Random path representation and sharp correlations asymptotics at high-temperatures

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Abstract.

We recently introduced a robust approach to the derivation of sharp asymptotic formula for correlation functions of statistical mechanics models in the high-temperature regime. We describe its application to the nonperturbative proof of Ornstein-Zernike asymptotics of 2-point functions for self-avoiding walks, Bernoulli percolation and ferromagnetic Ising models. We then extend the proof, in the Ising case, to arbitrary odd-odd correlation functions. We discuss the fluctuations of connection paths (invariance principle), and relate the variance of the limiting process to the geometry of the equidecay profiles. Finally, we explain the relation between these results from Statistical Mechanics and their counterparts in Quantum Field Theory.

§1. Introduction

In many situations, various quantities of interest can be represented in terms of path-like structures. This is the case, e.g., of correlations in various lattice systems, either in perturbative regimes (through a suitable expansion), or non-perturbatively, as in the ferromagnetic Ising models at supercritical temperatures. Many important questions about the fine asymptotics of these quantities can be reformulated as local limit theorems for these (essentially) one-dimensional objects. In [7], building upon the earlier works [15, 6], we proposed a robust non-perturbative approach to such a problem. It has already been applied successfully in the case of self-avoiding walks, Bernoulli percolation and Ising models.

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We briefly review the results that have been thus obtained (see also [8] for a short description of the main ideas of the proof).

Self-avoiding walks. A self-avoiding path ω from 0 to $x \neq 0$ is a sequence of distinct sites $t_0 = 0, t_1, t_2, \ldots, t_n = x$ in $\mathbb{Z}^d$, with $|t_i - t_{i-1}| = 1$, $i = 1, \ldots n$ (the restriction to nearest-neighbor jumps can be replaced by arbitrary, possibly weighted, jumps of finite range). Let $\beta < 0$, we are interested in the following quantity:

$$G_\beta^{\mathrm{SAW}}(x) \stackrel{\triangle}{=} \sum_{\omega : 0 \to x} e^{\beta|\omega|} ,$$

where the sum runs over all self-avoiding paths from 0 to x, and $|\omega|$ denotes the length of the path. $G_\beta^{\mathrm{SAW}}(x)$ is finite for all $\beta < \beta_{\mathrm{c}}^{\mathrm{SAW}}$, with $\beta_{\mathrm{c}}^{\mathrm{SAW}} > -\infty$. Actually, $\sum_{x \in \mathbb{Z}^d} G_\beta^{\mathrm{SAW}}(x)$ is finite if and only if $\beta < \beta_{\mathrm{c}}^{\mathrm{SAW}}$.

Bernoulli bond percolation. Let $\beta > 0$. We consider a family of i.i.d. $\{0,1\}$-valued random variables n_e, indexed by the bonds e between two nearest-neighbor sites of $\mathbb{Z}^d$ (again, restriction to nearest-neighbor sites can be dropped); $\mathrm{Prob}_\beta(n(e) = 1) = 1 - e^{-\beta}$. We say that 0 is connected to x $(0 \leftrightarrow x)$ in a realization n of these random variables if there is a self-avoiding path ω from 0 to x such that $n_e = 1$ for all increments e along the path. We are interested in the following quantity:

$$G_\beta^{\mathrm{perc}}(x) \stackrel{\triangle}{=} \mathrm{Prob}_\beta(0 \leftrightarrow x) .$$

The high-temperature region $\beta < \beta_{\mathrm{c}}^{\mathrm{perc}}$ is defined through

$$\beta_{\mathrm{c}}^{\mathrm{perc}} \stackrel{\triangle}{=} \sup\{\beta : \sum_{x \in \mathbb{Z}^d} G_\beta^{\mathrm{perc}}(x) < \infty\} > 0.$$

It is a deep result of [2] that the percolation transition is sharp, i.e.

$$\beta_c = \inf\{\beta : \mathrm{Prob}_\beta(0 \leftrightarrow \infty) > 0\}.$$

Ising model. Let $\beta > 0$. We consider a family of $\{-1,1\}$-valued random variables σ_x, indexed by the sites $x \in \mathbb{Z}^d$. Let $\Lambda_L = \{-L, \ldots, L\}^d$. The probability of a realization σ of the random variables $(\sigma_x)_{x \in \Lambda_L}$, with boundary condition $\overline{\sigma} \in \{-1,1\}^{\mathbb{Z}^d}$, is given by

$$\mu_{\beta,\overline{\sigma},L}(\sigma) \stackrel{\triangle}{=} (Z_{\beta,\overline{\sigma},L})^{-1} \exp\bigl[\beta \sum_{\substack{\{x,y\} \subset \Lambda_L \\ |x-y|=1}} \sigma_x \sigma_y + \beta \sum_{\substack{x \in \Lambda_L, y \notin \Lambda_L \\ |x-y|=1}} \sigma_x \overline{\sigma}_y\bigr] .$$

(As for the two previous models, the nearest-neighbor restriction can be replaced by a – possibly weighted – finite-range assumption.) The set of limiting measures, as $L \to \infty$ and for any boundary conditions, is a simplex, whose extreme elements are the Gibbs states of the model. We define the high-temperature region as $\beta < \beta_{\rm c}^{\rm Ising}$, where

$$\beta_{\rm c}^{\rm Ising} = \sup\{\beta : \text{There is a unique Gibbs state at parameter } \beta\} > 0\,.$$

We are interested in the following quantity:

$$G_\beta^{\rm Ising}(x) \triangleq \mathbb{E}_{\mu_\beta}[\sigma_0 \sigma_x]\,,$$

where the expectation is computed with respect to any translation invariant Gibbs state μ_β (it is independent of which one is chosen). It is a deep result of [3] that the high-temperature region can also be characterized as the set of all β such that

$$\sum_{x\in\mathbb{Z}^d} G_\beta^{\rm Ising}(x) < \infty\,.$$

We now discuss simultaneously these three models; to that end, we simply forget the model-specific superscripts, and simply write $\beta_{\rm c}$ or G_β. It can be shown that for all three models, for all $\beta < \beta_{\rm c}$, the function $G_\beta(x)$ is actually exponentially decreasing in $|x|$, i.e. the corresponding inverse correlation length $\xi_\beta : \mathbb{R}^d \to \mathbb{R}$ satisfy

$$\xi_\beta(x) \triangleq \lim_{k\to\infty} -\frac{1}{k}\log G_\beta(\lfloor kx \rfloor) > 0\,,$$

where $\lfloor x \rfloor$ is the componentwise integer part of x. Obviously, ξ_β is positive-homogeneous, and it is not difficult to prove that it is convex; it is thus an equivalent norm on $\mathbb{R}^d$ (for $\beta < \beta_{\rm c}$).

The main result of [15, 6, 7] is the derivation of the following sharp asymptotics for $G_\beta(x)$, as $|x| \to \infty$, for these three models, in the corresponding high-temperature regions.

Theorem 1.1. *Consider one of the models above, and let $\beta < \beta_{\rm c}$. Then, uniformly as $|x| \to \infty$,*

$$G_\beta(x) = \frac{\Psi_\beta(n_x)}{\sqrt{|x|^{d-1}}}\, e^{-\xi_\beta(n_x)\,|x|}\,(1+o(1))\,,$$

where $n_x = x/|x|$, and Ψ_β is strictly positive and analytic. Moreover, ξ_β is also an analytic function.

As a by-product of the proof of Theorem 1.1, we obtain the following results on the shape of the equidecay profiles,

$$\mathbf{U}_\beta \stackrel{\triangle}{=} \{x \in \mathbb{R}^d \,:\, \xi_\beta(x) \leq 1\}$$

and their polar, the Wulff shapes

$$\mathbf{K}_\beta \stackrel{\triangle}{=} \bigcap_{n \in \mathbb{S}^{d-1}} \{t \in \mathbb{R}^d \,:\, (t,n)_d \leq \xi_\beta(n)\}\,.$$

Theorem 1.2. *Consider one of the models above, and let $\beta < \beta_{\mathrm{c}}$. Then $\mathbf{K}_\beta$ has a locally analytic, strictly convex boundary. Moreover, the Gaussian curvature κ_β of $\mathbf{K}_\beta$ is uniformly positive,*

$$\overline{\kappa}_\beta \stackrel{\triangle}{=} \min_{t \in \partial \mathbf{K}_\beta} \kappa_\beta(t) > 0\,. \tag{1}$$

By duality, $\partial \mathbf{U}_\beta$ is also locally analytic and strictly convex.

