Resolvent Convergence in Norm for Dirac Operator with Aharonov–Bohm Field

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Abstract We consider the Hamiltonian for relativistic particles moving in the Aharonov–Bohm magnetic field in two dimensions. The field has δ–like singularity at the origin, and the Hamiltonian is not necessarily essentially self–adjoint. The self–adjoint realization requires one parameter family of boundary conditions at the origin. We approximate the point–like field by smooth ones and study the problem of norm resolvent convergence to see which boundary condition is physically reasonable among admissible boundary conditions. We also study the effect of perturbations by scalar potentials. Roughly speaking, the obtained result is that the limit self–adjoint realization is different even for small perturbation of scalar potentials according to the values of magnetic fluxes. It changes at half–integer fluxes. The method is based on the resolvent analysis at low energy on magnetic Schrödinger operators with resonance at zero energy and the resonance plays an important role from a mathematical point of view. However it has been neglected in earlier physical works. The emphasis here is placed on this natural aspect.

1. Introduction

The relativistic Hamiltonian of massless particle moving in the Aharonov–Bohm field is formally defined as the operator

$$H^D = \sigma_1\pi_1 + \sigma_2\pi_2 \tag{1.1}$$

acting on the space $[L^2]^2 = [L^2(\boldsymbol{R}^2)]^2$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices and

$$\pi_j = -i\partial_j - a_{j\alpha}, \quad A_{0\alpha} = (a_{1\alpha}, a_{2\alpha}) = \alpha(-x_2/|x|^2, x_1/|x|^2), \quad \partial_j = \partial/\partial x_j,$$

for $j = 1, 2$. For notational brevity, we here confine ourselves to the massless Hamiltonian H^D throughout the entire discussion. However, the obtained results easily extend to the Hamiltonian $\sigma_1\pi_1 + \sigma_2\pi_2 + m\beta,\ \beta = \sigma_3$, with mass $m > 0$. Since the vector potential $A_{0\alpha} : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is written as

$$A_{0\alpha} = \alpha(-\partial_2 \log |x|, \partial_1 \log |x|),$$

it has the point-like magnetic field

$$\nabla \times A_{0\alpha} = \partial_1 a_{2\alpha} - \partial_2 a_{1\alpha} = \alpha\Delta \log |x| = 2\pi\alpha\delta(x)$$

with flux α. Let $\theta = \theta(x)$ be the azimuth angle from the positive x_1 axis. Then $A_{0\alpha}$ takes the form $A_{0\alpha} = \alpha\nabla\theta$, and hence

$$H^D = e^{i\alpha\theta} \left(-i\sigma_1\partial_1 - i\sigma_2\partial_2\right) e^{-i\alpha\theta}$$

in a formal way. When $\alpha \notin \boldsymbol{Z}$ is not an integer, $e^{i\alpha\theta}$ is a multi–valued function, and the magnetic effect comes from the multi–valuedness. We always assume that

$$0 < \alpha < 1. \tag{1.2}$$

If we replace $A_{0\alpha}(x)$ by $-A_{0\alpha}(x)$, it is easily seen that a similar argument applies to the case $-1 < \alpha < 0$ under a natural modification. However, if $|\alpha| > 1$, then the norm resolvent convergence can not be in general expected. We will later make a brief comment on the matter (Remarks 3.2 and 5.1 (3)). According to [12] (see also [7, 8, 9, 18]), we know that H^D is not essentially self–adjoint in $[C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})]^2$ and it has the deficiency indices (1,1). If $u \in [L^2]^2$ is in the domain of some self–adjoint extension, then it behaves like

$$u = \begin{pmatrix} (u_{-1} + o(1))r^{-\alpha} \\ (u_{-2} + o(1))e^{i\theta}r^{-(1-\alpha)} \end{pmatrix} + o(1), \qquad r = |x| \to 0, \tag{1.3}$$

for some pair (u_{-1}, u_{-2}) in the polar coordinate system (r, θ) under assumption (1.2). The ratio $\kappa = iu_{-1}/u_{-2}$ can be shown to be a real number independent of u, and the self-adjoint extension H_κ has the domain

$$\mathcal{D}(H_\kappa) = \{u \in [L^2]^2 : H^D u \in [L^2]^2,\ u_{-1} + i\kappa\, u_{-2} = 0\} \tag{1.4}$$

parameterized by κ, $\ -\infty < \kappa \le \infty$, where

$$u_{-1} = \lim_{|x|\to 0} |x|^\alpha u_1(x), \qquad u_{-2} = \lim_{|x|\to 0} |x|^{1-\alpha} e^{-i\theta} u_2(x)$$

for $u = {}^{\mathrm{t}}(u_1, u_2) \in [L^2]^2$, and $H^D u$ is understood in the distribution sense. If $\kappa = \infty$, then $u_{-2} = 0$ and the lower component has a weak singularity near the origin for $u \in \mathcal{D}(H_\infty)$, while the singularity of upper component is weak in the case $\kappa = 0$. It is impossible to impose the boundary condition in which both components of wave functions remain bounded near the origin.

We approximate $2\pi\alpha\delta(x)$ by a smooth field with compact support. Let $b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be given field with flux $\alpha = \int b(x)\,dx/2\pi$ and set

$$b_\varepsilon(x) = \varepsilon^{-2} b(x/\varepsilon)$$

for $\varepsilon > 0$ small enough, where the integration without domain is taken over the whole space. Then b_ε preserves the flux $\int b_\varepsilon(x)\,dx = 2\pi\alpha$ and it approximates $b_\varepsilon \to 2\pi\alpha\delta(x)$ as $\varepsilon \to 0$ in the distribution sense. We define the potential $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ associated with field b by

$$A(x) = (-\partial_2\varphi(x), \partial_1\varphi(x)) \tag{1.5}$$

and set $A_\varepsilon(x) = \varepsilon^{-1}A(x/\varepsilon)$, where

$$\varphi(x) = (2\pi)^{-1}\int \log|x-y|\,b(y)\,dy. \tag{1.6}$$

As is easily seen, A and A_ε satisfy $\nabla \times A = \Delta\varphi = b$ and $\nabla \times A_\varepsilon = b_\varepsilon$. We further define

$$H_\varepsilon^D = \sigma_1\nu_{1\varepsilon} + \sigma_2\nu_{2\varepsilon} \tag{1.7}$$

as the Hamiltonian approximating H^D, where $(\nu_{1\varepsilon}, \nu_{2\varepsilon}) = -i\nabla - A_\varepsilon$. This Hamiltonian admits a unique self–adjoint realization with domain $\mathcal{D}\,(H_\varepsilon^D) = [H^1(\boldsymbol{R}^2)]^2$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s. As the first main theorem (Theorem 3.1), we obtain that H_ε^D is convergent to H_∞ in the sense (norm resolvent sense) that

$$\left(H_\varepsilon^D + i\right)^{-1} \to (H_\infty + i)^{-1}, \qquad \varepsilon \to 0,$$

in norm. The attempt of approximation to point–like fields has been often done in physical literatures. For example, the approximation by the bounded potential

$$\tilde{A}(x) = \alpha(-x_2/|x|^2, x_1/|x|^2) \text{ on } \{|x| > 1\}, \quad \tilde{A}(x) = 0 \text{ on } \{|x| < 1\}$$

has been studied in a series of works [14, 15, 17]. A similar problem has been studied for the non–relativistic particles. We write $L(p,q) = (-i\nabla - p)^2 + q$ for the Schrödinger operator with vector and scalar potentials $p : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ and $q : \boldsymbol{R}^2 \to \boldsymbol{R}$. Let $A_{0\alpha}$, A and A_ε be as above. According to [1, 11], $L(A_{0\alpha}, 0)$ is known to have the deficiency indices (2,2) as a symmetric operator over $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, and one of possible self–adjoint extensions has the domain

$$\mathcal{D} = \{u \in L^2 : L(A_{0\alpha}, 0)u \in L^2, \ \lim_{|x|\to 0} |u(x)| < \infty\}. \tag{1.8}$$

If $u \in \mathcal{D}$, then the wave function u remains bounded around the origin. This is not the case for the relativistic particles, as stated above. The self–adjoint operator with domain $\mathcal{D}$ is often called the Aharonov–Bohm Hamiltonian ([3]) and we denote it by L_{AB}. We have proved in [20] that $L(A_\varepsilon, 0)$ converges to L_{AB} in the norm resolvent sense. The difference between the non–relativistic and relativistic cases can be explained as follows. Since the spin matrices satisfy the anti–commutator relation $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}$ and since $\sigma_1\sigma_2 = i\sigma_3$ and

$$[\nu_{1\varepsilon}, \nu_{2\varepsilon}] = \nu_{1\varepsilon}\nu_{2\varepsilon} - \nu_{2\varepsilon}\nu_{1\varepsilon} = ib_\varepsilon,$$

we take the square to obtain the Pauli operator

$$\left(H_\varepsilon^D\right)^2 = \begin{pmatrix} \nu_{1\varepsilon}^2 + \nu_{2\varepsilon}^2 - b_\varepsilon & 0 \\ 0 & \nu_{1\varepsilon}^2 + \nu_{2\varepsilon}^2 + b_\varepsilon \end{pmatrix} = \begin{pmatrix} L(A_\varepsilon, -b_\varepsilon) & 0 \\ 0 & L(A_\varepsilon, b_\varepsilon) \end{pmatrix}.$$

Then $L(A_\varepsilon, \mp b_\varepsilon)$ have additional δ–like potentials $\mp b_\varepsilon \sim \mp 2\pi\alpha\delta(x)$ for $\varepsilon > 0$ small enough. The repulsive potential b_ε makes the lower component vanish near the origin, but the attractive one $-b_\varepsilon$ makes the upper component singular.

We define the unitary operator $J_\varepsilon : L^2 \to L^2$ or $[L^2]^2 \to [L^2]^2$ by

$$(J_\varepsilon f)(x) = \varepsilon^{-1} f(x/\varepsilon). \tag{1.9}$$

Then we have $L(A_\varepsilon, \mp b_\varepsilon) = \varepsilon^{-2} J_\varepsilon L(A, \mp b) J_\varepsilon^*$ and

$$(L(A_\varepsilon, \mp b_\varepsilon) + 1)^{-1} = \varepsilon^2 J_\varepsilon \left(L(A, \mp b) + \varepsilon^2\right)^{-1} J_\varepsilon^*.$$

Thus the proof of Theorem 3.1 is reduced to the analysis on the behavior of resolvent $(L(A, \mp b) + \varepsilon^2)^{-1}$ as $\varepsilon \to 0$. In particular, it is significant that $L(A, -b)$ has a resonance state at zero energy. We can represent $L(A, \pm b)$ as

$$L(A, \pm b) = \nu_\pm \nu_\mp = \nu_\mp^* \nu_\mp, \tag{1.10}$$

so that the operators are both non–negative, where $\nu_\pm = \nu_1 \pm i\nu_2$ with $(\nu_1, \nu_2) = -i\nabla - A$. A simple calculation yields the relation

$$\nu_+ = -ie^{-\varphi}\left(\partial_1 + i\partial_2\right) e^{\varphi} \tag{1.11}$$

for φ defined by (1.6). Hence the equation $L(A, -b)u = 0$ has a bounded solution behaving like

$$\tilde{\rho} = e^{-\varphi(x)} = |x|^{-\alpha}(1 + O(|x|^{-1})), \qquad |x| \to \infty.$$

Such a solution is called a resonance state. On the other hand, $L(A, b)$ can be shown to have no resonance state at zero energy. Thus the two operators $L(A, \pm b)$ have the different spectral property at zero energy. It seems that this essential aspect has not been recognized in earlier physical literatures.

We discuss the effect of perturbations by scalar potentials in the latter sections. The obtained result (Theorem 5.1) strongly depends on flux α, and the limit self–adjoint extension H_κ changes at $\alpha = 1/2$. What is interesting is that it occurs even for small perturbation of scalar potentials. We set

$$V_\varepsilon(x) = \varepsilon^{-1} V(x/\varepsilon)$$

for $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$, and we define $H_\varepsilon(V_\varepsilon)$ by

$$H_\varepsilon(V_\varepsilon) = H_\varepsilon^D + V_\varepsilon = \sigma_1 \nu_{1\varepsilon} + \sigma_2 \nu_{2\varepsilon} + V_\varepsilon. \tag{1.12}$$

Roughly speaking, the operator $H+V = \sigma_1\nu_1+\sigma_2\nu_2+V$ is said to have a resonance state at zero energy, if the equation $(H+V)u = 0$ admits a bounded but not square integrable solution. The precise definition is formulated in section 5 (Definition 5.1). If $H + V$ do not have a resonance state at zero energy, then $H_\varepsilon(V_\varepsilon)$ is proved to converge to H_∞ for $0 < \alpha < 1/2$ and to H_0 for $1/2 < \alpha < 1$ in the norm resolvent sense. If $\alpha = 1/2$, then $H_\varepsilon(V_\varepsilon)$ is convergent to H_κ for some κ determined from the above resonance state $\tilde{\rho}$ of $L(A, -b)$. If, on the other hand, $H + V$ has a resonance state at zero energy, then the situation is completely reversed. We obtain that $H_\varepsilon(V_\varepsilon)$ is convergent to H_0 for $0 < \alpha \leq 1/2$ and to H_∞ for $1/2 < \alpha < 1$.

Our motivation of the study on the convergence to point–like fields comes from an application to the scattering of Dirac particles by magnetic fields ([2, 7, 13]). For example, the asymptotic behavior of the amplitude in the scattering by the small disk has been calculated under a certain impenetrable boundary condition in the book (Afanasiev [2, Sec. 7.10]). The behavior changes discontinuously at half–integer fluxes. Another motivation is to study the scattering of Dirac particles by electromagnetic fields with small support in the interaction of cosmic string with matter (Alford, March–Rusell and Wilczek [7]). The amplitude there has been also explicitly calculated for the electric potential of characteristic function of the small disk, and its asymptotic behavior has been shown to take a different form at half–integer fluxes as the support of electromagntic field shrinks. This phenomenon also seems to be closely related to the resonance at zero energy of magnetic Schrödinger operators. However, such a discontinuity at half–integer fluxes in the asymptotic form is completely hidden behind this explicit calculation using the Bessel functions. The argument developed here seems to account for this phenomenon partly. The present work is the first step toward analyzing this problem from a mathematical point of view.

