A SZEGŐ CONDITION FOR A MULTIDIMENSIONAL SCHRÖDINGER OPERATOR

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ABSTRACT. We consider spectral properties of a Schrödinger operator perturbed by a potential vanishing at infinity and prove that the corresponding spectral measure satisfies a Szegő type condition.

1. INTRODUCTION

In this paper we consider a multidimesional Schrödinger operator in $L^2(\mathbb{R}^d)$ and introduce a version of a Szegő condition (2.5), which is well known for spectral measures of Jacobi matrices (see, for example [4], [10], [11], [12], [13], [15]). Our condition seems to be comparable with the classical Szegő integrability only for small energies. The corresponding condition for large energies is much weaker than the expected one and cannot be obtained by the methods of this paper.

One of the motivations of this article is the conjecture of B. Simon formulated in [13].

Conjecture: Let V be a real function on $\mathbb{R}^d$, $d \geq 2$, which obeys

$$\int |x|^{-d+1}|V(x)|^2\, dx < \infty. \tag{1.1}$$

Then $-\Delta + V$ has the a.c. spectrum of infinite multiplicity essentially supported by $[0, \infty)$.

Note that for spherically symmetric potentials this result follows from the paper by P. Deift and R. Killip [3], where this conjecture is solved for $d = 1$. For $d \geq 2$ it is still open.

The assumptions imposed on the potential in our main Theorem 2.1 are much more restrictive than those in (1.1). In fact they are close to the conditions under which absolute continuity of the spectrum can be proven by the methods of the scattering theory. However our work differs from the results obtained in the scattering theory in a critical way: we prove a certain estimate showing that the spectral measure of the Schrödinger operator can not be too small and this estimate turns to be of an independent interest. It

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is known that there are so called Lieb-Thiring bounds for the eigenvalues of the operator $-\Delta + V$:

$$\sum_j |\lambda_j|^\gamma \le C \int |V(x)|^{d/2+\gamma}\, dx, \quad \gamma > 0.$$

The Szegő condition (2.5) can be interpreted as a version of Lieb-Thirring inequalities for the a.c. part of the spectrum.

Although we start the proof considering a class of smooth potentials V, the final result does not assume any smoothness of V.

When proving the main result we are able to use an analog of Buslaev-Faddeev-Zakharov trace formulae well known for one-dimensional Schrödinger operators. The multidimensional case is reduced to a problem for a second order elliptic intergo-differential operator. One of the main difficulties of this appoach is the treatment of the “potential” type term which appears to be a dissipative integral operator depending on the spectral parameter. The corresponding Fredholm equation for the Jost functions might be not solvable for a discrete subset of the complex upper half plane. There is a hope that the corresponding contribution into trace formulae coming from this subset can be controled by some Lieb-Thirring ineqaulities. Fortunately the positivity of the imaginary parts of the points from this subset appears together with the ”right” sign in the so-called ”first” trace formula. The contribution of these points in the ”second” trace formula is distructive and requires some upper estimates. This explains why in our main theorem we obtain condition involving the first power of the potential V rather than V^2.

2. The main result

Let Ω_1 be the unit ball in $\mathbb{R}^d$ and V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$. We consider the operator $H = H_0 + V = -\Delta + V$ on $L^2(\mathbb{R}^d \setminus \Omega_1)$, with the Dirichlet boundary conditions on $\partial\Omega_1 = \mathbb{S}^{d-1}$. We can assume without loss of generality that there is $c_1 > 1$ such that

$$V + \frac{\alpha_d}{|x|^2} = 0 \quad \text{for} \quad 1 < |x| < c_1, \tag{2.1}$$

where $\alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}$. Let $E_H(\delta)$, $\delta \subset \mathbb{R}$, be the spectral projection of the operator H. We construct a measure μ on the the real line such that for spherically symetric functions f

$$(E_H(\delta)f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda), \quad \delta \subset \mathbb{R}_+ = (0, \infty), \tag{2.2}$$

where
(2.3)

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\, r^{(d-1)/2}dr, \quad \mathrm{supp} f \subset \{x:\ 1 < |x| < c_1\}.$$

and $k^2 = \lambda > 0$.

Let $\mathbb{Q} = [0,1)^d$. Then cubes $\mathbb{Q}_n = \mathbb{Q}+n,\ n \in \mathbb{Z}^d$, form a partition of $\mathbb{R}^d$ with which we associate the classes of functions u such that the sequence of (quasi)norms $\{\|u\|_{L^q(\mathbb{Q}_n)}\}_{n=1}^{\infty}$ belongs to ℓ^p, $0 < p, q \leq \infty$. These classes are denoted by $\ell^p(\mathbb{Z}^d; L^q(\mathbb{Q}))$. When proving the main result we need the boundedness of the operators in (7.4). For example this can be provided by the following local condition on V from [2]

$$V \in \ell^\infty(\mathbb{Z}^d; L^q(\mathbb{Q})), \quad q > d/2, \tag{2.4}$$

which can be weakened by using the characterization of weak Hardy's weights in terms of capacities obtained by V.Maz'ya (see [8]). Note that if (2.4) is satisfied then the operator H can be defined in the sense of quadratic forms (see [2]).