Remark 1.3. *In two dimensions $\mathbf{K}_\beta$ is reminiscent of the Wulff shape (and is exactly the low-temperature Wulff shape in the cases of the nearest-neighbor Ising and percolation models). Equation* (1) *is then called the positive stiffness condition; it is known to be equivalent to the following sharp triangle inequality* [14, 20]*: Uniformly in $u, v \in \mathbb{R}^2$*

$$\xi_\beta(u) + \xi_\beta(v) - \xi_\beta(u+v) \geq \overline{\kappa}_\beta \left(|u| + |v| - |u+v|\right)\,.$$

Theorem 1.1 can in fact easily be extended to arbitrary odd-odd correlation functions. We show this here in the most difficult case of ferromagnetic Ising models; namely, we establish exact asymptotic formula for correlation functions of the form $\mathbb{E}_{\mu_\beta}[\sigma_A \sigma_{B+x}]$, where A, B are finite subsets of $\mathbb{Z}^d$ with $|A|$ and $|B|$ odd, and for any $C \subset \mathbb{Z}^d$, $\sigma_C \stackrel{\triangle}{=} \prod_{y \in C} \sigma_y$. Notice that even-odd correlations are necessarily zero by symmetry. The case of even-even correlations is substantially more delicate though (already for the much simpler SAW model), in particular in low dimensions; we hope to come back to this issue in the future.

Theorem 1.4. *Consider the Ising model. Let $\beta < \beta_{\mathrm{c}}^{\mathrm{Ising}}$, and let A and B be finite odd subsets of $\mathbb{Z}^d$. Then, uniformly in $|x| \to \infty$,*

$$\mathbb{E}_{\mu_\beta}[\sigma_A \sigma_{B+x}] = \frac{\Psi_\beta^{A,B}(n_x)}{\sqrt{|x|^{d-1}}}\, e^{-\xi_\beta(n_x)\,|x|}\,(1 + o(1))\,,$$

where $n_x = x/|x|$, and $\Psi_\beta^{A,B}$ is strictly positive and analytic.

We sketch the proof of this theorem in Section 4.

The main feature shared by the three models discussed above is that the function $G_\beta(x)$ can each time be written in the form

$$G_\beta(x) = \sum_{\lambda:\, 0\to x} q_\beta(\lambda)\,, \tag{2}$$

where the sum runs over admissible path-like objects (SAW paths, percolation clusters, random-lines, see Section 3, respectively). The weights $q_\beta(\,\cdot\,)$ are supposed to be strictly positive and to possess a variation of the following four properties:

- **Strict exponential decay of the two-point function:** There exists $C_1 < \infty$ such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$g(x) \;=\; \sum_{\lambda:\, 0\to x} q(\lambda) \le C_1\, e^{-\xi(x)}\,, \tag{3}$$

  where $\xi(x) = -\lim_{k\to\infty}(k)^{-1}\log g(\lfloor kx\rfloor)$ is the inverse correlation length.
- **Finite energy condition:** For any pair of compatible paths λ and η define the conditional weight

$$q(\lambda\,|\,\eta) = q(\lambda \amalg \eta)/q(\eta)$$

  where $\lambda \amalg \eta$ denotes the concatenation of λ and η. Then there exists a universal finite constant $C_2 < \infty$ such that the conditional weights are controlled in terms of path sizes $|\lambda|$ as:

$$q(\lambda\,|\,\eta) \;\ge\; e^{-C_2|\lambda|}\,. \tag{4}$$

- **BK-type splitting property:** There exists $C_3 < \infty$, such that, for all $x, y \in \mathbb{Z}^d \setminus \{0\}$ with $x \neq y$,

$$\sum_{\lambda:\, 0\to x\to y} q(\lambda) \le C_2 \sum_{\lambda:\, 0\to x} q(\lambda) \sum_{\lambda:\, x\to y} q(\lambda)\,. \tag{5}$$

- **Exponential mixing :** There exists $C_4 < \infty$ and $\theta \in (0,1)$ such that, for any four paths λ, η, γ_1 and γ_2, with $\lambda \amalg \eta \amalg \gamma_1$ and $\lambda \amalg \eta \amalg \gamma_2$ both admissible,

$$\frac{q(\lambda\,|\,\eta \amalg \gamma_1)}{q(\lambda\,|\,\eta \amalg \gamma_2)} \le \exp\{C_4 \sum_{\substack{x\in\lambda\\ y\in\gamma_1\cup\gamma_2}} \theta^{|x-y|}\}\,. \tag{6}$$

Many other models enjoy a graphical representation of correlation functions of the form (2). In perturbative regimes, cluster expansions provide a generic example. Non-perturbative examples include the random-cluster representation for Potts (and other) models [10], or random walk representation of N-vector models [11], etc... However, it might not always be easy, or even possible, to establish properties (3), (4), (5) and (6) for the corresponding weights, especially (5) which is probably the less robust one. It should however be possible to weaken the latter so that it only relies on some form of locally uniform mixing properties.

Road-map to the paper

In Section 2 we review and explain our probabilistic approach to the analysis of high temperature correlation functions. The point of departure is the random path representation formula (2), and the whole theory is built upon a study of the local fluctuation structure of the corresponding connection paths. One of the consequences is the validity of the invariance principle under the diffusive scaling, which we formulate in Theorem 2.2 below. For simplicity the discussion in Section 2 is restricted to the case of SAW-s, and hence the underlying local limit results are those about the sums of independent random variables. In the case of high temperature ferromagnetic Ising models the random line representation, which we shall briefly recall in Section 3, gives rise to path weights q_β which do not possess appropriate factorization properties. Nevertheless these weights satisfy conditions (3)-(6) and we conclude Section 3 with an explanation of how the problem of finding correlation asymptotics can be reformulated in terms of local limit properties of one dimensional systems generated by Ruelle operators for full shifts on countable alphabets. The proof of Theorem 1.4 is discussed in Section 4. Finally, in Section 5, we explain the relation between the problems discussed here, inspired by Statistical Physics, and their counterparts originating from the corresponding lattice Quantum Field Theories.

§2. Fluctuations of connection paths

In this section we describe local structure and large scale properties of connection paths conditioned to hit a distant point. In all three models above (SAW, percolation, Ising) the distribution of the connection paths converges, after the appropriate rescaling, to the $(d-1)$-dimensional Brownian bridge, and, from the probabilistic point of view, these results belong to the realm of classical Gaussian local limit analysis of one dimensional systems based on uniform analytic expansions of finite volume log-moment generating functions. An invariance principle for the sub-critical Bernoulli bond percolation has been established in [17] and for the phase separation line in the 2D nearest neighbour Ising model at any $\beta > \beta_c$ in [13]. In both cases the techniques and the ideas of [6] and [7] play the crucial role, and, in fact, the renormalization and the fluctuation analysis developed in the latter papers pertains to a large class of models which admit a random path type representation with path weights enjoying a suitable variation of (3)-(6). In particular, it should lead to a closed form theory of low temperature phase boundaries in two dimensions [16]. Note that different tools have been early employed in [9, 12].

For the sake of simplicity we shall sketch here the case of self-avoiding walks and shall try to stipulate the impact of the geometry of $\mathbf{K}_\beta$ on the magnitude of paths fluctuations in the corresponding directions.

Let $\hat{x} \in \mathbb{S}^{d-1}$ and the dual point $\hat{t} \in \partial\mathbf{K}_\beta$; $(\hat{t}, \hat{x}) = \xi_\beta(\hat{x})$, be fixed for the rest of the section. Consider the set $\mathcal{P}^n$ of all self-avoiding paths $\gamma : 0 \to \lfloor n\hat{x} \rfloor$, where for $y \in \mathbb{R}^d$ we define $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \ldots, \lfloor y_d \rfloor) \in \mathbb{Z}^d$. Finally, consider the following probability measure $\mathbb{P}^n_\beta$ on $\mathcal{P}^n$:

$$\mathbb{P}^n_\beta(\gamma) \;=\; \frac{1}{\mathbf{Z}^n_\beta} \mathrm{e}^{\beta|\gamma|} \mathbb{1}_{\{\gamma \in \mathcal{P}^n\}}. \tag{7}$$

In order to explain and to formulate the invariance principle which holds under $\mathbb{P}^n_\beta$ we need, first of all, to readjust the notion of irreducible splitting of paths $\gamma \in \mathcal{P}^n$;

$$\gamma \;=\; \lambda_L \amalg \lambda_1 \amalg \cdots \amalg \lambda_M \amalg \lambda_R. \tag{8}$$

Fix $\delta \in (0,1)$ and a large enough renormalization scale K. Given a path $\lambda = (u_0, u_1, \ldots, u_m)$ let us say that a point u_l; $0 < l < m$, is $\hat{x}$-correct break point of λ if the following two conditions hold:

- **A)** $(u_j, \hat{x}) < (u_l, \hat{x}) < (u_i, \hat{x})$ for all $j < l < i$.
- **B)** The remaining sub-path $(u_{l+1}, \ldots, u_m)$ lies inside the set

$$2K\mathbf{U}_\beta(u_l) + \mathcal{C}_\delta(\hat{t}),$$

where $\mathbf{U}_\beta(z) = z + \mathbf{U}_\beta$, and the forward cone $\mathcal{C}_\delta(\hat{t})$ is defined as

$$\mathcal{C}_\delta(\hat{t}) \;=\; \left\{ y \in \mathbb{R}^d \,:\, (y, \hat{t}) > (1-\delta)\xi_\beta(y) \right\}. \tag{9}$$

Note that this definition depends on the parameters K and δ; as they are usually kept constant, we only write them explicitly when needed.

