SINGULARLY PERTURBED HAMILTONIANS OF A QUANTUM RAYLEIGH GAS DEFINED AS QUADRATIC FORMS

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Abstract. We consider a class of singular, zero-range perturbations of the quantum hamiltonian of a system consisting of a test particle and N harmonic oscillators (Rayleigh gas). Using the theory of quadratic forms we construct the self-adjoint and bounded from below perturbed hamiltonian and we give representation formulas for the resolvent and the unitary group. The one dimensional, two dimensional and three dimensional cases are discussed.

1. Introduction

In this paper we study the quantum hamiltonian of a Rayleigh gas made of a test particle interacting with N harmonic oscillators via zero-range or point interactions.
We shall use the theory of quadratic forms to construct the hamiltonian as a self adjoint and semibounded operator in the Hilbert space $L^2(\mathbb{R}^{d(N+1)})$, where $d = 1, 2, 3$ is the space dimension.
At least formally, the perturbed hamiltonian reads

$$\begin{aligned} H_\alpha &= -\frac{1}{2}\Delta_x + \sum_{i=1}^{N}\left(-\frac{1}{2}\Delta_{y_i} + \frac{1}{2}(y_i - y_i^0)^2\right) + \alpha\sum_{i=1}^{N}\delta(x - y_i) \\ &\equiv H_0 + \alpha\sum_{i=1}^{N}\delta(x - y_i) \end{aligned} \tag{1.1}$$

where we have denoted by x the coordinate of the test particle, by y_i the coordinates of the oscillators and by y_i^0 their equilibrium positions.
The hamiltonian H_0 refers to the free system, i.e. the system of the test particle and the oscillators without point interactions, while the interaction term in (1.1) is chosen in the form of a zero-range or point interaction between the test particle and each harmonic oscillator.
Hamiltonians with zero-range or point interactions have been introduced in Physics since the thirties (see e.g. [KP] and [F]) for the analysis of the scattering of low-energy neutrons from a target of condensed matter.
The first and more basic model of point interaction hamiltonian refers to the case of a test particle interacting with fixed scattering centers, corresponding to the infinite mass limit for the nuclei of the target.
Here we consider a more refined model in which the nuclei of the target are quantum particles harmonically bounded to their equilibrium positions. This kind of model is widely used

in Physics to reconstruct the structure and the properties of the target from the measured scattering data (see e.g. [LA],[D],[LO], [DO]).

¿From the point of view of computations, point interactions are interesting non trivial model interactions since they are indexed by a minimal set of parameters, i.e. the strengths and the positions of the interaction centers. As a consequence, it is relatively easy to find simple representations of the solution of the Schroedinger equation and then to compute physical relevant quantities.

On the other hand, it should be noted that in all the applications one can find in the physical literature the computation of the relevant quantities in dimension two and three are only given in the Born approximation. This is due to the fact that the interaction term formally introduced in (1.1) is too singular and the second order term in the perturbation series diverges.

¿From the mathematical point of view this means that point interactions cannot be considered as small perturbation of the free hamiltonian, even in the sense of quadratic forms.

Then the problem naturally arises to give (if possible) a rigorous meaning to the formal hamiltonian as a self adjoint and bounded from below operator in the appropriate Hilbert space. In particular this would open the possibility to have a well defined perturbation theory and then to compute the relevant quantities at any order of approximation.

This program started in the sixties when Berezin and Faddeev gave the first rigorous definition ([BF]) for the hamiltonian of a particle undergoing the influence of a single delta interaction and it has been successfully completed (see e.g. [AGH-KH]) for the case of one particle subject to point interactions placed at fixed positions.

The case of a system of N particles interacting via a two-body local point interaction is more difficult. In fact, in the case of three particles in dimension three it was shown in ([MF1],[MF2]) that the most natural self adjoint realization of the hamiltonian is in fact unbounded from below and then physically unreasonable.

Surprisingly enough, the same collapse phenomenon occurs in the case of one particle interacting via a point interaction with a system of N non interacting fermions, if N is sufficiently large ([DFT], [M1]).

For the case of harmonically bounded nuclei, which we approach here, it seems that no rigorous results have been obtained so far.

In dimension one, since the interaction term is small in the sense of quadratic forms with respect to the free hamiltonian H_0, perturbation theory can be applied and the construction of the hamiltonian and representation formulas for the resolvent and the unitary group can be given following standard methods.

In the two dimensional case perturbation theory cannot be used and we define the singularly perturbed quadratic form exploiting a (heuristic) renormalization procedure, following the line of [T],[DFT].

We can prove that the form is closed and bounded from below and we can characterize domain, action and resolvent of the associated self adjoint and semibounded hamiltonian.

Moreover it can be seen that the defined point interactions are local, in a sense which will be specified later.

In the three dimensional case the situation is more delicate and we can construct a self adjoint and semibounded hamiltonian only in the case $N = 1$.
For $N > 1$, the problem of defining a physically reasonable hamiltonian remains open. The hope is that the presence of the harmonic potentials could prevent the nuclei from being too close, at least if the equilibrium positions y_i^0 are sufficiently far apart from each other, and this could avoid the collapse phenomenon, which is typical of the case without harmonic potentials. We plan to come back to this question in a further work.
The rest of the paper is organized as follows.
In section 2 we construct the hamiltonian (1.1) in dimension one starting from the naturally associated quadratic form and we give representation formulas for the resolvent and the unitary group.
In section 3 we analyze the two dimensional case. We construct the quadratic form using a renormalization procedure and we prove that it is closed and bounded from below. The characterization of the hamiltonian and a representation for the resolvent is also given.
In section 4 we consider the three dimensional case and, following the scheme of section 3, we construct the hamiltonian (1.1) when $N = 1$.
Finally in the appendix we derive an integral representation for the unperturbed resolvent used in the paper.

2. The One Dimensional Case

In this section we introduce the quadratic form F_α associated to (1.1) in dimension one and we prove that it is closed and bounded from below; moreover we construct the corresponding self adjoint operator and we give representation formulas for the resolvent and the unitary group.
The analysis in dimension one is considerably simplified by the fact that the interaction term in (1.1) is a small perturbation in the sense of quadratic forms of the free term and then standard perturbative arguments can be used.
Let's consider the quadratic form F_0 associated to H_0

$$F_0[u] = \int dx dy_1 \dots dy_N \left\{ \frac{1}{2}|\partial_x u|^2 + \sum_{i=1}^{N} \left(\frac{1}{2}|\partial_{y_i} u|^2 + \frac{1}{2}(y_i - y_i^0)^2 |u|^2 \right) \right\} \tag{2.1}$$

$$\mathscr{D}(F_0) = \left\{ u \in L^2(\mathbb{R}^{N+1}) \ s.t. \ F_0[u] < +\infty \right\} \tag{2.2}$$

It is well known that F_0 is a closed, positive quadratic form and that (2.2) reduces to

$$\mathscr{D}(F_0) = \left\{ u \in L^2(\mathbb{R}^{N+1}) \ s.t. \ u \in H^1(\mathbb{R}^{N+1}), \ y_i u \in L^2(\mathbb{R}^{N+1}), \ \ i = 1, \dots, N \right\}$$

We shall often use y without under script to indicate $(y_1, \dots, y_N)$ and we shall omit summations over the index i to make formulas less heavy.

Then formula (2.1) can be written as

$$F_0[u] = \int dxdy \left\{ \frac{1}{2}|\partial_x u|^2 + \frac{1}{2}|\partial_y u|^2 + \frac{1}{2}(y-y^0)^2|u|^2 \right\} \tag{2.3}$$

We can now introduce the quadratic form F_α.

Definition 2.1. *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ is defined as follows*

$$\begin{aligned} F_\alpha[u] = & \int dxdy \left\{ \frac{1}{2}|\partial_x u|^2 + \frac{1}{2}|\partial_y u|^2 + \frac{1}{2}(y-y^0)^2|u|^2 \right\} + \alpha \sum_{i=1}^N \int dy\, |u(y_i, y_1, \dots, y_N)|^2 \\ & \equiv F_0[u] + \alpha F_{int}[u] \end{aligned} \tag{2.4}$$

$$\mathscr{D}(F_\alpha) = \left\{ u \in L^2(\mathbb{R}^{N+1}) \ s.t. \ F_\alpha[u] < +\infty \right\} \tag{2.5}$$

The main properties of F_α are proved in the following theorem.

Theorem 2.2. *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ is closed and bounded from below on $\mathscr{D}(F_\alpha) = \mathscr{D}(F_0)$.*

Proof

By KLMN theorem (see e.g. [RSII]) it is sufficient to show that F_{int} is infinitesimal with respect to F_0 in Kato sense. By Sobolev trace theorems we have:

$$F_{int}[u] \le c\|u\|^2_{H^{\frac{1}{2}}}$$

We denote by $\tilde{u}$ the Fourier transform of u; using the Cauchy-Schwartz inequality

$$\int dk\, |\tilde{u}(k)|^2(1+k^2)^{\frac{1}{4}} \le \varepsilon \int dk\, |\tilde{u}(k)|^2(1+k^2)^{\frac{1}{2}} + \frac{1}{\varepsilon}\int dk\, |\tilde{u}(k)|^2$$

and the estimate $\|u\|^2_{H^1} \le 2F_0[u]$ we conclude the proof.

□

Let us denote by $(H_\alpha, \mathscr{D}(H_\alpha))$ the self adjoint and bounded from below operator in $L^2(\mathbb{R}^{N+1})$ defined by (2.4) and (2.5).

Let $G^\lambda(x, y; x', y')$ be the integral kernel of the Green's function of the unperturbed system (an integral representation for it is given in the appendix) and denote π_i the hyperplane $\{(x, y) \in \mathbb{R}^{N+1} \mid x = y_i\}$, $i = 1, \dots, N$.

For given functions $q = (q_1, \dots, q_N)$, $q_i : \pi_i \to \mathbb{C}$, $i = 1, \dots, N$, which we call charges, we shall consider the potential produced by q_i in $\mathbb{R}^{N+1}$

$$\left(G^\lambda q_i\right)(x,y) = \int_{\pi_i} dy' G^\lambda(x,y;y'_i,y')q_i(y')$$

and also its restriction on π_j

$$\left(G^\lambda_j q_i\right)(y) = \left(G^\lambda q_i\right)(y_j,y) = \int_{\pi_i} dy' G^\lambda(y_j,y;y'_i,y')q_i(y')$$

Concerning the resolvent of H_α we have the following result.

