ADIABATIC CHARGE TRANSPORT AND THE KUBO FORMULA FOR LANDAU TYPE HAMILTONIANS

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Abstract. The adiabatic charge transport is investigated in a two dimensional Landau model perturbed by a bounded potential at zero temperature. We show that if the Fermi level lies in a spectral gap then in the adiabatic limit the accumulated excess Hall transport is given by the linear response Kubo-Štreda formula. The proof relies on the expansion of Nenciu, some generalized phase space estimates, and a bound on the speed of propagation.

1. Introduction

In this work we prove the validity of the linear response theory in the context of the Landau type model of non-interacting electrons. The framework of the linear response was used extensively to explain the quantization of conductance in the Quantum Hall Effect (QHE) [9, 20, 4], first observed in the celebrated experiment by von Klitzing et. al. [11]. Most of the results derived in this paper can be proved for more general families of magnetic Schrödinger operators in $\mathbb{R}^2$ than the Landau model, but the latter is *primus inter pares* for us due to its accessibility. It is also worthwhile to note that as far as we know, the Landau model is the only model in $\mathbb{R}^2$ for which the (quantized) conductivity was actually computed (see, for example [3] and references therein).

The model we consider here is described by a one-particle Hamiltonian on $L^2(\mathbb{R}^2)$:

$$H_\lambda \;=\; (p_1 - \frac{B}{2}x_2)^2 \;+\; (p_2 + \frac{B}{2}x_1)^2 \;+\; \lambda V \,,$$

where the potential V is smooth and relatively weak: $\|V\|_{n,\infty} \le C_n$ for some integer n large enough ($n = 5$ will do), and $\|V\|_\infty < B/\lambda$. The operator H_0 has been introduced by Lev Landau, and has a number of nice features. In particular, its spectrum consists of the odd multiples of the strength of the magnetic field B. The corresponding eigenvalues are infinitely degenerate. Consequently, the spectrum of the Hamiltonian H_λ, under the above assumptions, contains a (infinite) sequence of bands, separated from each other by finite gaps. In order to investigate the transport properties of the above system, we consider the transverse current induced by a time-dependent potential gradient. The full Hamiltonian is

$$H_\lambda(t) \;=\; H_\lambda \;+\; \frac{1}{\tau}\, g(t/\tau)\, \Lambda_1 \,, \tag{1.1}$$

where $\dot{g}(\cdot)$ is a smooth function supported in $(0, 1)$ (without loss of generality we will assume that g vanishes for negative values of the argument). The variable t here stands for the time, and the large parameter τ is a convenient tool to control the rate at which the system changes. We study here the evolution up to time

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τ. The potential Λ_1 is taken here to be of the form of a smooth step function depending only on the x_1 component of the position, which is 0 on $[-\infty, -m]$ and 1 on $[m, +\infty]$ for some $m > 0$. The coefficient $1/\tau$ in front of the time dependent potential tells us that the external field is weak. In fact, we will be interested in the limiting behavior $\tau \to \infty$.

A well known weakness of standard linear response theory, stressed by van Kampen [21], is that it takes the limit of weak field first and only then the thermodynamic limit. The correct order is, of course, the reverse. Theorem 1 below is free of this criticism in that one starts with a system that has infinite extent. The linear response limit is then realized by our adiabatic limit $\tau \to \infty$.

We will compute the induced current across the line $x_2 = 0$, in the state $\varrho_\tau(t)$ which evolves from the zero temperature state $P_\lambda = \chi(H_\lambda < E_f)$, with E_f the Fermi-energy. Our results here deal with values of the Fermi energy which lie in a gap of the spectrum of H_λ.

The expectation value of the excess current is given by

$$\mathcal{J}_\tau(t) \;=\; -\mathrm{i}\ \mathrm{Tr}\,(\varrho_\tau(t) - P_\lambda)\ [H_\lambda\,, \Lambda_2]\ , \tag{1.2}$$

where Λ_2 is the characteristic function of the upper half plane ($x_2 \geq 0$). The subtraction eliminates the contribution present even without the external field (persistent current).

We will prove that, if the Fermi energy falls in a spectral gap, then the combination appearing in Eq. (1.2) forms a trace-class operator (which implies that the r.h.s. of Eq. (1.2) is well defined). Moreover, the main result of this paper will be

$$\lim_{\tau\to\infty} \tau \mathcal{J}_\tau(t) \;=\; -\mathrm{i} g(t/\tau)\, \mathrm{Tr}\, P_\lambda\,[\,[P_\lambda\,, \Lambda_1]\ , [P_\lambda\,, \Lambda_2]\,]\, P_\lambda\ . \tag{1.3}$$

This agrees with the Kubo-Štreda formula for the bulk Hall conductance [12, 19, 13], confirming the validity of the linear response calculation in this context. The equality of the bulk and the edge conductance was rigorously established in [18, 6].