Theorem 2.1. *Let V be a real valued function on $\mathbb{R}^d \setminus \Omega_1$ which obeys* (2.4) *and such that*

$$\int_{\mathbb{R}^d\setminus\Omega_1} V_-^{(d+1)/2}(x)\, dx < \infty, \qquad \int_{\mathbb{R}^d\setminus\Omega_1} V_+|x|^{-d+1}\, dx < \infty,$$

where $2V_\pm = |V| \pm V$. Then

$$\int_0^\infty \frac{\log(1/\mu'(t))\, dt}{(1+t^{3/2})\sqrt{t}} < \infty, \tag{2.5}$$

where μ is defined in (2.2). *If* (2.1) *is satisfied then* (2.5) *is equivalent to*

$$\int_0^\infty \frac{\log\big(\frac{d}{d\lambda}(E_H(\lambda)f,f)\big)\, d\lambda}{(1+\lambda^{3/2})\sqrt{\lambda}} > -\infty, \tag{2.6}$$

for any bounded spherically symmetric function $f \neq 0$ with $\mathrm{supp} f \subset \{x : 1 < |x| < c_1\}$.

Remark 1. The inequality (2.5) guaranties that the a.c. spectrum of H is essentially supported by $[0,\infty)$, since $\mu' > 0$ almost everywhere and gives a quantative information about the measure μ.

Remark 2. The equivalence of (2.5) and (2.6) follows from the fact that if F is defined as in (2.3), then the function $(1+\lambda^2)^{-1}\log(|F(\lambda)|)$ is in $L^1(\mathbb{R}_+)$ see, for example, P. Koosis [5] (section IIIG2).

3. Reduction to a one-dimensional problem

In this section we assume that $V \in C_0^\infty$ and often use polar coordinates (r,θ), $x = r\theta \in \mathbb{R}^d$, $\theta \in \mathbb{S}^{d-1}$. Denote by $\{Y_j\}_{j=0}^\infty$ the orthonormal in $L^2(\mathbb{S}^{d-1})$ basis of (real) spherical funcions and let P_j be the orthogonal projection given by

$$P_j u(r,\theta) = Y_j(\theta)\int_{\mathbb{S}^{d-1}} Y_j(\theta')u(r,\theta')\,d\theta'.$$

Clearly $P_0 u$ depends only on r. Denote

$$V_1 = P_0 V P_0, \quad H_{0,1} = P_0 H_0 P_0,$$

$$V_{1,2} = P_0 V(I-P_0), \quad V_{2,1} = V_{1,2}^*,$$

$$V_2 = (I-P_0)V(I-P_0), \quad H_{0,2} = (I-P_0)H_0(I-P_0).$$

Then the operator $H - z$ can be represented as a matrix:

$$H - z = \begin{pmatrix} H_{0,1}+V_1-z & V_{1,2} \\ V_{2,1} & H_{0,2}+V_2-z \end{pmatrix},$$

and the equation

$$(H-z)u = P_0 f, \quad \operatorname{Im} z \neq 0,$$

is equivalent to

$$(H_{0,1}+T_z-z)P_0 u = P_0 f, \quad (H_{0,2}+V_2-z)^{-1}V_{2,1}P_0 u = (P_0 - I)u. \tag{3.1}$$

Here the operator T_z is defined by

$$T_z = V_1 - V_{1,2}(H_{0,2}+V_2-z)^{-1}V_{2,1}$$

on $L^2((1,\infty), r^{d-1}\,dr)$.

By using the unitary operator from $L^2((1,\infty),dr)$ to $L^2((1,\infty), r^{d-1}\,dr)$,

$$Uu(r) = r^{-(d-1)/2}u,$$

we reduce (3.1) to the problem for the following one-dimensional Schrödinger operator in $L^2(1,\infty)$

$$L_z u(r) = -\frac{d^2u}{dr^2} + Q_z u, \quad u \in L^2(1,\infty),\ u(1) = 0, \tag{3.2}$$

where

$$Q_z = V_1 + \frac{\alpha_d}{r^2} - V_{1,2}(U^* H_{0,2} U + V_2 - z)^{-1}V_{2,1}, \quad \alpha_d = \frac{(d-1)^2}{4} - \frac{d-1}{2}.$$

By considering the potential

$$V - \frac{\alpha_d}{r^2} \quad \text{instead of} \quad V$$

without loss of generality we can assume that

$$Q_z = V_1 - V_{1,2}(S + V_2 - z)^{-1}V_{2,1}, \tag{3.3}$$

where

$$Su = -\frac{d^2u}{dr^2} - \frac{\Delta_\theta u}{r^2}, \quad u(1,\theta) = 0. \tag{3.4}$$

Note that the conditions of Theorem 2.1 on V won't be changed.

According to (3.1) we obtain

$$P_0(H - z)^{-1}P_0 = U(L_z - z)^{-1}U^*. \tag{3.5}$$

We see also that if $\operatorname{supp} V \subset \{x \in \mathbb{R}^d : \quad c_1 < |x| < c_2\}$, $c_1 > 1$, then for the operator (3.3) we have

$$Q_z = Q_z\chi = \chi Q_z,$$

where χ is an operator of multiplication by the characteristic function of the interval (c_1, c_2), $c_1 > 0$. It is impotant for us that Q_z is an analytic operator valued function of z with a negative imaginary part in the upper half plane and which has a positive imaginary part in the lower half plane.