With $\hat{x} \in \mathbb{S}^{d-1}$, $\hat{t} \in \partial \mathbf{K}_\beta$, K and δ fixed as above let us say that a path λ is irreducible if it does not contain $\hat{x}$-correct break points. We use $\mathcal{S}$ to denote the set of all irreducible paths (modulo $\mathbb{Z}^d$-shifts). Define also the following three subsets of $\mathcal{S}$:

$$
\begin{aligned}
&\mathcal{S}_L = \{\lambda = (u_0, \dots, u_m) \in \mathcal{S} \, : \, \forall l > 0 \, (u_l, \hat{x}) < (u_m, \hat{x})\} \\
&\mathcal{S}_R = \left\{ \begin{array}{r} \lambda = (u_0, \dots, u_m) \in \mathcal{S} \, : \, \forall l > 0 \, (u_l, \hat{x}) > (u_0, \hat{x}) \ \text{ and} \\ \\ \gamma \subset K\mathbf{U}_\beta(u_0) + \mathcal{C}_\delta(\hat{t}). \end{array} \right\} \qquad (10) \\
&\mathcal{S}_0 = \mathcal{S}_L \cap \mathcal{S}_R
\end{aligned}
$$

For any $\gamma \in \mathcal{P}_n$ which has at least two $\hat{x}$-correct break points the decomposition (8) is unambiguously defined by the following set of conditions:

$$
\lambda_L \in \mathcal{S}_L, \ \lambda_R \in \mathcal{S}_R \text{ and } \lambda_1, \dots, \lambda_M \in \mathcal{S}_0.
$$

The only difference between (8) and the irreducible decomposition employed in [7] is that the break points here are defined with respect to the $\hat{x}$-orthogonal hyper-planes instead of $\hat{t}$-orthogonal hyper-planes. This is to ensure that the displacements along all the λ-paths which appear in (8) have positive projection on the direction of $\hat{x}$. More precisely, given a SAW path $\lambda = (u_0, \dots, u_m)$ let us define the displacement along λ as $V(\lambda) = u_m - u_0$. By the very definition of (8) all

$$
V_L \stackrel{\Delta}{=} V(\lambda_L), \ V_1 \stackrel{\Delta}{=} V(\lambda_1), \dots, \ V_M \stackrel{\Delta}{=} V(\lambda_M), \ V_R \stackrel{\Delta}{=} V(\lambda_R).
$$

belong to the (lattice) half-space $\{y \in \mathbb{Z}^d \, : \, (y, \hat{x}) > 0\}$. The renormalization calculus developed in [6, 7] implies:

Lemma 2.1. *For every $\beta < \beta_c$ and for any $\delta > 0$ there exists a finite scale $K_0 = K_0(\delta, \beta)$ and a number $\nu = \nu(\delta, \beta) > 0$, such that*

$$
\sum_{\lambda \in \mathcal{S} \, : \, V(\lambda) = y} \mathrm{e}^{\beta|\lambda|} \ \le \ \exp\left\{ -(\hat{t}, y) - \nu|y| \right\} , \qquad (11)
$$

uniformly in $y \in \mathbb{Z}^d$.

Going back to the decomposition (8) notice that

$$
V_L + V_1 + \cdots + V_M + V_R = \lfloor n\hat{x} \rfloor \qquad (12)
$$

for any $\gamma : 0 \to \lfloor n\hat{x} \rfloor$. Therefore, Lemma 2.1 and the Ornstein-Zernike formula of Theorem 1.1 yield:

$$\mathbb{P}^n_\beta \left(\max\{|V_L|, |V_1|, \dots, |V_M|, |V_R|\} > (\log n)^2 \right) \; = \; o\left(\frac{1}{n^\rho} \right), \tag{13}$$

for any $\rho > 0$. In particular, if for given $\gamma : 0 \to \lfloor n\hat{x} \rfloor$ one considers the piece-wise constant trajectory $\widehat{\gamma}$ through the vertices $0, V_L, V_L + V_1, \dots, \lfloor n\hat{x} \rfloor$, then the $\mathbb{R}^d$-Hausdorff distance between γ and $\widehat{\gamma}$ is bounded above as:

$$\mathbb{P}^n_\beta \left(\mathrm{d_H}\left(\gamma, \widehat{\gamma} \right) > (\log n)^2 \right) \; = \; o\left(\frac{1}{n^\rho} \right), \tag{14}$$

as well. Indeed, one needs only to control the fluctuation of λ_L in (8), the traversal deviations of paths in $\mathcal{S}_R$ are automatically under control by the cone confinement property (10).

Estimate (14) enables a formulation of the invariance principle for SAW γ in terms of the effective path $\widehat{\gamma}$. In its turn the invariance principle for $\widehat{\gamma}$ is a version of the conditional invariance principle for paths of random walks in $(d-1)$-dimensions with the direction of the target point $\hat{x}$ playing the role of time. It happens to be natural to choose the frame of the remaining $(d-1)$ spatial dimensions according to principal directions of curvature $\mathfrak{v}_1, \dots, \mathfrak{v}_{d-1}$ of $\partial \mathbf{K}_\beta$ at $\hat{t}$. In this way, in view of the positive $\hat{x}$-projection property of all the λ-path displacements in (8), the effective path $\widehat{\gamma} \subset \mathbb{R}^d$ could be parametrized in the orthogonal frame $(\hat{x}, \mathfrak{v}_1, \dots, \mathfrak{v}_{d-1})$ as a function $\widehat{X} : [0, n] \to \mathbb{R}^{d-1}$. As usual define the diffusive scaling $\widehat{X}_n(\cdot)$ of $\widehat{X}(\cdot)$ as

$$\widehat{X}_n(\tau) \; = \; \frac{1}{\sqrt{n}} \widehat{X}(\lfloor n\tau \rfloor).$$

Theorem 2.2. *The distribution of $\widehat{X}_n(\cdot)$ under $\mathbb{P}^n_\beta$ weakly converges in $C_{0,0}[0,1]$ to the distribution of*

$$\left(\sqrt{\kappa_1} B_1(\cdot), \dots, \sqrt{\kappa_{d-1}} B_{d-1}(\cdot) \right), \tag{15}$$

where $B_1(\cdot), \dots, B_{d-1}(\cdot)$ are independent Brownian bridges on $[0,1]$ and $\kappa_1, \dots, \kappa_{d-1}$ are the principal curvatures of $\partial \mathbf{K}_\beta$ at $\hat{t}$.

Let us dwell on the probabilistic picture behind Theorems 1.1, 1.2 and 2.2: First of all, note that by Lemma 2.1

$$\mathbb{Q}_0(y) \; = \; \mathrm{e}^{(\hat{t}, y)} \sum_{\lambda \in \mathcal{S}_0 \, : \, V(\lambda) = y} \mathrm{e}^{\beta |\lambda|} \; \triangleq \; \mathrm{e}^{(\hat{t}, y)} W_0(y) \tag{16}$$

is a (non-lattice) probability distribution on $\mathbb{Z}^d$ with exponentially decaying tails. Indeed, an alternative important way to think about $\mathbf{K}_\beta$ is as of the closure of the domain of convergence of the series

$$t \in \mathbb{R}^d \;\mapsto\; \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{(t,y)} G_\beta(y). \tag{17}$$

On the other hand, Lemma 2.1 ensures that the series $\mathbb{W}_0(t) \stackrel{\Delta}{=} \sum \mathrm{e}^{(t,y)} W_0(y)$ converges in the ν-neighbourhood $B_\nu(\hat{t}) = \{t \,:\, |t-\hat{t}| < \nu\}$ of $\hat{t}$. In view of the decomposition (8) and Lemma 2.1,

$$G_\beta(n\hat{x}) \;=\; O\left(\mathrm{e}^{-n\xi_\beta(\hat{x})-\nu n}\right) \;+\; \sum_{M=1}^{\infty} W_L * W_0^{*M} * W_R(n\hat{x}), \tag{18}$$

where we have assumed for the convenience of notation that $n\hat{x} \in \mathbb{Z}^d$, and

$$W_L(y) = \sum_{\lambda\in\mathcal{S}_L\,:\,V(\lambda)=y} \mathrm{e}^{\beta|\lambda|} \quad \text{and} \quad W_R(y) = \sum_{\lambda\in\mathcal{S}_R\,:\,V(\lambda)=y} \mathrm{e}^{\beta|\lambda|}.$$