Moreover it turns out that the quantity

$$K_\lambda := \mathrm{Tr}\, P_\lambda\,[\,[P_\lambda\,, \Lambda_1]\ , [P_\lambda\,, \Lambda_2]\,]\, P_\lambda \tag{1.4}$$

defined in Eq. (1.3) remains invariant under variation of λ, and in particular

$$K_\lambda = K_0\ , \tag{1.5}$$

with K_0 being the Hall conductance at energy E_F for the Landau model. The latter was studied extensively by a number of authors [16, 3, 5], and can be computed explicitly. Whenever the Fermi energy E_F lies in the gap between the jth and $j+1$th Landau level, the corresponding value of K_0 is equal to j.

A key role in the proof is played by an asymptotic expansion developed by Nenciu [15] for an adiabatically evolved projection operator, and by the propagation estimate of the evolution operator, generated by the Hamiltonian $H_\lambda(t)$. The physical intuition behind the proof will be presented in details in Section 3

Our results are essentially parallel to the ones derived for lattice models in [1]. The important difference however lies in the absence of the ultraviolet cutoff in the continuous case, which affects both the trace class properties of the relevant operators and the propagation estimates.

There is a large amount of literature related to the different features of Landau type systems in both physical (e.g. [7, 8]) and mathematical literature, see for example [22] and references therein. This work is not the first one where the linear response theory is discussed in this context. We are aware of two works on this subject, [2, 14]. In [2] the validity of the linear response theory is proven for finite dimensional spectral projection of magnetic Hamiltonian in the torus geometry. The method derived there enables one to compute the charge transport in the system, where ours also gives a pointwise value for a current. In [14] the author derives the Hall conductance of the infinite sample under assumption of the validity of the linear response theory following Kubo [12], and also proves the stability of the Hall conductance assuming that the strength of the (random) potential is weak enough and that states near the edges of the Landau bands are sufficiently localized. We don't use the latter condition.

The flow of the paper is organized as follows: in Section 2 we state our main results, in Section 3 we outline the Nenciu expansion and present the basic propositions needed for our analysis: Generalized space-momentum inequalities, finite speed of propagation for certain initial data, and trace class estimates. Equipped with these tools, we prove our main result, Theorem 1. In Sections 4 and 5 we outline the proofs of the aforementioned bounds.

2. Statement of the main result

We require certain technical assumptions on the Hamiltonian and the perturbing potential.

The external potential considered here will be in the form of a *switch function*, Λ_1, in the terminology of ref. ([2]).

Definition 1. A *switch function in the* j^{th}*-direction* with $j = 1$ or 2 is a smooth function $\Lambda_j : \mathbb{R}^2 \to [0,1]$ which depends only on the variable x_j and satisfies

$$\begin{cases} \Lambda_j \; = 0 & x_j < -m \\ 1 - \Lambda_j \; = 0 & x_j > m \end{cases}, \tag{2.1}$$

for some $m > 0$.

Remark: The condition above could be replaced with sufficiently fast power decay of the derivative of Λ_j at infinities without affecting the results which appear below.

The following theorem consists of three parts. The first statement shows that the current defined in (1.2) is well defined, the second proves the convergence of the charge, in the adiabatic limit, to the value given by the Kubo formula, and the third one deals with the stability of the limiting value of the charge under changes of λ.

Theorem 1. *Let H_λ be as above, with $a := \lambda \, \|V\|_\infty < B$. Assume in addition that the Sobolev norm of the potential, $\|V\|_{N,\infty}$ is bounded for N large enough ($N = 6$ will do). Let ϱ_τ be the solution to the evolution equation, for $t \in [0, \tau]$:*

$$\begin{cases} \mathrm{i}\dot{\varrho}_\tau(t) &= \left[H_\lambda + \frac{1}{\tau} g(t/\tau)\,\Lambda_1 \,,\, \varrho_\tau(t)\right] \\ \varrho_\tau(0) &= \chi(H_\lambda < E_F) \;\; =: \;\; P_\lambda \end{cases}, \tag{2.2}$$

with Λ_1 a switch function in the 1^{st} *direction, $(2j-1)B + a < E_F < (2j+1)B - a$ for some integer j, and $\dot{g} \in C^k([0,1])$, $g(0) = 0$ ($k = 4$ will be enough). Then for any switch function in the* 2^{nd} *direction, Λ_2,* *(1) [Well-posedness] The observable whose trace is the induced Hall current,*

$$\widehat{\mathcal{J}}_\tau(t) := -\mathrm{i}(\varrho_\tau(t) - P_0)\,[H_\lambda\,, \Lambda_2]\ , \tag{2.3}$$

is trace class for all $\tau \geq 0$ and $t \in [0, \tau]$.