The spectral problems for Dirac Hamiltonians with Aharonov-Bohm fields in two dimensions has been recently discussed from various aspects by several authors. For example, Sitenko [21] has studied the vacuum polarization effects in the background o singular magnetic vortex for all self–adjoint extensions, and Falomir and Pisani [22] have determined the spectrum in the presence of a uniform magnetic field.

2. Krein theory on self–adjoint extension

In this section we discuss the self–adjointness of the symmetric operator H^D defined by (1.1). This is defined as an operator acting on $\left[C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})\right]^2$. We write

$$H^D = \begin{pmatrix} 0 & \pi_- \\ \pi_+ & 0 \end{pmatrix}, \qquad \pi_\pm = \pi_1 \pm i\,\pi_2,$$

and denote by $\overline{H}^D$ and $\left(H^D\right)^*$ the closure and adjoint of H^D in $[L^2]^2$. Let

$$\Sigma_\pm = \mathrm{Ker}\,(\left(H^D\right)^* \mp i) = \{u \in [L^2]^2 : \left(\left(H^D\right)^* \mp i\right) u = 0\}.$$

The pair (n_+, n_-), $n_\pm = \dim \Sigma_\pm$, is called the deficiency indices of H^D. As is well known, H^D has self–adjoint extensions if and only if $n_+ = n_-$.

We shall show that $n_+ = n_- = 1$. Suppose that

$$\left(\left(H^D\right)^* - i\right) u = 0, \quad u = {}^t(u_1, u_2) \in [L^2]^2. \tag{2.1}$$

Then u_2 satisfies

$$(\pi_+\pi_- + 1)\, u_2 = \left(\pi_1^2 + \pi_2^2 + 1\right) u_2 = 0$$

in $\boldsymbol{R}^2 \setminus \{0\}$, and u_1 is given by $u_1 = -i\pi_- u_2$. The operator $\pi_\pm$ has the representation

$$\pi_+ = e^{i\theta}\left(-i\partial_r + r^{-1}(\partial_\theta - i\alpha)\right), \quad \pi_- = e^{-i\theta}\left(-i\partial_r - r^{-1}(\partial_\theta - i\alpha)\right)$$

in terms of the polar coordinates (r, θ), and

$$\pi_1^2 + \pi_2^2 = (-i\,\nabla - A_{0\alpha})^2 = -\partial_r^2 - r^{-1}\partial_r + r^{-2}\left(-i\partial_\theta - \alpha\right)^2.$$

We write $H_\nu(z) = H_\nu^{(1)}(z)$ for the Hankel function of first kind. All Hankel functions are throughout understood to be of the first kind. Since

$$(z^\nu H_\nu(az))' = az^\nu H_{\nu-1}(az) \tag{2.2}$$

by formula, it follows that

$$\left(H_{1-\alpha}(ir)e^{i\theta}\right) = H'_{1-\alpha}(ir) - i(1-\alpha)r^{-1}H_{1-\alpha}(ir) = H_{-\alpha}(ir).$$

If we make use of this relation, then the solution $u \in [L^2]^2$ to (2.1) takes the form

$$u_2 = cH_{1-\alpha}(ir)e^{i\theta}, \qquad u_1 = -icH_{-\alpha}(ir) = -ice^{i\alpha\pi}H_\alpha(ir)$$

with constant c. This is the solution corresponding to angular momentum $l = 0$. The solutions with momentum $l \geq 1$ are obtained by

$$u_2 = cH_{l+1-\alpha}(ir)e^{i(l++1)\theta}, \qquad u_1 = -icH_{l-\alpha}(ir)e^{il\theta},$$

but these are not in $[L^2]^2$ because of the strong singularity near the origin. The same reasoning applies to solutions with momentum $l < 0$. Thus we see that $\Sigma_+ = \{u \in [L^2]^2 : u = c\psi_+\}$ is the one dimensional space spanned by

$$\psi_+ = N_\alpha \begin{pmatrix} -i^{1+2\alpha}H_\alpha(ir) \\ H_{1-\alpha}(ir)e^{i\theta} \end{pmatrix}, \qquad \|\psi_+\|_{L^2} = 1, \tag{2.3}$$

where

$$N_\alpha = (\sin \alpha\pi)^{1/2}/2. \tag{2.4}$$

The normalization constant N_α is calculated by use of the formula

$$2\pi \int_0^\infty r|H_\alpha(ir)|^2\,dr = 4\alpha/(\sin \alpha\pi).$$

Similarly $\Sigma_- = \{u \in [L^2]^2 : u = c\psi_-\}$ is spanned by

$$\psi_- = N_\alpha \begin{pmatrix} i^{1+2\alpha} H_\alpha(ir) \\ H_{1-\alpha}(ir)e^{i\theta} \end{pmatrix}, \qquad \|\psi_-\|_{L^2} = 1, \tag{2.5}$$

with the same constant N_α.

The Krein theory determines a family of all possible self–adjoint extensions. According to Theorem X.2 and its Corollary of [19], there is a one–to–one correspondence between the unitary mapping $U : \Sigma_+ \to \Sigma_-$ and the self–adjoint extension H_U, which acts as

$$H_U u = \overline{H}^D v + iv_+ - iUv_+$$

on the domain $\mathcal{D}(H_U) = \{u \in [L^2]^2 : u = v + v_+ + Uv_+,\ \ v \in \mathcal{D}(\overline{H}^D),\ \ v_+ \in \Sigma_+\}$.

Lemma 2.1 *Let J be the isometry operator defined by*

$$J = -\left(\overline{H}^D - i\right)\left(\overline{H}^D + i\right)^{-1} : \Sigma_+^\perp \to \Sigma_-^\perp$$

where $\Sigma_\pm^\perp = \mathrm{Ran}\,(\overline{H}^D \pm i)$ is the orthogonal complement of $\Sigma_\pm$. Then the resolvent $(H_U + i)^{-1}$ is represented as

$$(H_U + i)^{-1} = (1/2i)\,(1 + (U \oplus J)) : [L^2]^2 = \Sigma_+ \oplus \Sigma_+^\perp \to [L^2]^2 = \Sigma_- \oplus \Sigma_-^\perp.$$

Proof. Let $f \in [L^2]^2$ and set $u = (1/2i)\,(1 + (U \oplus J))\,f$. We decompose f into $f = h + g$, where $h \in \Sigma_+$ and $g \in \Sigma_+^\perp$. Then

$$u = (1/2i)\left((1 + J)g + (1 + U)h\right).$$

Since $g \in \Sigma_+^\perp = \mathrm{Ran}(\overline{H}^D + i)$, we can write $g = (\overline{H}^D + i)\tilde{g}$ for some $\tilde{g} \in \mathcal{D}(\overline{H}^D)$. Thus $(1 + J)\,g = 2i\tilde{g} \in \mathcal{D}(\overline{H}^D)$, and hence $u \in \mathcal{D}(H_U)$. We calculate

$$(H_U + i)\,u = (1/2i)\left((\overline{H}^D + i)(1 + J)g + (H_U + i)(1 + U)h\right).$$

Then we obtain $(H_U + i)\,u = g + h = f$, so that $u = (H_U + i)^{-1}f$. This proves the lemma. $\square$

We examine what boundary condition H_U satisfies at the origin. When $\nu > 0$ is not an integer, $H_\nu(z)$ is represented as

$$H_\nu(z) = J_\nu(z) + iN_\nu(z) = (i/\sin\nu\pi)\left(e^{-i\nu\pi}J_\nu(z) - J_{-\nu}(z)\right)$$

in terms of Bessel functions, and it obeys

$$H_\nu(z) = -\left(i/\sin\nu\pi\right)\left(1/\Gamma(1-\nu)\right)(z/2)^{-\nu}\left(1 + O(|z|^{2\nu}) + O(|z|^2)\right), \qquad |z| \to 0.$$

We now use the abstract lemma in [19] (p. 138). According to the notation there, we have

$$\mathcal{D}\left(\left(H^D\right)^*\right) = \mathcal{D}(\overline{H}^D) \oplus_{\overline{H}^D} \Sigma_+ \oplus_{\overline{H}^D} \Sigma_-.$$

This implies that $v = {}^{\mathrm{t}}(v_1, v_2) \in \mathcal{D}(\overline{H}^D)$ obeys

$$\int_0^{2\pi} v_1(r,\theta)\,d\theta = o(r^{-\alpha}), \qquad \int_0^{2\pi} v_2(r,\theta)e^{-i\theta}\,d\theta = o(r^{-(1-\alpha)})$$

as $r = |x| \to 0$. The other Fourier coefficients vanish at the origin. Hence $u = {}^{\mathrm{t}}(u_1, u_2) \in \mathcal{D}(H_U)$ satisfies

$$\int_0^{2\pi} u_1(r,\theta)\,d\theta = 2\pi(u_{-1} + o(1))r^{-\alpha}, \quad \int_0^{2\pi} u_2(r,\theta)e^{-i\theta}\,d\theta = 2\pi(u_{-2} + o(1))r^{-(1-\alpha)}$$

for some pair (u_{-1}, u_{-2}) as in (1.3). The unitary mapping U acts as the multiplication $e^{i\zeta}\times$ for some ζ, $-\pi < \zeta \le \pi$, between the one dimensional spaces $\Sigma_\pm$. If we take account of the above behavior of Hankel functions, then the ratio $\kappa = iu_{-1}/u_{-2}$ is calculated as the real number

$$\kappa = \left(2^{2\alpha-1}\Gamma(\alpha)/\Gamma(1-\alpha)\right)\tan(\zeta/2) \tag{2.6}$$

independent of $u \in \mathcal{D}(H_U)$. Thus we can obtain the family $\{H_\kappa\}$ of self–adjoint extensions with domain $\mathcal{D}(H_\kappa)$ parameterized as in (1.4). If $\zeta = \pi$ and hence $\kappa = \infty$, then

$$\mathcal{D}(H_\infty) = \{u = {}^{\mathrm{t}}(u_1, u_2) \in [L^2]^2 : H^D u \in [L^2]^2,\ \lim_{|x|\to 0} |x|^{1-\alpha}e^{-i\theta}u_2(x) = 0\},$$

and if $\zeta = 0$ and hence $\kappa = 0$, then

$$\mathcal{D}(H_0) = \{u = {}^{\mathrm{t}}(u_1, u_2) \in [L^2]^2 : H^D u \in [L^2]^2,\ \lim_{|x|\to 0} |x|^{\alpha}u_1(x) = 0\}.$$

Roughly speaking, only one component of Dirac wave functions has a weak singularity near the origin in the two special cases $\kappa = \infty$ and $\kappa = 0$, while both the components have in general strong singularities in the other cases. Thus it is not allowed as a boundary condition that both the components remain bounded near the origin.

We denote by $(\ ,\ )_{L^2}$ the scalar product in L^2 or $[L^2]^2$. We write

$$f \otimes g = (\,\cdot\,, g)_{L^2} f : L^2 \to L^2$$

for the integral operator with kernel $f(x)\,\overline{g}(y)$, and use a similar notation for the operator $u \otimes v : [L^2]^2 \to [L^2]^2$ defined by $(u \otimes v)\,w = (w, v)_{L^2} u$. This operator is represented in the matrix form

$$u \otimes v = \left(u_j \otimes v_k\right)_{1 \le j,k \le 2}, \quad u_j \otimes v_k : L^2 \to L^2,$$

for $u = {}^t(u_1, u_2)$ and $v = {}^t(v_1, v_2)$ in $[L^2]^2$.

We end the section by deriving the relation between the two resolvents $(H_\kappa + i)^{-1}$ and $(H_\infty + i)^{-1}$. Since $U : \Sigma_+ \to \Sigma_-$ takes the form $U = e^{i\zeta}\,(\psi_- \otimes \psi_+)$, it follows from Lemma 2.1 that

$$(H_\kappa + i)^{-1} - (H_{\kappa'} + i)^{-1} = (1/2i)\left(e^{i\zeta} - e^{i\zeta'}\right)(\psi_- \otimes \psi_+),$$

where κ and κ' are related to ζ and ζ' through (2.6) respectively. If, in particular, $\zeta' = \pi$, then we obtain

$$(H_\kappa + i)^{-1} = (H_\infty + i)^{-1} + (1/2i)\left(e^{i\zeta} + 1\right)(\psi_- \otimes \psi_+). \tag{2.7}$$

This relation plays a basic role in the future argument. This relation is a special case of the Krein formula between resolvents. We refer to [5, Chap.VII] for the general Krein theory for self–adjoint extensions.

3. Norm resolvent convergence

We consider the problem as to which self–adjoint extension is realized as a norm resolvent limit of Hamiltonians with smooth magnetic fields having small support when the support is shrinking. Let H_ε^D be defined by (1.7). The aim here is to prove the following theorem.

Theorem 3.1 *Assume (1.2). Then H_ε^D is convergent to H_∞ in the norm resolvent sense as $\varepsilon \to 0$.*

Remark 3.1. The potential associated with given field $b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ is not uniquely determined. Let $\tilde{A} \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ be another potential having b as a field. Then it takes the form $\tilde{A} = A + \nabla g$ with some $g \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$. If $g(x) \to 0$ at infinity, we can prove the norm resolvent convergence for $\tilde{A}_\varepsilon = \varepsilon^{-1}\tilde{A}(x/\varepsilon)$ also. This can be shown in the course of the proof of the theorem.

Before going into the proof, we fix the notation and introduce several auxiliary operators. We assume for brevity that the field $b \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ has support in the unit disk. According to [20, Lemma 2.1] (see (2.2) there), we can construct a potential $a \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ associated with field b such that

$$a(x) = (a_1(x), a_2(x)) = A_{0\alpha}(x) = \alpha \left(-x_2/|x|^2, x_1/|x|^2\right) \tag{3.1}$$

for $|x| > 2$. Then we define

$$p_j = -i\partial_j - a_j, \qquad p_{j\varepsilon} = -i\partial_j - a_{j\varepsilon}$$

for $j = 1, 2$, and

$$K = \sigma_1 p_1 + \sigma_2 p_2, \qquad K_\varepsilon = \sigma_1 p_{1\varepsilon} + \sigma_2 p_{2\varepsilon},$$

where $a_\varepsilon(x) = \varepsilon^{-1} a(x/\varepsilon) = (a_{1\varepsilon}(x), a_{2\varepsilon}(x))$. Both K and K_ε are self–adjoint operators with domain $[H^1(\boldsymbol{R}^2)]^2$.