4. Green's function.

Let us consider the equation

$$-\frac{d^2}{dr^2}\psi(r) + (Q_z\psi)(r) = z\psi(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{4.1}$$

with Q_z given by (3.3) and let $\psi_k(r)$ be the solution of the equation (4.1) satisfying

$$\psi_k(r) = \exp{(ikr)}, \quad k^2 = z,\ \operatorname{Im} k > 0,\ \forall r > c_2.$$

Then this solution also satisfies the following "integral" equation

$$\psi_k(r) = e^{ikr} - k^{-1}\int_r^\infty \sin k(r-s)(Q_z\psi_k)(s)\,ds. \tag{4.2}$$

According to the analytic Fredholm theorem (see, for example, Theorem VI.14, [9]) we conclude that the equation (4.2) is uniquely solvable for all k except perhaps a discret sequence of points and its solution ψ_k is a meromorphic with respect to k function, $\operatorname{Im} k \geq 0$.

Consider the resolvent operator $R(z) = (L_z - z)^{-1}$, where L_z is defined in (3.2). If χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then $R(z)\chi_{c_1}$ is an integral operator with the kernel:

$$G_z(r,s) = \begin{cases} \frac{\psi_k(s)}{\psi_k(1)}\frac{\sin(k(r-1))}{k}, & \text{for } r < s < c_1, \\ \frac{\psi_k(r)}{\psi_k(1)}\frac{\sin(k(s-1))}{k}, & \text{for } s < \min\{c_1, r\}. \end{cases} \tag{4.3}$$

Indeed, assuming that $\mathrm{supp}(f) \subset (1, c_1)$ we can easily check that the function

$$u(r) = \frac{1}{\psi_k(1)}\Big\{\int_r^\infty \frac{\sin(k(r-1))}{k}\psi_k(s)f(s)\,ds + \int_1^r \psi_k(r)\frac{\sin(k(s-1))}{k}f(s)\,ds\Big\}$$

satisfies the equation

$$-\frac{d^2}{dr^2}u(r) + (Q_z u)(r) - zu(r) = f(r), \quad r \geq 1,\ z \in \mathbb{C}, \tag{4.4}$$

and moreover $u(1) = 0$.

5. Wronskian and properties of the M-function.

Let as in (3.3)

$$Q_z = V_1 - V_{1,2}(S + V_2 - z)^{-1}V_{2,1}.$$

The function

$$M(k) = \frac{\psi_k'(1)}{\psi_k(1)}$$

is called the Weyl M-function of the operator (4.1). Let us consider the Wronskian

$$W[\overline{\psi_k}, \psi_k](r) = \overline{\psi_k'}(r)\psi_k(r) - \overline{\psi_k}(r)\psi_k'(r). \tag{5.1}$$

Note that $\overline{\psi_k}$ satisfies the equation (4.1) with $Q_{\overline{z}}$ and $\overline{z}$ instead of Q_z and z. Since ψ_k is a solution of the equation (4.1) we find

$$\frac{d}{dr}W[\overline{\psi_k}, \psi_k](r) = (z-\overline{z})\overline{\psi_k}(r)\psi_k(r) + (Q_{\overline{z}}\overline{\psi_k})(r)\psi_k(r) - \overline{\psi_k}(r)(Q_z\psi_k)(r).$$

So we obtain

$$\pm\mathrm{Im}\,\{W[\overline{\psi_k}, \psi_k](c_2) - W[\overline{\psi_k}, \psi_k](c_1)\} \geq 0, \quad \text{for } \pm\,\mathrm{Im}\,z \geq 0+, \tag{5.2}$$

which means that for all k we have the following inequality

$$\frac{k}{\mathrm{Im}\,M(k)} \leq |\psi_k(1)|^2.$$

Moreover, if we represent the solution ψ_k for real k in the form

$$\psi_k(x) = a(k)e^{ikx} + b(k)e^{-ikx}, \quad x < c_1, \tag{5.3}$$

then it follows from (5.2) that

$$|a|^2 - |b|^2 \geq 1.$$

Then for $k^2 = z$

$$M(k) = \psi_k'(1)(\psi_k(1))^{-1} = ik(1-\rho(k))(1+\rho(k))^{-1}, \quad \rho(k) := e^{-2ik}b(k)a(k)^{-1}.$$

The latter implies

$$\rho(k) = (ik - M(k))(ik + M(k))^{-1}.$$

Since $|a|^2 - |b|^2 \geq 1$ we obtain that for real k

$$|a(k)|^{-2} \leq 1 - |\rho(k)|^2 = \frac{4k\text{Im } M}{|ik + M(k)|^2}.$$

Note that since Im $M \geq 0$, then for any $k > 0$ we have

$$|ik + M(k)|^2 = k^2 + |M|^2 + 2k\text{Im } M \geq k^2$$

and therefore

$$|a(k)|^{-2} \leq 4k^{-1}\big(\text{Im } M\big), \quad k > 0. \tag{5.4}$$

Note also that

$$\text{Im } M(k) > 0 \quad \text{if Im } k^2 > 0. \tag{5.5}$$

Thus, there are constants $C_0 \in \mathbb{R}$ and $C_1 \geq 0$ and a positive measure μ, such that