(2) [Adiabatic limit and the Kubo formula] For every $\tau > 0$ and $t \in [0, \tau]$ we have

$$\left| \mathrm{Tr}\, \widehat{\mathcal{J}}_\tau(t) \ + \ \frac{1}{\tau} \mathrm{i} g(t/\tau) K_\lambda \right| \ \leq \ \frac{C}{\tau^2}\,, \tag{2.4}$$

where K_λ is defined in Eq. (1.4).

(3) [Stability] The value of K_λ is independent of the value of λ whenever $a < B$. If E_F satisfies the conditions above, then $K_\lambda = j$.

Remarks

(1) Eq. (2.4) implies that the limiting value of the charge is given by

$$\lim_{\tau \to \infty} \int_0^\tau \mathrm{Tr}\, \widehat{\mathcal{J}}_\tau(t)\, \mathrm{d}t \ = \ -\mathrm{i} \left(\int_0^1 g(u) du \right) \mathrm{Tr}\, P_\lambda \left[\, [P_\lambda\,, \Lambda_1]\ , [P_\lambda\,, \Lambda_2]\, \right] P_\lambda\ , \tag{2.5}$$

known as the Kubo formula.

(2) In order to prove (2.4) we will expand $\widehat{\mathcal{J}}(\tau s)$ in an asymptotic series $\sum_{j=1}^\infty \frac{1}{\tau^j} N_j(s)$. Assuming that g and its first $k+2$ derivatives vanish at 0, we will then prove that

$$\sup_{s \in [0,1]} \tau^{k-2}\ \mathrm{Tr} \left| \widehat{\mathcal{J}}(\tau s) \ - \ \sum_{j=1}^{k-1} \frac{1}{\tau^j} N_j(s) \right| \ < \ C_k\ . \tag{2.6}$$

This asymptotic series is derived from the Nenciu expansion for ϱ_τ [15].

(3) It is known that the quantization of the Hall conductance is measured with very high accuracy, so that one expect that the total charge transport $Q(\tau) - Q(0) = \int_0^\tau \mathrm{d}t \mathcal{J}_\tau(t)$ should coincide with the linear response result up to higher powers of $1/\tau$ when $g(1) = 0$. In the case of finite dimensional Fermi projection P_λ this behavior was established rigorously in [10]. It can be also proven in our setup, using the approach of Avron, Seiler, and Yaffe rather then using the Nenciu expansion.

In order to facilitate the proof of the theorem above we will use a *scaled time* $s = t/\tau$. Notice that the scaled time s changes from 0 to 1 when t changes from 0 to τ. Moreover we will work in the so called interaction picture; Under the time-dependent gauge transformation

$$P_\tau(s) \ = \ \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\, \varrho_\tau(\tau s) \mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1} \tag{2.7}$$

the evolution (2.2) is translated into the initial value problem:

$$\begin{cases} \mathrm{i}\dot{P}_\tau(s) \ = \ \tau\, [H(s)\,, P_\tau(s)] \\ P_\tau(0) \ = \ P_\lambda \end{cases}, \tag{2.8}$$

with $\phi(s) = \int_0^s g(u)\, du$ and the time dependent Hamiltonian $H(s)$ defined by:

$$H(s) \ = \ \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\, H_\lambda\, \mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}\ . \tag{2.9}$$

The utility of the working with $H(s)$ in place of $H_\lambda(s) = H_\lambda + 1/\tau g(s)\Lambda_1$ is related to the iso-spectrality of the former family of Hamiltonians.