We again denote by $L(p, q) = (-i\nabla - p)^2 + q$ the Schrödinger operator with vector potential p and scalar one q. According to this notation, we further define

$$L = L(a, 0) = p_1^2 + p_2^2, \qquad L_\varepsilon = L(a_\varepsilon, 0) = p_{1\varepsilon}^2 + p_{2\varepsilon}^2 \tag{3.2}$$

and

$$L_\pm = L(a, \pm b) = p_1^2 + p_2^2 \pm b, \qquad L_{\pm\varepsilon} = L(a_\varepsilon, \pm b_\varepsilon).$$

These are self–adjoint operators with domain $H^2(\boldsymbol{R}^2)$. If we set

$$p_\pm = p_1 \pm i p_2, \qquad p_{\pm\varepsilon} = p_{1\varepsilon} \pm i p_{2\varepsilon},$$

then $L_\pm = p_\pm p_\mp = p_\mp^* p_\mp$ and $L_{\pm\varepsilon} = p_{\pm\varepsilon} p_{\mp\varepsilon}$ as in (1.10). We can easily show

$$p_{+\varepsilon} \left(L_{-\varepsilon} + 1\right)^{-1} = \left(L_{+\varepsilon} + 1\right)^{-1} p_{+\varepsilon}, \quad p_{-\varepsilon} \left(L_{+\varepsilon} + 1\right)^{-1} = \left(L_{-\varepsilon} + 1\right)^{-1} p_{-\varepsilon} \tag{3.3}$$

and the resolvent $(K_\varepsilon + i)^{-1}$ is calculated as

$$(K_\varepsilon + i)^{-1} = \begin{pmatrix} -i(L_{-\varepsilon} + 1)^{-1} & p_{-\varepsilon}(L_{+\varepsilon} + 1)^{-1} \\ (L_{+\varepsilon} + 1)^{-1} p_{+\varepsilon} & -i(L_{+\varepsilon} + 1)^{-1} \end{pmatrix}. \tag{3.4}$$

We now assert that

$$(K_\varepsilon + i)^{-1} \to (H_\infty + i)^{-1}, \qquad \varepsilon \to 0, \tag{3.5}$$

in norm. To prove the assertion, we analyze the behavior as $\varepsilon \to 0$ of $(L_{\pm\varepsilon} + 1)^{-1}$. By (3.1), $a_\varepsilon(x) \to A_{0\alpha}(x)$ as $\varepsilon \to 0$ for $x \neq 0$, and hence we have

$$L_{\pm\varepsilon} \to L(A_{0\alpha}, 0) = (-i\nabla - A_{0\alpha})^2$$

on $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. As stated in section 1, the limit operator $L(A_{0\alpha}, 0)$ has the deficiency indices (2,2). The Krein theory yields a family $\{L_U\}$ of all possible self–adjoint extensions which is parameterized by 2×2 unitary mapping U from one deficiency subspace to the other one. The operator L_U is realized as a differential operator with some boundary conditions at the origin. If w belongs to $\mathcal{D}(L_U)$, then w behaves like

$$w = \left(w_{-0}r^{-\alpha} + w_{+0}r^{\alpha} + o(r^{\alpha})\right) + \left(w_{-1}r^{-(1-\alpha)} + w_{+1}r^{1-\alpha} + o(r^{1-\alpha})\right)e^{i\theta} + o(r)$$

near the origin for some coefficients $w_{\pm l}$, $l = 0, 1$, and there exist 2×2 matrices $B_\pm$ for which the boundary condition is described as the relation

$$B_- \begin{pmatrix} w_{-0} \\ w_{-1} \end{pmatrix} + B_+ \begin{pmatrix} w_{+0} \\ w_{+1} \end{pmatrix} = 0$$

between these four coefficients. For later references, we distinguish the two special operators among the family of self–adjoint extensions. One is the Aharonov–Bohm Hamiltonian L_{AB} with domain

$$\mathcal{D}(L_{AB}) = \{w \in L^2 : L(A_{0\alpha}, 0)w \in L^2,\ \ w_{-0} = w_{-1} = 0\}$$

and the other is L_Z with domain

$$\mathcal{D}(L_Z) = \{w \in L^2 : L(A_{0\alpha}, 0)w \in L^2,\ \ w_{+0} = w_{-1} = 0\}.$$

Let $J_\varepsilon : L^2 \to L^2$ be the unitary operator defined by (1.9). Then we have

$$(L_{\pm\varepsilon} + 1)^{-1} = \varepsilon^2 J_\varepsilon \left(L_\pm + \varepsilon^2\right)^{-1} J_\varepsilon^*,$$

so that the behavior as $\varepsilon \to 0$ of resolvents $(L_{\pm\varepsilon} + 1)^{-1}$ strongly depends on the spectral structure of $L_\pm$ at low energy. We explain the resonance state at zero energy in some details. We have assumed in (1.2) that $0 < \alpha < 1$. Hence it follows from the Aharonov–Casher theorem ([4]) that $L_\pm = p_\mp^* p_\mp$ has no bound states at zero energy. It should be noted that L_- has bound states in the case that $\alpha > 1$, even if $L_- \geq 0$ is non–negative. The resonance space $E_\pm$ at zero energy of $L_\pm$ is defined by

$$E_\pm = \{u \in C^\infty(\boldsymbol{R}^2) : L_\pm u = 0,\ \ u \text{ is bounded}\}.$$

We look at the structure of the resonance spaces. We consider E_- only. A similar reasoning applies to E_+ also. We recall the facts obtained in [20, Section 5] : (1) $\dim E_- \leq 2$. (2) If $\dim E_- = 2$, then E_- is spanned by the linearly independent pair

$$e_l(x) = r^{-\nu}e^{il\theta} + g_l, \quad \nu = |l - \alpha|, \quad l = 0, 1,$$

where the remainder term $g_l \in L^2 \cap C^\infty$ falls off like $\partial^\beta g_l = O(|x|^{-\nu-1-|\beta|})$. (3) If $\dim E_- = 1$, then E_- is spanned by the linear combination

$$e(x) = c_l r^{-\alpha} + c_2 r^{-1+\alpha}e^{i\theta} + g, \quad |c_1| + |c_2| \neq 0,$$

where $g \in L^2 \cap C^\infty$ obeys the bound as above at infinity. If $u \in E_-$, then

$$L_- u = p_+^* p_+ u = p_1^2 u + p_2^2 u - bu = 0, \quad b \in C_0^\infty(\boldsymbol{R}^2),$$

by definition, and we have by Lemma 4.3 in [20] (or by the argument used for its proof) that $p_1 u$ and $p_2 u$ are in L^2. This implies that $p_\pm u$ are also in L^2 and hence it follows by use of partial integration that $p_+ u = 0$ for $u \in E_-$. Let φ be defined by (1.6). The function φ behaves like $\varphi(x) = \alpha \log |x| + O(|x|^{-1})$ at infinity, so that $e^{\varphi(x)} = |x|^\alpha (1 + O(|x|^{-1}))$. Recall that $a \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ has the property (3.1). The potential a takes the form

$$a = (-\partial_2 \varphi(x), \partial_1 \varphi(x)) + \nabla h = A + \nabla h \tag{3.6}$$

for some $h \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$, where h obeys $h = O(|x|^{-1})$ at infinity. We can write

$$p_+ = -ie^{ih} e^{-\varphi} \left(\partial_1 + i\partial_2 \right) e^{\varphi} e^{-ih} \tag{3.7}$$

as in (1.11) and similarly for $p_- = p_+^*$. Thus we see that $e^{\varphi} e^{-ih} u$ is an entire function for $u \in E_-$. This implies that $E_- = \{c\rho\}$ is the one dimensional space spanned by $\rho = e^{ih} e^{-\varphi}$, and the resonance state ρ behaves like

$$\rho(x) = |x|^{-\alpha} \left(1 + O(|x|^{-1}) \right), \qquad |x| \to \infty. \tag{3.8}$$

On the other hand, we can show in a similar way that $\dim E_+ = 0$. If we take account of these facts, then Theorem 6.1 in [20] implies the following lemma.

Lemma 3.1 *As $\varepsilon \to 0$, one has*

$$(L_{+\varepsilon} + i)^{-1} \to (L_{AB} + i)^{-1}, \qquad (L_{-\varepsilon} + i)^{-1} \to (L_Z + i)^{-1}$$

in norm.

Remark 3.2. If $\alpha > 1$, then it follows from the Aharonov–Casher theorem in [4] that L_- has the zero eigenvalue with multiplicity $[\alpha]$, $[\alpha]$ being the greatest integer not exceeding α. Hence $L_{-\varepsilon} = \varepsilon^{-2} J_\varepsilon L_- J_\varepsilon^*$ has also eigenstates at zero energy. But L_Z never has an eigenstate at zero energy. Thus $L_{-\varepsilon}$ is not convergent to L_Z in norm resolvent sense. It seems that $(L_{-\varepsilon} + i)^{-1}$ is strongly convergent to $(L_Z + i)^{-1}$, so that Theorem 3.1 remains true in the strong sense even for $\alpha > 1$. We do not go into the details here.

We calculate the resolvent $(H_\infty + i)^{-1}$. Consider the equation $(H_\infty + i)\, u = f$ for $f = {}^t(f_1, f_2) \in [C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})]^2$. Then $u = {}^t(u_1, u_2)$ satisfies

$$(L(A_{0\alpha}, 0) + 1)\, u_2 = \pi_+ f_1 - i f_2, \qquad u_1 = i\pi_- u_2 - i f_1$$

for the Aharonov–Bohm potential $A_{0\alpha}(x)$. We see that $u \in \mathcal{D}(H_\infty)$ implies $u_2 \in \mathcal{D}(L_{AB})$. Thus $u_2 = (L_{AB}+1)^{-1}(\pi_+ f_1 - i f_2)$ and

$$u_1 = -i\left(1 - \pi_-(L_{AB}+1)^{-1}\pi_+\right) f_1 + \pi_-(L_{AB}+1)^{-1} f_2,$$

so that we have

$$(H_\infty + i)^{-1} = \begin{pmatrix} i\pi_-(L_{AB}+1)^{-1}\pi_+ - i & \pi_-(L_{AB}+1)^{-1} \\ (L_{AB}+1)^{-1}\pi_+ & -i(L_{AB}+1)^{-1} \end{pmatrix}. \tag{3.9}$$

Lemma 3.2 *One has the relation*

$$1 - \pi_-(L_{AB}+1)^{-1}\pi_+ = (L_Z+1)^{-1}.$$

Proof. We note that the operator on the left side of the relation is bounded. Let $f,\ g \in C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. By (3.1), there exists $\delta > 0$ such that $\pi_\pm f = p_{\pm\varepsilon} f$ and $\pi_\pm g = p_{\pm\varepsilon} g$ for $0 < \varepsilon < \delta$. Since

$$(\pi_-(L_{AB}+1)^{-1}\pi_+ f, g)_{L^2} = \lim_{\varepsilon\to 0}\left((L_{+\varepsilon}+1)^{-1} p_{+\varepsilon} f, p_{+\varepsilon} g\right)_{L^2}$$

by Lemma 3.1, it follows from (3.3) that

$$\left((1 - \pi_-(L_{AB}+1)^{-1}\pi_+) f, g\right)_{L^2} = \lim_{\varepsilon\to 0}\left((L_{-\varepsilon}+1)^{-1} f, g\right)_{L^2}.$$

This, together with Lemma 3.1 again, proves the lemma. $\square$

We denote by $Op(\varepsilon^\nu)$ and $op(\varepsilon^\nu)$ the classes of bounded operators B_ε acting on L^2 or $[L^2]^2$ such that the operator norm of B_ε obeys the bound $\|B_\varepsilon\| = O(\varepsilon^\nu)$ and $\|B_\varepsilon\| = o(\varepsilon^\nu)$ respectively as $\varepsilon \to 0$. We now prove Theorem 3.1, accepting the three lemmas below as proved. The proof of the lemmas is done in the next section.

Lemma 3.3 $\chi(L_+ + \varepsilon^2)^{-1}\chi = Op(\varepsilon^0)$ *for* $\chi \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$.

Lemma 3.4 $p_{\pm\varepsilon}(L_{+\varepsilon}+1)^{-1} \to \pi_\pm (L_{AB}+1)^{-1}$ *in norm as* $\varepsilon \to 0$.

Lemma 3.5 $\chi(L_- + \varepsilon^2)^{-1}\chi = Op(\varepsilon^{-2\alpha})$ *for* $\chi \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$.

Proof of Theorem 3.1. We first prove (3.5). The two resolvents $(K_\varepsilon + i)^{-1}$ and $(H_\infty + i)^{-1}$ are represented by (3.4) and (3.9) respectively. If we use Lemma 3.2, then

$$(H_\infty + i)^{-1} = \begin{pmatrix} -i(L_Z + 1)^{-1} & \pi_-(L_{AB} + 1)^{-1} \\ (L_{AB} + 1)^{-1}\pi_+ & -i(L_{AB} + 1)^{-1} \end{pmatrix}.$$

By Lemma 3.4, we have $p_{-\varepsilon}\left(L_{+\varepsilon} + 1\right)^{-1} \to \pi_-\left(L_{AB} + 1\right)^{-1}$ and

$$(L_{+\varepsilon} + 1)^{-1}p_{+\varepsilon} = \left(p_{-\varepsilon}(L_{+\varepsilon} + 1)^{-1}\right)^* \to \left(\pi_-\left(L_{AB} + 1\right)^{-1}\right)^* = \left(L_{AB} + 1\right)^{-1}\pi_+.$$