Theorem 2.3. *There exists $\lambda_0 > 0$ s.t. for $\lambda > \lambda_0$ and for any $f \in L^2(\mathbb{R}^{N+1})$ one has*

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q^f \tag{2.6}$$

where the charges q^f are solutions of the following system

$$q^f_i(y) + \alpha\left(\sum_{j=1}^N G^\lambda_i q^f_j(y) + G^\lambda f(y_i,y)\right) = 0 \qquad i = 1,\ldots,N \tag{2.7}$$

Proof
Since H_α is bounded from below, the operator $(H_\alpha+\lambda)^{-1}$ is bounded for fixed $\lambda > 0$ sufficiently large.
If $u = (H_\alpha + \lambda)^{-1} f$ and $v \in C_0^\infty(\mathbb{R}^{N+1})$, by the first representation theorem of quadratic forms, one has

$$F_\alpha[u,v] + \lambda(u,v) = (f,v)$$

We define

$$R^\lambda f = G^\lambda f + G^\lambda q$$

and we look for conditions on q in order to satisfy

$$F_\alpha[R^\lambda f, v] + \lambda(R^\lambda f, v) = (f,v) \tag{2.8}$$

Equation (2.8) can be explicitly written as

$$\int dxdy\left[\frac{1}{2}\overline{\partial_x G^\lambda f}\,\partial_x v+\frac{1}{2}\overline{\partial_y G^\lambda f}\,\partial_y v+\frac{1}{2}(y-y^0)^2\overline{G^\lambda f}\,v+\lambda\overline{G^\lambda f}\,v\right]+$$
$$+\int dxdy\left[\frac{1}{2}\overline{\partial_x G^\lambda q}\,\partial_x v+\frac{1}{2}\overline{\partial_y G^\lambda q}\,\partial_y v+\frac{1}{2}(y-y^0)^2\overline{G^\lambda q}\,v+\lambda\overline{G^\lambda q}\,v\right]+$$
$$+\alpha\sum_{i=1}^N\int dy\left(\overline{G^\lambda f(y_i,y)+G^\lambda q(y_i,y)}\right)v(y_i,y)=\int dxdy\overline{f}v \tag{2.9}$$

If we integrate by parts and notice that $G^\lambda f\in\mathscr{D}(H_0)$, we have

$$\sum_{i=1}^N\int dy\left(\overline{q_i(y)+\alpha G^\lambda f(y_i,y)+\alpha G^\lambda q(y_i,y)}\right)v(y_i,y)=0 \tag{2.10}$$

for any v; then each term of the sum must be zero and we get (2.7).
The system (2.7) can be written in a more compact form as a fixed point equation in the Hilbert space $\bigoplus_{i=1}^N L^2(\pi_i)$; defining $\left(T_f^\lambda q\right)_i(y)=G^\lambda f(y_i,y)+\sum_j G^\lambda q_j(y_i,y)$, the system (2.7) is equivalent to the equation

$$q+\alpha T_f^\lambda q=0 \tag{2.11}$$

Using Schur's test (see e.g. [HS]) it is straightforward to see that for $i,j=1,\dots,N$ the following estimate holds

$$\int dy\left|G^\lambda q_i(y_j,y)\right|^2\le\left(\frac{1}{\sqrt{2\pi}}\int_0^1 d\nu\,\frac{\nu^{\lambda-\frac12}}{(\ln\frac{1}{\nu})^{\frac12}}\right)\int dy\,|q_i(y)|^2 \tag{2.12}$$

By estimate (2.12), T_f^λ becomes a contraction for $\lambda>0$ sufficiently large and then (2.11) has a unique solution $q^f\in\bigoplus_{i=1}^N L^2(\pi_i)$.
Now we prove that the domain of the operator R^λ is $L^2(\mathbb{R}^{N+1})$, that is for every f and $\lambda>0$ sufficiently large the solution q^f of equation (2.7) is such that $G^\lambda q_i^f\in L^2(\mathbb{R}^{N+1})$ for $i=1,\dots,N$.
It is straightforward to see that

$$\sup_{x,y}\int dy\,G^\lambda(x,y;y_i',y')\le\int_0^1 d\nu\,\nu^{\lambda-\frac12},\qquad \sup_{y'}\int dxdy\,G^\lambda(x,y;y_i',y')\le\frac{1}{\sqrt{2\pi}}\int_0^1 d\nu\,\frac{\nu^{\lambda-\frac12}}{(\ln\frac{1}{\nu})^{\frac12}} \tag{2.13}$$

and then Schur's test implies

$$\left\|G^\lambda q\right\|_{L^2(\mathbb{R}^{N+1})}\le c\,\|q\|_{L^2(\mathbb{R}^N)} \tag{2.14}$$

We show next that the operator R^λ is symmetric. The condition $(R^\lambda f, g) = (f, R^\lambda g)$ is equivalent to $(G^\lambda q^f, g) = (f, G^\lambda q^g)$ which can be explicitly written as

$$\sum_{i=1}^{N} \int dy \overline{q_i^f(y)} G^\lambda g(y_i, y) = \sum_{i=1}^{N} \int dy \overline{G^\lambda f(y_i, y)} q_i^g(y) \tag{2.15}$$

If we solve (2.7) with respect to $G^\lambda f(y_i, y)$ and to $G^\lambda g(y_i, y)$ and we substitute into (2.15) we easily check that the equality holds.
Being everywhere defined and symmetric, the operator R^λ is also bounded and self adjoint, by Hellinger-Toeplitz theorem.
The last step is to show that R^λ has an empty kernel, which is equivalent to say that H_α is densely defined. If $R^\lambda f = 0$ then also $Tr_{\pi_i} R^\lambda f = 0, \;\; i = 1, \dots, N$ (notice that the range of R^λ is contained in $H^1(\mathbb{R}^{N+1})$ and then the traces of $R^\lambda f$ on π_i is well defined).
From (2.7), this means $q^f = 0$. Then $G^\lambda f = 0$ which implies $f = 0$.

□

¿From the knowledge of the resolvent we can easily reconstruct the domain and the action of the operator itself.

Theorem 2.4. *The domain and the action of H_α are the following*

$$\mathscr{D}(H_\alpha) = \left\{ u \in L^2(\mathbb{R}^{N+1}) \, s.t. \, u = \varphi^\lambda + G^\lambda q, \, \varphi^\lambda \in \mathscr{D}(H_0), \, q_i + \alpha Tr_{\pi_i} u = 0, \;\; i = 1, \dots N, \right\} \tag{2.16}$$

$$(H_\alpha + \lambda) u = (H_0 + \lambda) \varphi^\lambda \tag{2.17}$$

Proof
It is sufficient to observe that $u \in \mathscr{D}(H_\alpha)$ if and only $u = (H_\alpha + \lambda)^{-1} f$ for some $f \in L^2(\mathbb{R}^{N+1})$. Denoting $\varphi^\lambda = G^\lambda f$, (2.16) and (2.17) immediately follow from theorem (2.3).

□

It is straightforward to see that H_α is an extension of $\widetilde{H_0}$ defined by

$$\mathscr{D}(\widetilde{H_0}) = \left\{ u \in C_0^\infty(\mathbb{R}^{N+1} \setminus \cup_{i=1}^N \pi_i) \right\}$$

$$\widetilde{H_0} u = \left[-\frac{1}{2}\Delta_x + \sum_{i=1}^{N} \left(-\frac{1}{2}\Delta_{y_i} + \frac{1}{2}(y_i - y_i^0)^2 \right) \right] u$$

Then, by definition, H_α is a perturbation of H_0 supported by the null set $\cup_i \pi_i$, i.e. a rigorous counterpart of (1.1). It immediately follows from a general argument (see lemma C.2 of [AGH-KH]) that H_α is a local operator, i.e. if $u = 0$ in an open set Ω then $H_\alpha u = 0$ in Ω.
The effect of the interacion is expressed by the boundary condition $q_i + \alpha Tr_{\pi_i} u = 0$ satisfied by $u \in \mathscr{D}(H_\alpha)$ on each π_i (see (2.16)).

Such boundary condition is manifestly local, i.e. the value of q_i at a given point on π_i is proportional to the value of $Tr_{\pi_i}u$ at the same point.
In this sense the constructed hamiltonian H_α defines the so-called local point interactions.
Finally we consider the unitary group e^{-itH_α}. In order to construct an integral representation for e^{-itH_α} it is convenient to introduce the unperturbed propagator

$$e^{-itH_0}(x,y;x',y') = \left(\frac{1}{2\pi it}\right)^{\frac{1}{2}} e^{\frac{i}{2t}(x-x')^2} \left(\frac{1}{\pi(1-e^{-2it})}\right)^{\frac{N}{2}} e^{-\frac{1}{2i\tan(t)}(y^2+y'^2)} e^{\frac{1}{i\sin(t)}yy'}$$

and the integral operators W^t_{jk} defined by the restriction of $e^{-itH_0}(x,y;x',y')$ over the planes π_j, π_k

$$W^t_{j,k}(y;y') = \left(\frac{1}{2\pi it}\right)^{\frac{1}{2}} e^{\frac{i}{2t}(y_j-y_k)^2} \left(\frac{1}{\pi(1-e^{-2it})}\right)^{\frac{N}{2}} e^{-\frac{1}{2i\tan(t)}(y^2+y'^2)} e^{\frac{1}{i\sin(t)}yy'}$$

Theorem 2.5. *For any initial datum $\psi_0 \in C_0^\infty\left(\mathbb{R}^{N+1}\setminus \cup_{i=1}^N \pi_i\right)$ we have the following representation for $\psi(t) = e^{-itH_\alpha}\psi_0$*

$$\psi(t;x,y) = \left(e^{-itH_0}\psi_0\right)(x,y) + i\int_0^t ds \sum_{j=1}^N \int dy' e^{-i(t-s)H_0}(x,y;y'_j,y')q_j(s;y') \tag{2.18}$$

where the charges $q_i(t)$ are solutions of the system of integral equations

$$q_j(t) + i\alpha\sum_{k=1}^N \int_0^t ds\, W^{t-s}_{jk} q_k(s) + \alpha Tr_{\pi_j}\left(e^{-itH_0}\psi_0\right) = 0 \qquad j = 1\dots N \tag{2.19}$$

Proof
For any $v \in L^2(\mathbb{R}^{N+1})$ we define

$$\hat{v}(k,\underline{n}) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int dx\, e^{-ikx}\int dy' e_{\underline{n}}(y')v(x,y') \tag{2.20}$$

where $e_{\underline{n}}(y) = e_{n_1}(y_1)e_{n_2}(y_2)\cdots e_{n_N}(y_N)$ and e_{n_j} is the j-th normalized eigenvector of the one dimensional harmonic oscillator. Obviously (2.20) defines a unitary map which diagonalizes the unperturbed hamiltonian H_0. Using this map, the ansatz (2.18) can be written as

$$\hat{\psi}(t;k,\underline{n}) = e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)t}\hat{\psi}_0(k,\underline{n}) + i\int_0^t ds\, e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)} \sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \tag{2.21}$$

$$\check{q}_j(t;k,\underline{n}) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int dy'\, e^{-iky'_j} e_{\underline{n}}(y')q_j(t;y') \tag{2.22}$$

In (2.22) we have used the standard notation $|\underline{n}| = n_1 + \ldots n_N$. Let us assume that (2.19) has a unique solution in $C(0,T;\oplus_{i=1}^N L^2(\pi_i))$ with $\dot{q} \in L^1(0,T;\oplus_{i=1}^N L^2(\pi_i))$ (we shall prove this fact later on). Then an integration by parts gives

$$\hat{\psi}(t;k,\underline{n}) = e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)t}\hat{\varphi}_0^\lambda(k,\underline{n}) - \frac{1}{\frac{k^2}{2}+|\underline{n}|+\lambda}\int_0^t ds\, e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)}\sum_{i=j}^N \dot{\check{q}}_j(s;k,\underline{n}) +$$
$$+ i\frac{\lambda}{\frac{k^2}{2}+|\underline{n}|+\lambda}\int_0^t ds e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) + \frac{1}{\frac{k^2}{2}+|\underline{n}|\ +\lambda}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \quad (2.23)$$

