Tilings Colourful — Only Even More So

Michael Baake [a,1], Uwe Grimm [b], Max Scheffer [c]

[a] *Institut für Mathematik, Universität Greifswald, Jahnstr. 15a, 17487 Greifswald, Germany*

[b] *Applied Mathematics Department, The Open University, Walton Hall, Milton Keynes MK7 6AA, U.K.*

[c] *Institut für Physik, Technische Universität Chemnitz, 09107 Chemnitz, Germany*

Abstract

We consider colour symmetries for planar tilings of certain n-fold rotational symmetry. The colourings are such that one colour occupies a submodule of n-fold symmetry, while the other colours encode the cosets. We determine the possible number of colours and count inequivalent colouring solutions with those numbers. The corresponding Dirichlet series generating functions are zeta functions of cyclotomic fields. The cases with $\phi(n) \leq 8$, where ϕ is Euler's totient function, have been completely presented in previous publications. The same methods can be employed to extend the classification to all cases where the cyclotomic integers have class number one. Several examples for symmetries with $\phi(n) > 8$ are discussed here.

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1 Introduction

Colour symmetries of crystals and quasicrystals continue to attract a lot of attention, simply because so little is known about their classification [7]. A first step in this analysis consists in answering the question how many different colourings of an infinite point set exist which are compatible with its underlying symmetry. More precisely, one has to determine the possible numbers of colours, and to count the corresponding possibilities to distribute the colours over the point set (up to permutations).

To simplify the situation, one demands that one colour occupies a subset which is of the same Bravais type as the original set, while the other colours encode the cosets. Several results are known in this case and can be given in closed form [1,4,5,7,8]. Of particular interest are planar cases because, on the one hand, they show up in quasicrystalline T-phases, and, on the other hand, they are linked to the rather interesting classification [9] of planar Bravais classes with n-fold symmetry (if n is odd, the symmetry is $2n$-fold).

The symmetry cases are grouped into classes with equal value of the Euler function

$$\phi(n) \;=\; \{1 \leq k \leq n \mid \gcd(k,n) = 1\}\,. \qquad (1)$$

Note that $\phi(n) = 2$ are the crystallographic cases $n = 3$ (triangular lattice) and $n = 4$ (square lattice), while $\phi(n) \geq 4$ are the non-crystallographic ones.

2 Number Theoretic Formulation

The standard Bravais colouring of planar modules with n-fold symmetry is most efficiently described in terms of the Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\xi_n)$, where ξ_n is a primitive n-th root of unity (e.g., $\xi_n = e^{2\pi i/n}$), provided this field has class number one [10,9]. If $a_n(k)$ denotes the number of colourings of the module $\mathcal{M}_n = \mathbb{Z}[\xi_n]$ with k colours (up to permutations

Email addresses:
`mbaake@uni-greifswald.de` (Michael Baake),
`u.g.grimm@open.ac.uk` (Uwe Grimm),
`max.scheffer@physik.tu-chemnitz.de` (Max Scheffer).
URLs:
`http://schubert.math-inf.uni-greifswald.de/` (Michael Baake),
`http://mcs.open.ac.uk/ugg2/` (Uwe Grimm).
[1] Corresponding author: Telephone +49 3834 86 4609, Fax +49 3834 86 4615

of the colours), one obtains

$$\zeta_{\mathcal{M}_n}(s) := \sum_{k=1}^{\infty} \frac{a_n(k)}{k^s} = \zeta_{\mathbb{Q}(\xi_n)}(s) \; = \prod_{p \text{ prime}} E(p^{-s}) \, , \tag{2}$$

where the Euler factors have the form

$$E(p^{-s}) = \frac{1}{(1-p^{-\ell s})^m} = \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} \frac{1}{(p^s)^{k\ell}} \tag{3}$$

with characteristic numbers ℓ and m that depend on p and n. This allows to calculate $a_n(p^r)$ for $r \geq 0$, and then $a_n(k)$ for arbitrary k by means of the multiplicativity of this arithmetic function (one has $a_n(kk') = a_n(k)\, a_n(k')$ for coprime integers k, k').