After these transformations, the current (1.2) becomes

$$\begin{aligned}\operatorname{Tr}\widehat{\mathcal{J}}(\tau s) &= -\,\mathrm{i}\operatorname{Tr}(\varrho_\tau(\tau s) - P_\lambda)\,[H_\lambda\,,\Lambda_2] \\ &= -\,\mathrm{i}\operatorname{Tr}\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}(P_\tau(s) - P(s))\,[H(s)\,,\Lambda_2]\,\mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1} \\ &= -\,\mathrm{i}\operatorname{Tr}(P_\tau(s) - P(s))\,[H(s)\,,\Lambda_2]\ ,\end{aligned} \tag{2.10}$$

where

$$P(s) \ := \ \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\,P_\lambda\,\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1} \ = \ \chi(H(s) < E_F)\,. \tag{2.11}$$

In order to facilitate the writing of inequalities we shall adopt the following convention: $C_{n,m,..}$ will denote a general constant (not necessary the same at different occurrences), which depends only on the integers $n, m, ..$, on the Sobolev norm of the potential V, and on the strength of magnetic field B.

3. Asymptotic expansion for P_τ and the Kubo formula

In 1993 G. Nenciu [15] found a general form of the solution of the Heisenberg equation

$$\mathrm{i}\dot{P}_\tau(s) \ = \ \tau[H(s), P_\tau(s)]\,, \tag{3.1}$$

where $P_\tau(0)$ is a spectral projection of the operator $H(0)$. The idea was to look for an asymptotic series of the form

$$P_\tau(s) \sim B_0(s) \ + \ \frac{1}{\tau}B_1(s) \ + \ \frac{1}{\tau^2}B_2(s) \ + \ \dots\ . \tag{3.2}$$

The substitution of (3.2) into (3.1) leads to a sequence of differential equations

$$\mathrm{i}\dot{B}_j(s) \ = \ [H(s)\,, B_{j+1}(s)] \quad j = 0, \dots \tag{3.3a}$$

In addition, using that $P_\tau(s)$ is a projection for each s, we get $P_\tau(s)^2 = P_\tau(s)$, which generates the following sequence of algebraic relations:

$$B_j(s) \ = \ \sum_{m=0}^{j} B_m(s)B_{j-m}(s) \quad j = 0, \dots \tag{3.3b}$$

In particular: $B_0(s)^2 = B_0(s)$, so $B_0(s)$ is a projection for each s.

It turns out that the system of hierarchical relations (3.3a) and (3.3b) has a unique solution, which is given by the following recursive construction:

$$\begin{cases} B_0(s) &= P(s) \\ B_j(s) &= \frac{1}{2\pi}\int_\Gamma R_z(s)\left[P(s)\,, \dot{B}_{j-1}(s)\right]R_z(s)\mathrm{d}z + \ S_j(s) - 2P(s)S_j(s)P(s)\ , \end{cases} \tag{3.4}$$

where $R_z(s) = (H(s) - z)^{-1}$,

$$S_j(s) \ = \ \sum_{m=1}^{j-1} B_m(s)B_{j-m}(s)\ , \tag{3.5}$$

and the contour Γ encircles the spectrum below the Fermi energy. In particular the first order (and most prominent for the linear response) term is given by

$$B_1(s) \ = \frac{1}{2\pi}\int_\Gamma R_z(s)\left[P(s)\,, \dot{P}(s)\right]R_z(s)\mathrm{d}z\ . \tag{3.6}$$

One can truncate the expansion (3.2) at some finite order $k > 0$ by observing that[1]:

$$P_\tau(s) \;=\; B_0(s) + \frac{1}{\tau} B_1(s) \;+\; \ldots \;+\; \frac{1}{\tau^k} B_k(s) - \;\frac{1}{\tau^k}\int_0^s U_\tau(s,r)\dot{B}_k(r)U_\tau(r,s)\mathrm{d}r\,. \tag{3.7}$$

where $U_\tau(s,t)$ are the Schrödinger unitary propagators, satisfying

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial s}U_\tau(s,r) \;=\; \tau H(s)U_\tau(s,r) \\ U_\tau(s,s) \;=\; \mathbf{1}\,. \end{cases} \tag{3.8}$$

The observable whose trace gives the induced current can thus be expanded according to

$$\begin{aligned} i\tau\widehat{\mathcal{J}}(\tau s) = B_1(s)[H(s),\Lambda_2] + \frac{1}{\tau}B_2(s)[H(s),\Lambda_2] + \cdots + \frac{1}{\tau^{k-1}}B_k(s)[H(s),\Lambda_2] \\ + \frac{1}{\tau^{k-1}}\int_0^s \mathrm{d}r U_\tau(s,r)\dot{B}_k(r)U_\tau(r,s)[H(s),\Lambda_2]. \end{aligned} \tag{3.9}$$