The assumed regularity of $q(t)$ implies that the second and the third term in the r.h.s. of (2.23) belong to $\mathscr{D}(H_0)$, while the last term in the r.h.s. of (2.23) coincides with $G^\lambda q(t)$, which doesn't belong to $\mathscr{D}(H_0)$. Imposing the boundary condition on each π_i, i.e. the last equality in (2.16), we obtain (2.22) and hence $\psi(t) \in \mathscr{D}(H_\alpha)$. It is now straightforward to verify that $\psi(t)$ satisfies the Schroedinger equation. In fact from (2.18) and (2.23) we have

$$\frac{d\hat{\psi}}{dt}(t) = \left(\frac{k^2}{2}+|\underline{n}|\right)e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)t}\hat{\psi}_0(k,\underline{n}) - \sum_{j=1}^N \check{q}_j(s;k,\underline{n}) +$$
$$+ i\left(\frac{k^2}{2}+|\underline{n}|\right)\int_0^t ds\, e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \quad (2.24)$$

$$\left(\widehat{H_\alpha\psi}\right)(t;k,\underline{n}) = \left(\frac{k^2}{2}+|\underline{n}|\right)e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)t}\hat{\varphi}_0^\lambda(k,\underline{n}) +$$
$$-\frac{\frac{k^2}{2}+|\underline{n}|}{\frac{k^2}{2}+|\underline{n}|+\lambda}\int_0^t ds\, e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)}\sum_{i=1}^N \dot{\check{q}}_i(s;k,\underline{n}) +$$
$$+ i\lambda\frac{\left(\frac{k^2}{2}+|\underline{n}|\right)}{\frac{k^2}{2}+|\underline{n}|+\lambda}\int_0^t ds\, e^{-i\left(\frac{k^2}{2}+|\underline{n}|\right)(t-s)}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) - \frac{\lambda}{\frac{k^2}{2}+|\underline{n}|+\lambda}\sum_{j=1}^N \check{q}_j(s;k,\underline{n}) \quad (2.25)$$

and the r.h.s. of (2.24) and (2.25) coincide.
It remains to show that (2.19) has the required global regular solution. We write (2.19) as a linear equation in $C(0,T;\oplus_{i=1}^N L^2(\pi_i))$.

$$q(t) + i\alpha\int_0^t ds W^{t-s}q(s) + \alpha\Psi_0(t) = 0 \quad (2.26)$$

where

$$\left(\Psi_0(t)\right)_j = Tr_{\pi_j}e^{-itH_0}\psi_0 \qquad \left(W^{t-s}q(s)\right)_j = \sum_{k=1}^N W_{jk}^{t-s}q_k(s) \quad (2.27)$$

Using standard results on Fourier Integral Operators (see e.g. [Fu]) one easily obtains the following estimate of W^t_{jk} as an operator in $L^2(\mathbb{R}^N)$

$$\|W^t_{jk}\| \leq C\frac{1}{\sqrt{t}} \tag{2.28}$$

This implies the existence of $T_0 > 0$ such that $\int_0^t ds\, W^{t-s}q(s)$ is a contraction for $t < T_0$ and then (2.26) has a local solution. Then the solution can be extended to any fixed T by iteration. Concerning the time derivative of $q(t)$ one has the following equation

$$\dot{q}(t) + i\alpha\int_0^t dsW^{t-s}\dot{q}(s) + i\alpha W^t q(0) + \alpha\dot{\Psi}_0(t) = 0$$

Using the same argument one sees that $\dot{q}(t) \in L^1(0,T;\oplus_{i=1}^N L^2(\pi_i))$ and this is sufficient to justify the integration by parts in (2.23) and the decomposition of the r.h.s. of (2.23) into a regular part belonging to $\mathscr{D}(H_0)$ and the singular part $G^\lambda q(t)$, which we used before. This concludes the proof of the theorem.

□

3. The Two Dimensional Case

In this section we consider the two dimensional case. Even if we are still concerned with a perturbation of the free quadratic form F_0 supported by a null set, i.e. the planes π_i, the situation is different from the previous case. The perturbation has now support on a set of codimension two and this gives stronger singularities which make the construction of the quadratic form more involved. We shall adapt to our case the techniques developed in [T] and [DFT].

3.1 THE QUADRATIC FORM AND A USEFUL HEURUSTIC DERIVATION

The definition of F_α in dimension two is the following

Definition 3.1.

$$\mathscr{D}(F_\alpha) = \left\{u \in L^2(\mathbb{R}^{2N+2}) \; s.t. \; \exists q \in \mathscr{D}(\Phi^\lambda_\alpha)\,,\, \varphi^\lambda \equiv u - G^\lambda q \in \mathscr{D}(F_0)\right\}$$

$$F_\alpha[u] = \mathcal{F}^\lambda[u] + \Phi^\lambda_\alpha[u]$$

$$\mathcal{F}^\lambda[u] = \int dxdy\left\{\frac{1}{2}|\nabla_x\varphi^\lambda|^2 + \frac{1}{2}|\nabla_y\varphi^\lambda|^2 + \lambda|\varphi^\lambda|^2 - \lambda|u|^2 + \frac{1}{2}(y-y^0)^2|\varphi^\lambda|^2\right\}$$

$$\mathscr{D}(\Phi^\lambda_\alpha) = \left\{q \; s.t. \; q_i \in L^2(\pi_i)\,,\, \Phi^\lambda_\alpha[q] < +\infty\right\}$$

$$\Phi^\lambda_\alpha[q] = \alpha\sum_{i=1}^N \int dy|q_i(y)|^2 - 2\Re\sum_{i<j}\int dy\overline{q_i(y)}(G^\lambda_i q_j)(y) +$$
$$+ \sum_{i=1}^N \int dy\, a^\lambda_i(y)|q_i(y)|^2 + \frac{1}{2}\sum_{i=1}^N \int dy dy' G^\lambda(y_i, y; y'_i, y')|q_i(y) - q_i(y')|^2$$

In the previous definition we have introduced the functions $a^\lambda_i : \pi_i \to \mathbb{R}$

$$a^\lambda_i(y) = \frac{1}{4\pi}\Big\{C + \int_0^1 d\nu \frac{1}{(1-\nu)}\Big[1 - \nu^{\lambda+N-1}\left(\frac{2}{1+\nu^2}\right)^{N-1}\left(\frac{2}{\frac{1+\nu^2}{2}\frac{\ln\frac{1}{\nu}}{1-\nu} + \frac{1+\nu}{2}}\right)$$
$$\exp\left(-\frac{1}{2}\frac{1-\nu^2}{1+\nu^2}\sum_{\substack{j=1\\ j\neq i}}^N (y_j - y^0_j)^2\right)\exp\left(-\frac{\frac{1}{4} + \frac{1-\nu}{1+\nu}\frac{1}{2\ln\frac{1}{\nu}}}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2} + \frac{1}{2\ln\frac{1}{\nu}}}(y_i - y^0_i)^2\right)\Big]\Big\} \quad (3.1)$$

$$C = -\left(\int_0^1 d\nu\,\frac{1}{\nu}e^{-\frac{1}{\nu}} + \int_1^\infty d\nu\,\frac{1}{\nu^2}\ln\frac{1}{\nu}e^{-\frac{1}{\nu}}\right)$$

For later use, it is convenient to decompose the quadratic form Φ^λ_α into a diagonal part $\Phi^\lambda_{\alpha,d}$ and a non diagonal part $\Phi^\lambda_{\alpha,nd}$.

$$\Phi^\lambda_{\alpha,d}[q] = \alpha\sum_{i=1}^N \int dy|q_i(y)|^2 + \sum_{i=1}^N \int dy\, a^\lambda_i(y)|q_i(y)|^2 + \frac{1}{2}\sum_{i=1}^N \int dy dy' G^\lambda(y_i, y; y'_i, y')|q_i(y) - q_i(y')|^2$$

$$\Phi^\lambda_{\alpha,nd}[q] = -2\Re\sum_{i<j}\int dy\overline{q_i(y)}(G^\lambda_i q_j)(y)$$

In order to give a motivation for definition (3.1) one can proceed as follows.
The naive quadratic form associated to H_γ is

$$\tilde{F}_\gamma[u] = F_0[u] + \sum_j \gamma\int dx dy|u|^2\delta(x - y_j) \quad (3.2)$$

which can be written as

$$\tilde{F}_\gamma[u] = F_0[u - G^\lambda q] + \lambda\int dx dy|u - G^\lambda q|^2 - \lambda\int dx dy|u|^2 + \big((H_0+\lambda)G^\lambda q, u\big) +$$
$$+ \big(u, (H_0+\lambda)G^\lambda q\big) - \big((H_0+\lambda)G^\lambda q, G^\lambda q\big) + \sum_j \gamma\int dx dy|u|^2\delta(x - y_j) \quad (3.3)$$

for any $\lambda > 0$ and any charge q. Now we fix $q_i = -\gamma\, u|_{\pi_i}$ and denote $\varphi^\lambda = u - G^\lambda q$. Then we use the fact the $(H_0 + \lambda)G^\lambda q = q$ and we obtain

$$\tilde{F}_\gamma[u] + \lambda \int dxdy|u|^2 = F_0[\varphi^\lambda] + \lambda \int dxdy|\varphi^\lambda|^2 + \tilde{\Phi}_\gamma[q]$$

$$\tilde{\Phi}_\gamma[q] = \sum_i \int_{\pi_i} dy\, \overline{q_i}(y) \left[-\frac{q_i(y)}{\gamma} - G_i^\lambda q_i(y) \right] - 2\Re \sum_{i<j} \int dy \overline{q_i}(y) G_i^\lambda q_j(y) \tag{3.4}$$

It's clear that the form $\tilde{\Phi}_\gamma[q]$ on the charges is ill defined due to the presence of the first term of the r.h.s. of (3.4) (diagonal terms) since the potential $G^\lambda q_i$ is divergent on π_i. The singularity at $y \in \pi_i$ is only due to the value of q_i in y, i.e. it has the same singularity of the potential generated by the uniform charge distribution $q_i(y)$ on U_ε, a small neighborhood of y. Then we can write

$$G^\lambda q_i(x, y) = q_i(y)\zeta_{1_\varepsilon}(x, y) + \tau_{q_i}(x, y)$$

where ζ_{1_ε} is the divergent part of $G^\lambda 1_\varepsilon$ and $\tau_{q_i}(x, y)$ remains finite in the limit $x \to y_i$; we have denoted the characteristic function of U_ε with 1_ε. Moreover we define $G^\lambda_{i,ren} q_i(y) = \lim_{x \to y_i} \tau_{q_i}(x, y)$.
The idea is to separate the divergent part and the finite part of the quadratic form and to compensate the divergence through a renormalization of the coupling constant γ. We only outline here the procedure.
Define $\pi_i^\delta = \{(x, y)\, s.t.\, \frac{1}{\sqrt{2}}|x - y_i| \le \delta\}$ and consider a charge distribution q_i^δ on π_i^δ such that $q_i^\delta \to q_i$ and $G^\lambda q_i^\delta \to G^\lambda q_i$ for $\delta \to 0$. Moreover $G^\lambda q_i^\delta(x, y) = q_i^\delta(y)\zeta^\delta(x, y) + \tau^\delta_{q_i}(x, y)$ where $\zeta^\delta \to \zeta_{1_\varepsilon}$ and $\tau^\delta_{q_i} \to \tau_{q_i}$ for $\delta \to 0$. Introducing a new coupling constant $\gamma = \gamma^\delta(x, y)$ which depends on x, y and δ, the ill defined diagonal term can now be replaced by

$$\lim_{\delta \to 0} \int_{\pi_i^\delta} \overline{q_i^\delta}(x, y) \left[-\frac{q_i^\delta(x, y)}{\gamma^\delta(x, y)} - G^\lambda q_i^\delta(x, y) \right] =$$

$$\lim_{\delta \to 0} \int_{\pi_i^\delta} \overline{q_i^\delta}(x, y) \left[-\frac{q_i^\delta(x, y)}{\gamma^\delta(x, y)} - q_i^\delta(x, y)\zeta^\delta(x, y) \right] - \int_{\pi_i^\delta} \overline{q_i^\delta}(x, y)\tau^\delta_{q_i}(x, y) \tag{3.5}$$

In order to get a finite limit in (3.5) it is sufficient to define

$$-\frac{1}{\gamma^\delta(x, y)} = \zeta^\delta(x, y) + \alpha, \qquad \alpha \in \mathbb{R}$$

so that the diagonal term reduces to

$$\alpha \int_{\pi_i} |q_i|^2 - \int_{\pi_i} \overline{q_i}(y) G^\lambda_{i,ren} q_i(y) \tag{3.6}$$