3 Examples

All cases with $\phi(n) \leq 8$ are given in [1,3,4,7,8]. The general theory is explained in [5,7]. Our first example is the only case with $\phi(n) = 10$, which is $n = 11$:

$$\begin{aligned}
\zeta_{\mathcal{M}_{11}}(s) &= \frac{1}{1-11^{-s}} \prod_{p \equiv 1\,(11)} \frac{1}{(1-p^{-s})^{10}} \\
&\quad \times \prod_{p \equiv -1\,(11)} \frac{1}{(1-p^{-2s})^5} \\
&\quad \times \prod_{p \equiv 3,4,5,9\,(11)} \frac{1}{(1-p^{-5s})^2} \\
&\quad \times \prod_{p \equiv 2,6,7,8\,(11)} \frac{1}{1-p^{-10s}} \\
&= 1 + \frac{1}{11^s} + \frac{10}{23^s} + \frac{10}{67^s} + \frac{10}{89^s} + \frac{1}{121^s} \\
&\quad + \frac{10}{199^s} + \frac{2}{243^s} + \frac{10}{253^s} + \frac{10}{331^s} + \ldots
\end{aligned} \tag{4}$$

An example is shown in Figure 1.

There are four solutions to the equation $\phi(n) = 12$ with $n \not\equiv 2 \bmod 4$, namely $n \in \{13, 21, 28, 36\}$. The first few terms of the corresponding Dirichlet series read

$$\begin{aligned}
\zeta_{\mathcal{M}_{13}}(s) = 1 &+ \frac{1}{13^s} + \frac{4}{27^s} + \frac{12}{53^s} + \frac{12}{79^s} + \frac{12}{131^s} + \frac{12}{157^s} \\
&+ \frac{1}{169^s} + \frac{12}{313^s} + \frac{4}{351^s} + \frac{12}{443^s} + \ldots
\end{aligned} \tag{5}$$

$$\begin{aligned}
\zeta_{\mathcal{M}_{21}}(s) = 1 &+ \frac{2}{7^s} + \frac{12}{43^s} + \frac{3}{49^s} + \frac{2}{64^s} + \frac{12}{127^s} + \frac{6}{169^s} \\
&+ \frac{12}{211^s} + \frac{24}{301^s} + \frac{12}{337^s} + \frac{4}{343^s} + \ldots
\end{aligned} \tag{6}$$

$$\begin{aligned}
\zeta_{\mathcal{M}_{28}}(s) = 1 &+ \frac{2}{8^s} + \frac{12}{29^s} + \frac{1}{49^s} + \frac{3}{64^s} + \frac{12}{113^s} + \frac{6}{169^s} \\
&+ \frac{12}{197^s} + \frac{24}{232^s} + \frac{12}{281^s} + \frac{12}{337^s} + \ldots
\end{aligned} \tag{7}$$

$$\begin{aligned}
\zeta_{\mathcal{M}_{36}}(s) = 1 &+ \frac{1}{9^s} + \frac{12}{37^s} + \frac{1}{64^s} + \frac{12}{73^s} + \frac{1}{81^s} + \frac{12}{109^s} \\
&+ \frac{12}{181^s} + \frac{6}{289^s} + \frac{12}{333^s} + \frac{6}{361^s} + \ldots
\end{aligned} \tag{8}$$

It is obvious that, with increasing symmetry, also the minimum number of colours required increases. The only exceptions come from primes which divide n, i.e. the ramified primes.

4 Outlook

As indicated above and in previous publications [1,5], the above strategy works for all cyclotomic fields with class number one, of which there is a finite list [10]. In a forthcoming paper [2], we will give a complete treatment, together with further details on the calculations.

The next step is then to embark on the classification of the corresponding colour symmetry groups, very much in the spirit of [7,8]. This should be possible for the planar cases, as the underlying rotations commute.

Much more involved and ambitious would be an extension to higher dimensions. Here, the combinatorial part for the underlying modules is already available for all cases with irreducible point symmetries in dimensions 3 and 4, compare [1,5,6], but much less is known about the colour symmetry groups.