As we will see below, the first term of this expansion exactly yields the Kubo formula

$$\mathrm{Tr}\, B_1(s)\,[H(s)\,,\Lambda_2] \;=\; g(s)\,\mathrm{Tr}\, P_\lambda\,[[P_\lambda\,,\Lambda_1]\;,[P_\lambda\,,\Lambda_2]] \tag{3.10}$$

and directly proves the first part of Theorem 1, provided that we can control the other terms in the expansion (3.9), showing that their contribution vanishes in the limit $\tau \to \infty$. Most of the paper deals in fact with the control of these terms. In the following we illustrate the strategy and state the propositions used at the end of this section to prove Theorem 1. In order to keep the attention of the readers on the main physical ideas we defer the quite technical proofs of these propositions to later sections.

First, we need to derive some bounds, applicable for a broad class of time dependent Hamiltonians with sufficiently smooth potential. These bounds will be very useful to handle products of functions of the Hamiltonian with different functions of the space and the momentum coordinates. From here on we use the notations $\langle x\rangle = \sqrt{1+x^2}$ and $\mathbf{p_A} = (\mathbf{p} - \mathbf{A}(\mathbf{x}))$, with $\mathbf{A}(x_1,x_2) = B/2(-x_2,x_1)$.

Proposition 1 (Generalized space–momentum bounds)**.** *Consider the Hamiltonian $H_t = p_A^2 + W(t)$ acting on $L^2(\mathbb{R}^2,dx)$, where $W(t)$ is a time dependent multiplication operator. Fix $m,n \in \mathbb{Z}/2$ and assume that $W(t) \in H_{2|m|,\infty}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$ so that $D_m = \sup_{t\in\mathbb{R}} \|W(t)\|_{2|m|,\infty} < \infty$. Without loss of generality we can also assume that $\inf_{t\in\mathbb{R}}\inf\sigma(H_t) \geq 1$.*

a) *There is a constant C_m, depending on B and on D_m, such that, for any function $f \in H_{2|m|,\infty}(\mathbb{R}^2)$ we have*

$$\text{i) } \|H_t^{m-1/2}\mathbf{p_A}H_t^{-m}\| \leq C_m, \tag{3.11}$$

$$\text{ii) } \|H_t^m f(x) H_t^{-m}\| \leq C_m\|f\|_{2|m|,\infty}. \tag{3.12}$$

b) *There is a constant $C_{n,m}$, depending on B and on D_m such that, for $i = 1,2$,*

$$\|H_t^m\langle x_i\rangle^n H_t^{-m}\langle x_i\rangle^{-n}\| \leq C_{n,m}. \tag{3.13}$$

All these bounds are uniform in $t \in \mathbb{R}$.

[1] We owe this observation to Jeff Schenker.

Remarks

(1) Proposition 1 holds true also for $m \in \mathbb{Z}/4$, if we assume that $f, W(t) \in H_{2\tilde{m},\infty}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$ and that $\sup_{t\in\mathbb{R}} \|W(t)\|_{2\tilde{m},\infty} < \infty$, where $\tilde{m}$ is the smallest half integer larger or equal to $|m|$. The proof is then very similar to the one given below, and is omitted here.

(2) In particular the result applies to the Hamiltonian $H_\lambda(t)$ defined in Eq. (1.1), if the potential V has the required smoothness assumptions.

(3) If $H(s) = e^{i\phi(s)\Lambda_1} H_\lambda e^{-i\phi(s)\Lambda_1}$ is the gauged Hamiltonian introduced in Section 2, we also have

$$\|H(s)^m f(x) H(s)^{-m}\| \le C_m \|f\|_{2|m|,\infty} \quad \text{and} \tag{3.14}$$

$$\|H(s)^m \langle x_i\rangle^n H(s)^{-m} \langle x_i\rangle^{-n}\| \le C_{n,m}\,, \tag{3.15}$$

because $e^{i\phi(s)\Lambda_1}$ commutes with the operators $f(x)$ and $\langle x_i\rangle$.

According to Eq. (3.9) we have to control two different types of terms

$$\frac{1}{\tau^{j-1}} \operatorname{Tr} B_j(s)\,[H(s)\,,\Lambda_2] \quad \text{with } 2 \le j \le k, \tag{3.16}$$

and

$$\frac{1}{\tau^{k-1}} \operatorname{Tr} \int_0^s \mathrm{d}r\, U_\tau(s,r)\dot{B}_k(r) U_\tau(r,s)\,[H(s)\,,\Lambda_2]\;. \tag{3.17}$$

In order to estimate these traces we use the following result.