We shall now rewrite the renormalized potential $G^{\lambda}_{i,ren}q_i$ in a more convenient form

$$
\begin{aligned}
G^{\lambda}_{i,ren}q_i(y) = \lim_{x\to y_i}\left[G^{\lambda}q_i(x,y) - q_i(y)\zeta_{1_\varepsilon}(x,y)\right] = \lim_{x\to y_i}\Bigg[\int_{|y'-y|>\varepsilon} dy' q_i(y')G^{\lambda}(x,y;y'_i,y') + \\
+\int_{|y'-y|<\varepsilon} dy' G^{\lambda}(x,y;y'_i,y')\big(q_i(y')-q_i(y)\big) + \\
+q_i(y)\left(\int_{|y'-y|<\varepsilon} dy' G^{\lambda}(x,y;y'_i,y') - \zeta_{1_\varepsilon}(x,y)\right)\Bigg]
\end{aligned}
$$

Then the second term of (3.6) becomes

$$
\begin{aligned}
-\int dy\overline{q_i(y)}G^{\lambda}_{i,ren}q_i(y) = &-\int_{|y'-y|>\varepsilon} dy dy'\overline{q_i(y)}q_i(y')G^{\lambda}(y_i,y;y'_i,y') + \\
&-\int_{|y'-y|<\varepsilon} dy dy'\overline{q_i(y)}(q_i(y')-q_i(y))G^{\lambda}(y_i,y;y'_i,y') + \\
&-\int dy|q_i(y)|^2\tau_{1_\varepsilon}(y_i,y) \\
=&\int dy dy'\overline{q_i(y)}(q_i(y)-q_i(y'))G^{\lambda}(y_i,y;y'_i,y') + \\
&+\int dy|q_i(y)|^2\left(-\tau_{1_\varepsilon}(y_i,y) - \int_{|y'-y|>\varepsilon} dy' G^{\lambda}(y_i,y;y'_i,y')\right) \\
=&\frac{1}{2}\int dy dy'|q_i(y)-q_i(y')|^2 G^{\lambda}(y_i,y;y'_i,y') + \int dy\, a^{\lambda}_i(y)|q_i(y)|^2
\end{aligned}
$$

where we have denoted

$$
a^{\lambda}_i(y) = -\lim_{x\to y_i}\left(\tau_{1_\varepsilon}(x,y) + \int_{|y'-y|>\varepsilon} dy' G^{\lambda}(x,y;y'_i,y')\right)
$$

and τ_{1_ε} is the finite part of the potential generated by 1_ε.
It remains to analyze a^{λ}_i. To be concrete we fix $i=1$, put $y^0=0$ and introduce the coordinates ξ and η defined in the following way

$$
\begin{cases}
\xi = \dfrac{1}{\sqrt{2}}(x-y_1) \\
\eta = \dfrac{1}{\sqrt{2}}(x+y_1)
\end{cases}
$$

With this choice π_1 is defined by $\xi=0$. We have to study the behavior for small ξ of

$$
\left(G^{\lambda}1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) = \int_{U_\varepsilon} d\eta' dy'_2\dots dy'_N\, G^{\lambda}(\xi,\eta,y_2,\dots,y_N;0,\eta',y'_2,\dots,y'_N) \tag{3.7}
$$

The choice of U_ε is irrelevant, indeed a_i^λ does not depend on ε; we choose

$$U_\varepsilon = \{|\eta' - \eta| < \varepsilon, |y_2' - y_2| < \varepsilon, \dots, |y_N' - y_N| < \varepsilon\}$$

Using (5.1), equation (3.7) can be written as:

$$\begin{aligned}
\left(G^\lambda 1_\varepsilon\right)(\xi, \eta, y_2, \dots, y_N) &= \frac{1}{2\pi}\frac{1}{\pi^N}\int_0^1 d\nu \frac{\nu^{\lambda+N-1}}{(1+\nu)^N}\frac{1}{(1-\nu)^N(\log\frac{1}{\nu})} \\
&\prod_{i=2}^N \left\{ e^{-\frac{1-\nu}{1+\nu}y_i^2}\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\right)^{-1}\int_{|y_i'|<\varepsilon\left(\frac{(1+\nu^2)}{2(1-\nu^2)}\right)^{\frac{1}{2}}} dy_i'\, e^{-y_i'^2} e^{-y_i\cdot y_i'\left(\frac{2(1-\nu^2)}{(1+\nu^2)}\right)^{\frac{1}{2}}\frac{1-\nu}{1+\nu}} \right\} \\
&e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}(\eta^2 - \eta\cdot\xi)} e^{-\xi^2\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{1}{4\ln\frac{1}{\nu}}\right)}\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2} + \frac{1}{4\ln\frac{1}{\nu}}\right)^{-1} \\
&\int_{|\eta'|<\varepsilon\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}+\frac{1}{4\ln\frac{1}{\nu}}\right)^{\frac{1}{2}}} d\eta' e^{-\eta'^2}\exp\left\{\frac{2\eta'\cdot\left[\left(\frac{1}{4\ln\frac{1}{\nu}} - \frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\right)\xi + \frac{1}{4}\frac{1-\nu}{1+\nu}\eta\right]}{\left(\frac{1}{4}\frac{1-\nu^2}{1+\nu^2}\frac{1}{4\ln\frac{1}{\nu}}\right)^{\frac{1}{2}}}\right\}
\end{aligned} \tag{3.8}$$

With the changes of variable $\nu = 1 - \nu'$ and $\nu' = \xi^2\nu''$ we recover a logarithmic singularity for the potential

$$\left(G^\lambda 1_\varepsilon\right)(\xi, \eta, y_2, \dots, y_N) \simeq_{\xi\to 0} \frac{1}{2\pi}\ln\left(\frac{2}{\xi^2}\right) \tag{3.9}$$

If we subtract (3.8) from (3.9) and we add $\int_{|y'-y|>\varepsilon} dy' G^\lambda(y_i, y; y_i', y')$, we obtain (3.1).
We believe that the above limit procedure can be made rigorous in the framework of $\Gamma-$convergence of quadratic forms (see e.g. [DFT] for a similar result) but here we do not give any proof and we simply consider (3.1) as our starting point.

3.2 WELL-POSEDNES OF THE DEFINITION

Definition (3.1) requires a preliminary analysis before proving that F_α is closed and bounded from below.
First we prove that the decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique. Next we prove that even if $(\mathcal{F}^\lambda, \mathscr{D}(\mathcal{F}^\lambda))$ and $(\Phi_\alpha^\lambda, \mathscr{D}(\Phi_\alpha^\lambda))$ depend on λ, the quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ doesn't depend on λ.

Proposition 3.2. *If $q_i \in L^2(\mathbb{R}^{2N})$ then $G^\lambda q_i \in L^2(\mathbb{R}^{2N+2})$.*

Proof
Fix $i = 1$ and put $y^0 = 0$ to be concrete.

$$\int dx\, dy\, |G^\lambda q_1(x,y)|^2 = \frac{1}{(2\pi)^2}\frac{1}{\pi^{2N}}\int dxdy \int dz\, \overline{q_1(z)} \int_0^1 d\nu\, \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}$$
$$e^{-\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-z_1)^2}e^{-\frac{\nu}{1-\nu^2}(y-z)^2}\int dw\, q_1(w)\int_0^1 d\mu\, \mu^{\lambda+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}$$
$$e^{-\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}e^{-\frac{1}{2\ln\frac{1}{\mu}}(x-w_1)^2}e^{-\frac{\mu}{1-\mu^2}(y-w)^2}$$

$$= \frac{1}{(2\pi)^2}\frac{1}{\pi^{2N}}\int dzdw\, \overline{q_1}(z)q_1(w)\int_0^1 d\nu\, \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}\int_0^1 d\mu\, \mu^{\lambda+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}$$
$$\int dxdy e^{-\frac{\nu}{1-\nu^2}(y-z)^2}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-z_1)^2}e^{-\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}e^{-\frac{1}{2\ln\frac{1}{\mu}}(x-w_1)^2}e^{-\frac{\mu}{1-\mu^2}(y-w)^2}e^{-\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$

It's then sufficient to show that

$$T(z,w) = \int_0^1 d\nu\, \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}\int_0^1 d\mu\, \mu^{\lambda+N-1}\frac{1}{(1-\mu^2)^N\left(\ln\frac{1}{\mu}\right)}$$
$$\int dxdy\, e^{-\frac{\nu}{1-\nu^2}(y-z)^2}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-z_1)^2}e^{-\frac{1}{2\ln\frac{1}{\mu}}(x-w_1)^2}e^{-\frac{\mu}{1-\mu^2}(y-w)^2}e^{-\frac{1-\nu}{2(1+\nu)}(y^2+z^2)}e^{-\frac{1-\mu}{2(1+\mu)}(y^2+w^2)}$$

is the integral kernel of a bounded operator in $L^2(\mathbb{R}^{2N})$. This kernel can be pointwise estimated in the following way

$$T(z,w) \le \pi^{N+1}\int_0^1 d\nu d\mu \left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-(N+1)}\nu^{\lambda+N-1}$$
$$\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\mu^{\lambda+N-1}\frac{1}{(1-\mu^2)^N}\frac{1}{\left(\ln\frac{1}{\mu}\right)}e^{-\frac{1}{4\left(\frac{1-\nu^2}{\nu}+\frac{1-\mu^2}{\mu}\right)}(w-z)} \tag{3.10}$$

Using the Schur's test, and (3.10) we have

$$\|T\| \le \sup_w \int dw\, T(z,w) \le \pi^{N+1}\int_0^1 d\nu d\mu\, \nu^{\lambda-1}\mu^{\lambda-1}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\frac{1}{\left(\ln\frac{1}{\mu}\right)}\left(\frac{\nu}{1-\nu^2}+\frac{\mu}{1-\mu^2}\right)^{-1}$$
$$\le \pi^{N+1}\int_0^1 d\nu d\mu\, \nu^{\lambda-1}\mu^{\lambda-1}\frac{1}{\left[\frac{\nu}{1+\nu}\frac{\ln\frac{1}{\nu}}{1-\nu}\ln\frac{1}{\mu}+\frac{\mu}{1+\mu}\frac{\ln\frac{1}{\mu}}{1-\mu}\ln\frac{1}{\nu}\right]} < +\infty$$

and this concludes the proof.

□

Proposition 3.3. *If $q_i \in \mathscr{D}(\Phi_\alpha^\lambda)$ and $q_i \neq 0$ then $G^\lambda q_i \notin \mathscr{D}(F_0)$.*

Proof
Fix $i = 1$ and put $y^0 = 0$. Since $G^\lambda q_1$ is divergent near π_1, we shall regularize F_0 restricting the domain of the integration to $\mathbb{R}^{2(N+1)} \setminus \pi_1^\delta$.
Let's show that the following identity holds

$$\begin{aligned}\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{\frac{1}{2}|\nabla_x G^\lambda q_1|^2 + \frac{1}{2}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}y^2 + \lambda\right)\left|G^\lambda q_1\right|^2\right\} = \\ = -\frac{1}{2}\int dydy'|q_1(y) - q_1(y')|^2 \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1, y)\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1', y') + \\ + \int_{\partial\pi_1^\delta} dxdy \left(G^\lambda 1 - \zeta_1\right)\frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right) + \int_{\partial\pi_1^\delta} dxdy\, \zeta_1 \frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right)\end{aligned} \tag{3.11}$$

In the previous formula we have introduced the kernel $\frac{\partial G^\lambda}{\partial n}(x, y; x', y')$ given by the normal derivative with respect to π_1^δ of $G^\lambda(x, y; x', y')$; it is understood that the normal derivative acts only on the (x, y) variable. Formula (3.11) is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{\frac{1}{2}|\nabla_x G^\lambda q_1|^2 + \frac{1}{2}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}y^2 + \lambda\right)\left|G^\lambda q_1\right|^2\right\} = \\ = -\frac{1}{4}\int dydy'|q_1(y) - q_1(y')|^2 \left(\int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1, y)\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1', y') + \right. \\ \left. + \int_{\partial\pi_1^\delta} dx''dy'' G^\lambda(x'', y''; y_1', y')\frac{\partial G^\lambda}{\partial n}(x'', y''; y_1, y)\right) + \int_{\partial\pi_1^\delta} dxdy\, G^\lambda 1 \frac{\partial G^\lambda}{\partial n}\left(|q_1|^2\right)\end{aligned} \tag{3.12}$$

which can be further written as

$$\int_{\mathbb{R}^{2N+2}\setminus\pi_1^\delta} dxdy \left\{\frac{1}{2}|\nabla_x G^\lambda q_1|^2 + \frac{1}{2}|\nabla_y G^\lambda q_1|^2 + \left(\frac{1}{2}y^2 + \lambda\right)\left|G^\lambda q_1\right|^2\right\} = \int_{\partial\pi_1^\delta} dxdy\, \overline{G^\lambda q_1}\frac{\partial G^\lambda}{\partial n}(q_1) \tag{3.13}$$

Formula (3.13) holds true by a direct application of Stoke's theorem and then (3.11) is verified. In order to prove that $F_0[G^\lambda q_1] = +\infty$ we shall consider the r.h.s. of (3.11) and we shall prove that the sum of the first and the second term converge to $-2\Phi_\alpha^\lambda[q]$ as $\delta \to 0$ while the third term diverges; by proposition (3.2), the l.h.s. of (3.11) differs from $F_0[G^\lambda q_1]$ only by a bounded term and then we conclude that $F_0[G^\lambda q_1] = +\infty$.