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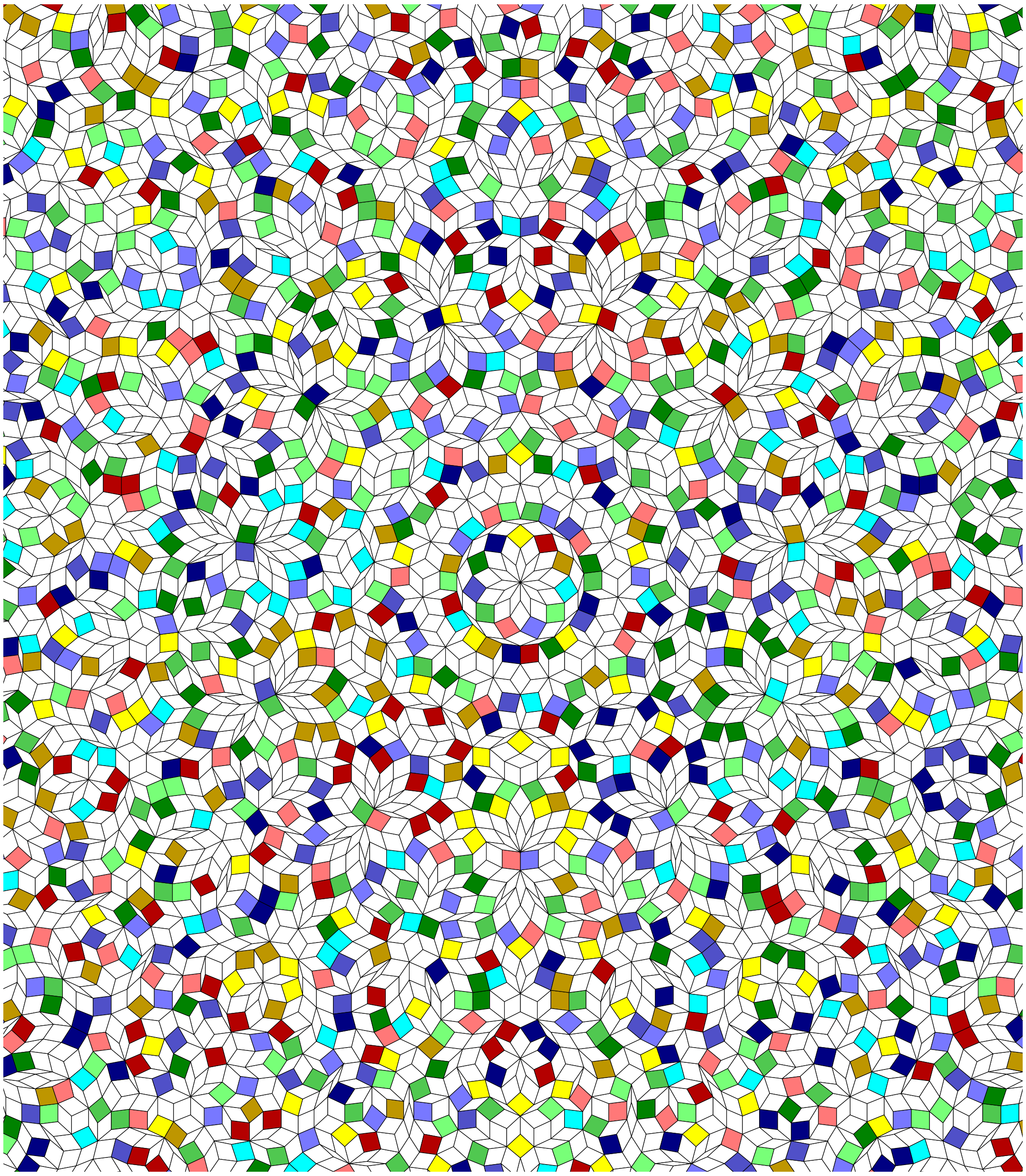

Fig. 1. The figure shows the colouring of an elevenfold tiling with eleven colours. For clarity, the colours are displayed for one kind of rhombs only.