Proposition 2 (Trace class estimates)**.** *Suppose that the operators A, B acting on $L^2(\mathbb{R}^2, dx)$ satisfy the two conditions*

$$\|A\,\langle \mathbf{x}\rangle^{\mathbf{3}}\| < \infty\,; \quad \|\mathbf{H}_\lambda^{\mathbf{3/2}}\,\mathbf{D}\| < \infty\,. \tag{3.18}$$

Then AD is a trace class operator.

Using the last proposition the problem of showing that the traces (3.16) and (3.17) are finite reduce to the problem of proving that the corresponding operators decay sufficiently fast in the energy and in the space coordinates. Consider first the term that appears in Eq. (3.16). The corresponding decay property follows from the fact that the operator $B_j(s)$ is localized in the energy and in the x_1 coordinate, while the commutator $[H(s), \Lambda_2]$ decays in the x_2 coordinate. Here are the corresponding claims.

Proposition 3 (Fast decay away from the x_2 axis)**.** *Fix $N \in \mathbb{N}$ and assume that $V \in H_{2N,\infty}(\mathbb{R}^2)$. Then, for all $j = 1, 2, \ldots$ we have*

$$\sup_{s\in[0,1]} \|B_j(s)\, H^N(s) \langle x_1\rangle^N\| < C_{j,N}\;, \tag{3.19}$$

and for all $j = 0, 1, \ldots$

$$\sup_{s\in[0,1]} \|\dot{B}_j(s)\, H^N(s) \langle x_1\rangle^N\| < C_{j,N}\;, \tag{3.20}$$

Proposition 4 (Localization of the current operator near the x_1 axis)**.** *The operator $[H(s)\,,\Lambda_2]$ is supported in a strip of width $2m$ around x axis. Moreover, if $V \in H_{1,\infty}(\mathbb{R}^2)$, we have*

$$\|H^{-1/2}(s)\,\langle x_2\rangle^N\;[H(s)\,,\Lambda_2]\,\| \;\le\; C_N\,.$$

Remark The proof of this claim is a trivial consequence of the locality of the Hamiltonian $H(s)$ and Proposition 1.

Propositions 2-3 imply that the trace

$$\operatorname{Tr} B_j(s)[H(s), \Lambda_2]$$

is finite and since it is τ-independent, we observe that each term (3.16) vanishes in the limit $\tau \to \infty$. What about the remainder (3.17)? Also in this case the operator $\dot{B}_k(r)$ decays in the energy and in the x_1 coordinate, and the commutator $[H(s), \Lambda_2]$ decays in the x_2 coordinate. But the two operators are separated by the time evolution $U_\tau(r, s)$. Knowing that $[H(s), \Lambda_2]$ is localized in a strip of length $2m$ around the x_1 axis, what can be said about $U_\tau(r, s)[H(s), \Lambda_2]$? To answer this question we use the fact that the electrons cannot propagate faster than ballistically. Since the energy of the electrons is essentially bounded by the Fermi energy (modulo the spread due to time dependent potential), we observe that electrons initially confined inside the strip of width $2m$ can, in the course of their evolution, propagate onto a strip of width $O(\tau)$ (because the time difference $|r - s|$ can be of order τ) around the x_1 axis. This is the content of Proposition 5 below. Using this result we can then prove that the operator

$$\int_0^s \mathrm{d}r\, U_\tau(s, r)\dot{B}_k(r)U_\tau(r, s)\,[H(s)\,, \Lambda_2] \tag{3.21}$$

is trace class, and that the corresponding norm is proportional to some power of τ. Thus choosing the order k of the expansion (3.9) sufficiently large, also the term (3.17) vanishes in the limit $\tau \to \infty$.

Proposition 5. *[Finite speed of propagation] Consider the Hamiltonian $H_\lambda(t)$ as defined in Eq. (1.1) and denote by $U(t, s)$ the corresponding Schrödinger evolution. Fix $n, m \in \mathbb{N}/2$, and assume that $V \in H_{2(n+m),\infty}(\mathbb{R}^2)$. Then there is a constant $D = D(n, m)$ such that, for $i = 1, 2$,*

$$\|\langle x_i\rangle^n H_\lambda(t_1)^m U(t_1, t_2) H_\lambda(t_2)^{-m-n}\varphi\| \leq D\langle t_1 - t_2\rangle^n \|\langle x_i\rangle^n \varphi\|, \tag{3.22}$$

for all $t_1, t_2 \in \mathbb{R}$.