As a result, the piece of the boundary $\partial\mathbf{K}_\beta$ inside $B_\nu(\hat{t})$ is implicitly given by

$$\partial\mathbf{K}_\beta \cap B_\nu(\hat{t}) \;=\; \{t \in B_\nu(\hat{t}) \,:\, \mathbb{W}_0(t) = 1\}.$$

In order to obtain the full claim of Theorem 1.2 one needs only to check the non-degeneracy of $\mathrm{Hess}(\mathbb{W}_0)$ at $\hat{t}$, which, in the case of SAW-s, is a direct consequence of the finite energy condition (4). Note, by the way, that since $\hat{x}$ is the normal direction to $\partial\mathbf{K}_\beta$ at $\hat{t}$, there exists a number $\alpha \in (0,\infty)$, such that

$$\nabla\mathbb{W}_0(\hat{t}) \;=\; \alpha\hat{x}. \tag{19}$$

Multiplying both sides of (18) by $\mathrm{e}^{n\xi_\beta(\hat{x})} = \mathrm{e}^{(\hat{t},n\hat{x})}$ we arrive to the following key representation of the two point function G_β:

(20)

$$\begin{aligned}\mathrm{e}^{n\xi_\beta(\hat{x})} G_\beta(n\hat{x}) \;&=\; \mathrm{e}^{n(\hat{t},\hat{x})} G_\beta(n\hat{x}) \;=\; O\left(\mathrm{e}^{-n\nu}\right) \\ &+ \sum_{v_L,v_R\in\mathbb{Z}^d} \mathbb{Q}_L(v_L)\mathbb{Q}_R(v_R) \sum_{M=1}^{\infty} \mathbb{Q}_0(V_1 + \cdots + V_M = n\hat{x} - v_L - v_R),\end{aligned}$$

where, similar to (16), we have defined $\mathbb{Q}_L(v) = \mathrm{e}^{(\hat{t},v)} W_L(v)$ and, accordingly, $\mathbb{Q}_R(v) = \mathrm{e}^{(\hat{t},v)} W_R(v)$.

Unlike $\mathbb{Q}_0$ the measures $\mathbb{Q}_L$ and $\mathbb{Q}_R$ are in general not probability but, by Lemma 2.1, they are finite and have exponentially decaying tails:

$$\sum_{|y|>n} \left(\mathbb{Q}_L(y) + \mathbb{Q}_R(y)\right) \le \mathrm{e}^{-\nu n/2}.$$

Since by (19) the expectation of V_l under $\mathbb{Q}_0$ equals to $\alpha\hat{x}$, the usual local limit CLT for $\mathbb{Z}^d$ random variables and the Gaussian summation formula imply that the right hand side in (20) equals to $c_1/\sqrt{n^{d-1}}$. Actually, a slightly more careful analysis along these line leads to the full analytic form of the Ornstein-Zernike formula as claimed in Theorem 1.1.

Let us explain now how the principal curvatures $\kappa_1, \dots, \kappa_{d-1}$ of $\partial\mathbf{K}_\beta$ at $\hat{t}$ enter the picture: By the irreducible path representation and arguments completely similar to those just reproduced above, the total weight of all piece-wise constant paths $\widehat{\gamma} = \widehat{\gamma}(V_L, V_1, \dots, V_M, V_R)$; $M = 1, 2, \dots$, which pass through a point $v_n \in \mathbb{Z}^d$;

$$v_n = \lambda n\hat{x} + \sqrt{n}\sum_{l=1}^{d-1} a_l \mathfrak{v}_l \triangleq \lambda n\hat{x} + \sqrt{n}\mathfrak{v},$$

equals to

$$\frac{c_2}{\sqrt{\left(\lambda(1-\lambda)n^2\right)^{(d-1)}}} \mathrm{e}^{-\xi_\beta(v_n) - \xi_\beta(n\hat{x}-v_n)} \left(1 + o(1)\right),$$

where $c_2 > 0$ does not depend on $\lambda \in (0,1)$ and the coefficients $a_1, \dots, a_{d-1}$. Comparing with the OZ formula for the full partition function G_β, we infer that

$$\mathbb{P}^n_\beta\left(v_n \in \widehat{\gamma}\right) = \frac{c_3 \exp\left\{-\left(\xi_\beta(v_n) + \xi_\beta(n\hat{x}-v_n) - \xi_\beta(n\hat{x})\right)\right\}}{\sqrt{(\lambda(1-\lambda)n)^{d-1}}} \left(1 + o(1)\right).$$

From now on we refer to Chapter 2.5 in [22] for the missing details in the arguments below. ξ_β is the support function of $\mathbf{K}_\beta$ and by Theorem 1.2 it is a smooth function. Thus, for every $v \in \mathbb{R}^d$ the gradient $\nabla\xi_\beta(v) \in \partial\mathbf{K}_\beta$ and $\xi_\beta(v) = (\nabla\xi_\beta(v), v)$ (in particular $\hat{t} = \nabla\xi_\beta(\hat{x}) = \nabla\xi_\beta(n\hat{x})$). Principal radii of the curvature $1/\kappa_1, \dots, 1/\kappa_{d-1}$ of $\partial\mathbf{K}_\beta$ at $\hat{t}$ are the eigenvalues of the linear map

$$\mathrm{d}^2\xi_\beta|_{\hat{x}} : T_{\hat{x}}\mathbb{S}^{d-1} \mapsto T_{\hat{x}}\mathbb{S}^{d-1},$$

and $\mathfrak{v}_1, \dots, \mathfrak{v}_{d-1} \in T_{\hat{x}}\mathbb{S}^{d-1}$ are the corresponding eigenvectors. Therefore,

$$\begin{aligned}
\xi_\beta(v_n) + \xi_\beta(n\hat{x} - v_n) - \xi_\beta(n\hat{x}) \;&=\; n\lambda \left(\xi_\beta \left(\hat{x} + \frac{1}{\lambda\sqrt{n}}\mathfrak{v} \right) - \xi_\beta(\hat{x}) \right) \\
&\quad + n(1-\lambda) \left(\xi_\beta \left(\hat{x} - \frac{1}{(1-\lambda)\sqrt{n}}\mathfrak{v} \right) - \xi_\beta(\hat{x}) \right) \\
&= \frac{1}{2\lambda} \left(\mathrm{d}^2\xi_\beta|_{\hat{x}}\mathfrak{v}, \mathfrak{v} \right) \;+\; \frac{1}{2(1-\lambda)} \left(\mathrm{d}^2\xi_\beta|_{\hat{x}}\mathfrak{v}, \mathfrak{v} \right) \;+\; O\left(\frac{1}{\sqrt{n}} \right) \\
&= \frac{1}{2\lambda(1-\lambda)} \sum_{l=1}^{d-1} \frac{a_l^2}{\kappa_l} \;+\; O\left(\frac{1}{\sqrt{n}} \right).
\end{aligned}$$

Computations for higher order finite dimensional distributions follow a completely similar pattern.

§3. Random-line representation of Ising correlations

Correlation functions of ferromagnetic Ising models admit a very useful representation in terms of sums over weighted random paths, which is especially convenient for our purposes here. The two-point function formula (2) is a particular case. In this section, we recall how this representation is derived; we refer to [20] for details and additional results. In the end of the section we shall briefly indicate how (20) and, accordingly, the whole local limit analysis should be re-adjusted in order to incorporate the (dependent) case of Ising paths.

Although we use it for the infinite-volume Gibbs measure, it is convenient to derive the random path representation first for finite volumes, and then take the limit. As there is a single Gibbs state for the values of β we consider, it suffices to consider free boundary conditions (i.e. no interactions between spins inside the box and spins outside).

Given a set of edges B of the lattice $\mathbb{Z}^d$, we define the associated set of vertices as $V_B \stackrel{\triangle}{=} \{x \in \mathbb{Z}^d \,:\, \exists\, e \in B \text{ with } x \in e\}$ ($x \in e$ means that x is an endpoint of e). For any vertex $x \in V_B$, we define the *index* of x in B by $\mathrm{ind}(x, B) \stackrel{\triangle}{=} \sum_{e\in B} \mathbf{1}_{\{e\ni x\}}$. The *boundary* of B is defined by $\partial B \stackrel{\triangle}{=} \{x \in V_B \,:\, \mathrm{ind}(x, B) \text{ is odd}\}$.