Now we show that

$$\lim_{\delta\to 0}\int dy' \left[\int_{\partial\pi_1^\delta} dx dy \left(G^\lambda 1(x,y)-\zeta_1(x,y)\right)\frac{\partial G^\lambda}{\partial n}(x,y;y_1',y')\right]|q_1(y')|^2 = -2\int dy' a_1^\lambda(y')|q_1(y')|^2 \tag{3.14}$$

Since $\frac{\partial}{\partial n}=-\frac{\partial}{\partial|\xi|}$, the integral kernel of $\frac{\partial G^\lambda}{\partial n}$ can be easily obtained from (5.1).

$$\begin{aligned}\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\dots,y_n;y_1',y') &= \frac{1}{2\pi}\frac{1}{\pi^N}\int_0^1 d\nu\,\frac{\nu^{\lambda+N-1}}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)} \\ & e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}\left[\frac{1}{2}(\eta-\xi)^2+y_2^2+\dots y_N^2+y'^2\right]}\, e^{-\frac{\nu}{1-\nu^2}\left[(\frac{1}{\sqrt{2}}(\eta-\xi)-y_1')^2+(y_2-y_2')^2+\dots+(y_N-y_N')^2\right]}\, e^{-\frac{1}{2\ln\frac{1}{\nu}}(\frac{1}{\sqrt{2}}(\eta+\xi)-y_1')^2} \\ & \left\{\left[|\xi|+\sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta+y_1'\right)\right]\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\left[|\xi|+\sqrt{2}\frac{\xi}{|\xi|}\cdot\left(\frac{1}{\sqrt{2}}\eta-y_1'\right)\right]\frac{1}{2\ln\frac{1}{\nu}}\right\}\end{aligned} \tag{3.15}$$

The potential $G^\lambda 1$ and its divergent part ζ_1 have the following expressions

$$\begin{aligned}\left(G^\lambda 1\right)(\xi,\eta,y_2,\dots,y_n) &= \frac{1}{2\pi}\int_0^1 d\mu\,\frac{1}{(1-\mu^2)}\mu^{\lambda+N-1}\left(\frac{2}{1+\mu^2}\right)^{N-1}\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu}+\frac{1+\mu}{2}\right)^{-1} \\ & \exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\sum_{j=2}^N y_j^2\right)\exp\left(-\frac{\frac{1}{4}+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}+\frac{1}{2\ln\frac{1}{\mu}}}\frac{1}{2}\eta^2\right)\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}{\frac{1}{4\ln\frac{1}{\mu}}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}\xi^2\right) \\ & \exp\left(-\frac{\frac{1}{8}}{\frac{1}{4\ln\frac{1}{\mu}}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}\eta\cdot\xi\right)\end{aligned}$$

$$\zeta_1=\frac{1}{2\pi}\ln\left(\frac{2}{\xi^2}\right)$$

It is useful to write $G^\lambda 1 - \zeta_1$ in the following way

$$\left(G^\lambda 1 - \zeta_1\right)(\xi,\eta,y_2\dots y_N) = \frac{1}{2\pi}\left\{\left[\int_0^1 d\mu\, \exp\left(-\frac{\xi^2}{2\mu}\right) - C - \ln\left(\frac{2}{\xi^2}\right)\right] + \right.$$
$$+C + \int_0^1 d\mu\, \frac{1}{(1-\mu)^{\frac{3}{2}}}\left[\mu^{\lambda+N-1}\left(\frac{2}{1+\mu^2}\right)^{N-1}\left(\frac{1+\mu^2}{2}\frac{\ln\frac{1}{\mu}}{1-\mu} + \frac{1+\mu}{2}\right)^{-1}\right.$$
$$\exp\left(-\frac{1}{2}\frac{1-\mu^2}{1+\mu^2}\sum_{j=2}^N y_j^2\right)\exp\left(-\frac{\frac{1}{4}+\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}}{\frac{1}{2}\frac{1+\mu^2}{1-\mu^2}+\frac{1}{2\ln\frac{1}{\mu}}}\frac{1}{2}\eta^2\right)\exp\left(-4\frac{\frac{1}{4\ln\frac{1}{\mu}}\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}{\frac{1}{4\ln\frac{1}{\mu}}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}\xi^2\right)$$
$$\left.\left.\exp\left(-\frac{\frac{1}{2}\frac{1-\mu}{1+\mu}\frac{1}{2\ln\frac{1}{\mu}}}{\frac{1}{4\ln\frac{1}{\mu}}+\frac{1}{4}\frac{1+\mu^2}{1-\mu^2}}\eta\cdot\xi\right) - \exp\left(-\frac{1}{1-\mu}\frac{\xi^2}{2}\right)\right]\right\} \tag{3.16}$$

Equation (3.16) shows that $a_1^\lambda(y) = -2\left(G^\lambda 1 - \zeta_1\right)(0,\eta,y_2\dots y_N)$.
If we interchange the limit and the integrals over the parameter μ and over (x,y) in (3.14), by dominated convergence we see that it is sufficient to prove

$$\int_{|\xi|=\delta} d\xi \int d\eta dy_2\dots dy_N\, e^{-(a\xi^2+b\eta^2+c\eta\cdot\xi)}e^{-d(y_2^2+\ldots+y_N^2)}\frac{\partial G^\lambda}{\partial n}(\xi,\eta,y_2,\dots,y_n;y_1',y') = e^{-b\eta^2}e^{-d(y_2^2+\ldots+y_N^2)} \tag{3.17}$$

where a, b, c and d are suitable functions of μ. This can be proved by direct calculations. Using (3.15) and integrating over $\eta, y_2,\dots,y_N$, we have

$$\lim_{\delta\to 0}\frac{1}{\pi^{N+1}}\int_0^1 d\nu\, \frac{\nu^{\lambda+N-1}}{(1-\nu^2)^N\left(\ln\frac{1}{\nu}\right)}\left[\frac{1}{\pi}\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+d\right)\right]^{-(N-1)}$$
$$\exp\left(-(y_2'+\dots y_N'^2)\frac{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}d+\frac{1}{4}}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+d}\right)\left(\frac{1}{4}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{4\ln\frac{1}{\nu}}b\right)^{\frac{3}{2}}$$
$$\exp\left\{-\frac{1}{2}\delta^2\left[\frac{\frac{1+\nu^2}{1-\nu^2}\frac{1}{\ln\frac{1}{\nu}}+2b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}\right)+2c\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}-\frac{1}{2\ln\frac{1}{\nu}}\right)+(c)^2}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}+2b}\right]\right\}$$
$$\exp\left\{-y_1'^2\left[\frac{\frac{1}{4}+\frac{1}{2\ln\frac{1}{\nu}}\frac{1}{2}\frac{1-\nu}{1+\nu}+2b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}+2b}\right]\right\}$$

$$\int_{|\xi|=\delta} d\xi \exp\left\{\sqrt{2}\xi\cdot y_1'\left[\frac{\frac{1-\nu}{1+\nu}\frac{1}{2\ln\frac{1}{\nu}}+2b\left(\frac{1}{2\ln\frac{1}{\nu}}-\frac{\nu}{1-\nu^2}\right)+b\left(\frac{1}{2\ln\frac{1}{\nu}}+\frac{\nu}{1-\nu^2}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}+2b}\right]\right\}$$

$$\left\{|\xi|\frac{4\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}\frac{1}{2\ln\frac{1}{\nu}}+2b\left(\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}\right)+c\left(\frac{1}{2\ln\frac{1}{\nu}}-\frac{\nu}{1-\nu^2}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}+2b}+\right.$$

$$\left.-\sqrt{2}\frac{\xi}{|\xi|}\cdot y_1'\frac{\frac{1-\nu}{1+\nu}\frac{1}{2\ln\frac{1}{\nu}}+b\left(\frac{1}{2\ln\frac{1}{\nu}}-\frac{\nu}{1-\nu^2}\right)}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}+2b}\right\} \tag{3.18}$$

With the change of variables $\nu = 1-\nu'$ and $\nu' = \delta^2\nu''$ one sees that (3.18) equals $2e^{-2by_1'^2}e^{-d(y_2'^2+\ldots+y_N'^2)}$. Since $\eta = \sqrt{2}y_1$ on π_1 we have proved (3.14).
In a similar way it can be proved that

$$\lim_{\delta\to 0}\int dydy'\left[\int_{\partial\pi_1^\delta} dx''dy''G^\lambda(y_1,y;x'',y'')\frac{\partial G^\lambda}{\partial n}(;x'',y'';y_1',y')\right]|q_1(y)-q_1(y')|^2 =$$
$$= \int dydy'G^\lambda(y_1,y;y_1',y')|q_1(y)-q_1(y')|^2$$

The third term on the r.h.s. of (3.11) is equal to

$$\frac{1}{2\pi}\ln\left(\frac{2}{\delta^2}\right)\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) \tag{3.19}$$

¿From equation (3.17) we have

$$\lim_{\delta\to 0}\int_{\partial\pi_1^\delta}\frac{\partial G^\lambda}{\partial n}(|q_1|^2) = 2\int dy'|q_1|^2 < +\infty$$

which shows that (3.19) diverges when $\delta\to 0$. The proof is then concluded.

□

Notice that we have shown that Φ_α^λ can be viewed as a renormalized energy of the potential $G^\lambda q$.

Proposition 3.4. *The decomposition $u = \varphi^\lambda + G^\lambda q$ is well defined and unique.*

Proof
By proposition (3.2) $G^\lambda q \in L^2(\mathbb{R}^{2N+2})$. Uniqueness follows from proposition (3.3). If $u = \varphi^\lambda + G^\lambda q = \varphi^{\lambda\prime} + G^\lambda q'$ then we have also $0 = \varphi^\lambda - \varphi^{\lambda\prime} = G^\lambda(q'-q)$; but then $q'-q=0$ because $G^\lambda q = 0$ if and only if $q=0$.