Remarks

(1) The proposition actually holds also for $m \in \mathbb{Z}/2$. The proof in this case is identical to the one given below for the case $m \geq 0$ and is therefore omitted.

(2) It follows from this proposition that, for all $n \in \mathbb{N}/2$ and $m \in \mathbb{Z}/2$,

$$\|\langle x_i\rangle^n H_\lambda(t_1)^m U(t_1, t_2) H_\lambda(t_2)^{-m-n}\langle x_i\rangle^{-n}\| \leq D\langle t_1 - t_2\rangle^n.$$

(3) After rescaling the time $s = t/\tau$ and introducing the gauged Hamiltonian $H(s) = e^{i\phi(s)\Lambda_1} H_\lambda e^{-i\phi(s)\Lambda_1}$ the last equation implies that, for $0 < s_1, s_2 < 1$, and for all $n \in \mathbb{N}/2$ and $m \in \mathbb{Z}/2$, we have

$$\|\langle x_i\rangle^n H(s_1)^m U_\tau(s_1, s_2) H(s_2)^{-m-n}\langle x_i\rangle^{-n}\| \leq D\langle \tau\rangle^n, \tag{3.23}$$

where $U_\tau(t, s)$ denotes the time evolution generated by $H(s)$. This is the bound explicitly used in the applications.

Using Propositions 1-5 and following the strategy outlined above, we can now prove our main result, Theorem 1.

Proof of Theorem 1. Part (1): We use the expansion (3.9) of the operator $\tau\widehat{\mathcal{J}}(s\tau)$, and we estimate the trace of the different terms in the expansion. Clearly,

$$\begin{aligned}\|B_j\,[H(s)\,,\Lambda_2]\,\|_1 \le \|B_j(s)\,H^N(s)\langle x_1\rangle^N\|\|\langle x_1\rangle^{-N}H^{-N}(s)\langle x_2\rangle^{-N}H^{1/2}(s)\|_1\\ \cdot\,\|H^{-1/2}(s)\langle x_2\rangle^N\,[H(s)\,,\Lambda_2]\,\|\,.\end{aligned} \tag{3.24}$$

The first and the last factors are finite by Propositions 3 and 4. Since

$$H(s) \;=\; \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}\,H_\lambda\,\mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}\;, \tag{3.25}$$

we obtain that

$$\|\langle x_1\rangle^{-N}H^{-N}(s)\langle x_2\rangle^{-N}H^{1/2}(s)\|_1 = \|\langle x_1\rangle^{-N}H_\lambda^{-N}\langle x_2\rangle^{-N}H_\lambda^{1/2}\|_1\,. \tag{3.26}$$

Proposition 1 implies in turn that

$$\begin{aligned}&\|\langle x_1\rangle^{-N}H_\lambda^{-N}\langle x_2\rangle^{-N}H_\lambda^{1/2}\|_1\\ &\quad\le \|\langle x_1\rangle^{-N}H_\lambda^{-N+1/2}\langle x_2\rangle^{-N}\|_1\|\langle x_2\rangle^{N}H_\lambda^{-1/2}\langle x_2\rangle^{-N}H_\lambda^{1/2}\|\\ &\quad\le C_N\,\|\langle x_1\rangle^{-N}H_\lambda^{-N+1/2}\langle x_2\rangle^{-N}\|_1\\ &\quad\le C_N\,\|\langle x_1\rangle^{-N}\langle x_2\rangle^{-N}H_\lambda^{-N+1/2}\|_1\|H_\lambda^{N-1/2}\langle x_2\rangle^{N}H_\lambda^{-N+1/2}\langle x_2\rangle^{-N}\|\\ &\quad\le C_N\,\|\langle\mathbf{x}\rangle^{-\mathbf{N}}\mathbf{H}_\lambda^{-\mathbf{N}+\mathbf{1/2}}\|_{\mathbf{1}}\,.\end{aligned} \tag{3.27}$$

The latter norm is finite by Proposition 2.