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty,$$

where

$$M(k) = C_0 + C_1 z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big) d\mu(t), \quad k^2 = z. \tag{5.6}$$

Finally, note that $R(z) = P_0(U^*H_0U + V - z)^{-1}P_0$ and therefore we can write formally that

$$M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)} = (P_0(U^*H_0U + V - z)^{-1}P_0\delta_1', \delta_1'),$$

where δ_1' is the derivative of $\delta(r-1)$. Let χ_{c_1} be the characteristic function of $(1, c_1)$. The representation (4.3) for the resolvent operator gives us the representation for the operator $\chi_{c_1}P_0E_{U^*H_0U+V}(\delta)P_0\chi_{c_1}$, where $E_{U^*H_0U+V}(\delta)$ is the spectral measure of $U^*H_0U + V$:

$$(P_0E_{U^*H_0U+V}(\delta)P_0f, f) = \int_\delta |F(\lambda)|^2 d\mu(\lambda) \tag{5.7}$$

and where

$$F(\lambda) = \frac{1}{k}\int_0^{c_1} \sin(k(r-1))f(r)\, dr, \quad \text{supp} f \subset (1, c_1), \;\; k^2 = \lambda.$$

Since F is a boundary value of an analytic function, we obtain that $F(\lambda) \neq 0$ for a.e. λ. This means that $E_H(\delta) \neq 0$ if $\mu' > 0$ a.e. on δ.

6. A TRACE INEQUALITY

In this section we assume that V is not a potential but the operator $\sum_{j=0}^n P_j V \sum_{j=0}^n P_j$, which approximates V for large n. It can be interpreted as an operator of multiplication by a matrix valued function of r. In this case the function V_1 remains the same as before. Since P_j are projections on real spherical functions, this matrix is real and therefore the the inequality (5.4) holds true. Moreover we substitute the term $-\Delta_\theta/r^2$ on $(1,\infty)$ by a "compactly supported" approximation $-\zeta_\varepsilon(r)\Delta_\theta/r^2$ where $\zeta_\varepsilon \in C_0^\infty(1,\infty)$ and $\zeta_\varepsilon(r)/r^2 \to 1/r^2$ in $L^1(1,\infty)$ as $\varepsilon \to 0$. Then the coefficient $a(k)$ introduced in (5.3) will depend on ε and we shall write $a_\varepsilon(k)$ instead of $a(k)$. From (4.2) and (3.3) we find that

$$\exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int_r^\infty (1 - e^{2ik(s-r)})V_1(s)\,ds + o(1/k)$$

and thus

$$a_\varepsilon(k) = \lim_{r\to-\infty} \exp(-ikr)\psi_k(r) = 1 - \frac{1}{2ik}\int V_1\,dr + o(1/k),$$

as $k \to \infty$. Now let $i\beta_m$ and γ_j be zeros and poles of $a_\varepsilon(k)$. Note that $-\overline{\gamma_j}$ are also poles of $a_\varepsilon(k)$ (this will follow from (6.2)). We shall see in a moment that $\beta_m > 0$. Let $\mathfrak{B}$ be the corresponding Blaschke product

$$\mathfrak{B}(k) = \prod_m \frac{(k - i\beta_m)}{(k + i\beta_m)} \prod_j \frac{(k - \overline{\gamma_j})}{(k - \gamma_j)}.$$

Clearly $|\mathfrak{B}(k)| = 1$, $\overline{\mathfrak{B}(k)} = \mathfrak{B}(-k)$, $k \in \mathbb{R}$, and we obtain

$$\text{(6.1)}\qquad \int_{-\infty}^{+\infty} \log(a_\varepsilon(k)/\mathfrak{B}(k))\,dk = \frac{\pi}{2}\int V_1\,dr + 2\pi\Big(\sum \beta_n - \sum \operatorname{Im}\gamma_j\Big),$$

provided that for some integer $l \geq 0$ the coefficient $a_\varepsilon(k)$ has an expansion $a_\varepsilon(k) = \sum_{j\geq -l} c_j k^j$ at zero. The existance of such an expansion as well as the condition $|a_\varepsilon(k)| - 1 = O(1/|k|^2)$ as $k \to \pm\infty$ will be proven later.

In order to prove that $\beta_m > 0$ let us show that $-\beta_m^2$ are the eigenvalues of a certain selfadjoint operator of a Schrödinger type. Namely, let $P = \sum_{j=0}^n P_j$ and let $\hat{H}_\varepsilon$ be the operator in $L^2(\mathbb{R}, PL^2(\mathbb{S}^{d-1}))$

$$\hat{H}_\varepsilon u = -\frac{d^2u}{dr^2} - \zeta_\varepsilon \frac{\Delta_\theta u}{r^2}, \quad (I - P_0)u(1,\cdot) = 0, \quad u(r) \in PL^2(\mathbb{S}^{d-1}),\ \forall r,$$

where ζ_ε is the same as above. Obviously, if $s < c_1 < c_2 < r$, then the kernel of the operator $P_0(\hat{H}_\varepsilon + V - z)^{-1}P_0$ equals