□

At least formally the quadratic form (3.1) has been obtained from (3.2) by subtracting a divergent term which is independent of the parameter λ. This is made precise by the following proposition

Proposition 3.5. *The quadratic form $(F_\alpha, \mathscr{D}(F_\alpha))$ doesn't depend on λ.*

Proof
First we notice that $\mathscr{D}(F_\alpha)$ does not depend on λ. If $u = \varphi^\lambda + G^\lambda q$ it is sufficient to show that also $u = \varphi^{\lambda'} + G^{\lambda'} q$ for $\lambda' \neq \lambda$. Since $G^\lambda q - G^{\lambda'} q \in \mathscr{D}(H_0)$ one can define $\varphi^{\lambda'} \equiv \varphi^\lambda + G^\lambda q - G^{\lambda'} q$. Now we show that

$$\mathcal{F}^\lambda[u] - \mathcal{F}^{\lambda'}[u] = \Phi^{\lambda'}[u] - \Phi^\lambda[u] \tag{3.20}$$

Integrating by parts and using the identity

$$-\frac{1}{2}\Delta_x(G^{\lambda'}q - G^\lambda q) - \frac{1}{2}\Delta_y(G^{\lambda'}q - G^\lambda q) = -\frac{1}{2}(y-y^0)^2(G^{\lambda'}q - G^\lambda q) + \lambda G^\lambda q - \lambda' G^{\lambda'} q$$

the l.h.s. of (3.20) can be written as

$$\mathcal{F}^\lambda[u] - \mathcal{F}^{\lambda'}[u] = (\lambda' - \lambda)\int dxdy\, \overline{G^\lambda q}\, G^{\lambda'} q \tag{3.21}$$

Now we analyze the r.h.s. of (3.20).
Using the identity $\big(a_i^{\lambda'}(y) - a_i^\lambda(y)\big) + \int dy' \big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\big) = 0$, see (3.1) and (3.16), the diagonal terms can be written in the following way

$$\begin{aligned}
\frac{1}{2}\int dydy' \Big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\Big) |q_i(y) - q_i(y')|^2 + \int dy\, \Big(a_i^{\lambda'}(y) - a_i^\lambda(y)\Big) |q_i(y)|^2 = \\
-\int dydy' \Big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\Big) \overline{q_i(y)} q_i(y') + \\
+\int dy \left[\Big(a_i^{\lambda'}(y) - a_i^\lambda(y)\Big) + \int dy' \Big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\Big)\right] |q_i(y)|^2 = \\
-\int dydy' \Big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\Big) \overline{q_i(y)} q_i(y')
\end{aligned} \tag{3.22}$$

Then we can write

$$\begin{aligned}
\Phi^\lambda[u] - \Phi^{\lambda'}[u] = -\sum_{i=1}^N \int dydy' \Big(G^{\lambda'}(y_i, y; y_i', y') - G^\lambda(y_i, y; y_i', y')\Big) \overline{q_i(y)} q_i(y') + \\
-\sum_{i,j=1}^N \int dy \overline{q_i(y)} \big((G_j^{\lambda'} - G_j^\lambda) q_i\big)(y)
\end{aligned} \tag{3.23}$$

By equation (3.23) and the first resolvent identity applied to (3.21), we conclude the proof.

$\square$

Proposition (3.3) shows that $G^\lambda q_i$ are the most singular part of u; the converse is also true: q_i is determined by the singular behaviour of u at π_i.

Proposition 3.6. *The following limit holds in* $L^2(\pi_i)$

$$\lim_{\delta\to 0}\frac{2}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\, u = q_i \tag{3.24}$$

where $B_{\delta,i} = \{x\, s.t.\, |x-y_i| = \delta\}$

Proof
Using Sobolev embedding theorems we first prove that

$$\lim_{\delta\to 0}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\, u = \lim_{\delta\to 0}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\, q_i \tag{3.25}$$

and then it suffices to show that

$$\lim_{\delta\to 0}\frac{2}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\, q_i = q_i \tag{3.26}$$

Since $\varphi^\lambda(\cdot,y)\in H^1$ for a.e. y, its trace over $B_{\delta,i}$ belongs to $H^{\frac{1}{2}}$; due to the embedding $H^{\frac{1}{2}}(B_\delta)\hookrightarrow L^\infty(B_\delta)$ we obtain the following inequalities

$$\int_{B_{\delta,i}} dx|\varphi^\lambda(x,y)| \le c\|\varphi^\lambda(\cdot,y)\|_{L^\infty(B_\delta)}\|1\|_{L^1(B_\delta)} \le c\delta\|\varphi^\lambda(\cdot,y)\|_{H^{\frac{1}{2}}(B_\delta)} \le c\delta\|\varphi^\lambda(\cdot,y)\|_{H^1(\mathbb{R}^2)} \tag{3.27}$$

If we take the L^2 norm of the first and the last term in (3.27) we see that the regular part of u gives no contribution to the limit in (3.24). The restriction of $G^\lambda q_j$ to π_i is a smooth function away from $\pi_i\cap\pi_j$, which has measure zero as subset of π_i. It is then clear that (3.25) holds. Let us introduce the following two integral operators

$$T_\delta(y;y') = \frac{1}{2\pi}\frac{1}{\pi^N}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\int_0^1 d\nu\ \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}\cdot e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}(y^2+y'^2)}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-y_i')^2}e^{-\frac{\nu}{1-\nu^2}(y-y')^2}$$

$$T'_\delta(y;y') = \frac{1}{2\pi}\frac{1}{\pi^N}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{B_{\delta,i}} dx\int_0^1 d\nu\ \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-y_i')^2}e^{-\frac{\nu}{1-\nu^2}(y-y')^2}$$

We shall prove that $\lim_{\delta\to 0}\int dy' T'_\delta(y;y')q(y') = \frac{1}{2}q(y)$ which is equivalent to (3.26).
It is straightforward to prove the estimate $\int dy' T'_\delta(y;y') \le \frac{1}{2}$. Since $T_\delta(y;y')\le T'_\delta(y;y')$, $\{T_\delta\}$ and $\{T'_\delta\}$ are two equibounded family of operators by Schur's lemma.

Now we prove that $\int dy' T_\delta(y;y')$ converges uniformly on compact sets to $\frac{1}{2}$ as δ goes to zero. With a change of variables we have

$$\lim_{\delta\to 0}\int dy'\, T_\delta(y;y') = \lim_{\delta\to 0}\frac{1}{2\pi}\frac{1}{\pi^N}\frac{1}{\ln\frac{1}{\delta^2}}\int_{|x|=1} dx \int dy'$$
$$\int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\lambda+N-1}}{(\nu)^N(2-\delta^2\nu)^N}\left(\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}\right) e^{-\frac{1}{2}\frac{\delta^2\nu}{2-\delta^2\nu}\left[y^2+(\delta y+y')^2\right]} e^{-\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}(x-y_i')^2} e^{-\frac{1-\delta^2\nu}{2-\delta^2\nu}\frac{1}{\nu}y'^2} \tag{3.28}$$

Taking the limit under the integral by dominated convergence we prove the previous claim. The family $\{T_\delta'\}$ is an approximation of the identity. First we notice that T_δ' is a convolution operator.

$$T_\delta'(y-y') = \frac{1}{2\pi}\frac{1}{\pi^N}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{|x|=\delta} dx\int_0^1 d\nu\, \nu^{\lambda+N-1}\frac{1}{(1-\nu^2)^N}\frac{1}{\left(\ln\frac{1}{\nu}\right)} e^{-\frac{1}{2\ln\frac{1}{\nu}}(x+y_i-y_i')^2} e^{-\frac{\nu}{1-\nu^2}(y-y')^2} \tag{3.29}$$

Its kernel is pointwise positive and its L^1 norm is uniformly bounded with respect to δ. With a change of variable we have

$$T_\delta'(y-y') = \frac{1}{2\pi}\frac{1}{\pi^N}\frac{1}{\delta\ln\frac{1}{\delta^2}}\int_{|x|=1} dx\int_0^{\frac{1}{\delta^2}} d\nu\, \frac{(1-\delta^2\nu)^{\lambda+N-1}}{\nu^N(2-\delta^2\nu)^N}\left(\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}\right)$$
$$e^{-\frac{\delta^2}{\ln\frac{1}{1-\delta^2\nu}}(x+y_i-y_i')^2} e^{-\frac{1-\delta^2\nu}{2-\delta^2\nu}\frac{1}{\nu}(y-y')^2} \tag{3.30}$$

and it is straightforward to prove that the integral outside any compact set containing the origin goes to zero.
Since, by previous remarks, $q_i(y)\int dy' T_\delta(y;y') \to \frac{1}{2}q_i(y)$ in $L^2(\mathbb{R}^{3N})$ in order to conclude the proof it is sufficient to show that $\int dy' T_\delta(y;y')(q(y)-q(y')) \to 0$. This is straightforward since

$$\lim_{\delta\to 0}\int dy\left|\int dy'\, T_\delta(y;y')\,(q(y)-q(y'))\right|^2 \le \lim_{\delta\to 0}\int dy\left[\int dy'\, T_\delta'(y;y')\,|q(y)-q(y')|\right]^2 = 0 \tag{3.31}$$

The second limit in (3.31) holds because we can interchange the limit and the integral over y by dominated convergence theorem and T_δ' is an approximation of the identity.

□

3.3 CONSTRUCTION OF THE HAMILTONIAN

Here we prove the closure and the boundedness from below of the quadratic form F_α. The real object to study is Φ_α^λ which contains most of the information about the interaction. Once we have proved that Φ_α^λ is closed and coercive it is quite straightforward to prove that F_α is closed and bounded from below. Indeed one has

Theorem 3.7. *Let Φ_α^λ be closed and coercive, then F_α is closed and bounded from below.*

Proof
It is sufficient to prove that $F_\alpha^\lambda[u] = F_\alpha[u] + \lambda\|u\|^2$ is positive and closed. The positivity of $F_\alpha^\lambda[u]$ is trivial and then we discuss the closure.
If $\{u_n = \varphi_n^\lambda + G^\lambda q_n\} \subset \mathscr{D}(F_\alpha)$ is a sequence converging to u in $L^2(\mathbb{R}^{2N+2})$ such that $\lim_{n,m} F_\alpha^\lambda[u_n - u_m] = 0$, one has

$$\lim_{n,m\to\infty} F_0[\varphi_n^\lambda - \varphi_m^\lambda] = 0 \qquad \lim_{n,m\to\infty} \Phi_\alpha^\lambda[q_n - q_m] = 0$$

Since Φ_α^λ is coercive then $\{q_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^{2N})$ and there exists $q \in L^2(\mathbb{R}^{2N})$ such that $q_n \to q$. By proposition (3.2) $G^\lambda q_n \to G^\lambda q$ and then $\varphi_n^\lambda \to \varphi^\lambda \equiv u - G^\lambda q$.
Since F_0 and Φ_α^λ are closed, we have $\varphi^\lambda \in \mathscr{D}(F_0)$, $q \in \mathscr{D}(\Phi_\alpha^\lambda)$ and

$$\lim_{n\to\infty} F_0[\varphi_n^\lambda - \varphi^\lambda] = 0 \qquad \lim_{n\to\infty} \Phi_\alpha^\lambda[q_n - q] = 0$$

This implies $u \in \mathscr{D}(F_\alpha)$ and $F_\alpha[u_n - u] \to 0$.

□

In the following proposition we study Φ_α^λ.