In this context, the finite volume Gibbs measure is defined by

$$\mu_{B,\beta}(\sigma) \stackrel{\triangle}{=} Z_\beta(B)^{-1} \exp[-\beta \sum_{e=(x,y)\in B} \sigma_x\sigma_y]\,,$$

and we use the standard notation $\langle \cdot \rangle_{B,\beta}$ to denote expectation w.r.t. this probability measure.

We fix an arbitrary total ordering of $\mathbb{Z}^d$. At each $x \in \mathbb{Z}^d$, we fix (in an arbitrary way) an ordering of the x-incident edges of the graph:

$$B(x) \triangleq \{e \in B \,:\, \mathrm{ind}(x, \{e\}) > 0\} = \{e_1^x \ldots, e_{\mathrm{ind}(x,B)}^x\},$$

and for two incident edges $e = e_i \in B(x)$, $e' = e_j \in B(x)$ we say that $e \leq e'$ if the corresponding inequality holds for their sub-indices; $i \leq j$.

Let $A \subset V_B$ be such that $|A|$ is even; we write $\sigma_A \triangleq \prod_{i \in A} \sigma_i$. Using the identity $e^{\beta\sigma_x\sigma_y} = \cosh(\beta)\big(1+\sigma_x\sigma_y \tanh(\beta)\big)$, we obtain the following expression for the correlation function $\langle \sigma_A \rangle_{B,\beta}$,

$$\langle \sigma_A \rangle_{B,\beta} = Z_\beta(B)^{-1} \sum_{\substack{D \subset B \\ \partial D = A}} \prod_{e \in D} \tanh \beta\,,$$

where

$$Z_\beta(B) \triangleq \sum_{\substack{D \subset B \\ \partial D = \emptyset}} \prod_{e \in D} \tanh \beta\,.$$

From $D \subset B$ with $\partial D = A$, we would like to extract a family of $|A|/2$ “self-avoiding paths” connecting pairs of sites of A. We apply the following algorithm:

STEP 0 Set $k = 1$ and $\Delta = \emptyset$.

STEP 1 Set $z_0^{(k)}$ to be the first site of A in the ordering of $\mathbb{Z}^d$ fixed above, $j = 0$, and update $A \triangleq A \setminus \{z_0^{(k)}\}$.

STEP 2 Let $e_j^{(k)} = (z_j^{(k)}, z_{j+1}^{(k)})$ be the first edge in $B(z_j^{(k)}) \setminus \Delta$ (in the ordering of $B(z_j^{(k)})$ fixed above) such that $e_j^{(k)} \in D$. This defines $z_{j+1}^{(k)}$.

STEP 3 Update $\Delta \triangleq \Delta \cup \{e \in B(z_j^{(k)}) \,:\, e \leq e_j^{(k)}\}$. If $z_{j+1}^{(k)} \in A$, then go to STEP 4. Otherwise update $j \triangleq j + 1$ and return to STEP 2.

STEP 4 Set $n^{(k)} = j + 1$ and stop the construction of this path. Update $A \triangleq A \setminus \{z_{j+1}^{(k)}\}$, $k \triangleq k + 1$ and go to STEP 1.

This procedure produces a sequence $(z_0^{(1)}, \ldots, z_{n^{(1)}}^{(1)}, z_0^{(2)}, \ldots, z_{n^{(|A|/2)}}^{(|A|/2)})$. Let $\bar{i} \triangleq |A|/2 + 1 - i$, and set $w_k^{(i)} \triangleq z_{n-k}^{(\bar{i})}$.

We, thus, constructed $|A|/2$ paths, γ_i, $i = 1, \dots, |A|/2$, given by[1]

$$\gamma_i \triangleq \gamma_i(D) \triangleq (w_0^{(i)}, \dots, w_{n^{(i)}}^{(i)})$$

connecting distinct pairs of points of A, and such that

- $(w_k^{(i)}, w_{k+1}^{(i)}) \in B$, $k = 0, \dots, n^{(i)} - 1$, $i = 1, \dots, |A|/2$
- $(z_k^{(i)}, z_{k+1}^{(i)}) \neq (z_l^{(j)}, z_{l+1}^{(j)})$ if $i \neq j$, or if $i = j$ but $k \neq l$.

(but $z_k^{(i)} = z_l^{(j)}$ is allowed). A family of contours $\underline{\gamma} = (\gamma_1, \dots, \gamma_{|A|/2})$ is (A, B)-admissible if it can be obtained from a set $D \subset B$ with $\partial D = A$, using this algorithm; in that case we write $\underline{\gamma} \sim (A, B)$. Notice that here the order of the paths is important: if γ_k is a path from x_k to y_k then we must have $y_1 > y_2 > \dots > y_{|A|/2}$. This is to ensure that we do not count twice the same configuration of paths.

The construction also yields a set of edges $\Delta(\underline{\gamma}) \triangleq \Delta$. Observe that $\Delta(\underline{\gamma})$ is entirely determined by $\underline{\gamma}$ (and the order chosen for the sites and edges). In particular the sets $D \subset B$ giving rise to an (A, B)-admissible family $\underline{\gamma}$ are characterized by $\partial D = A$ and

$$D \cap \Delta(\underline{\gamma}) = \bigcup_{i=1}^{|A|/2} \gamma_i \,.$$

Therefore, for such sets, $\partial(D \setminus \Delta(\underline{\gamma})) = \emptyset$, and we can write

$$\langle \sigma_A \rangle_{\beta,B} = \sum_{\underline{\gamma} \sim (A,B)} q_{\beta,B}(\underline{\gamma}) \,,$$

where

$$q_{\beta,B}(\underline{\gamma}) = w(\underline{\gamma}) \, \frac{Z_\beta(B \setminus \Delta(\underline{\gamma}))}{Z_\beta(B)} \,,$$

with

$$w(\underline{\gamma}) = \prod_{i=1}^{|A|/2} \prod_{k=1}^{n^{(i)}} \tanh \beta \,.$$

This is an instance of the *random-line representation* for correlation functions of the Ising model in B. It has been studied in detail in [19, 20] and is essentially equivalent (though the derivations are quite different) to the random-walk representation of [1]. We'll need a version

[1] This backward construction of the lines turns out to be convenient for the reformulation in terms of Ruelle's formalism, see [7].

of this representation when B is replaced by the set $\mathcal{E}(\mathbb{Z}^d)$ of all edges of $\mathbb{Z}^d$. To this end, we use the following result ([20], Lemmas 6.3 and 6.9): For all $\beta < \beta_{\mathrm{c}}$,

$$\langle \sigma_A \rangle_\beta = \sum_{\underline{\gamma} \sim A} q_\beta(\underline{\gamma}) \,, \tag{21}$$

where $q_\beta(\underline{\gamma}) \stackrel{\triangle}{=} \lim_{B_n \nearrow \mathcal{E}(\mathbb{Z}^d)} q_{\beta,B_n}(\underline{\gamma})$ is well defined.

It will also be useful to work with a more relaxed definition of admissibility, since we want to cut our paths into pieces, and the order of the resulting pieces might not correspond with the order of their endpoints. In general, given a path $\gamma = (x_1, x_2, \ldots x_n)$, we define $\Delta(\gamma) = \bigcup_{k=1}^n \{e \in B(x_k) \,:\, e \leq (x_{k-1}, x_k)\}$. We say that a path $\gamma = (x_1, \ldots, x_n)$ is admissible if $\{(x_1, x_2), \ldots, (x_{k-1} x_k)\} \cap \Delta((x_k, \ldots, x_n)) = \emptyset$ for all $2 \leq k \leq n-1$. Given a family of paths $\underline{\gamma} = (\gamma_1, \ldots, \gamma_n)$, we define $\Delta(\underline{\gamma}) = \bigcup_{k=1}^n \Delta(\gamma_k)$. A family of admissible paths $\underline{\gamma}$ is then admissible if $(\gamma_1, \ldots, \gamma_k) \cap \Delta((\gamma_{k+1}, \ldots, \gamma_n)) = \emptyset$ for all $1 \leq k \leq n-1$. Notice that the order of the paths is still important ((γ_1, γ_2) can be admissible while (γ_2, γ_1) is not), but there are no constraint on the order of their endpoints. Indeed, they can even share endpoints. Observe that these definitions are identical to those above when restricted to the same setting.