$$\text{(6.2)}\qquad g(r,s,k) = -\frac{\exp ik(r-s)}{2ika_\varepsilon(k)}.$$

The proof of the latter relation is a couterpart of the proof of (4.3). On the other hand we can consider the expansion of g near the eigenvalue $-\beta_m^2$. Denote by $\phi_{m,j}(r,\theta)$, $j = 1, 2 \dots n$ the orthonormal system of eigenfunctions corresponding to $-\beta_m^2$. If $\phi_{m,j}^{(0)} = \int_{\mathbb{S}^{d-1}} \phi_{m,j}(r,\theta)\, d\theta$ then

$$g(r,s,k) = \frac{\sum_{j=1}^n \phi_{m,j}^{(0)}(r)\overline{\phi_{m,j}^{(0)}(s)}}{k^2+\beta_m^2} + g_0(r,s,k), \qquad s < c_1 < c_2 < r,$$

where $g_0(r,s,k) = O(1)$, as $k \to i\beta_m$. This proves that $a_\varepsilon(k)$ is a meromorphic funcion in the upper half plane and its zeros correspond to the eigenvalues $-\beta_m^2$ of the operator $\hat{H}_\varepsilon + V$. Moreover the multiplicities of these zeros are equal to one. The latter arguments were inspired by [6].

Let us introduce martices $A(k)$ and $B(k)$ defined in the space $PL^2(\mathbb{S}^{d-1})$, such that the solution of the equation (for the matrix valued function Φ)

$$-\frac{d^2\Phi}{dr^2} - \zeta_\varepsilon \frac{\Delta_\theta \Phi}{r^2} + V\Phi = k^2\Phi, \quad \Phi = \exp(ikr)P, \quad r > c_2, \tag{6.3}$$

equals $\exp(ikr)A(k) + \exp(-ikr)B(k)$ for $r < c_1$.

We prove in the appendix of the paper that

$$\frac{1}{a_\varepsilon(k)} P_0 = P_0\big(A(k) + (I - P_0)e^{-2ik}B(k)\big)^{-1} P_0. \tag{6.4}$$

We shall also see that $A(k)$ and $B(k)$ both have at most a simple pole at zero and therefore by (6.4) $a_\varepsilon(k)$ could also have a pole at zero. Moreover we shall prove that

$$|a_\varepsilon(k)| - 1 = O\Big(\frac{1}{|k|^2}\Big),$$

as $k \to \pm\infty$, which, in particular, means that $\log|a_\varepsilon(k)| \in L^1(\mathbb{R})$.

Observe that when $\varepsilon \to 0$ the eigenvalues of $\hat{H}_\varepsilon + V$ converge to the eigenvalues of the operator $\hat{H} + V$, where $\hat{H}$ is the following operator in $L^2(\mathbb{R}, L^2(\mathbb{S}^{d-1}))$

$$\hat{H} = -\frac{d^2u}{dr^2} - \zeta_\varepsilon \frac{\Delta_\theta u}{r^2}, \quad (I - P_0)u(1,\cdot) = 0.$$

Denote the eigenvalues of $\hat{H}_\varepsilon$ by $-\big(\beta_m^{(0)}\big)^2$, where $\beta_m^{(0)} > 0$. Let us prove that by using Lieb-Thirring inequalities [7] we can obtain

$$\sum \beta_m^{(0)} \le C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2} dx + \int_{\mathbb{R}^d} V_-|x|^{-d+1} dx\Big). \tag{6.5}$$

Indeed, let $W_- = \sqrt{V_-}$. Then

$$W_-(\hat{H} - z)^{-1}W_- = W_-(S - z)^{-1}W_- + W_-\Theta(z)W_-,$$

where S is defined in (3.4) and $\Theta(z)$ is the operator of rank one with the integral kernel $e^{ik(r+s-2)}/2ik$, $k^2 = z$. Therefore

$$||W_-\Theta(z)W_-|| \le \frac{C}{\sqrt{|z|}} \int V_-|x|^{-d+1}\, dx, \quad z < 0. \tag{6.6}$$

Now for any compact operator T and $s > 0$ denote $n_+(s,T) = \operatorname{rank} E_T(s,\infty)$. Then

$$\sum \beta_m^{(0)} = \int_0^\infty n_+(1, W_-(\hat{H}+t)^{-1}W_-)\frac{dt}{2\sqrt{t}} \le$$
$$\le \int_0^\infty \big(n_+(1/2, W_-(S+t)^{-1}W_-) + n_+(1/2, W_-\Theta(-t)W_-)\big)\frac{dt}{2\sqrt{t}}.$$

Now the inequality (6.5) follows from

$$\int_0^\infty n_+(1/2, W_-(S+t)^{-1}W_-)\frac{dt}{2\sqrt{t}} \le C\int_{\mathbb{R}^d} V_-^{(d+1)/2} dx,$$

which is the classical Lieb-Thirring inequality and from

$$\int_0^\infty n_+(1/2, W_-\Theta(-t)W_-)\frac{dt}{2\sqrt{t}} \le C\int V_-|x|^{-d+1}\, dx,$$

which is implied by (6.6). Consequently, since Im $\gamma_j \ge 0$ the trace formula (6.1) together with (6.5) leads to the inequality

$$\begin{aligned} \limsup_{\varepsilon\to 0} \int_{-\infty}^{+\infty} \log|a_\varepsilon(k)|\, dk \le \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\, dr + \\ +C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2} dx + \int_{\mathbb{R}^d} V_-|x|^{-d+1} dx\Big). \end{aligned} \tag{6.7}$$

