathfrak{G}$; we have $(AG) \cup G_0 = X$. But A is non-void. For $x \in A$ one has $G_0 = x \cdot \left[x^{-1}G_0\right] \subset A\left[x^{-1}G_0\right]$, thus $X = (AG) \cup \left(A[x^{-1}G_0]\right) = A\left(G \cup [x^{-1}G_0]\right)$ and $G \cup [x^{-1}G_0] \in \mathfrak{G}$. ■

It is easy to check that these two classes introduced above satify a result analogue to Lemma 3.2, by a similar proof.

Lemma 3.21 *(a) A is $(\mathfrak{G}, \mathfrak{F}')$-syndetic iff $\complement A$ is not $(\mathfrak{G}^{-1}, \mathfrak{F}')$-replete.*
(b) B is $(\mathfrak{G}^{-1}, \mathfrak{F}')$-replete iff $\complement B$ is not $(\mathfrak{G}, \mathfrak{F}')$-syndetic.

Since in many particular cases the class $\mathfrak{M}$ of all the $(\mathfrak{G}, \mathfrak{F}')$-syndetic subsets of X is indeed $\mathfrak{F}$-admissible, one can apply the machine developed in Section 2 in order to extend the results of this Section. To be specific, by a proof similar to that of Theorem 3.3 we get

Proposition 3.22 *Assume that the filter $\mathfrak{F}$ verifies (H1) and (H2) and that $\mathfrak{G} \cap \mathfrak{F}' = \emptyset$. Then the family of $(\mathfrak{G}, \mathfrak{F}')$-syndetic sets is $\mathfrak{F}$-admissible.*

Note that the condition $\mathfrak{G} \cap \mathfrak{F}' = \emptyset$ is verified as soon as $\mathfrak{F}$ and $\mathfrak{F}'$ are comparable.

We leave to the reader the task of reformulating Theorem 3.3 and Corollary 3.6 for the present, more general case, as well as finding relevant examples. We end this Section by indicating a rather large class of $(\mathfrak{G}, \mathfrak{F}')$-syndetic functions that can be constructed from $\mathfrak{G}$-syndetic functions.

Proposition 3.23 *Let $\varphi \in BC$ and $\lambda \in \mathbb{C}$. Assume that there exists a function $\varphi_0 \in BC$ which is $\mathfrak{G}$-syndetic at λ such that $\lim_{\mathfrak{F}'}(\varphi - \varphi_0) = 0$. Then φ is $(\mathfrak{G}, \mathfrak{F}')$-syndetic at λ.*

Proof: For any $x \in X$ we have

$$|\varphi(x) - \lambda| \leq |\varphi(x) - \varphi_0(x)| + |\varphi_0(x) - \lambda|,$$

which implies that $M_{\varepsilon_1}(\varphi - \varphi_0; 0) \cap M_{\varepsilon_2}(\varphi_0; \lambda) \subset M_{\varepsilon_1+\varepsilon_2}(\varphi; \lambda)$ for all $\varepsilon_1, \varepsilon_2 > 0$. So it is enough to show that if $M_1 \in \mathfrak{F}'$ and M_2 is $\mathfrak{G}$-syndetic, then $M_1 \cap M_2$ is $(\mathfrak{G}, \mathfrak{F}')$-syndetic.

Let $G \in \mathfrak{G}$ such that $M_2 G = X$; there is no loss of generality to suppose that $e \in G$. Then

$$(M_1 \cap M_2)G = (M_1 G) \cap (M_2 G) = M_1 G \supset M_1 \in \mathfrak{F}',$$

thus $(M_1 \cap M_2)G \in \mathfrak{F}'$ too. ■

Corollary 3.24 *If $\mathfrak{F}' \subset \mathfrak{F}$, φ_0 is $\mathfrak{G}$-syndetic and $\varphi - \varphi_0 \in BC^{\mathfrak{F}'}$ then φ is $(\mathfrak{G}, \mathfrak{F}')$-syndetic (at any value of its $\mathfrak{F}$-range).*

Proof: This follows easily from Proposition 3.23 and the fact that if $\varphi - \varphi_0 \in BC^{\mathfrak{F}'} \subset BC^{\mathfrak{F}}$, then φ and φ_0 have the same $\mathfrak{F}$-range. ■

So let $\mathcal{A}$ a $\mathfrak{G}$-syndetic algebra. Then if $\mathfrak{F}' \subset \mathfrak{F}$, then $\mathcal{A} + BC^{\mathfrak{F}'}$ will be a $(\mathfrak{G}, \mathfrak{F}')$-syndetic C^*-algebra.

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