We then have the following crucial inequality: Let $\underline{\gamma}$ be an admissible family of paths. Then

$$\sum_{\substack{\gamma_0 : x \to y \\ \gamma_0 \cap \Delta(\underline{\gamma}) = \emptyset}} q_\beta(\gamma_0, \underline{\gamma}) \leq q_\beta(\underline{\gamma}) \sum_{\gamma_0 : x \to y} q_\beta(\gamma_0) \,. \tag{22}$$

We give a brief proof. It is enough to consider the analogous statement in finite volumes B. Since $\Delta(\gamma_0, \underline{\gamma}) = \Delta(\gamma_0) \cup \Delta(\underline{\gamma})$ and $\gamma_0 \cap \Delta(\underline{\gamma}) = \emptyset$, we have

$$q_{\beta,B}(\gamma_0, \underline{\gamma}) = q_{\beta, B \setminus \Delta(\underline{\gamma})}(\gamma_0) \, q_{\beta,B}(\underline{\gamma}) \,.$$

Hence (22) follows simply from Griffiths' second inequality since

$$\sum_{\gamma_0 \subset B \setminus \Delta(\underline{\gamma})} q_{\beta, B \setminus \Delta(\underline{\gamma})}(\gamma_0) = \langle \sigma_x \sigma_y \rangle_{\beta, B \setminus \Delta(\underline{\gamma})} \leq \langle \sigma_x \sigma_y \rangle_\beta \,.$$

If the set A in (21) contains only two points, $A = \{x, y\}$, then we recover (2). The main difference between the SAW case considered in

Section 2 and the case of sub-critical ferromagnetic Ising models is that the path weights q_β in (2) do not factorize: In general,

$$q_\beta(\gamma \amalg \lambda) \neq q_\beta(\gamma) q_\beta(\lambda).$$

Consequently, the displacement variables $V_1, V_2, \dots$ fail to be independent and the underlying local limit analysis should be generalized. The appropriate framework is that of the statistical mechanics of one dimensional systems generated by Ruelle operators for full shifts on countable alphabets. We refer to [7] for all the background material and here only sketch how the construction leads to the claims of Theorems 1.1, 1.2 and, after an appropriate re-definition of the measures $\mathbb{P}^n_\beta$, to the invariance principle stated in Theorem 2.2: As in the case of SAW-s fix a direction $\hat{x} \in \mathbb{S}^{d-1}$. The key renormalization result (Theorem 2.3 in [7]) which implies that the rate of decay of the irreducible connections is strictly larger than the rate of decay of the two point function G_β. In view of (8) this validates a representation of G_β as a sum of dependent random variables $V_L + V_1 + \cdots + V_M + V_R$ with exponentially decaying tails. Namely, as in Section 2 let $\mathcal{S} = \mathcal{S}(K)$ be the set of all $\hat{x}$-irreducible paths. Then the following Ising analog of Lemma 2.1 holds:

Lemma 3.1. *For every $\beta < \beta_c$ and for any $\delta > 0$ there exists a finite scale $K_0 = K_0(\delta, \beta)$ and a number $\nu = \nu(\delta, \beta) > 0$, such that*

$$\sum_{\lambda \in \mathcal{S}\,:\,V(\lambda) = y} q_\beta(\lambda) \ \leq\ \exp\left\{-(\hat{t}, y) - \nu|y|\right\}, \tag{23}$$

uniformly in $y \in \mathbb{Z}^d$.

The above Lemma suggests that the main contribution to the sharp asymptotics of G_β comes from the weights of the paths $\lambda_1, \dots, \lambda_M$ in the decomposition (8). Accordingly, consider now the set $\mathcal{S}_0$ of cylindrical $\hat{x}$-irreducible paths which was introduced in Section 2. Given a finite collection $\lambda, \lambda_1, \dots, \lambda_M \in \mathcal{S}_0$ define the conditional weight

$$q_\beta(\lambda \,|\, \underline{\lambda}) \ =\ q_\beta(\lambda \,|\, \lambda_1 \amalg \cdots \amalg \lambda_M) \ =\ \frac{q_\beta(\lambda \amalg \lambda_1 \amalg \cdots \amalg \lambda_M)}{q_\beta(\lambda_1 \amalg \cdots \amalg \lambda_M)}.$$

By the crucial exponential mixing property (6) one is able to control the dependence of the conditional weights $q_\beta(\lambda \,|\, \underline{\lambda})$ on λ_M as follows:

$$\sup_{\lambda, \lambda_1, \dots, \lambda_{M-1} \in \mathcal{S}_0} \ \sup_{\lambda_M, \tilde{\lambda}_M \in \mathcal{S}_0} \frac{q_\beta(\lambda \,|\, \lambda_1 \amalg \cdots \amalg \lambda_M)}{q_\beta(\lambda \,|\, \lambda_1 \amalg \cdots \amalg \tilde{\lambda}_M)} \leq \mathrm{e}^{c_1 \theta^M}. \tag{24}$$

In our formalism the set $\mathcal{S}_0$ plays the role of a countable alphabet. The estimate (24) enables the extension of the conditional weights $q_\beta(\lambda \,|\, \underline{\lambda})$ to the case of infinite strings $\underline{\lambda} = (\lambda_1, \lambda_2, \dots)$. Let $\mathfrak{S}_{0,\theta}$ be the set of all such strings endowed with the metrics

$$\mathrm{d}_\theta(\underline{\lambda}, \tilde{\underline{\lambda}}) \;=\; \theta^{\inf\{k:\lambda_k \neq \tilde{\lambda}_k\}}.$$

and let $\mathfrak{F}_{0,\theta}$ be the set of all bounded Lipschitz continuous functions on $\mathfrak{S}_{0,\theta}$.

As before we choose $\hat{t} \in \partial\mathbf{K}_\beta$ to be the dual direction to $\hat{x}$. Given a path $\lambda \in \mathcal{S}_0$ and a string $\underline{\lambda} \in \mathfrak{S}_{0,\theta}$ define the potential

$$\psi_\beta(\lambda \,|\, \underline{\lambda}) \;=\; \log q_\beta(\lambda \,|\, \underline{\lambda}) + \big(\hat{t}, V(\lambda)\big)\,.$$

By (6) and Lemma 3.1 the operator

$$\mathcal{L}_z f(\underline{\lambda}) \;=\; \sum_{\lambda \in \mathcal{S}_0} \mathrm{e}^{\psi_\beta(\lambda \,|\, \underline{\lambda}) + (z, V(\lambda))} f(\lambda \,\mathrm{II}\, \underline{\lambda}), \tag{25}$$

is well defined and bounded on $\mathfrak{F}_{0,\theta}$ for every $z \in \mathbb{C}^d$ with $|z| < \nu$.

The dependent Ising analog of (20) is then given (see Section 3 of [7]) by

$$\begin{aligned} \mathrm{e}^{n\xi_\beta(\hat{x})} G_\beta(n\hat{x}) \;&=\; O\left(\mathrm{e}^{-n\nu}\right) \\ &+ \sum_{\mu \in \mathcal{S}_L} \sum_{\eta \in \mathcal{S}_R} q_\beta(\mu) q_\beta(\eta) \sum_{M=1}^{\infty} \mathbb{Q}_{0,M}^{\mu,\eta}(n\hat{x} - v_L - v_R). \end{aligned} \tag{26}$$

For each $M = 1, 2, \dots$ the family of weights $\{\mathbb{Q}_{0,M}^{\mu,\eta}\}$ is related to the family of operators $\{\mathcal{L}_z\}$ via the Fourier transform:

$$\sum_{y \in \mathbb{Z}^d} \mathrm{e}^{(z,y)} \mathbb{Q}_{0,M}^{\mu,\eta}(y) \;=\; \mathcal{L}_z^M w_{\mu,\eta}, \tag{27}$$

where the family $\{w_{\mu,\eta}\}$ is uniformly positive and uniformly bounded in $\mathfrak{F}_{0,\theta}$. In this way the analytic perturbation theory of the leading (that is lying on the spectral circle) eigenvalue of $\mathcal{L}_z$ enables the expansion of the logarithm of the right hand side in (26) which, in its turn, leads to classical Gaussian local limit results for the dependent sums $V_1 + \cdots + V_M$.

§4. Asymptotics of odd-odd correlations

In this section, we sketch the proof of Theorem 1.4. We do not give a complete, self-contained argument, since this would be too long, and would involve many repetitions from [7]. Instead, we provide the only required update as compared to the proof for 2-point functions given in the latter work. As such, this section should be considered as a complement, and we shall give exact references to the formulas in [7] whenever required.