Proposition 3.8. *If $\lambda > 0$ is sufficiently large then Φ_α^λ is closed and positive on*

$$\mathscr{D}(\Phi_{\alpha,d}^\lambda) = \{q \in L^2(\mathbb{R}^{2N}) \ s.t. \ \Phi_{\alpha,d}^\lambda[q] < +\infty\}$$

Proof
First we prove that the diagonal part of the quadratic form is closed and coercive for $\lambda > 0$ sufficiently large. It is sufficient to consider the case $N = 1$.

$$\Phi_{\alpha,1}^\lambda[q_1] = \alpha \int dy|q_1(y)|^2 + \int dy\, a_1^\lambda(y)|q_1(y)|^2 + \frac{1}{2}\int dydy' G^\lambda(y,y;y',y')|q_1(y) - q_1(y')|^2$$

Let's define $C_\alpha^\lambda = \alpha + \inf_y a_1^\lambda(y)$. Since $\Phi_{\alpha,d}^\lambda \geq C_\alpha^\lambda\|q\|^2$ and $\lim_{\lambda\to+\infty} C_\alpha^\lambda = +\infty$ the form $\Phi_{\alpha,d}^\lambda$ is coercive for sufficiently large λ.
In order to prove the closure of $\Phi_{\alpha,1}^\lambda$ we can mimic the Fischer-Riesz proof of the completeness of L^2. Given a Cauchy sequence $\{q_n\} \subset \mathscr{D}(\Phi_{\alpha,1}^\lambda)$ we can pick a subsequence, still denoted by $\{q_n\}$, such that $\Phi_{\alpha,1}^\lambda[q_n - q_{n+1}] < 2^{-n}$ and prove the convergence of such a subsequence.
Let's consider $Q_N(y) = \sum_{i=1}^N |q_i(y) - q_{i+1}(y)|$ and $Q(y) = \sum_{i=1}^\infty |q_i(y) - q_{i+1}(y)|$; by the monotone convergence theorem we have $Q \in L^2(\mathbb{R}^3)$ and $Q < +\infty$ almost everywhere.

In order to prove that $Q \in \mathscr{D}(\Phi^\lambda_{\alpha,11})$ we notice that $\Phi^\lambda_{\alpha,1}[Q_N]$ is uniformly bounded with respect to N, indeed one has

$$\sqrt{\Phi^\lambda_{\alpha,1}[Q_N]} \le \sum_{i=1}^{N} \sqrt{\Phi^\lambda_{\alpha,1}[|q_i - q_{i+1}|]} \le c$$

By Fatou's lemma we get $Q \in \mathscr{D}(\Phi^\lambda_{\alpha,1})$.
Then $q_1 + \sum_i (q_{i+1} - q_i)$ is absolutely convergent to a sum q with $|q - q_1| \le Q$ so that $q \in \mathscr{D}(\Phi^\lambda_\alpha)$.
It is straightforward to show that $\Phi^\lambda_{\alpha,1}[q - q_n] \to 0$.
Now we prove

$$\left| \int dy\, \overline{q_i} G^\lambda q_j \right| \le C \|q_i\| \|q_j\| \tag{3.32}$$

Take $i = 1$ $j = 2$ and put $y_0 = 0$ to be concrete. Since

$$G^\lambda(y_1, y; y_2', y') \le \frac{1}{2\pi^{N+1}} \int_0^1 d\nu \frac{\nu^{\lambda+N-1}}{(1-\nu^2)^N \ln \frac{1}{\nu}} e^{-\frac{\nu}{1-\nu^2}(y-y')^2} e^{-\frac{1}{2\ln\frac{1}{\nu}}(y_1 - y_2')^2} \tag{3.33}$$

it is sufficient to prove that the r.h.s. of (3.33) is a bounded operator. Using the Fourier transform, we are concerned with the estimate of

$$\begin{aligned}
&\Bigg| 2^{N-5} \int dk_1 dk_2 dk_2' dk_3 \dots dk_N\, \overline{q_1(k_1, \dots, k_N)} q_2(k_1 + k_2 - k_2', k_2', k_3, \dots, k_N) \cdot \\
&\qquad \int_0^1 d\nu \nu^{\lambda-1} e^{-\frac{1}{2}\ln\frac{1}{\nu}(k_2 - k_2')^2} e^{-\frac{1-\nu^2}{4\nu}(k_2^2 + (k_1 + k_2 - k_2')^2)} \Bigg| \\
&\le 2^{N-5} \int dk dk_2' |q_1(k_1, k_2 + k_2', k_3, \dots, k_N)| |q_2(k_1 + k_2, k_2', k_3, \dots, k_N)| \cdot \\
&\qquad \int_0^1 d\nu \nu^{\lambda-1} e^{-\frac{1-\nu}{2}\left[k_2^2 + (k_2 + k_2')^2 + (k_1 + k_2)^2\right]}
\end{aligned} \tag{3.34}$$

With the following change of variables

$$\begin{cases} k_1 = -2p_1 + p_2 \\ k_2 = p_1 - p_2 - p_3 \\ k_2' = -2p_1 + p_3 \end{cases}$$

we arrive at

$$9\ \left(2^{N-5}\right)\int dp_1 dp_2 dp_3 dk_3 \dots dk_N |q_1(-2p_1+p_2,-p_1-p_2,k_3,\dots,k_N)|\cdot$$
$$|q_2(-p_1-p_3,-2p_1+p_3,k_3,\dots,k_N)|\int_0^1 d\nu\nu^{\lambda-1}e^{-\frac{1-\nu}{2}\left[3p_1^2+2p_2^2+2p_3^2+2p_2\cdot p_3\right]}$$
$$\leq 9\ \left(2^{N-5}\right)\int dp_1 dp_2 dp_3 dk_3 \dots dk_N |q_1'(p_1,p_2,k_3,\dots,k_N)||q_2'(p_1,p_3,k_3,\dots,k_N)|\cdot$$
$$\int_0^1 d\nu\nu^{\lambda-1}e^{-(1-\nu)\left[p_2^2+p_3^2+p_2\cdot p_3\right]}$$

In the previous formula we have defined $q_1(-2p_1+p_2,-p_1-p_2,k_3,\dots,k_N) = q_1'(p_1,p_2,k_3,\dots,k_N)$ and $q_2(-p_1-p_3,-2p_1+p_3,k_3,\dots,k_N) = q_2'(p_1,p_3,k_3,\dots,k_N)$. By the Cauchy-Schwartz inequality it is sufficient to prove that

$$S(p_2;p_3) = \int_0^1 d\nu\nu^{\lambda-1}e^{-(1-\nu)\left[p_2^2+p_3^2+p_2\cdot p_3\right]}$$

defines a bounded operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$. Using Schur's test we have

$$\|S\| \leq \sup_{p_2}|p_2|^{\frac{1}{2}}\int dp_3\int_0^1 d\nu\nu^{\lambda-1}e^{-(1-\nu)\left[p_2^2+p_3^2+p_2\cdot p_3\right]}\frac{1}{|p_3|^{\frac{1}{2}}}$$
$$\leq \sup_{p_2}|p_2|^{\frac{1}{2}}\int dp_3\int_0^1 d\nu\nu^{\lambda-1}e^{-\frac{1-\nu}{2}\left[p_2^2+p_3^2\right]}\frac{1}{|p_3|^{\frac{1}{2}}} \leq \pi 2^{\frac{3}{4}}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

Using the estimate

$$\left|\Phi^\lambda_{\alpha,nd}[q]\right| \leq \frac{C(N-1)}{C^\lambda_\alpha}\Phi^\lambda_{\alpha,d}[q]$$

and applying the KLMN theorem, we conclude that Φ^λ_α is closed and coercive on $\mathscr{D}(\Phi^\lambda_{\alpha,d})$. □

We state, for the sake of completeness, the theorem following from proposition (3.8) and theorem (3.7).

Theorem 3.9. *The quadratic form F_α is closed and bounded from below*

Let us denote by Γ^λ_α the positive self adjoint operator in $L^2(\mathbb{R}^{2N})$ associated to Φ^λ_α and by H_α the hamiltonian defined by F_α. Then we have

Theorem 3.10. *The domain and the action of H_α are the following*

$$\mathscr{D}(H_\alpha) = \left\{u \in L^2(\mathbb{R}^{2N+2})\, s.t.\, u = \varphi^\lambda + G^\lambda q,\ \varphi^\lambda \in \mathscr{D}(H_0), q \in \mathscr{D}(\Gamma^\lambda_\alpha), \Gamma^\lambda_\alpha q = Tr_\pi \varphi^\lambda\right\} \quad (3.35)$$

$$(H_\alpha + \lambda)u = (H_0 + \lambda)\varphi^\lambda \tag{3.36}$$

and the resolvent of H_α can be represented as

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q \tag{3.37}$$

where q is the solution of

$$\Gamma_\alpha^\lambda q = Tr_\pi G^\lambda f \tag{3.38}$$

Proof

A function u belongs to $\mathscr{D}(H_\alpha)$ if and only if there exists $w \in L^2$, such that

$$F_\alpha[u, v] = (w, v) \qquad\qquad \forall v \in \mathscr{D}(F_\alpha) \tag{3.39}$$

In particular for $v \in C_0^\infty$, which means $q_v \equiv 0$, (3.39) reduces to:

$$\int dxdy \left[\frac{1}{2}\overline{\partial_x \varphi_u^\lambda}\partial_x v + \frac{1}{2}\overline{\partial_y \varphi_u^\lambda}\partial_y v + \frac{1}{2}(y - y^0)^2 \overline{\varphi_u^\lambda} v + \lambda \overline{\varphi_u^\lambda} v\right] = \int dxdy\, \overline{(H_\alpha + \lambda)u}\, v \tag{3.40}$$

Equation (3.40) implies

$$\varphi_u^\lambda \in \mathscr{D}(H_0) \qquad\qquad (H_\alpha + \lambda)u = (H_0 + \lambda)\varphi_u^\lambda \tag{3.41}$$

If we now take a generic element of the form domain as test function we obtain

$$\int dxdy \left[\frac{1}{2}\overline{\partial_x \varphi_u^\lambda}\partial_x \varphi_v^\lambda + \frac{1}{2}\overline{\partial_y \varphi_u^\lambda}\partial_y \varphi_v^\lambda + \left(\frac{1}{2}(y - y^0)^2 + \lambda\right)\overline{\varphi_u^\lambda}\varphi_v^\lambda\right] + \\ + \Phi_\alpha^\lambda[q_u, q_v] = \int dxdy \overline{(H_\alpha + \lambda)u} v$$

Using (3.41), we arrive at

$$\Phi_\alpha^\lambda[q_u, q_v] = \int dxdy \overline{(H_0 + \lambda)\varphi_u^\lambda}\, G^\lambda q_v$$

which is equivalent to

$$\Phi_\alpha^\lambda[q_u, q_v] = \int dxdy\, \overline{Tr_\pi \varphi_u^\lambda}\, q_v$$

The previous equation implies

$$q_u \in \mathscr{D}(\Gamma_\alpha^\lambda) \qquad\qquad \Gamma_\alpha^\lambda q_u = Tr_\pi \varphi_u^\lambda \tag{3.42}$$

Equation (3.42) is a boundary condition which must be satisfied by the elements of the domain. □

As in the one dimensional case the hamiltonian H_α is a local operator. The operator domain (3.35) is characterized by the boundary condition $\Gamma_\alpha^\lambda q = Tr_\pi \varphi^\lambda$ connecting the singular and the regular part of $u \in \mathscr{D}(H_\alpha)$. As in the one dimensional case, such boundary condition is in fact

local and it is the natural generalization of the boundary condition defining the Schroedinger operator with point interactions at fixed positions in $\mathbb{R}^2$ (see [AGH-KH]).
More precisely we can show that for $u \in \mathscr{D}(H_\alpha)$ the condition $\Gamma_\alpha^\lambda q = Tr_\pi \varphi^\lambda$ is equivalent to

$$p_i = \alpha q_i, \qquad i = 1, \dots, N \tag{3.43}$$

where

$$p_i(y) = \lim_{x \to y_i} \left[u(x,y) - \frac{1}{4\pi} \ln\left(\frac{1}{|x - y_i|} \right) q_i(y) \right], \qquad y \in \pi_i \tag{3.44}$$