Therefore for any pair of finite numbers $r_2 > r_1 \ge 0$

(6.8)

$$\begin{aligned} \int_{r_1}^{r_2} \frac{1}{2}\log\frac{k}{4\mathrm{Im}\, M(k)}\, dk \le \limsup_{\varepsilon\to 0}\int_{-\infty}^{+\infty} \log|a_\varepsilon(k)|\, dk \le \frac{\pi}{2}\int_{-\infty}^{+\infty} V_1\, dr \\ +C\Big(\int_{\mathbb{R}^d} V_-^{(d+1)/2} dx + \int_{\mathbb{R}^d} V_-|x|^{-d+1} dx\Big), \end{aligned}$$

where the first inequality follows from Corollary 5.3 [4].

7. The end of the proof of Theorem 2.1

Assume that our perturbation V is an arbitrary function satisfying the conditions of Theorem 0.1. Then the Weyl function M can be defined for example as $M(k) = \frac{\partial^2}{\partial r \partial s} G_z(r,s)|_{(1,1)}$ where G_z is the integral kernel of the operator $P_0(U^*HU - z)^{-1}P_0$. The next proposition allows us to approximate V by compactly supported smooth functions V_n.

Proposition 7.1. *Let V satisfy the conditions of Theorem 2.1. Then there exists a sequence V_n of compactly supported smooth functions converging to V so that*

$$\int (V_n)_-^{(d+1)/2}\,dx < C(V) \quad \text{and} \quad \int (V_n)_+\,|x|^{-d+1}\,dx < C(V) \tag{7.1}$$

and such that the Weyl functions M_n corresponding to V_n converge uniformly when k^2 belongs to any compact subset of the upper half plane:

$$M_n(k) \to M(k).$$

Therefore the sequence of measures μ_n converges weakly to the spectral measure μ.

Proof. Let $W_\pm = \sqrt{V_\pm}$. Since the class C_0^∞ is dense in L^p for any $p > 0$, we can find a pair of sequences W_n^- and $W_n^+ \in C_0^\infty$ satisfying

$$\begin{gathered} W_n^- \to W_- \;\text{ in } L^{(d+1)}(\mathbb{R}^d); \quad W_n^+ \to W_+ \text{ in } L^2(\mathbb{R}^d, |x|^{-d+1}dx) \\ W_n^\pm \to W_\pm \text{ in } \ell^\infty(\mathbb{Z}^d; L^p(\mathbb{Q})), \quad p > d. \end{gathered} \tag{7.2}$$

Introduce a sequence of functions $\{V_n\}_{n=1}^\infty$

$$V_n = (W_n^+)^2 - (W_n^-)^2.$$

Then $V_n \in C_0^\infty$ and the relations (7.4) hold true. Suppose now that $\Gamma_0(z)$ and $\Gamma_n(z)$ are the resolvent operators of $S = U^*(-\Delta)U - \alpha_d/r^2$ and $S_n = S + V_n$ respectively. Denote by δ_1' the derivative of the delta function $\delta(r-1)$. The expression $\Gamma_0(z)\delta_1'$, $\operatorname{Im} z \neq 0$, can be understood as the function

$$\Gamma_0(z)\delta_1' = -\exp(ik(r-1)).$$

According to assumptions (7.2) we have that

$$W_n^\pm \Gamma_0(z)\delta_1' \to W_\pm \Gamma_0(z)\delta_1',$$

in $L^2(\mathbb{R}^d)$ (with respect to any weight). Thus in order to prove that the Weyl functions

$$\begin{aligned} M_n(k) &= \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_n(z)\delta_1', \delta_1') \\ &= (\Gamma_0(z)\delta_1', \delta_1') - ((W_n^+ - W_n^-)\Gamma_0(z)\delta_1', (W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1') \end{aligned}$$

converge, it is sufficient to show that

$$(W_n^+ + W_n^-)\Gamma_n(\overline{z})\delta_1' \to (W_+ + W_-)(S + V - \overline{z})^{-1}\delta_1' \tag{7.3}$$

in $L^2(\mathbb{R}^d)$.

Let us denote $W_n = W_n^+ + W_n^-$ and $W_n{}^{(0)} = W_n^+ - W_n^-$. Clearly, if $W_n^\pm \to W_\pm$ in the class (2.4) with $q > d$, as $n \to \infty$, then

$$W_n \Gamma_0(\overline{z}) W_n^{(0)} \to (W_+ + W_-)\Gamma_0(\overline{z})(W_+ - W_-) \tag{7.4}$$

in the operator norm topology.

Then (7.3) follows from the identity

$$W_n\Gamma_n(\overline{z})\delta_1' = (I + W_n\Gamma_0(\overline{z})W_n^{(0)})^{-1}W_n\Gamma_0(\overline{z})\delta_1'.$$

□

Similarly we can prove the following

Proposition 7.2. *Let V be a compactly supported smooth function. Then the Weyl functions M_l corresponding to $\sum_{j=0}^l P_j V \sum_{j=0}^l P_j$ converge uniformly to M when k^2 belongs to any compact subset K of the upper half plane*

$$M_l(k) \to M(k)$$

and therefore the sequence of measures μ_l converges weakly to the spectral measure μ constructed for V.