As explained in Section 3, the correlation function $\langle \sigma_A \sigma_{B+x} \rangle_\beta$ admits a random-line representation of the form

$$\langle \sigma_A \sigma_{B+x} \rangle_\beta = \sum_{\underline{\gamma} \sim A \cup (B+x)} q_\beta(\underline{\gamma}) \,,$$

where $\underline{\gamma}$ runs over families of compatible open contours connecting all the sites of $A \cup (B+x)$. Among the $\frac{1}{2}(|A|+|B|)$ paths of $\underline{\gamma}$, at least one must connect a site of A to a site of $B+x$. We first show that one can ignore the contribution of $\underline{\gamma}$ with more than one such connection (i.e. at least three of them). The first observation is that we have the following lower bound on the correlation function: By the second Griffiths' inequality,

$$\langle \sigma_A \sigma_{B+x} \rangle_\beta \geq \langle \sigma_{A \setminus \{y\}} \rangle_\beta \, \langle \sigma_{B \setminus \{z\}} \rangle_\beta \, \langle \sigma_y \sigma_{z+x} \rangle_\beta \,,$$

where y and z are arbitrarily chosen sites of A and B respectively. Another application of the second Griffiths' inequality implies that

$$\langle \sigma_{A \setminus \{y\}} \rangle_\beta \, \langle \sigma_{B \setminus \{z\}} \rangle_\beta > 0 \,.$$

Moreover, we already know that

$$\langle \sigma_y \sigma_{z+x} \rangle_\beta = \Psi_\beta(n_x) \, |x|^{-(d-1)/2} \, e^{-\xi_\beta(x)} \, (1 + o(1)) \,.$$

But, applying (22), we obtain immediately that the contribution of families of paths $\underline{\gamma}$ with three or more connections between A and $B+x$ is bounded above by $C(A,B) \, e^{-3\xi_\beta(x)}$ and is therefore negligible.

We can henceforth safely assume that there is a single connection between A and $B+x$; we denote the corresponding path by γ, while the remaining paths are denoted by $\underline{\gamma}_A$ and $\underline{\gamma}_B$. We want to show that we can repeat the argument used for the two-point function in [7] in this more general setting. This is indeed quite reasonable since the paths in $\underline{\gamma}_A$ and $\underline{\gamma}_B$ should remain localized, and therefore the picture is still that of a single very long path as for 2-point functions. The main point is thus to prove sufficiently strong localization properties for the paths $\underline{\gamma}_A$ and $\underline{\gamma}_B$, so as to ensure that an appropriate version Lemma 3.1 (see also Theorem 2.3 of [7]) still holds. The import of the latter lemma was to assert nice decay and decoupling properties of the integrated weights of the irreducible pieces in the decomposition of connection paths (8). Notice first that exactly the same decomposition can still be used here, provided we attach the paths in $\underline{\gamma}_A$ and $\underline{\gamma}_B$ to the corresponding leftmost and rightmost extremal pieces λ_L and λ_R, and keep the remaining intermediate cylindrical irreducible pieces unchanged. Apart from the compatibility requirements one then has to check that $\underline{\gamma}_B$ stays inside the forward cone containing λ_R, so that the crucial estimate (3.9) in [7] remains valid.

Let $y \in A$, $z \in B + x$, $\gamma : y \to z$, and let $\underline{\gamma}_A$, resp. $\underline{\gamma}_B$, denote the collections of remaining paths connecting pairs of sites in $A \setminus \{y\}$, respectively $x + B \setminus \{z\}$. For given collections $\underline{\gamma}_A$ and $\underline{\gamma}_B$ we define the irreducible decomposition of γ in precisely the same way as in (8), except for the extremal pieces $\lambda_L = (u_0^L, \dots, u_m^L)$ and $\lambda_R = (u_0^R, \dots, u_n^R)$, which have to satisfy the following modified set of conditions:

- $(u_k^L, \hat{x}) < (u_m^L, \hat{x}) \ \forall\, k = 0, \dots, m-1$
- $(u_k^R, \hat{x}) > (u_0^R, \hat{x}) \ \forall\, k = 1, \dots, n$
- $\underline{\gamma}_A$ must belong to the same $\hat{x}$-halfspace as λ_L and for any $\hat{x}$-break point u_k^L of λ_L the $\hat{x}$-orthogonal hyperplane through u_k^L intersects $\underline{\gamma}_A$.
- $\underline{\gamma}_B$ must belong to the same $\hat{x}$-halfspace as λ_R and for any $\hat{x}$-break point u_k^R of λ_R the $\hat{x}$-orthogonal hyperplane through u_k^R intersects $\underline{\gamma}_B$.
- $\underline{\gamma}_B$ must belong to $2K\mathbf{U}_\beta(u_0^R) + \mathcal{C}_\delta(t)$ (see (9)).

With a slight ambiguity of notation let us call compatible pairs $(\underline{\gamma}_A, \lambda_L)$ and $(\underline{\gamma}_B, \lambda_R)$ $\hat{x}$-irreducible if they satisfy all the conditions above. We then only have to check that

$$\sum_{\substack{(\underline{\gamma}_A, \lambda_L)\ \hat{x}\text{-irreducible} \\ \lambda_L : y \to u}} q_\beta(\underline{\gamma}_A, \lambda) \le e^{-(\hat{t}, u-y) - \nu|u-y|}\,,$$

$$\sum_{\substack{(\underline{\gamma}_B, \lambda_R)\ \hat{x}\text{-irreducible} \\ \lambda_R : u \to z}} q_\beta(\underline{\gamma}_B, \lambda) \le e^{-(\hat{t}, z-u) - \nu|z-u|}$$

for some $\nu > 0$ and any $u \in \mathbb{Z}^d$. We only check the second statement since it is the most complicated one. Fix a large enough scale K. A site u of γ is a $(\hat{x}, \gamma_B, \delta)$-admissible break point if it is a $\hat{x}$-break point of γ and, in addition,

$$\underline{\gamma}_B \subset 2K\mathbf{U}_\beta(u) + \mathcal{C}_\delta(\hat{t})\,.$$

Lemma 4.1. *Fix a forward cone parameter $\delta \in (0, 1/4)$ and a set $B = \{y_1, z_1, \dots, y_n, z_n\}$; $B \subset \mathbb{Z}^d \setminus \{0\}$. There exist a renormalization scale K_0 and positive numbers $\epsilon = \epsilon(\delta, \beta)$, $\nu = \nu(\delta, \beta)$ and $N = N(\beta) < \infty$, such that for all $K \geq K_0$, the upper bound*

$$\sum_{\substack{\lambda : -x \to 0 \\ \partial \underline{\gamma}_B}} q_\beta(\lambda, \underline{\gamma}_B) \mathbf{1}_{\{\lambda \text{ has no } (\hat{x}, \underline{\gamma}_B)\text{-admissible break points}\}} \leq N \mathrm{e}^{-(t,x)_d - \nu |x|} ,$$

holds uniformly in the dual directions $t \in \partial \mathbf{K}_\beta$ and in the starting points $x \in \mathbb{Z}^d$. In the first sum $\underline{\gamma}_B, \lambda$ runs over all admissible family of paths such that $\lambda : -x \to 0$, while $\underline{\gamma}_B = (\gamma_1, \dots, \gamma_n)$ satisfies $\gamma_k : y_k \to z_k$.

Proof. Applying (22), we can assume that $x \in \mathcal{C}'_\nu(t)$, see the remark after Theorem 2.3 of [7]. Let $Q_C(x) = \{v \in \mathbb{Z}^d : |v| \leq |x|/C\}$ where C is some large enough constant. To simplify notations, we suppose that $B = \{y, z\}$, i.e. that $\underline{\gamma}_B \equiv \gamma : y \to z$. The general case is treated in the same way.

We first show that, typically, $\gamma \subset Q_C(x)$. Indeed, using again (22), we have that

$$\begin{aligned}
\sum_{\substack{(\lambda,\gamma) \\ \lambda : -x \to 0 \\ \gamma : y \to z}} q_\beta(\lambda, \gamma) \mathbf{1}_{\{\gamma \not\subset Q_C(x)\}} &\leq \sum_{u \in \partial Q_C(x)} \sum_{\substack{(\lambda,\gamma_1,\gamma_2) \\ \lambda : -x \to 0 \\ \gamma_1 : y \to u,\, \gamma_2 : u \to z}} q_\beta(\lambda, \gamma_1, \gamma) \\
&\leq \sum_{u \in \partial Q_C(x)} \langle \sigma_{-x} \sigma_0 \rangle_\beta \, \langle \sigma_y \sigma_u \rangle_\beta \, \langle \sigma_u \sigma_v \rangle_\beta \\
&\leq \frac{c_d |x|}{C} \, \mathrm{e}^{-c|x|} \, e^{-\xi_\beta(x)} .
\end{aligned}$$

We can therefore suppose that $\gamma \subset Q_C(x)$. Observe now that in the latter case

$$\begin{aligned}
&\{\lambda \text{ has no } (x, \gamma, 2\delta)\text{-break points}\} \\
&\qquad \subset \{\lambda \text{ has no } x\text{-break point } u \text{ with } (t, u) \leq -\tfrac{1}{2}(t, x)\} \\
&\qquad\qquad \stackrel{\triangle}{=} \mathcal{A}(t, K, \delta, x) ,
\end{aligned}$$

provided that C is taken large enough. Indeed, would such a x-break point u exist then the cone $u + \mathcal{C}_{2\delta}(t)$ must contain the box $Q_C(x)$, hence also γ.