Hence Lemma 3.1 implies (3.5). Next we prove the norm resolvent convergence for the operator H_ε^D in question. Let $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ be defined by (1.5). Then $A = a - \nabla h$ by (3.6), and

$$\left(H_\varepsilon^D + i\right)^{-1} = \exp(-ih_\varepsilon)\left(K_\varepsilon + i\right)^{-1}\exp(ih_\varepsilon),$$

where $h_\varepsilon(x) = h(x/\varepsilon)$. By definition, $K = \varepsilon J_\varepsilon^* K_\varepsilon J_\varepsilon$ is independent of ε. We have

$$(K + i\varepsilon)^{-1} = \begin{pmatrix} -i\varepsilon(L_- + \varepsilon^2)^{-1} & p_-(L_+ + \varepsilon^2)^{-1} \\ (L_{+\varepsilon} + \varepsilon^2)^{-1}p_+ & -i\varepsilon(L_+ + \varepsilon^2)^{-1} \end{pmatrix} \tag{3.10}$$

and $(K_\varepsilon + i)^{-1} = \varepsilon J_\varepsilon(K + i\varepsilon)^{-1}J_\varepsilon^*$. For any $\delta > 0$ small enough, we can take $M > 0$ so large that $|h_\varepsilon(x)| < \delta$ for $|x| \geq \varepsilon M$, and hence $|\exp(\pm ih_\varepsilon) - 1| < 2\delta$ for x as above. On the other hand, we can prove that

$$\varepsilon\|\chi_M(K + i\varepsilon)^{-1}\chi_M\| = O(\varepsilon) + O(\varepsilon^{2(1-\alpha)})$$

for the characteristic function χ_M of $\{|x| < M\}$. In fact, Lemma 3.3 yields

$$\|\chi_M(L_+ + i\varepsilon^2)^{-1}p_+\chi_M\| + \|\chi_M p_-(L_+ + i\varepsilon^2)^{-1}\chi_M\| = O(1), \qquad \varepsilon \to 0,$$

by elliptic estimate. Note that $\chi_{\varepsilon M} = J_\varepsilon \chi_M J_\varepsilon^*$ for $\chi_{\varepsilon M}(x) = \chi_M(x/\varepsilon)$. Hence

$$\begin{aligned} \|\left(K_\varepsilon + i\right)^{-1}\chi_{\varepsilon M}\|^2 &= \|\left(K_\varepsilon + i\right)^{-1}J_\varepsilon\chi_M\|^2 \\ &= \|\chi_M J_\varepsilon^*(K_\varepsilon - i)^{-1}(K_\varepsilon + i)^{-1}J_\varepsilon\chi_M\| \\ &= O(\varepsilon)\,\|\chi_M\left((K + i\varepsilon)^{-1} - (K - i\varepsilon)^{-1}\right)\chi_M\| \to 0 \end{aligned}$$

by lemmas 3.3 and 3.5, and also $\|\left(H_\varepsilon^D + i\right)^{-1}\chi_{\varepsilon M}\| \to 0$ by gauge transformation. Thus we obtain

$$\|\left(H_\varepsilon^D + i\right)^{-1} - \left(K_\varepsilon + i\right)^{-1}\| \to 0.$$

This, together with (3.5), completes the proof. $\square$

4. Resolvent analysis at low energy on Schrödinger operators I

We shall prove Lemmas 3.3, 3.4 and 3.5 which remain unproved. Throughout the section, wee denote by $\chi \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ a real function such that $\operatorname{supp}\chi \subset B_M$ and $\chi = 1$ on $B_{M/2}$, where $B_M = \{|x| < M\}$ for $M \gg 1$ fixed arbitrarily. We further denote by $k \in \boldsymbol{C}$ a small complex number with $\operatorname{Im} k > 0$.

Let $L = L(a, 0)$ be as in (3.2). We first make a brief review on the spectral property of L from [20]. Let $L_c^2 = \{u \in L^2 : \operatorname{supp} u \subset B_M\}$ be the space of L^2 functions with support in B_M. Then we know that there exists a limit

$$G_0 = \lim_{k\to 0} (L - k^2)^{-1}$$

and the limit operator $G_0 : L_c^2 \to L_{-1}^2$ is bounded into the weighted L^2 space $L_{-1}^2 = L^2(\boldsymbol{R}^2; \langle x\rangle^{-2}\, dx)$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. By (3.1), L coincides with

$$L = L(A_{0\alpha}, 0) = -\partial_r^2 - r^{-1}\partial_r + r^{-2}\left(-i\partial_\theta - \alpha\right)^2$$

on $\{|x| > 2\}$, and hence $u_l(x) = r^\nu e^{il\theta}$, $\nu = |l - \alpha|$, satisfies $Lu_l = 0$ there. If we define ω_l by

$$\omega_l = c_l(1 - \chi)u_l + c_l G_0[L, \chi]u_l, \quad c_l = (l + \nu)^{1/2}/2\nu, \tag{4.1}$$

for $l = 0, 1$, then ω_l behaves like

$$\omega_l = c_l r^\nu e^{il\theta} + O(1) \tag{4.2}$$

as $r = |x| \to \infty$, and it solves $L\omega_l = 0$. Such a solution is shown to be unique (see [20, Lemma 4.3]). The argument here is based on the proposition below, which has been proved as Proposition 4.2 in [20].

Proposition 4.1 *Let the notation be as above and let $\nu = |l - \alpha|$ again. Then*

$$\chi(L - k^2)^{-1}\chi = \chi G_0 \chi + \sum_{l=0,1} \gamma_l(k) i^{-2\nu} k^{2\nu} \left(\chi\omega_l \otimes \chi\omega_l\right) + Op(|k|^2), \quad |k| \to 0,$$

for some coefficients $\gamma_l(k)$, and $\gamma_l(k)$ behaves like

$$\gamma_l(k) = \gamma_l + o(1) \tag{4.3}$$

as $|k| \to 0$, where

$$\gamma_l = -\left(2^{1-2\nu}\Gamma(1 - \nu)\right) / \left((l + \nu)\Gamma(\nu)\right). \tag{4.4}$$

The precise value of γ_l is not required to prove the lemmas, but it is important in studying the effect of perturbation by scalar potentials in the latter sections.

Next we consider the operator $L_+ = L(a, b) = L + b$. We shall show that $(L_+ - k^2)^{-1}$ admits an expansion similar to that in Proposition 4.1. We have the relation

$$(L_+ - k^2)^{-1} = (L - k^2)^{-1}\left(1 + b(L - k^2)^{-1}\right)^{-1} \tag{4.5}$$

by the resolvent identity. Since L_+ has neither bound states nor resonance states at zero energy, $1+bG_0 : L^2_c \to L^2_c$ has the bounded inverse

$$T = (1+bG_0)^{-1} : L^2_c \to L^2_c.$$

We see that there exists a limit

$$G_+ = \lim_{k\to 0} (L_+ - k^2)^{-1}.$$

and the limit operator $G_+ = G_0 T : L^2_c \to L^2_{-1}$ is bounded. The inverse T is represented as

$$T = LG_+ = 1 - bG_+ : L^2_c \to L^2_c \tag{4.6}$$

and it follows that the adjoint operator $T^* = 1 - G_+ b : L^2_{\text{loc}} \to L^2_{\text{loc}}$ is well defined as an operator from the space L^2_{loc} of locally square integrable functions into itself. If we define

$$\omega_{+l} = T^*\omega_l = \omega_l - G_+ b\omega_l \tag{4.7}$$

for $l = 0, 1$, then ω_{+l} solves $L_+\omega_{+l} = 0$ and obeys (4.2) at infinity.

Proposition 4.2 *Let the notation be as above. Then*

$$\chi(L_+ - k^2)^{-1}\chi = \chi G_+\chi + \sum_{l=0,1} \gamma_{+l}(k) i^{-2\nu} k^{2\nu} \left(\chi\omega_{+l} \otimes \chi\omega_{+l}\right) + Op(|k|^2)$$

for some coefficients $\gamma_{+l}(k)$ obeying $\gamma_{+l}(k) = \gamma_l + o(1)$ as in Proposition 4.1.

Proof. We start with the relation

$$\chi(L_+ - k^2)^{-1}\chi = \chi(L - k^2)^{-1} \left(1 + b(L-k^2)^{-1}\right)^{-1} \chi$$

obtained from (4.5). Set

$$E(k) = \sum_{l=0,1} \gamma_l(k) i^{-2\nu} k^{2\nu} \left(\omega_l \otimes \omega_l\right) : L^2_c \to L^2_{\text{loc}}.$$

Then we have

$$\left(1 + b(L-k^2)^{-1}\right)^{-1} = \left(1 + TbE(k)\right)^{-1} T + Op(|k|^2)$$

by Proposition 4.1, and hence $(L_+ - k^2)^{-1}$ is expanded as

$$\chi(L_+ - k^2)^{-1}\chi = G_1 + G_2(k) + Op(|k|^2),$$

where $G_1 = \chi G_0 T\chi = \chi G_+\chi$ and

$$G_2(k) = \chi E(k) \left(1 + TbE(k)\right)^{-1} T\chi + \chi G_0 \left(\left(1 + TbE(k)\right)^{-1} - 1\right) T\chi.$$

We can calculate $G_2(k)$ as

$$
\begin{aligned}
G_2(k) &= \chi\,(1-G_0Tb)\,E(k)\,(1+TbE(k))^{-1}\,T\chi \\
&= \chi T^* E(k)\left(\sum_{j=0}^{\infty}(-1)^j(TbE(k))^j\right)T\chi.
\end{aligned}
$$

Hence it follows from (4.7) that

$$
G_2(k) = \sum_{0\le l,m\le 1}\gamma_{lm}(k)\,(\chi\omega_{+l}\otimes\chi\omega_{+m})
$$

for some 2×2 matrix $(\gamma_{lm}(k))_{0\le l,m\le 1}$. The components obey $\gamma_{lm}(k)=O(|k|^2)$ for $l\ne m$, and $\gamma_{ll}(k)=\gamma_l i^{-2\nu}k^{2\nu}(1+o(1))$. This completes the proof. $\square$

Proof of Lemma 3.3. The lemma follows from Proposition 4.2. $\square$

Proof of Lemma 3.4. We have

$$
\chi\left((L_+ + i\varepsilon^2)^{-1}-(L_+ - i\varepsilon^2)^{-1}\right)\chi = Op(\varepsilon^{2\alpha})+Op(\varepsilon^{2(1-\alpha)})
$$

by Proposition 4.2. Since $J_\varepsilon^*(L_{+\varepsilon}+i)^{-1}J_\varepsilon=\varepsilon^2(L_+ + i\varepsilon^2)^{-1}$, it follows that

$$
\|\chi_\varepsilon(L_{+\varepsilon}+i)^{-1}\|^2 = \|\chi J_\varepsilon^*(L_{+\varepsilon}+i)^{-1}(L_{+\varepsilon}-i)^{-1}J_\varepsilon\chi\| = O(\varepsilon^{2+2\alpha})+O(\varepsilon^{2+2(1-\alpha)}),
$$

where $\chi_\varepsilon(x)=\chi(x/\varepsilon)$. Hence

$$
\|\chi_\varepsilon(L_{+\varepsilon}+1)^{-1}\| = O(\varepsilon^{1+\alpha})+O(\varepsilon^{1+(1-\alpha)}).
$$

If we set $u_\varepsilon=(L_{+\varepsilon}+1)^{-1}f$ for $f\in L^2$, then u_ε satisfies the equation

$$
(L_{+\varepsilon}+1)u_\varepsilon = (p_{1\varepsilon}^2+p_{2\varepsilon}^2)u_\varepsilon + b_\varepsilon u_\varepsilon + u_\varepsilon = f
$$

and obeys the bound

$$
\|\chi_\varepsilon u_\varepsilon\|_{L^2} = \left(O(\varepsilon^{1+\alpha})+O(\varepsilon^{1+(1-\alpha)})\right)\|f\|_{L^2}. \tag{4.8}
$$

Thus we have

$$
\|\chi_\varepsilon p_{j\varepsilon}u_\varepsilon\|_{L^2} = \left(O(\varepsilon^{\alpha})+O(\varepsilon^{1-\alpha})\right)\|f\|_{L^2}. \tag{4.9}
$$

by elliptic estimate. A similar argument applies to $v=(L_{AB}+1)^{-1}f$. Note that

$$
J_\varepsilon^*(L_{AB}+1)^{-1}J_\varepsilon = \varepsilon^2(L_{AB}+\varepsilon^2)^{-1}.
$$

The Green kernel of $(L_{AB}-k^2)^{-1}$ is explicitly represented in terms of the Bessel functions after separation into the polar coordinates, and it admits an expansion similar to $(L_+ - k^2)^{-1}$. Thus we obtain

$$
\|\chi_\varepsilon v\|_{L^2} = \left(O(\varepsilon^{1+\alpha})+O(\varepsilon^{1+(1-\alpha)})\right)\|f\|_{L^2} \tag{4.10}
$$

and

$$\|\chi_\varepsilon \pi_j v\|_{L^2} = \left(O(\varepsilon^\alpha) + O(\varepsilon^{1-\alpha})\right) \|f\|_{L^2}. \tag{4.11}$$

By (3.1), $p_j = \pi_j$ on $\{|x| > 2\}$, so that $(1-\chi_\varepsilon)(\pi_j - p_{j\varepsilon}) = 0$. Since $\chi_\varepsilon = 1$ on $\operatorname{supp} b_\varepsilon$, $w_\varepsilon = u_\varepsilon - v$ satisfies

$$(1-\chi_\varepsilon)\left(p_{1\varepsilon}^2 + p_{2\varepsilon}^2 + 1\right) w_\varepsilon = 0$$

and it follows from (4.8) and (4.10) that

$$\|(1-\chi_\varepsilon) p_{j\varepsilon} w_\varepsilon\|_{L^2} = \left(O(\varepsilon^\alpha) + O(\varepsilon^{1-\alpha})\right) \|f\|_{L^2}.$$

This, together with (4.9) and (4.11), yields

$$\|p_{j\varepsilon}(L_{+\varepsilon} + 1)^{-1} - \pi_j (L_{AB} + 1)^{-1}\| = O(\varepsilon^\alpha) + O(\varepsilon^{1-\alpha})$$

and the proof is complete. □

We move to the proof of Lemma 3.5. We analyze the behavior as $|k| \to 0$ of $(L_- - k^2)^{-1}$. Decompose b into the product $b = b^{1/2}|b|^{1/2}$, where $b^{1/2} = |b|^{1/2}\operatorname{sgn} b$ with $\operatorname{sgn} b = b/|b|$. Then we have

$$(L_- - k^2)^{-1} = (L - k^2)^{-1} + (L - k^2)^{-1}|b|^{1/2} Z(k)^{-1} b^{1/2} (L - k^2)^{-1} \tag{4.12}$$

by repeated use of the resolvent identity, where

$$Z(k) = 1 - b^{1/2}(L - k^2)^{-1}|b|^{1/2}.$$

Since L_- has a resonance state at zero energy, $1 - b^{1/2} G_0 |b|^{1/2} : L^2 \to L^2$ is not invertible. Let

$$X = \{v \in L^2 : v = b^{1/2} G_0 |b|^{1/2} v\}.$$

According to [20], $\dim X = 1$, and $v \in X,\ v \neq 0$, satisfies $(v, |b|^{1/2}\omega_0)_{L^2} \neq 0$ and $(v, |b|^{1/2}\omega_1)_{L^2} = 0$ for ω_l defined by (4.1). We normalize $\eta_0 \in X$ as

$$(\eta_0, |b|^{1/2}\omega_0)_{L^2} = 1, \qquad (\eta_0, |b|^{1/2}\omega_1)_{L^2} = 0 \tag{4.13}$$

and define

$$\rho_0 = G_0 |b|^{1/2} \eta_0. \tag{4.14}$$

Then ρ_0 solves $L_- \rho_0 = 0$ and behaves like

$$\rho_0(x) = (2\pi\alpha)^{-1/2} r^{-\alpha} + g, \qquad r \to \infty, \tag{4.15}$$

with some $g \in L^2$ (see (5.8) and (5.9) in [20]). Thus ρ_0 spans the resonance space E_- of L_- at zero energy, because $\dim E_- = l$ as shown already in section 3.