Proof. Let us denote $V_l = \sum_{j=0}^l P_j V \sum_{j=0}^l P_j$ let $\Gamma_0(z)$ and let $\Gamma_l(z)$ be the resolvent operators of $S = U^*(-\Delta)U - \alpha_d/r^2$ defined in (3.4) and $S_l = S + V_l$ respectively. As in Proposition 7.1 the expression $\Gamma_0(z)\delta_1'$, Im $z \neq 0$, is understood as the function $\Gamma_0(z)\delta_1' = -\exp(ik(r-1))$. According to our assumptions

$$V_l\Gamma_0(z)\delta_1' = \sum_{j=0}^{l} P_jV\Gamma_0(z)\delta_1' \to V\Gamma_0(z)\delta_1'$$

in $L^2(\mathbb{R}^d)$. Thus in order to prove that the Weyl functions

$$M_l(k) = \frac{\partial^2}{\partial r \partial s} G_{n,z}(r,s)|_{(1,1)} = (\Gamma_l(z)\delta_1', \delta_1')$$
$$= (\Gamma_0(z)\delta_1', \delta_1') - (V_l\Gamma_0(z)\delta_1', \Gamma_l(\overline{z})\delta_1')$$

converge, it is sufficient to show that $\Gamma_l(\overline{z})\delta_1'$ converges to $(S + V - \overline{z})^{-1}\delta_1'$ in $L^2(\mathbb{R}^d)$ uniformly on compact subsets K of the complex plane. The latter follows from the identity

$$\Gamma_l(\overline{z})\delta_1' = (S + V - \overline{z})^{-1}\delta_1' - \Gamma_l(\overline{z})(V_l - V)(S + V - \overline{z})^{-1}\delta_1' =$$

$$= (S + V - \overline{z})^{-1}\delta_1' + \Gamma_l(\overline{z})(I - \sum_{j=0}^{l} P_j)V(S + V - \overline{z})^{-1}\delta_1' +$$

$$+\Gamma_l(\overline{z})\sum_{i=0}^{l} P_iV(I - \sum_{j=0}^{l} P_j)(S + V - \overline{z})^{-1}\delta_1'$$

and from the bound

$$||\Gamma_l(\overline{z})|| \leq \frac{1}{\text{Im } z} \leq C, \quad z \in K.$$

□

Finally according to inequality (6.8) and Propositions 7.1, 7.2 we observe that there exists a sequence of measures μ_l weakly convergent to μ, such that for any fixed $c > 0$

$$\int_0^c \frac{\log(1/\mu_l'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < C(V), \quad \forall l,$$

where $C(V)$ is independent of c. Therefore due to the statement on the upper semicontinuity of an entropy (see [4]) we obtain

$$\int_0^\infty \frac{\log(1/\mu'(t))\,dt}{(1+t^{3/2})\sqrt{t}} < \infty.$$

The proof of Theorem 2.1 is complete.

8. Appendix

1. Let $G(r,s,k)$ be the kernel of the operator $(\hat{H}_\varepsilon + V - z)^{-1}\chi_{c_1}$, where χ_{c_1} is the operator of multiplication by the characteristic function of $(1, c_1)$. Then

$$G(r,s,k) = \begin{cases} \Psi(r,k)Z_1(s,k), & \text{as} \quad r < s < c_1 \\ -\Phi(r,k)Z_2(s,k), & \text{as} \quad s < c_1,\ s < r. \end{cases}$$

Here $\Psi(r,k) = e^{-ikr}P_0 + k^{-1}\sin(k(r-1))(P - P_0)$ for $r < c_1$ and $\Phi(r,k) = e^{ikr}P$ for $r > c_2$. The matrices $Z_1(s,k)$ and $Z_2(s,k)$ are chosen such that $G(r,s,k)$ is continuous at the diagonal and

$$\lim_{r\to s-0} G_r'(r,s,k) = \lim_{r\to s+0} G_r'(r,s,k) + P.$$

The two latter equations are equivalent to

$$\begin{aligned} &[e^{-ikr}P_0 + k^{-1}\sin(k(r-1))(P-P_0)]Z_1 + \\ &\qquad [e^{-ikr}B(k) + e^{ikr}A(k)]Z_2 = 0; \\ &[-ike^{-ikr}P_0 + \cos(k(r-1))(P-P_0)]Z_1 + \\ &\qquad [-ike^{-ikr}B(k) + ike^{ikr}A(k)]Z_2 = P \end{aligned} \tag{8.1}$$

and are uniquely solvable if and only if k^2 is not an eigenvalue of $\hat{H}_\varepsilon + V$. The first equation of the system (8.1) gives

$$Z_1 = -\big[e^{ikr}P_0 + \frac{k}{\sin(k(r-1))}(P-P_0)\big]\big[e^{-ikr}B(k) + e^{ikr}A(k)\big]Z_2.$$