In order to prove this statement, we introduce the regularized kernel G_ε^λ, $\varepsilon > 0$, obtained from (5.1) restricting the integration in the variable ν to $[0, 1-\varepsilon]$; If ew denote by $G_\varepsilon^\lambda q_i$ the corresponding regularized potentials, it is straightforward to prove that

$$\|G_\varepsilon^\lambda q - G^\lambda q\|_{L^2} \to_{\varepsilon \to 0} 0$$

Next we consider

$$\begin{aligned}\sum_{i=1}^N \int dy\, \overline{q_i(y)} p_i(y) = \sum_{i=1}^N \int dy\, \overline{q_i(y)} \Bigg[\lim_{x \to y_i} \lim_{\varepsilon \to 0} \Big(\varphi^(x,y) + G_\varepsilon^\lambda q + \\ - \frac{1}{4\pi} \int_0^{1-\varepsilon} d\nu\, \frac{1}{1-\nu} e^{-\frac{1}{1-\nu}(x-y_i)^2} q_i(y) - \frac{1}{4\pi} C q_i(y) \Big) \Bigg]\end{aligned} \tag{3.45}$$

If we interchange the limit over ε with the integral and the limit over x, the latter limit can be taken due to the introduced regularization. Then equation (3.45) can be written in a more useful way repeating the manipulations formally introduced in the heuristic derivation of the quadratic form

$$\begin{aligned}\sum_{i=1}^N \int dy\, \overline{q_i(y)} p_i(y) = \lim_{\varepsilon \to 0} \sum_{i=1}^N \int dy\, \overline{q_i(y)} \Bigg[Tr_{\pi_i} \varphi^\lambda + G_\varepsilon^\lambda q - q_i \int dy'\, G_\varepsilon^\lambda(y_i, y; y_i', y') + \\ + q_i \Big(\int dy'\, G_\varepsilon^\lambda(y_i, y; y_i', y') - \frac{1}{4\pi} \int_0^{1-\varepsilon} d\nu\, \frac{1}{1-\nu} e^{-\frac{1}{1-\nu}(x-y_i)^2} q_i(y) - \frac{1}{4\pi} C q_i(y) \Big) \Bigg] \\ = \int dy\, \overline{q(y)} Tr_\pi \varphi^\lambda - \lim_{\varepsilon \to 0} \Phi_{\varepsilon\alpha}^\lambda[q] + \alpha \int dy\, |q(y)|^2\end{aligned} \tag{3.46}$$

In the previous formula we have denoted by $\Phi_{\varepsilon\alpha}^\lambda$ the regularization of Φ_α^λ. It is straightforward to prove that $\lim_{\varepsilon \to 0} \Phi_{\varepsilon\alpha}^\lambda[q] = \Phi_\alpha^\lambda[q]$ for any $q \in \mathscr{D}(\Phi_\alpha^\lambda)$. Then we arrive at

$$\int dy\, \overline{q(y)} p(y) = \int dy\, \overline{q(y)} Tr_\pi \varphi^\lambda - \Phi_\alpha^\lambda[q] + \alpha \int dy\, |q(y)|^2 \tag{3.47}$$

which proves the equivalence of the two boundary conditions.

4. The three dimensional case

The three dimensional case is many respects similar to the two dimensional case and, in particular, the heuristic derivation of the quadratic form is essentially the same. The only difference is that the potential produced by the charges on the planes π_i has a stronger singularity. In fact one has

$$\left(G^\lambda 1_\varepsilon\right)(\xi,\eta,y_2,\dots,y_N) \simeq_{\xi\to 0} \frac{1}{2\pi}\frac{1}{|\xi|}$$

As a result we arrive at the following definition.

Definition 4.1.

$$\mathscr{D}(F_\alpha) = \left\{u \in L^2(\mathbb{R}^{3N+3})\; s.t.\; \exists q \in \mathscr{D}(\Phi^\lambda_\alpha),\; \varphi^\lambda \equiv u - G^\lambda q \in \mathscr{D}(F_0)\right\}$$

$$F_\alpha[u] = \mathcal{F}^\lambda[u] + \Phi^\lambda_\alpha[u]$$

$$\mathcal{F}^\lambda[u] = \int dxdy \left\{\frac{1}{2}|\nabla_x\varphi^\lambda|^2 + \frac{1}{2}|\nabla_y\varphi^\lambda|^2 + \lambda|\varphi^\lambda|^2 - \lambda|u|^2 + \frac{1}{2}(y-y^0)^2|\varphi^\lambda|^2\right\}$$

$$\mathscr{D}(\Phi^\lambda_\alpha) = \left\{q\; s.t.\; q_i \in L^2(\pi_i)\,,\; \Phi^\lambda[q] < +\infty\right\}$$

$$\Phi^\lambda_\alpha[q] = \alpha\sum_{i=1}^N \int dy|q_i(y)|^2 - 2\Re\sum_{i<j}\int dy\overline{q_i(y)}(G_i^\lambda q_j)(y) +$$
$$+\sum_{i=1}^N \int dy\, a_i^\lambda(y)|q_i(y)|^2 + \frac{1}{2}\sum_{i=1}^N \int dydy' G^\lambda(y_i,y;y_i',y')|q_i(y)-q_i(y')|^2$$

where the functions a_i^λ are now given by

$$a_i^\lambda(y) = \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\frac{1}{2}\right)^{\frac{3}{2}}\left\{\frac{1}{2} + \int_0^1 d\nu \frac{1}{(1-\nu)^{\frac{3}{2}}}\left[1-\nu^{\lambda+\frac{3N}{2}-1}\left(\frac{2}{1+\nu^2}\right)^{\frac{3(N-1)}{2}}\right.\right.$$
$$\left.\left.\left(\frac{2}{\frac{1+\nu^2}{2}\frac{\ln\frac{1}{\nu}}{1-\nu}+\frac{1+\nu}{2}}\right)^{\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{1-\nu^2}{1+\nu^2}\sum_{\substack{j=1\\ j\neq i}}^N (y_j-y_j^0)^2\right)\exp\left(-\frac{\frac{1}{4}+\frac{1-\nu}{1+\nu}\frac{1}{2\ln\frac{1}{\nu}}}{\frac{1}{2}\frac{1+\nu^2}{1-\nu^2}+\frac{1}{2\ln\frac{1}{\nu}}}(y_i-y_i^0)^2\right)\right]\right\} \tag{4.1}$$

The well-posedness of definition (4.1) can be shown following the same line of the previous case and then we omit the proof

Proposition 4.2.

i) If $q_i \in L^2(\mathbb{R}^{3N})$ *then* $G^\lambda q_i \in L^2(\mathbb{R}^{3N+3})$.
ii) If $q_i \in \mathscr{D}(\Phi_\alpha^\lambda)$ *then* $G^\lambda q_i \notin \mathscr{D}(F_0)$.
iii) The decomposition $u = \varphi^\lambda + G^\lambda q$ *is well defined and unique.*
iv) The following limit holds in $L^2(\pi_i)$

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{B_{\delta,i}} dx u = q_i \tag{4.2}$$

The main difference between the two and the three dimensional case arises in the proof of the closure and the boundedness from below of the quadratic form. As a matter of fact in dimension three we can prove the result only in the simpler case $N = 1$. This is due to the fact that the non diagonal terms of Φ_α^λ are not bounded in L^2 and then they are not easily dominated by the (positive and coercive) diagonal terms which was the strategy used in dimension two. The problem is analogous to the case in which the harmonic potential is absent and the form is unbounded from below. Nevertheless we believe that a more careful analysis of the non diagonal terms, exploiting the confining properties of the harmonic potential, could show that the concentration phenomenon responsible for the wave function collapse is absent here at least if the equilibrium positions y_i^0 are sufficiently far apart from each other. This may lead to a form closed and bounded below and then to a physically reasonable hamiltonian. We plan to come back to this problem in a further work. Here we simply state the result for $N = 1$; the proof requires only minor changes with respect to the two dimensional case and it is omitted for the sake of brevity.

Theorem 4.3. *Let* $N = 1$ *then the quadratic form* F_α *is closed and bounded from below.*

Moreover concerning the hamiltonian and the resolvent, one can easily prove the following result.

Theorem 4.4. *The domain and the action of* H_α *are given by*

$$\mathscr{D}(H_\alpha) = \left\{ u \in L^2(\mathbb{R}^6) \; s.t. \; u = \varphi^\lambda + G^\lambda q, \; \varphi^\lambda \in \mathscr{D}(H_0), q \in \mathscr{D}(\Gamma_\alpha^\lambda), \Gamma_\alpha^\lambda q = Tr_\pi \varphi^\lambda \right\}$$

$$(H_\alpha + \lambda) u = (H_0 + \lambda) \varphi^\lambda$$

and the resolvent has the following representation

$$(H_\alpha + \lambda)^{-1} f = G^\lambda f + G^\lambda q$$

where q *is a solution of*

$$\Gamma_\alpha^\lambda q = Tr_\pi G^\lambda$$

5. Appendix: the Green's function of the free system

Here we derive the expression of the integral kernel of the resolvent of H_0 used in the paper which in the dimension $d = 1, 2, 3$ has the form

$$G^\lambda(x, y; x', y') = (H_0 + \lambda)^{-1}(x, y; x'y') = \frac{1}{(2\pi)^{\frac{d}{2}}}\frac{1}{\pi^{\frac{dN}{2}}}$$

$$\int_0^1 d\nu\, \nu^{\lambda+\frac{dN}{2}-1}\frac{1}{(1-\nu^2)^{\frac{dN}{2}}}\frac{1}{\left(\ln\frac{1}{\nu}\right)^{\frac{d}{2}}}e^{-\frac{1}{2}\frac{1-\nu}{1+\nu}(y^2+y'^2)}e^{-\frac{1}{2\ln\frac{1}{\nu}}(x-x')^2}e^{-\frac{\nu}{1-\nu^2}(y-y')^2} \tag{5.1}$$

Starting from the Mehler kernel and performing the Laplace transform with respect to time one finds the Green's function for the harmonic oscillator in $\mathbb{R}^n$

$$G^\lambda_{\rm osc}(y, y') = (H_{\rm osc} + \lambda)^{-1}(y, y') = \int_0^{+\infty} dt\, e^{-\lambda t}\left(\frac{1}{\pi(1-e^{-2t})}\right)^{\frac{n}{2}} e^{-\frac{y^2+y'^2}{2\tanh(t)}+\frac{y\cdot y'}{\sinh(t)}}e^{-t\frac{n}{2}}$$

With the change of variable $\nu = e^{-t}$ one obtains

$$G^\lambda{}_{\rm osc}(y, y') = (H_{\rm osc} + \lambda)^{-1}(y, y') = \frac{1}{\pi^{\frac{n}{2}}}e^{-\frac{1}{2}(y^2+y'^2)}\int_0^1 d\nu\, \nu^{\lambda+\frac{n}{2}-1}\frac{1}{(1-\nu^2)^{\frac{n}{2}}}e^{\frac{2\nu y\cdot y'-\nu^2\left(y^2+y'^2\right)}{(1-\nu^2)}} \tag{5.2}$$

Notice that formula (5.2) can be also derived using Hermite functions (see e.g. [BC]). To compute $(H_0 + \lambda)^{-1}$, we solve the following equation with respect to f for a given g

$$\left[\left(-\frac{1}{2}\Delta_x - \frac{1}{2}\Delta_y + \frac{1}{2}y^2 + \lambda\right) f\right](x, y) = g(x, y)$$

Where we consider $y^0 = 0$ for simplicity. Taking the Fourier transform with respect to the x variable we find

$$\begin{aligned}\tilde g(k, y) &= \int dy' G^{\lambda+\frac{k^2}{2}}_{\rm osc}(y, y')\tilde f(k, y')\\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int dx'dy' e^{-\imath k\cdot x'}G^{\lambda+\frac{k^2}{2}}_{\rm osc}(y, y')f(x', y')\end{aligned}$$

and then

$$(H_0 + \lambda)^{-1}(x, y; x', y') = \frac{1}{(2\pi)^d}\int dk e^{\imath k\cdot(x-x')}G^{\lambda+\frac{k^2}{2}}_{\rm osc}(y, y')$$

Substituting (5.2) in this equation we finally get (5.1).

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