The probability of $\mathcal{A}(t, K, \delta, x)$ is estimated exactly as in the proof of Theorem 2.3 of [7]. Indeed, the presence of the path γ only affects an arbitrarily small fraction of the slabs $\mathcal{S}_k(t)$ introduced in the latter proof, provided C is taken large enough, so that the argument given there applies with no modifications. Q.E.D.

§5. Relation to Quantum Field Theories

There is an abundant literature devoted to the relation between Ising and other ferromagnetic type models to the Euclidean lattice quantum field theories, see e.g. [21, 18] or more recently [5, 4]; the latter article contains also an extensive bibliography on the subject. In this works the spins live on the integer lattice $\mathbb{Z}^{d+1}$ with one special direction, say $\vec{e}_1$, being visualized as the imaginary time axis. Thus, for example, the analyticity properties of the mixed Fourier transform

$$\mathbb{G}_\beta(p_1, i\mathbf{p}) \;=\; \sum_{x_1\in\mathbb{Z}} \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{p_1x_1 + i(\mathbf{p},\mathbf{x})} G_\beta(x_1, \mathbf{x}), \tag{28}$$

$(p_1, \mathbf{p}) \in \mathbb{T} \times \mathbb{T}^d$, are related in this way to the question of existence of one particle states.

Below we shall briefly indicate how the the key probabilistic representation (20) leads to the following conclusion (see e.g Proposition 4.2 in [18], Theorem 2.3 in [21]): For every $\mathbf{p} \in \mathbb{T}^d$ define

$$\omega(\mathbf{p}) \;=\; -\lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{i(\mathbf{p},\mathbf{x})} G_\beta(n, \mathbf{x}), \tag{29}$$

$\omega(\mathbf{p})$ being interpreted as the energy of a particle with momentum $\mathbf{p}$.

Theorem 5.1. *There exists a neighbourhood $B_\delta = \{\mathbf{p} : |\mathbf{p}| < \delta\}$ of the origin in $\mathbb{R}^d$ such that the function $\mathbf{p} \mapsto \omega(\mathbf{p})$ is real analytic on B_δ.* $\mathrm{Hess}\,(\omega)\,(0)$ *is precisely the matrix of the second fundamental form of $\partial\mathbf{K}_\beta$ at $\hat{t} = (\xi_\beta(\vec{e}_1), 0)$. Furthermore, there exists $\epsilon > 0$ such that for every $\mathbf{p} \in B_\delta$ the function*

$$p_1 \;\mapsto\; \mathbb{G}_\beta(p_1, i\mathbf{p})$$

has a meromorphic extension to the disc $\{p_1 \in \mathbb{C} : |p_1 - \hat{p}_1| < \epsilon\}$; $\hat{p}_1 = \xi_\beta(\vec{e}_1)$, with the only simple pole at $p_1 = \omega(\mathbf{p})$.

In the sequel we use the notation introduced in Section 2. Because of the $\mathbb{Z}^d$-lattice symmetries the dual point $\hat{t} \in \partial\mathbf{K}_\beta$ of $\vec{e}_1$ is given by $\hat{t} = (\hat{p}_1, 0)$ with $\hat{p}_1 = \xi_\beta(\vec{e}_1)$. Given $y \in \mathbb{Z}^{d+1}$ define (see (10))

$$W(y) \;=\; \sum_{\substack{\gamma:0\to y\\ \gamma\in\mathcal{S}}} \mathrm{e}^{\beta|\gamma|},$$

and let W_L, W_0 and W_R be defined as in Section 2. Summing up all the weights of irreducible paths in (8) we arrive to the following representation of G_β:

$$
\begin{aligned}
G_\beta(y) \;&=\; W(y) + \sum_{y_L+y_R=y} W_L(y_L)W_R(y_R) \\
&\quad + \sum_{y_L,y_R}\sum_{M=1}^{\infty} W_L(y_L)W_R(y_R)W_0^{*M}(y-y_L-y_R).
\end{aligned} \tag{30}
$$

Consider the mixed Fourier transforms

$$
\mathbb{W}(p_1,\mathbf{p}) \;=\; \sum_{x_1\in\mathbb{Z}}\sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{p_1x_1+(\mathbf{p},\mathbf{x})}W(x_1,\mathbf{x})
$$

and

$$
\mathbb{W}_b(p_1,\mathbf{p}) \;=\; \sum_{x_1\in\mathbb{Z}}\sum_{\mathbf{x}\in\mathbb{Z}^d} \mathrm{e}^{p_1x_1+(\mathbf{p},\mathbf{x})}W_b(x_1,\mathbf{x})\,;\;\; b=0,L,R.
$$

By Lemma 2.1 all four functions above are analytic in the complex neighbourhood $B_\nu^{\mathbb{C}}(\hat{t})$ of $\hat{t}$; $B_\nu^{\mathbb{C}}(\hat{t}) = \{(p_1,\mathbf{p}) : \sqrt{|p_1-\hat{p}_1|^2+|\mathbf{p}|^2} < \nu\}$. Thus, the extension of $\mathbb{G}_\beta(p_1,\mathbf{p})$ to $B_\nu^{\mathbb{C}}(\hat{t})$ is given by:

$$
\mathbb{W}(p_1,\mathbf{p}) \;+\; \frac{\mathbb{W}_L(p_1,\mathbf{p})\mathbb{W}_R(p_1,\mathbf{p})}{1-\mathbb{W}_0(p_1,\mathbf{p})}.
$$

Consequently, the surface of poles of $\mathbb{G}_\beta(p_1,\mathbf{p})$ inside $B_\nu^{\mathbb{C}}(\hat{t})$ is given by the implicit equation

$$
\mathbb{W}_0(p_1,\mathbf{p}) = 1. \tag{31}
$$

As we have already seen in Section 2, the restriction of (31) to $(p_1,\mathbf{p}) \in \mathbb{R}\times\mathbb{R}^d$ defines the piece of the boundary $\partial\mathbf{K}_\beta$ inside $B_\nu(\hat{t})$. Since by (19) $\partial\mathbb{W}_0/\partial p_1(\hat{t}) \neq 0$ and, in addition, $\mathrm{Hess}(\mathbb{W}_0)(\hat{t})$ is non-degenerate, the analytic implicit function theorem implies that there exists $\delta > 0$, such that the equation (31) can be resolved for $\mathbf{p} \in B_\delta^{\mathbb{C}}(\hat{t}) \subset \mathbb{C}^d$ as

$$
p_1 \;=\; \widetilde{\omega}(\mathbf{p}). \tag{32}
$$

In particular, for $\mathbf{p} \in \mathbb{R}^d$ the equation (32) gives a parameterization of $\partial\mathbb{K}_\beta$ in the δ-neighbourhood of $\hat{t}$, and $\mathrm{Hess}(\widetilde{\omega})(0)$ is, indeed, the matrix of the second fundamental form of $\partial\mathbf{K}_\beta$ at $\hat{t}$. Finally, the 1-particle mass shell ω in (29) is recovered as $\omega(\mathbf{p}) = \widetilde{\omega}(i\mathbf{p})$.

In the case of ferromagnetic Ising models set $\vec{p} = (p_1,\mathbf{p})$ and readjust the definition (25) of the Ruelle operator $\mathcal{L}_z$ as $\widetilde{\mathcal{L}}_{\vec{p}} = \mathcal{L}_{\vec{p}-\hat{t}}$. Then $\widetilde{\mathcal{L}}_{\vec{p}}$ is well defined and bounded on $\mathfrak{F}_{0,\theta}$ for every $\vec{p} \in B_\nu^{\mathbb{C}}(\hat{t})$. It could be then shown that the surface of poles of $\mathbb{G}_\beta$ inside $B_\nu^{\mathbb{C}}(\hat{t})$ is implicitly given by

$$\tilde{\rho}_\beta(p_1, \mathbf{p}) \;=\; 1,$$

where $\tilde{\rho}_\beta(p_1, \mathbf{p})$ is the leading (lying on the spectral circle) eigenvalue of $\widetilde{\mathcal{L}}_{\vec{p}}$. Further analysis of the spectral properties of the family $\{\widetilde{\mathcal{L}}_{\vec{p}}\}$ reveals [7] that there exists $\epsilon > 0$, such that $\rho_\beta(p_1, \mathbf{p})$ is a simple pole of the corresponding resolvent for every $\vec{p} \in B_\epsilon^{\mathbb{C}}(\hat{t})$. In this way, the conclusion of Theorem 5.1 follows from the analytic perturbation theory of discrete spectra and from the conditional variance argument which ensures the non-degeneracy of $\mathrm{Hess}(\rho_\beta)(\hat{t})$.

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