Therefore we obtain from the second equation of (8.1) that

$$\begin{aligned} &\big[ikP_0 - k\operatorname{ctg}(k(r-1))(P-P_0)\big]\big[e^{-ikr}B(k) + e^{ikr}A(k)\big]Z_2 \\ &\qquad + [-ike^{-ikr}B(k) + ike^{ikr}A(k)]Z_2 = P, \end{aligned} \tag{8.2}$$

or equivalently

$$(P-P_0)\Big[\big(-k\,\mathrm{ctg}(k(r-1))-ik\big)e^{-ikr}B(k)+\big(-k\,\mathrm{ctg}(k(r-1))+ik\big)e^{ikr}A(k)\Big]Z_2$$
$$+2ikP_0e^{ikr}A(k)Z_2 = P.$$

Obviously

$$-k\,\mathrm{ctg}(k(r-1)) \pm ik = -\frac{ke^{\mp ik(r-1)}}{\sin k(r-1)}.$$

This implies

$$(P-P_0)\Big[\frac{-k}{\sin k(r-1)}\Big(e^{-ik}B(k)+e^{ik}A(k)\Big)\Big]Z_2+2ikP_0e^{ikr}A(k)Z_2 = P.$$

Multiplying both sides of this identity by

$$\frac{-\sin k(r-1)}{k}e^{-ik}(P-P_0)+\frac{e^{-ikr}}{2ik}P_0$$

we derive

$$P_0Z_2(r,k)P_0 = (2ik)^{-1}e^{-ikr}P_0\big(A(k)+e^{-2ik}(P-P_0)B(k)\big)^{-1}P_0.$$

Finally, since

$$P_0Z_2(r,k)P_0 = (2ika_\varepsilon)^{-1}e^{-ikr}P_0$$

we obtain (6.4).

2. In this subsection we adapt the argument from [6]. The solution $\Phi(r,k)$ of (6.3) satisfies the integral equation

$$\Phi(r,k) = e^{ikr}P - \int_r^\infty k^{-1}\sin k(r-s)\tilde V(s)\Phi(s,k)\,ds, \tag{8.3}$$

where $\tilde V = V - r^{-2}\zeta_\varepsilon\,P\,\Delta_\theta$. Denote

$$X(r,k) = e^{-ikr}\Phi(r,k) - P\,.$$

Then

$$X(r,k) = \int_r^\infty K(r,s,k)\,ds + \int_r^\infty K(r,s,k)X(s,k)\,ds, \tag{8.4}$$

where

$$K(r,s,k) = \frac{e^{2ik(s-r)}-1}{2ik}\tilde V(s)\,. \tag{8.5}$$

Note that

$$\|K(r,s,k)\| \le C_1(\tilde V,n)/(1+|k|) \tag{8.6}$$

for all k with Im $k \ge 0$ and all k with $1 < r \le s$. Here and below $\|\cdot\|$ denotes the norm of an operator in $PL^2(\mathbb{S}^{d-1})$.

Solving the Volterra equation (8.4) we obtain the following convergent series

$$X(r,k) = \sum_{m=1}^{\infty} \int\cdots\int_{r\le r_1\le\cdots\le r_m} \prod_{l=1}^{m} K(r_{l-1}, r_l, k)\, dr_1\cdots dr_m\,.$$

From (8.6) we see that $|X(r,k)| \le C_2(\tilde V)$ for all $1 < r$. Obviously $X(r,k)$ is an entire function in k. Inserting this estimate back into (8.4), we conclude that the inequality

$$\|X(r,k)\| \le C_3(\tilde V, n)(1+|k|)^{-1} \tag{8.7}$$

holds for all r with $1 < r$ and all k with Im $k \ge 0$.

If we rewrite (8.3) as follows

(8.8)

$$\Phi(r,k) = e^{ikr}\left[P - \frac{1}{2ik}\int_r^{\infty}\tilde V(s)\,ds - \frac{1}{2ik}\int_r^{\infty}\tilde V(s)X(s,k)\,ds\right] + \frac{e^{-ikr}}{2ik}\left[\int_r^{\infty} e^{2iks}\tilde V(s)\,ds + \int_r^{\infty} e^{2iks}\tilde V(s)X(s,k)\,dx\right],$$

then the expressions in the brackets in the r.h.s. do not depend on r for $r \le 1$. From (8.8) it follows that

$$A(k) = P - \frac{1}{2ik}\int_{-\infty}^{+\infty}\tilde V(s)\,ds - \frac{1}{2ik}\int_{-\infty}^{+\infty}\tilde V(s)X(s,k)\,ds\,, \tag{8.9}$$

$$B(k) = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iks}\tilde V(s)\,ds + \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2iks}\tilde V(s)X(s,k)\,ds\,. \tag{8.10}$$

Recall that supp $\tilde V \subset (1,\infty)$. Thus for sufficiently large $|k|$ the smoothness of V and (8.7) imply

$$\left\|A(k) - P + \frac{1}{2ik}\int_{-\infty}^{+\infty}\tilde V(s)ds\right\| \le C_4(\tilde V, n)|k|^{-2}, \quad \text{Im}\, k \ge 0\,, \tag{8.11}$$

$$\left\|e^{-2ik}B(k)\right\| \le C_5(\tilde V, n)|k|^{-2}\,, \quad \text{Im}\, k \ge 0\,. \tag{8.12}$$

Note that from (6.4), (8.11) and (8.12) we now obtain that $a_\varepsilon(k)$ is a meromorphic function in a neighborhood of zero and $|a_\varepsilon(k)|$ tends to 1 as $O(1/|k|^2)$ when $k \to \pm\infty$.

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