Proposition 4.3 *Let $\rho_0 \in E_-$ be the resonance state defined above. Then*

$$\chi(L_- - k^2)^{-1}\chi = \gamma_-(k) i^{2\alpha} k^{-2\alpha} (\chi\rho_0 \otimes \chi\rho_0) + Op(|k|^0), \qquad |k| \to 0,$$

for some coefficient $\gamma_-(k)$ obeying $\gamma_-(k) = -1/\gamma_0 + o(1)$, γ_0 being as in (4.4).

Proof. For brevity, we prove the lemma when $b(x) \geq 0$. If $b(x)$ is not non–negative, $b^{1/2}G_0|b|^{1/2}$ is not necessarily self–adjoint. The argument below undergoes a slight change, but it is not serious. We skip the detail. Let Q_0 and Q be the orthogonal projections defined by $Q_0 = \eta_1 \otimes \eta_1$ and $Q = 1 - Q_0$, where $\eta_1 = \eta_0/\|\eta_0\|_{L^2}$. Since $1 - b^{1/2}G_0 b^{1/2} : \operatorname{Ran} Q \to \operatorname{Ran} Q$ is invertible and since

$$Q_0 \left((b^{1/2}\omega_1) \otimes (b^{1/2}\omega_1) \right) = \left((b^{1/2}\omega_1) \otimes (b^{1/2}\omega_1) \right) Q_0 = 0$$

by (4.13), it follows from Proposition 4.1 that the inverse $Z(k)^{-1}$ takes the form

$$Z(k)^{-1} = \gamma(k) i^{2\alpha} k^{-2\alpha} (\eta_0 \otimes \eta_0) + Op(|k|^0)$$

for some coefficient $\gamma(k)$ having the property in the proposition. This yields the relation. $\square$

Proof of Lemma 3.5. The lemma follows from Proposition 4.3. $\square$

Remark 4.1 Proposition 4.1 remains true for $\chi\nabla(L - k^2)^{-1}\chi$. For example, we have

$$\chi\nabla(L - k^2)^{-1}\chi = \chi\nabla G_0\chi + \sum_{l=0,1} \gamma_l(k) i^{-2\nu} k^{2\nu} (\chi\nabla\omega_l \otimes \chi\omega_l) + Op(|k|^2).$$

This follows by elliptic estimate. A similar relation is true for $(L_+ - k^2)^{-1}$ in Proposition 4.2.

We end the section by making a brief comment on the limit as $\varepsilon \to 0$ of the resolvent $(K \pm i\varepsilon)^{-1}$ calculated in (3.10). We formally write $(K \pm i0)^{-1}$ for the limit $\lim_{\varepsilon \to 0}(K \pm i\varepsilon)^{-1}$. If we combine (3.10) with Propositions 4.2, 4.3 and Remark 4.1, then we see that these limits have the different properties according as $0 < \alpha < 1/2$, $\alpha = 1/2$ or $1/2 < \alpha < 1$:

if $0 < \alpha < 1/2$, $(K \pm i0)^{-1}$ exists and $(K + i0)^{-1} = (K - i0)^{-1}$,
if $\alpha = 1/2$, $(K \pm i0)^{-1}$exists but $(K + i0)^{-1} \neq (K - i0)^{-1}$,
if $1/2 < \alpha < 1$, $(K \pm i0)^{-1}$does not exist.

It is obvious by gauge transformation that $H = \sigma_1\nu_1 + \sigma_2\nu_2$ preserves the same property as above. This means that H has a property similar to the three dimensional Laplacian for the case $0 < \alpha < 1/2$ and to the one or two dimensional Laplacian for the case $1/2 < \alpha < 1$.

5. Effect of scalar potentials and resonance state

The remaining three sections are devoted to the study on the effect of perturbation by scalar potentials. We assume that $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ satisfies

$$\lambda_0 = (V\rho_0, \rho_0)_{L^2} \neq 0 \tag{5.1}$$

for ρ_0 spanning the resonance space of L_- at zero energy. We set $V_\varepsilon(x) = \varepsilon^{-1}V(x/\varepsilon)$ and define $K_\varepsilon(V_\varepsilon)$ by

$$K_\varepsilon(V_\varepsilon) = K_\varepsilon + V_\varepsilon = \sigma_1 p_{1\varepsilon} + \sigma_2 p_{2\varepsilon} + V_\varepsilon,$$

which is self–adjoint with domain $\mathcal{D}(K_\varepsilon(V_\varepsilon)) = [H^1(\boldsymbol{R}^2)]^2$. We study the norm resolvent convergence for this Hamiltonian. The obtained result is invariant under gauge transformations. In particular, it remains true for the Hamiltonian $H_\varepsilon^D + V_\varepsilon$.

We define the resonance state at zero energy of the operator

$$K + V = \sigma_1 p_1 + \sigma_2 p_2 + V = \sum_{j=1}^{2} \sigma_j \left(-i\partial_j - a_j\right) + V$$

before formulating the result. The definition is different according as $0 < \alpha \leq 1/2$ or $1/2 < \alpha < 1$. By (3.7), we have $p_+ = -2i\, e^{ih} e^{-\varphi} \overline{\partial} e^{\varphi} e^{-ih}$, where $\overline{\partial} = (1/2)\,(\,\partial_1 + i\partial_2)$. The Cauchy–Riemann operator $\overline{\partial}$ has the fundamental solution $(1/\pi)\,(x_1 + ix_2)^{-1}$. We denote by $\overline{\partial}^{-1}$ the convolution operator

$$\overline{\partial}^{-1} = (1/\pi)\ (x_1 + ix_2)^{-1} * .$$

According to this notation, we define

$$p_+^{-1} = -(2i)^{-1} e^{ih} e^{-\varphi} \overline{\partial}^{-1} e^{\varphi} e^{-ih}$$

and $p_-^{-1} = \left(p_+^{-1}\right)^*$. By definition, we have $p_\pm p_\pm^{-1} = 1$.

Lemma 5.1 *One has the relations*

$$p_- G_+ = p_+^{-1}, \qquad G_+ p_+ = p_-^{-1}$$

on bounded functions with compact support.

Proof. We prove only the first relation. The second one follows by taking the adjoint. We set $w_1 = p_+^{-1} f$ for a function f as in the lemma. Then $w_1 \in L^2$ and it solves $p_+ w_1 = f$. If, on the other hand, we set $w_2 = p_- G_+ f \in L^2_{-1}$, then w_2 satisfies

$$p_+ w_2 = p_+ p_- G_+ f = L_+ G_+ f = f$$

and $w_2 \in L^2$. In fact, we have $\|p_-G_+f\|_{L^2} < \infty$ by a simple use of partial integration. Set $w = e^{\varphi}e^{-ih}(w_1 - w_2)$. Then w is an entire function. Note that $e^{\varphi} = O(|x|^{\alpha})$ at infinity for $0 < \alpha < 1$. Since $w_1 - w_2 \in L^2$, we can easily show that $w = 0$, and hence $w_1 = w_2$. Thus the lemma is obtained. $\square$

We now recall that $G_+ : L^2_{\mathrm{c}} \to L^2_{-1}$ is bounded. Hence

$$|V|^{1/2}p_+^{-1}|V|^{1/2} = |V|^{1/2}p_-G_+|V|^{1/2} : L^2 \to L^2$$

is bounded by elliptic estimate, and also it follows by adjoint that

$$|V|^{1/2}p_-^{-1}|V|^{1/2} = |V|^{1/2}G_+p_+|V|^{1/2} : L^2 \to L^2$$

is bounded.

We first consider the case $0 < \alpha \le 1/2$. We write $V^{1/2} = |V|^{1/2}/V$ and define $\Phi = \{u \in [L^2]^2 : Z_0u = -u\}$, where

$$Z_0 = V^{1/2}F_0|V|^{1/2}, \qquad F_0 = \begin{pmatrix} 0 & p_-G_+ \\ G_+p_+ & 0 \end{pmatrix}. \tag{5.2}$$

Since $Z_0 : [L^2]^2 \to [L^2]^2$ is a compact operator, it follows that $\dim \Phi < \infty$. As is easily seen from Lemma 5.1, $v = F_0|V|^{1/2}u$ with $u \in \Phi$ belongs to $L^2 \times L^{\infty}$ and it solves $(K + V)\, v = 0$. We further define $\Phi_0 = \{u \in \Phi : (u, |V|^{1/2}\tilde{\rho}_0)_{L^2} = 0\}$ with $\tilde{\rho}_0 = (\rho_0, 0)$. By definition, we have $\dim \Phi/\Phi_0 \le 1$.

Lemma 5.2 *If $u \in \Phi_0$, then $v = F_0|V|^{1/2}u$ is in $[L^2]^2$ and hence v becomes a bound state associated with zero eigenvalue. If, conversely, v is a bound state, then $u = -V^{1/2}v$ belongs to Φ_0.*

Proof. Let $u = {}^{\mathrm{t}}(u_1, u_2) \in \Phi_0$ and $v = {}^{\mathrm{t}}(v_1, v_2) = F_0|V|^{1/2}u$. Then $(u_1, |V|^{1/2}\rho_0) = 0$ by assumption, and it is easy to see that $v_1 \in L^2$. Since $\rho_0 = ce^{-\varphi}e^{ih}$ for some $c \ne 0$, $v_2 = p_-^{-1}|V|^{1/2}u_1$ behaves like

$$v_2(x) = c\,(u_1, |V|^{1/2}\rho_0)_{L^2}e^{\varphi}e^{-ih}(x_1 - ix_2)^{-1} + O(|x|^{-2+\alpha}), \qquad |x| \to \infty, \tag{5.3}$$

for another $c \ne 0$. Hence it follows that

$$(u_1, |V|^{1/2}\rho_0)_{L^2} = 0 \Longleftrightarrow v \in [L^2]^2. \tag{5.4}$$

This proves the first statement. The second one is also easy to prove. If $v \in [L^2]^2$ satisfies $(K + V)v = 0$, then $v = -F_0Vv$. Hence we have

$$u = -V^{1/2}v = V^{1/2}F_0Vv = Z_0V^{1/2}v = -Z_0u,$$

which, together with (5.4), implies that $u \in \Phi_0$. $\square$

If $\dim \Phi/\Phi_0 = 1$, then the above lemma implies that $(K+V)\,v = 0$ has a solution $v = {}^{\mathrm{t}}(v_1, v_2) \in L^2 \times L^\infty$ with $(v, V\tilde{\rho}_0)_{L^2} \neq 0$, and hence it follows from (5.3) that v_2 behaves like

$$v_2(x) = cr^{-1+\alpha}e^{i\theta}\,(1+o(1))\,, \qquad r = |x| \to \infty,$$

for $c \neq 0$. The converse is also true.

We move to the case $1/2 < \alpha < 1$. Recall that $\lambda_0 \neq 0$ by assumption (5.1). This enables us to define the projection

$$P = \lambda_0^{-1}\left(V^{1/2}\tilde{\rho}_0 \otimes |V|^{1/2}\tilde{\rho}_0\right) : [L^2]^2 \to [L^2]^2. \tag{5.5}$$

Let $Q = 1 - P$. We further define

$$\Psi = \{u \in [L^2]^2 : QZ_0Qu = -u\}, \qquad \Psi_0 = \{u \in \Psi : (u, Z_0^*|V|^{1/2}\tilde{\rho}_0)_{L^2} = 0\}.$$

We again have $\dim \Psi/\Psi_0 \leq 1$.

Lemma 5.3 *If $u \in \Psi_0$, then $v = F_0|V|^{1/2}u$ becomes a bound state associated with zero eigenvalue. If, conversely, v is a bound state, then $u = -V^{1/2}v$ belongs to Ψ_0.*

Proof. If $u \in \Psi_0$, then $PZ_0u = 0$ and it follows that

$$u = -QZ_0Qu = -Z_0u = -V^{1/2}F_0|V|^{1/2}u = -V^{1/2}v.$$

This shows that $|V|^{1/2}u = -Vv$ and hence $(K+V)v = 0$. If we write $v = F_0|V|^{1/2}u = -F_0|V|^{1/2}Z_0u$, then the first statement follows from (5.4). To prove the second one, we repeat the same computation as in the proof of Lemma 5.2 to obtain the relation $u = -Z_0u$. Since $v = -F_0Vv \in [L^2]^2$, it again follows from (5.4) that

$$(u, |V|^{1/2}\tilde{\rho}_0)_{L^2} = -(v, V\tilde{\rho}_0)_{L^2} = 0.$$

This implies that $Pu = 0$ and we obtain $u \in \Psi_0$. $\square$

Lemma 5.4 *If $\dim \Psi/\Psi_0 = 1$, then $(K+V)v = 0$ has a solution such that $v = {}^{\mathrm{t}}(v_1, v_2) \in L^\infty \times L^2$ and*

$$v_1(x) = cr^{-\alpha}\,(1+o(1))\,, \qquad |x| \to \infty,$$

for $c \neq 0$. The converse is also true.

Proof. Let $u \in \Psi$ be a representative of Ψ/Ψ_0. We set $v = F_0|V|^{1/2}u - d\tilde{\rho}_0$, where

$$d = \lambda_0^{-1}(u, Z_0^*|V|^{1/2}\tilde{\rho}_0)_{L^2} \neq 0.$$

Then $(Vv, \tilde{\rho}_0)_{L^2} = 0$, and $v_1(x) = -d\rho_0(x)\,(1+o(1))$ at infinity. We show that $(K+V)v = 0$. To see this, we calculate

$$V^{1/2}v = Z_0 u - PZ_0 u = QZ_0 u = -u.$$

Since $K\tilde{\rho}_0 = 0$, we obtain $Kv = |V|^{1/2}u = -Vv$. We claim that $v_2 \in L^2$. This follows from (5.4), because $(u, |V|^{1/2}\tilde{\rho}_0)_{L^2} = -(Vv, \tilde{\rho}_0)_{L^2} = 0$. Thus the first statement is established. We shall prove the converse. Let $v \in L^\infty \times L^2$ be a solution to $(K+V)v = 0$ with the properties in the lemma. We assert that v takes the form $v = -F_0Vv + c_0\tilde{\rho}_0$ for some $c_0 \neq 0$. To see this, we set $w = v + F_0Vv$. Then $Kw = 0$ and the first component w_1 behaves like $w_1 = cr^{-\alpha}\,(1+o(1))$ for $c \neq 0$ in the lemma, because $v_1 \notin L^2$ behaves in a similar way by assumption. Thus $w = c_0\tilde{\rho}_0$. The constant $c_0 \neq 0$ is determined as

$$c_0 = \lambda_0^{-1}(F_0Vv, V\tilde{\rho}_0)_{L^2} = \lambda_0^{-1}(V^{1/2}v, Z_0^*|V|^{1/2}\tilde{\rho}_0)_{L^2}$$

by use of (5.4). In fact, we have

$$(-F_0Vv + c_0\tilde{\rho}_0, V\tilde{\rho}_0)_{L^2} = (Vv, \tilde{\rho}_0)_{L^2} = (Vv_1, \rho_0)_{L^2} = 0$$

because $v_2 \in L^2$. If we set $u = -V^{1/2}v$, then $u \in [L^2]^2$ satisfies the relation $u = -Z_0u + PZ_0u = -QZ_0u$. Hence we have $u \in \Psi$, and also it follows that

$$(u, Z_0^*|V|^{1/2}\tilde{\rho}_0)_{L^2} = -(V^{1/2}v, Z_0^*|V|^{1/2}\tilde{\rho}_0)_{L^2} = -\lambda_0 c_0 \neq 0.$$

This implies the second statement, and the proof is complete. □

We combine Lemmas 5.2, 5.3 and 5.4 to formulate the precise definition of resonance state at zero energy.

Definition 5.1 The operator $K+V$ is said to admit a resonance state at zero energy, if the following condition is fulfilled :

(1) Assume that $0 < \alpha \leq 1/2$. The equation $(K+V)\,v = 0$ has a solution such that $v = {}^{\mathrm{t}}(v_1, v_2) \in L^2 \times L^\infty$ and

$$v_2(x) = r^{-1+\alpha}e^{i\theta}\,(1+o(1))\,, \qquad |x| \to \infty.$$

(2) Assume that $1/2 < \alpha < 1$ and that (5.1) is fulfilled. The equation $(K+V)\,v = 0$ has a solution such that $v = {}^{\mathrm{t}}(v_1, v_2) \in L^\infty \times L^2$ and

$$v_1(x) = r^{-\alpha}\,(1+o(1))\,, \qquad |x| \to \infty.$$

We note that the definition is invariant under gauge transformations. We are now in a position to state the second main theorem.

Theorem 5.1 *Assume that the flux α satisfies (1.2) and that $K+V$ has no bound states at zero energy. Then one has the following three statements.*

(1) *Let $0<\alpha<1/2$. If $K+V$ does not have a resonance state at zero energy, then $K_\varepsilon(V_\varepsilon)$ converges to H_∞ in the norm resolvent sense, and if $K+V$ has a resonance state, then $K_\varepsilon(V_\varepsilon)$ converges to H_0.*

(2) *Let $\alpha=1/2$. Assume that $K+V$ does not have a resonance state at zero energy, so that $(1+Z_0)^{-1}:[L^2]^2\to[L^2]^2$ exists as a bounded operator. Define the real number λ_2 by*

$$\lambda_2=((1+Z_0)^{-1}V^{1/2}\tilde{\rho}_0,|V|^{1/2}\tilde{\rho}_0)_{L^2}$$

with $\tilde{\rho}_0={}^{\mathrm{t}}(\rho_0,0)\in[L^2]^2$. Then $K_\varepsilon(V_\varepsilon)$ converges to H_κ with $\kappa=\tan(\zeta/2)$ determined through (2.6) with

$$\zeta=\arg\left((i\lambda_2-\gamma_0)/(i\lambda_2+\gamma_0)\right),$$

where $\gamma_0=-2$ is as in Proposition 4.1. If $K+V$ has a resonance state, then $K_\varepsilon(V_\varepsilon)$ converges to H_0.

(3) *Let $1/2<\alpha<1$. Assume that* (5.1) *is fulfilled. If $K+V$ does not have a resonance state at zero energy, then $K_\varepsilon(V_\varepsilon)$ converges to H_0, and if $K+V$ has a resonance state, then $K_\varepsilon(V_\varepsilon)$ converges to H_∞.*

The theorem is proved in the next section. We follow the argument used in the proof of Theorem 1.2.5 in [6] and of Theorem 3.3 in [10]. The major part of the argument is again occupied by the resolvent analysis at low energy of magnetic Schrödinger operator with resonance state at zero energy. We end the section by making some brief comments on the theorem.

Remark 5.1 (1) If $V(x)\ge 0$ or $V(x)\le 0$ and if it is sufficiently small but not identically zero, then $K+V$ has neither bound state nor resonance state at zero energy. The theorem asserts that the limit Hamiltonian changes at $\alpha=1/2$ even for small perturbations of scalar potentials, and also it is interesting that the situation is completely reversed when $K+V$ has a resonance state. (2) The same reason as in Remark 3.2 applies to $K_\varepsilon(V_\varepsilon)=\varepsilon^{-1}J_\varepsilon(K+V)J_\varepsilon^*$. The norm convergence is not true without assuming that $K+V$ has no bound states at zero energy. (3) The restriction (1.2) is technical. The same results as in the theorem seem to remain true even for $\alpha>1$ in the strong convergence sense, so that the limit Hamiltonian changes at half–integer fluxes. (4) The assumption (5.1) also seems to be technical. This is used for proving the third statement only, and a similar assumption has been used in [10] for the analysis on resolvents at low energy of Schrödinger operators in two dimensions. However the idea in the recent work [16] may make it possible to remove the assumption.

6. Proof of Theorem 5.1

We prove Theorem 5.1 in this section. The proof requires several new lemmas on the behavior at low energy of resolvents of magnetic Schrödinger operators besides the propositions in section 4. In formulating these lemmas, we use $\chi \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ with the same meaning as ascribed in section 4 and denote by $o_2(\varepsilon^\sigma)$ remainder terms with L^2 norm obeying the bound $o(\varepsilon^\sigma)$. We further define $\xi_l(x)$ by

$$\xi_l(x) = H_\nu(ir)e^{il\theta}, \quad \nu = |l-\alpha|, \tag{6.1}$$

for $l = 0, 1$. According to this notation, we have

$$\psi_+ = N_\alpha \begin{pmatrix} -i^{1+2\alpha}\xi_0 \\ \xi_1 \end{pmatrix}, \qquad \psi_- = N_\alpha \begin{pmatrix} i^{1+2\alpha}\xi_0 \\ \xi_1 \end{pmatrix}$$

by (2.3) and (2.5).

Lemma 6.1 *Let ω_{+l} be defined by (4.7). Then*

$$(L_{+\varepsilon}+1)^{-1}J_\varepsilon\chi = \sum_{l=0,1} \beta_{+l}(\varepsilon) i^\nu \varepsilon^{1+\nu} \left(\xi_l \otimes \chi\omega_{+l}\right) + Op(\varepsilon^2)$$

for some coefficient $\beta_{+l}(\varepsilon)$, and $\beta_{+l}(\varepsilon)$ behaves like $\beta_{+l}(\varepsilon) = \beta_l + o(1)$ as $\varepsilon \to 0$, where

$$\beta_l = i(2\pi)^{-1/2}\left(2^{-\nu}\Gamma(1-\nu)/(l+\nu)^{1/2}\right)\sin\nu\pi. \tag{6.2}$$

Lemma 6.2 *Let β_0 be as above and let γ_0 be as in Proposition 4.1. Let ρ_0 be the resonance state defined by (4.14). Then*

$$(L_{-\varepsilon}+1)^{-1}J_\varepsilon\chi = -i^\alpha\left(\beta_0/\gamma_0\right)\varepsilon^{1-\alpha}\left((\xi_0 + o_2(1)) \otimes \chi\rho_0\right) + Op(\varepsilon).$$

Lemma 6.3 *One has*

$$p_{-\varepsilon}(L_{+\varepsilon}+1)^{-1}J_\varepsilon\chi = Op(\varepsilon).$$

Lemma 6.4 *One has*

$$p_{+\varepsilon}(L_{-\varepsilon}+1)^{-1}J_\varepsilon\chi = i^{-\alpha}\left(\beta_0/\gamma_0\right)\varepsilon^{1-\alpha}\left((\xi_1 + o_2(1)) \otimes \chi\rho_0\right) + Op(\varepsilon).$$

We proceed to proving Theorem 5.1, accepting the above lemmas as proved. The proof of these lemmas is done in section 7.

Proof of Theorem 5.1. For brevity, we assume throughout the proof that $V(x) \ge 0$, so that the operator Z_0 defined by (5.2) becomes self–adjoint. We write $R_\varepsilon = (K_\varepsilon(V_\varepsilon) + i)^{-1}$. Then we obtain

$$R_\varepsilon = (K_\varepsilon + i)^{-1} - (K_\varepsilon + i)^{-1} V_\varepsilon^{1/2} X_\varepsilon^{-1} V_\varepsilon^{1/2} (K_\varepsilon + i)^{-1}$$

from the resolvent identity, where $X_\varepsilon = 1 + V_\varepsilon^{1/2}(K_\varepsilon + i)^{-1} V_\varepsilon^{1/2}$. We define Y_ε by

$$Y_\varepsilon = J_\varepsilon^* X_\varepsilon J_\varepsilon = 1 + V^{1/2}(K + i\varepsilon)^{-1} V^{1/2}.$$

Then R_ε is rewritten as

$$R_\varepsilon = (K_\varepsilon + i)^{-1} - \varepsilon^{-1} \left((K_\varepsilon + i)^{-1} J_\varepsilon V^{1/2} \right) Y_\varepsilon^{-1} \left((K_\varepsilon - i)^{-1} J_\varepsilon V^{1/2} \right)^* .$$

We analyze the behavior as $\varepsilon \to 0$ of operators on the right side. By Theorem 3.1, $(K_\varepsilon + i)^{-1} \to (H_\infty + i)^{-1}$. If we make use of relation (3.3), then we can obtain

$$(K_\varepsilon \pm i)^{-1} = \begin{pmatrix} \mp i(L_{-\varepsilon} + 1)^{-1} & p_{-\varepsilon}(L_{+\varepsilon} + 1)^{-1} \\ p_{+\varepsilon}(L_{-\varepsilon} + 1)^{-1} & \mp i(L_{+\varepsilon} + 1)^{-1} \end{pmatrix}$$

from (3.4). By Lemmas 6.1 $\sim$ 6.4, we have

$$(K_\varepsilon + i)^{-1} J_\varepsilon V^{1/2} = (\beta_0/\gamma_0)\, N_\alpha^{-1} i^{-\alpha} \varepsilon^{1-\alpha} \left((\psi_- + o_2(1)) \otimes q_0 \right) + Op(\varepsilon), \tag{6.3}$$

where $q_0 = {}^{\mathrm{t}}(V^{1/2}\rho_0, 0)$. Similarly $\left((K_\varepsilon - i)^{-1} J_\varepsilon V^{1/2} \right)^*$ takes the form

$$\left((K_\varepsilon - i)^{-1} J_\varepsilon V^{1/2} \right)^* = \left(\overline{\beta_0}/\gamma_0 \right) N_\alpha^{-1} i^{\alpha} \varepsilon^{1-\alpha} \left(q_0 \otimes (\psi_+ + o_2(1)) \right) + Op(\varepsilon). \tag{6.4}$$

The operator $(K + i\varepsilon)^{-1}$ also has been calculated as

$$(K + i\varepsilon)^{-1} = \begin{pmatrix} -i\varepsilon(L_- + \varepsilon^2)^{-1} & p_-(L_+ + \varepsilon^2)^{-1} \\ (L_+ + \varepsilon^2)^{-1} p_+ & -i\varepsilon(L_+ + \varepsilon^2)^{-1} \end{pmatrix}$$

in (3.10). By (4.2) and (4.7), ω_{+0} obeys $\omega_{+0} = (1/2)\alpha^{-1/2} r^\alpha + g$ for some $g \in L^2_{-1}$, and it solves $L_+ \omega_{+0} = 0$ uniquely (see Lemma 4.3 of [20]). Since $L_+ = p_+ p_-$ with

$$p_- = p_+^* = i e^{ih} e^{-\varphi} \left(-\partial_1 + i\partial_2 \right) e^{-\varphi} e^{-ih},$$

we see that $p_- \omega_{+0} = 0$. In fact, ω_{+0} is given by $\omega_{+0} = c e^{+ih} e^{\varphi}$ for some constant $c \ne 0$. Thus we have

$$V^{1/2} p_- (L_+ + \varepsilon^2)^{-1} V^{1/2} = V^{1/2} \left(p_- G_+ + \varepsilon^{2(1-\alpha)} \gamma_{+1}(i\varepsilon) \left(p_- \omega_{+1} \otimes \omega_{+1} \right) \right) V^{1/2} + Op(\varepsilon^2)$$

by Proposition 4.2 (see Remark 4.1) and similarly for $V^{1/2}(L_+ + \varepsilon^2)^{-1} p_+ V^{1/2}$. Hence it follows from Proposition 4.3 that

$$Y_\varepsilon = 1 + Z_0 - i\gamma_-(i\varepsilon) \varepsilon^{1-2\alpha} \left(q_0 \otimes q_0 \right) + \varepsilon^{2(1-\alpha)} Z_1(\varepsilon) + Op(\varepsilon),$$

where $\gamma_-(i\varepsilon) = -1/\gamma_0 + o(1)$ as $\varepsilon \to 0$ and

$$Z_1(\varepsilon) = \gamma_{+1}(i\varepsilon)V^{1/2}\begin{pmatrix} 0 & (p_-\omega_{+1} \otimes \omega_{+1}) \\ (\omega_{+1} \otimes p_-\omega_{+1}) & 0 \end{pmatrix} V^{1/2}. \tag{6.5}$$

Note that $p_- = \pi_-$ on $\{|x| > 2\}$, and recall the form of operator π_- (see section 2). Then it follows from (4.7) that $p_-\omega_{+1} = cr^{-\alpha} + g$ for some $c \neq 0$. We see by Lemma 5.1 (or by the argument in the proof) that the remainder term g is in L^2. Since $p_-\omega_{+1}$ satisfies $p_+p_-\omega_{+1} = L_+\omega_{+1} = 0$, we have $p_-\omega_{+1} = c\rho_0$ for another $c \neq 0$, and hence

$$Z_1(\varepsilon) = c\gamma_{+1}(i\varepsilon)\,(q_0 \otimes q + q \otimes q_0) \tag{6.6}$$

with $q = {}^{\mathrm{t}}(0, V^{1/2}\omega_{+1})$. The argument is different from now on according as $0 < \alpha < 1/2$, $\alpha = 1/2$ or $1/2 < \alpha < 1$.

(1) Assume that $0 < \alpha < 1/2$. If $K + V$ has neither bound nor resonance state at zero energy, then the bounded inverse $(1 + Z_0)^{-1} : [L^2]^2 \to [L^2]^2$ exists, and hence Y_ε has an inverse bounded uniformly in ε. Thus it follows that $R_\varepsilon \to (H_\infty + i)^{-1}$ in norm. Next we consider the case when $K + V$ has a resonance state. Since $K + V$ has no bound states by assumption, $\dim \Phi = 1$ and $\dim \Phi_0 = 0$. Let Φ be spanned by $q_1 \in [L^2]^2$ with $\|q_1\|_{L^2} = 1$. Then $\lambda_1 = (q_1, q_0)_{L^2} \neq 0$ by definition. We denote by $\Pi_1 = q_1 \otimes q_1$ the projection on Φ and write $\Pi = 1 - \Pi_1$ for the projection on the orthogonal complement $\Phi^\perp$. Then $\Pi(1 + Z_0)\Pi : \Phi^\perp \to \Phi^\perp$ has an bounded inverse and $\Pi_1\,(q_0 \otimes q_0)\,\Pi_1 = |\lambda_1|^2\Pi_1$. Thus Y_ε^{-1} takes the form

$$Y_\varepsilon^{-1} = -i(\gamma_0/|\lambda_1|^2)\varepsilon^{2\alpha-1}\,(q_1 \otimes q_1) + op(\varepsilon^{2\alpha-1}). \tag{6.7}$$

Since $\Gamma(1-\alpha)\Gamma(\alpha) = \pi/(\sin\alpha\pi)$ by formula, a direct calculation yields

$$|\beta_0|^2/\gamma_0 = -\sin\alpha\pi/4 = -N_\alpha^2 \tag{6.8}$$

by (2.4). We combine (6.3), (6.4), (6.7) and (6.8) to obtain that

$$R_\varepsilon = (H_\infty + i)^{-1} - i\,(\psi_- \otimes \psi_+) + op(\varepsilon^0).$$

This determines $\zeta = 0$ by (2.7) and hence $\kappa = 0$ by (2.6). Thus we obtain that $R_\varepsilon \to (H_0 + i)^{-1}$.

(2) If $\alpha = 1/2$, then Y_ε takes the form

$$Y_\varepsilon = 1 + Z_0 + (i/\gamma_0)\,(q_0 \otimes q_0) + op(\varepsilon^0)$$

and $\gamma_0 = -2$. Assume that zero is not a resonance energy. If we set $q_2 = (1+Z_0)^{-1}q_0$, then the real number λ_2 in the theorem is represented as $\lambda_2 = (q_2, q_0)_{L^2}$, and Y_ε^{-1} is calculated as

$$\begin{aligned} Y_\varepsilon^{-1} &= (1 + (i/\gamma_0)\,(q_2 \otimes q_0))^{-1}\,(1 + Z_0)^{-1} + op(\varepsilon^0) \\ &= \Big(1 - (i/(\gamma_0 + i\lambda_2))\,(q_2 \otimes q_0)\Big)(1 + Z_0)^{-1} + op(\varepsilon^0). \end{aligned}$$

Thus we obtain

$$R_\varepsilon = (H_\infty + i)^{-1} + c_2\,(\psi_- \otimes \psi_+) + op(\varepsilon^0)$$

with $c_2 = \lambda_2/(\gamma_0 + i\lambda_2)$, and

$$e^{i\zeta} = 2ic_2 - 1 = (i\lambda_2 - \gamma_0)/(i\lambda_2 + \gamma_0).$$

This determines $\kappa = \tan(\zeta/2)$ through (2.6). If $K + V$ has a resonance state, the lemma below yields the desired relation

$$R_\varepsilon = (H_\infty + i)^{-1} - i\,(\psi_- \otimes \psi_+) + op(\varepsilon^0) = (H_0 + i)^{-1} + op(\varepsilon^0).$$

Lemma 6.5 *Under the above situation, one has*

$$\left(q_0, Y_\varepsilon^{-1} q_0\right)_{L^2} = i\gamma_0 + o(1), \qquad \varepsilon \to 0.$$

Proof. If we again denote by Π_1 and $\Pi = 1 - \Pi_1$ the projections on Φ and $\Phi^\perp$ respectively, then

$$Y_{11} = \Pi_1(1 + Z_0)\Pi_1 = 0, \quad Y_{01} = \Pi(1 + Z_0)\Pi_1 = 0, \quad Y_{10} = \Pi_1(1 + Z_0)\Pi = 0$$

and $Y_{00} = \Pi(1 + Z_0)\Pi : \Phi^\perp \to \Phi^\perp$ is invertible. We write

$$q_0 = \Pi q_0 + \Pi_1 q_0 = r_0 + r_1.$$

We set $D = (i/\gamma_0)\,(q_0 \otimes q_0)$ and decompose it into the sum

$$D = \sum_{0 \le j,k \le 1} D_{jk}, \qquad D_{jk} = (i/\gamma_0)\,(r_j \otimes r_k)\,.$$

We represent $Y_\varepsilon = 1 + Z_0 + D + op(\varepsilon^0)$ in the matrix form

$$Y_\varepsilon = \begin{pmatrix} Y_{00} + D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} + op(\varepsilon^0) : \begin{pmatrix} \Phi^\perp \\ \Phi \end{pmatrix} \to \begin{pmatrix} \Phi^\perp \\ \Phi \end{pmatrix}.$$

By (5.4), $r_1 \neq 0$, and hence $D_{11} : \Phi \to \Phi$ is invertible. We have

$$D_{11}^{-1} = -i\gamma_0 \|r_1\|_{L^2}^{-4}\,(r_1 \otimes r_1)$$

and $D_{00} - D_{01}D_{11}^{-1}D_{10} = 0$. If we make use of this relation, then

$$Y_\varepsilon^{-1} = \begin{pmatrix} Y_{00}^{-1} & -Y_{00}^{-1}D_{01}D_{11}^{-1} \\ -D_{11}^{-1}D_{10}Y_{00}^{-1} & D_{11}^{-1}(1 + D_{10}Y_{00}^{-1}D_{01}D_{11}^{-1}) \end{pmatrix} + op(\varepsilon^0).$$

We compute

$$(r_0, Y_{00}^{-1}D_{01}D_{11}^{-1}r_1)_{L^2} = (r_0, Y_{00}^{-1}r_0)_{L^2}, \quad (r_1, D_{11}^{-1}D_{10}Y_{00}^{-1}r_0)_{L^2} = (Y_{00}^{-1}r_0, r_0)_{L^2}$$

and

$$(r_1, D_{11}^{-1}(1 + D_{10}Y_{00}^{-1}D_{01}D_{11}^{-1})r_1)_{L^2} = i\gamma_0 + (Y_{00}^{-1}r_0, r_0)_{L^2}.$$

This completes the proof of the lemma. $\square$

(3) The final case is $1/2 < \alpha < 1$. The projection $P = \lambda_0^{-1}(q_0 \otimes q_0) : [L^2]^2 \to [L^2]^2$ is defined by (5.5). Set $Q = 1 - P$ again. Then

$$Y_\varepsilon = \tau(\varepsilon)P + Q + Z_0 + \varepsilon^{2(1-\alpha)}Z_1(\varepsilon) + Op(\varepsilon),$$

where

$$\tau(\varepsilon) = 1 - i\lambda_0\gamma_-(i\varepsilon)\varepsilon^{1-2\alpha} = \varepsilon^{1-2\alpha}\left(i\lambda_0/\gamma_0 + o(1)\right).$$

We can easily prove that

$$(\tau(\varepsilon)P + Q)^{-1} = \delta(\varepsilon)P + Q, \qquad (1 + QZ_0P)^{-1} = 1 - QZ_0P,$$

where $\delta(\varepsilon) = 1/\tau(\varepsilon) = \varepsilon^{2\alpha-1}(-i\gamma_0/\lambda_0 + o(1))$. The two relations yield

$$Y_\varepsilon = (\tau(\varepsilon)P + Q)(1 + QZ_0P)T_\varepsilon$$

and hence we have

$$Y_\varepsilon^{-1} = T_\varepsilon^{-1}\left(\delta(\varepsilon)(P - QZ_0P) + Q\right), \tag{6.9}$$

where

$$T_\varepsilon = 1 + QZ_0Q + \delta(\varepsilon)(1 - QZ_0)PZ_0 + \varepsilon^{2(1-\alpha)}QZ_1(\varepsilon) + Op(\varepsilon).$$

By (6.6), we see that $QZ_1(\varepsilon)Q = 0$. Let $\Sigma_0 = \operatorname{Ran} P$ and $\Sigma = \operatorname{Ran} Q$. The second factor on the right side of (6.9) has the matrix representation

$$\delta(\varepsilon)(P - QZ_0P) + Q = \begin{pmatrix} \delta(\varepsilon)P & 0 \\ -\delta(\varepsilon)QZ_0P & Q \end{pmatrix} : \begin{pmatrix} \Sigma_0 \\ \Sigma \end{pmatrix} \to \begin{pmatrix} \Sigma_0 \\ \Sigma \end{pmatrix}, \tag{6.10}$$

while T_ε takes the form $T_\varepsilon = (T_{jk}(\varepsilon))_{0\le j,k\le 1}$, where

$$T_{00} = P(1 + Op(\varepsilon^{2\alpha-1}))P, \quad T_{01} = P(\delta(\varepsilon)Z_0 + Op(\varepsilon))Q,$$

$$T_{10} = Q(Op(\varepsilon^{2\alpha-1}) + Op(\varepsilon^{2(1-\alpha)}))P, \quad T_{11} = Q(1 + Z_0 - \delta(\varepsilon)Z_0PZ_0 + Op(\varepsilon))Q.$$

The inverse T_ε^{-1} can be calculated as $T_\varepsilon^{-1} = (S_{jk}(\varepsilon))_{0\le j,k\le 1}$, where

$$S_{00} = \left(T_{00} - T_{01}T_{11}^{-1}T_{10}\right)^{-1}, \quad S_{01} = -T_{00}^{-1}T_{01}\left(T_{11} - T_{10}T_{00}^{-1}T_{01}\right)^{-1},$$

$$S_{10} = -T_{11}^{-1}T_{10}\left(T_{00} - T_{01}T_{11}^{-1}T_{10}\right)^{-1}, \quad S_{11} = \left(T_{11} - T_{10}T_{00}^{-1}T_{01}\right)^{-1}.$$

We now analyze the behavior as $\varepsilon \to 0$ of inverse Y_ε^{-1}. We first deal with the resonance case. We claim that Y_ε^{-1} takes the form

$$Y_\varepsilon^{-1} = \begin{pmatrix} Pop(\varepsilon^{2\alpha-1})P & POp(\varepsilon^0)Q \\ QOp(\varepsilon^0)P & QOp(\varepsilon^{1-2\alpha})Q \end{pmatrix}. \tag{6.11}$$

Note that

$$(f \otimes q_0)\, Q = 0, \qquad Q\,(q_0 \otimes f) = 0$$

for $f \in [L^2]^2$. Hence we combine (6.11) with (6.3) and (6.4) to obtain that $R_\varepsilon \to (H_\infty + i)^{-1}$. When $K + V$ has a resonance state at zero energy,

$$\dim \Psi = \dim \mathrm{Ker}(Q + QZ_0Q) = 1$$

and $\dim \Psi_0 = 0$. Let Ψ be spanned by $q_3 \in [L^2]^2$ with $\|q_3\|_{L^2} = 1$. We denote by $Q_3 = q_3 \otimes q_3$ the projection on Ψ and decompose $Q = Q_3 + Q(1 - Q_3) = Q_3 + Q_4$. Since $\dim \Psi_0 = 0$, $\lambda_3 = (Z_0 q_3, q_0)_{L^2} \neq 0$. The operator T_{11} is represented in the form

$$\begin{aligned} T_{11} &= Q_4(1 + Z_0 + Op(\varepsilon^{2\alpha-1}))Q_4 - \left(\delta(\varepsilon)|\lambda_3|^2/\lambda_0\right) Q_3 \\ &+ Q_4 Op(\varepsilon^{2\alpha-1})Q_3 + Q_3 Op(\varepsilon^{2\alpha-1})Q_4. \end{aligned}$$

Hence

$$T_{11}^{-1} = -\left(\tau(\varepsilon)\lambda_0/|\lambda_3|^2\right) Q_3 + QOp(\varepsilon^0)Q.$$

As is easily seen, $T_{00}^{-1} = P(1 + op(\varepsilon^0))P$. Similarly we have

$$\left(T_{11} - T_{10}T_{00}^{-1}T_{01}\right)^{-1} = -\left(\tau(\varepsilon)\lambda_0/|\lambda_3|^2\right) Q_3 + QOp(\varepsilon^0)Q$$

and $\left(T_{00} - T_{01}T_{11}^{-1}T_{10}\right)^{-1} = P(1 + op(\varepsilon^0))P$. Thus we obtain

$$T_\varepsilon^{-1} = \begin{pmatrix} P(1 + op(\varepsilon^0))P & (\lambda_0/|\lambda_3|^2)PZ_0Q_3 + Pop(\varepsilon^0)Q \\ Q(Op(\varepsilon^0) + Op(\varepsilon^{3-4\alpha}))P & QOp(\varepsilon^{1-2\alpha})Q \end{pmatrix}.$$

This, together with (6.9) and (6.10), implies that

$$Y_\varepsilon^{-1} = \begin{pmatrix} \delta(\varepsilon)P(1 - (\lambda_0/|\lambda_3|^2)Z_0Q_3Z_0 + op(\varepsilon^0))P & POp(\varepsilon^0)Q \\ QOp(\varepsilon^0)P & QOp(\varepsilon^{1-2\alpha})Q \end{pmatrix}.$$

Since $PZ_0Q_3Z_0P = (|\lambda_3|^2/\lambda_0)P$, (6.11) follows at once. If $K + V$ does not have a resonance state, then $\dim \Psi = 0$, and $T_{11}(\varepsilon) : \Sigma \to \Sigma$ admits an inverse bounded uniformly in ε. We repeat the same argument as above to obtain that

$$Y_\varepsilon^{-1} = \begin{pmatrix} \delta(\varepsilon)P(1 + op(\varepsilon^0))P & POp(\varepsilon^0)Q \\ QOp(\varepsilon^0)P & QOp(\varepsilon^{1-2\alpha})Q \end{pmatrix},$$

so that $Y_\varepsilon^{-1} \sim -i(\gamma_0/\lambda_0^2)\varepsilon^{2\alpha-1}\,(q_0 \otimes q_0)$. Hence we have

$$R_\varepsilon = (H_\infty + i)^{-1} - i\,(\psi_- \otimes \psi_+) + op(\varepsilon^0) = (H_0 + i)^{-1} + op(\varepsilon^0)$$

in the same way as in case (1). Thus the proof of the theorem is now complete. $\square$

7. Resolvent analysis at low energy on Schrödinger operators II

The last section is devoted to proving Lemmas 6.1 $\sim$ 6.4 which remain unproved. The argument here is based on the following proposition obtained as Proposition 7.1 in [20]. The proposition has been proved for $(L_\varepsilon - i)^{-1}$, but the argument there applies to $(L_\varepsilon - \sigma^2)^{-1}$, $\operatorname{Im}\sigma > 0$, without any serious modification.

Proposition 7.1 *Let L_ε be defined by (3.2) and let $\sigma \in \boldsymbol{C}$ be as above. Then*

$$(L_\varepsilon - \sigma^2)^{-1} J_\varepsilon \chi = \sum_{l=0,1} \beta_l(\varepsilon) \sigma^\nu \varepsilon^{1+\nu} \left(\mu_l \otimes \chi\omega_l\right) + Op(\varepsilon^2)$$

for some coefficients $\beta_l(\varepsilon)$, where $\mu_l(x) = H_\nu(\sigma r)e^{il\theta}$ and $\beta_l(\varepsilon)$ behaves in the same manner as in Lemma 6.1.

Proof of Lemma 6.1. We use the relations $J_\varepsilon^*(L_{+\varepsilon}+1)^{-1}J_\varepsilon = \varepsilon^2(L_+ + \varepsilon^2)^{-1}$ and $b_\varepsilon J_\varepsilon = \varepsilon^{-2} J_\varepsilon b$. Then we have

$$(L_{+\varepsilon}+1)^{-1}J_\varepsilon\chi = (L_\varepsilon+1)^{-1}J_\varepsilon\chi - (L_\varepsilon+1)^{-1}J_\varepsilon b(L_+ + \varepsilon^2)^{-1}\chi$$

by the resolvent identity. We apply Proposition 7.1 with $\sigma = i$ to $(L_\varepsilon+1)^{-1}J_\varepsilon b$ and Proposition 4.2 with $k = i\varepsilon$ to $\chi(L_+ + \varepsilon^2)^{-1}\chi$. Since $\omega_{+l} = \omega_l - G_+ b\omega_l$ by (4.7) and since $\mu_l = \xi_l$ for $\sigma = i$, the desired relation is obtained. $\square$

Proof of Lemma 6.2. We have

$$(L_{-\varepsilon}+1)^{-1}J_\varepsilon\chi = (L_\varepsilon+1)^{-1}J_\varepsilon\chi + (L_\varepsilon+1)^{-1}J_\varepsilon b(L_- + \varepsilon^2)^{-1}\chi$$

by the resolvent identity. The first operator on the right side is of class $Op(\varepsilon)$ by Proposition 7.1. We apply Proposition 4.3 to $\chi(L_- + \varepsilon^2)^{-1}\chi$ in the second operator. Since

$$(b\rho_0, \omega_0)_{L^2} = (b^{1/2}G_0|b|^{1/2}\eta_0, |b|^{1/2}\omega_0)_{L^2} = (\eta_0, |b|^{1/2}\omega_0)_{L^2} = 1$$

and since $(b\rho_0, \omega_1)_{L^2} = (\eta_0, |b|^{1/2}\omega_1)_{L^2} = 0$ by (4.13), the desired relation again follows from Proposition 7.1 with $\sigma = i$. $\square$

Proof of Lemma 6.3. Since $p_{-\varepsilon}^* p_{-\varepsilon} = p_{+\varepsilon}p_{-\varepsilon} = L_{+\varepsilon}$, we have

$$\left(p_{-\varepsilon}(L_{+\varepsilon}+1)^{-1}\right)^* p_{-\varepsilon}(L_{+\varepsilon}+1)^{-1} = (L_{+\varepsilon}+1)^{-1} - (L_{+\varepsilon}+1)^{-2}$$

and hence

$$\|p_{-\varepsilon}(L_{+\varepsilon}+1)^{-1}J_\varepsilon\chi\|^2 = \|\chi\left(\varepsilon^2(L_+ + \varepsilon^2)^{-1} - \varepsilon^4(L_+ + \varepsilon^2)^{-2}\right)\chi\|.$$

By Proposition 4.2, $\|\chi(L_+ + \varepsilon^2)^{-1}\chi\| = O(1)$ as $\varepsilon \to 0$. This implies that

$$\|\chi(L_+ + \varepsilon^2)^{-1/2}\| = O(1).$$

Thus the bound in the lemma is obtained. $\square$

Proof of Lemma 6.4. We prove the lemma only in the simple case that $b(x) \geq 0$. We assert that

$$p_{+\varepsilon}(L_{-\varepsilon} - i)^{-1} J_\varepsilon \chi = -i^{(3-\alpha)/2} \, (\beta_0/\gamma_0) \, \varepsilon^{1-\alpha} \, ((\tau_1 + o_2(1)) \otimes \chi\rho_0) + Op(\varepsilon), \qquad (7.1)$$

where $\tau_1(x) = H_{1-\alpha}(i^{1/2}r)e^{i\theta}$. This implies the lemma. In fact, we use (3.3) to obtain

$$p_{+\varepsilon}(L_{-\varepsilon} + 1)^{-1} = \left(1 - (1+i)(L_{+\varepsilon} + 1)^{-1}\right) p_{+\varepsilon}(L_{-\varepsilon} - i)^{-1}$$

and it follows from Lemma 3.1 that

$$\left(1 - (1+i)(L_{+\varepsilon} + 1)^{-1}\right) \tau_1 \to f_1 = (1 - (1+i)(L_{AB} + 1)^{-1})\tau_1$$

strongly in L^2. The function $f_1 \in L^2$ satisfies $(\pi_1^2 + \pi_2^2 + 1) \, f_1 = 0$. Hence it takes the form $f_1(x) = g(r)e^{i\theta}$, where g solves

$$-g'' - r^{-1}g' + \left((1-\alpha)^2 r^{-2} + 1\right) g = 0.$$

Thus $f_1 = cH_{1-\alpha}(ir)e^{i\theta} = c\xi_1$ for some coefficient c. Since $u \in \mathcal{D}(L_{AB})$ vanishes at the origin, $f_1 - \tau_1$ also vanishes, so that c is determined as $c = i^{(1-\alpha)/2}$. Hence we see that (7.1) implies the relation of the lemma.

We shall show (7.1). We make use of the relation

$$(L_{-\varepsilon} - i)^{-1} J_\varepsilon \chi = (L_\varepsilon - i)^{-1} J_\varepsilon \chi + (L_\varepsilon - i)^{-1} J_\varepsilon b (L_- - i\varepsilon^2)^{-1} \chi$$

to decompose

$$p_{+\varepsilon}(L_{-\varepsilon} - i)^{-1} J_\varepsilon \chi = I_1(\varepsilon) + I_2(\varepsilon),$$

where $I_1(\varepsilon) = p_{+\varepsilon}(L_\varepsilon - i)^{-1} J_\varepsilon \chi$ and

$$I_2(\varepsilon) = p_{+\varepsilon}(L_\varepsilon - i)^{-1} J_\varepsilon b (L_- - i\varepsilon^2)^{-1} \chi.$$

Since $p_{-\varepsilon} p_{+\varepsilon} = L_{-\varepsilon} = L_\varepsilon - b_\varepsilon$, it follows from Proposition 4.1 that

$$\|I_1(\varepsilon)\|^2 = \|\chi J_\varepsilon^* (L_\varepsilon + i)^{-1} \, (L_\varepsilon - b_\varepsilon) \, (L_\varepsilon - i)^{-1} J_\varepsilon \chi\| = O(\varepsilon^2).$$

To evaluate $I_2(\varepsilon)$, we use Proposition 4.3 with $k = i^{1/2}\varepsilon$ for $b(L_- - i\varepsilon^2)^{-1}\chi$. Then

$$I_2(\varepsilon) = p_{+\varepsilon}(L_\varepsilon - i)^{-1} J_\varepsilon \chi \left(\gamma_-(k) i^\alpha \varepsilon^{-2\alpha} \, (b\rho_0 \otimes \chi\rho_0) + Op(\varepsilon^0)\right),$$

where $\gamma_-(k)$ obeys $\gamma_-(k) = -1/\gamma_0 + o(1)$. Since $p_{+\varepsilon}(L_\varepsilon - i)^{-1} J_\varepsilon \chi = Op(\varepsilon)$ as is seen from the proof of Lemma 6.3, (7.1) follows from the lemma below at once. $\square$

Lemma 7.1 *One has the relation*

$$p_{+\varepsilon}(L_\varepsilon - i)^{-1} J_\varepsilon b\rho_0 = i^{3(1-\alpha)/2} \beta_0 \varepsilon^{1+\alpha} \tau_1 + o_2(\varepsilon^{1+\alpha}).$$

Proof. Since $b^{1/2}G_0b^{1/2}\eta_0 = \eta_0$, it follows from (4.14) that $b\rho_0 = b^{1/2}\eta_0$. We write

$$g_\varepsilon = \varepsilon^{-(1+\alpha)}p_{+\varepsilon}(L_\varepsilon - i)^{-1}J_\varepsilon b\rho_0 = \varepsilon^{-(1+\alpha)}\Lambda(\varepsilon)\eta_0,$$

where $\Lambda(\varepsilon) = p_{+\varepsilon}(L_\varepsilon - i)^{-1}J_\varepsilon b^{1/2}$. To prove the lemma, it suffices to show that

$$g_\varepsilon \to g_0 = i^{3(1-\alpha)/2}\beta_0\tau_1$$

strongly in L^2. We first show that $\|g_\varepsilon\|_{L^2} \to \|g_0\|_{L^2}$. If we make use of the relation $p_{+\varepsilon}^*p_{+\varepsilon} = L_{-\varepsilon} = L_\varepsilon - b_\varepsilon$, then a simple calculation yields

$$\Lambda(\varepsilon)^*\Lambda(\varepsilon) = \left(\varepsilon^2/2\right)(\Pi(\varepsilon) + \Pi(\varepsilon)^*) - \varepsilon^2\Pi(\varepsilon)^*\Pi(\varepsilon),$$

where $\Pi(\varepsilon) = b^{1/2}(L - i\varepsilon^2)^{-1}b^{1/2}$. We apply Proposition 4.1 with $k = i^{1/2}\varepsilon$ to $\Pi(\varepsilon)$. Then

$$\Pi(\varepsilon) = b^{1/2}G_0b^{1/2} + \sum_{l=0,1}\gamma_l(k)i^{-\nu}\varepsilon^{2\nu}\left(b^{1/2}\omega_l \otimes b^{1/2}\omega_l\right) + Op(\varepsilon^2).$$

Since $(\eta_0, b^{1/2}\omega_0)_{L^2} = 1$ and $(\eta_0, b^{1/2}\omega_1)_{L^2} = 0$ by (4.13), we have

$$\Pi(\varepsilon)\eta_0 = \eta_0 + \gamma_0(k)i^{-\alpha}\varepsilon^{2\alpha}b^{1/2}\omega_0 + o_2(\varepsilon^{1+\alpha})$$

and similarly for $\Pi(\varepsilon)^*\eta_0$ with coefficient $\gamma_0(k)i^{-\alpha}$ replaced by $\overline{\gamma_0(k)}i^{\alpha}$. Thus

$$\|g_\varepsilon\|_{L^2}^2 = \varepsilon^{-2(1+\alpha)}\|\Lambda(\varepsilon)\eta_0\|_{L^2}^2 = -\gamma_0\cos(\alpha\pi/2) + o(1).$$

On the other hand, we use the formula

$$\|\tau_1\|_{L^2}^2 = 2\pi\int_0^\infty r|H_{1-\alpha}(i^{1/2}r)|^2\,dr = 2/\sin(\alpha\pi/2)$$

to obtain that

$$\|g_0\|_{L^2}^2 = |\beta_0|^2\|\tau_1\|_{L^2}^2 = 2|\beta_0|^2/\sin(\alpha\pi/2).$$

Since $2|\beta_0|^2/\sin(\alpha\pi/2) = -\gamma_0\cos(\alpha\pi/2)$ by (6.8), it follows that $\|g_\varepsilon\|_{L^2} \to \|g_0\|_{L^2}$. Next we shall show that $g_\varepsilon \to g_0$ weakly. Let $f \in C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. Then $p_{-\varepsilon}f = \pi_-f$ for $\varepsilon > 0$ small enough, and

$$(g_\varepsilon, f)_{L^2} = \varepsilon^{-(1+\alpha)}((L_\varepsilon - i)^{-1}J_\varepsilon b^{1/2}\eta_0, \pi_-f)_{L^2}.$$

We have

$$(L_\varepsilon - i)^{-1}J_\varepsilon b^{1/2}\eta_0 = i^{\alpha/2}\beta_0\varepsilon^{1+\alpha}\tau_0 + o_2(\varepsilon^{1+\alpha})$$

by Proposition 7.1 with $\sigma = i^{1/2}$, where $\tau_0 = H_\alpha(i^{1/2}r)$. Hence it follows that

$$(g_\varepsilon, f)_{L^2} \to i^{\alpha/2}\beta_0(\pi_+\tau_0, f)_{L^2}.$$

By (2.2), we can calculate

$$\pi_+\tau_0 = -i^{3/2}H_{\alpha-1}(i^{1/2}r)e^{i\theta} = -i^{3/2}e^{i(1-\alpha)\pi}\tau_1 = i^{3/2-2\alpha}\tau_1.$$

This shows that $g_\varepsilon \to g_0$ weakly, and the proof is complete. $\square$

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