Now we need to estimate the trace of the last term in the expansion (3.9),

$$\frac{1}{\tau^{k-1}}\int_0^s \mathrm{d}r U_\tau(s,r)\dot{B}_k(r)U_\tau(r,s)[H(s),\Lambda_2]. \tag{3.28}$$

To this end we note that

$$\begin{aligned}&\|U_\tau(s,r)\dot{B}_k(r)U_\tau(r,s)\,[H(s)\,,\Lambda_2]\,\|_1\\ &\quad\le \|\dot{B}_k(r)\,H^N(r)\langle x_1\rangle^N\|\,\|\langle x_1\rangle^{-N}H^{-N}(r)\langle x_2\rangle^{-m}H^{m+1/2}(r)\|_1\\ &\quad\cdot\,\|H^{-m-1/2}(r)\langle x_2\rangle^m U_\tau(r,s)\langle x_2\rangle^{-m}H^{1/2}(s)\|\,\|H^{-1/2}(s)\langle x_2\rangle^m\,[H(s)\,,\Lambda_2]\,\|\,.\end{aligned}$$

The first and the forth factors are bounded by Propositions 3 and 4, accordingly. The third term is bounded by $C_m\tau^m$, because of Proposition 5. Choosing $N=2m$, and $m\ge 3$ the second term is bounded by Propositions 2 and 1 (this can be shown as in Eq.(3.27)).This implies that the absolute value of the trace of (3.28) is bounded by $C_{k,m}\tau^{m-(k-1)}$, for any $m\ge 3$ and for any $k\ge 1$. Thus choosing $k\ge 5$ and $m=3$ it follows that the trace of $\tau\widehat{\mathcal{J}}(s\tau)$ is finite, uniformly in τ and that

$$|\tau\mathcal{J}(s\tau)+i\,\mathrm{Tr}\,B_1(s)[H(s),\Lambda_2]|\le C\tau^{-1} \tag{3.29}$$

Part (2): We need to prove that

$$\mathrm{Tr}\,B_1(s)\,[H(s)\,,\Lambda_2] \;=\; g(s)\,\mathrm{Tr}\,P_\lambda\,[[P_\lambda\,,\Lambda_1]\,,[P_\lambda\,,\Lambda_2]]\;. \tag{3.30}$$

This relation follows from the explicit equation for $B_1(s)$ and the cyclicity of the trace. Indeed,

$$B_1(s) \;=\; \frac{1}{2\pi}\int_\Gamma R(s,z)\,[P(s)\,,[P(s)\,,\mathrm{i}g(s)\Lambda_1]]\,R(s,z)\mathrm{d}z\;. \tag{3.31}$$

One may now verify Eq. (3.30) using that

$$\begin{aligned}\mathrm{Tr}\, B_1(s)\,[H(s)\,,\Lambda_2] \;&= \frac{\mathrm{i}g(s)}{2\pi}\int_\Gamma \mathrm{d}z\;\mathrm{Tr}\,[P(s)\,,[P(s)\,,\Lambda_1]]\,R_z(s)\,[H(s)\,,\Lambda_2]\,R_z(s)\\ &= \frac{\mathrm{i}g(s)}{2\pi}\int_\Gamma \mathrm{d}z\;\mathrm{Tr}\,[P(s)\,,[P(s)\,,\Lambda_1]]\,[\Lambda_2\,,R_z(s)]\\ &= g(s)\;\mathrm{Tr}\,[[P(s)\,,\Lambda_1]\;,P(s)]\,[\Lambda_2\,,P(s)]\;\;,\end{aligned} \tag{3.32}$$

since $[P(s)\,,[P(s)\,,\Lambda_1]]\,R_z(s)\,[H(s)\,,\Lambda_2]$ is trace class, as follows from the proof of the first part of Theorem 1 (note that $g(s)\,[P(s)\,,\Lambda_1] = \dot{B}_0(s)$). It is also immediate from the proof of the first part that the product $[P(s)\,,\Lambda_1]\,[\Lambda_2\,,P(s)]$ is also trace class. Therefore, the cyclicity of the trace yields

$$\mathrm{Tr}\, B_1(s)\,[H(s)\,,\Lambda_2] \;=\; g(s)\,\mathrm{Tr}\,P(s)\,[[P(s)\,,\Lambda_1]\;,[P(s)\,,\Lambda_2]]\;\;, \tag{3.33}$$

which is equivalent to (3.30) since $\mathrm{e}^{\pm\mathrm{i}\phi(s)\Lambda_1}$ commutes with Λ_1 and Λ_2; recall that $P(s) = \mathrm{e}^{\mathrm{i}\phi(s)\Lambda_1}P_\lambda \mathrm{e}^{-\mathrm{i}\phi(s)\Lambda_1}$.

Part (3) - the stability problem. We are going to employ here the following tactics: First, we will demonstrate that K_λ is stable with respect to changes in the potential far away from the origin. Secondly, we show that the relative trace class perturbation - namely the change of the potential in a finite region around the origin - also leaves K_λ invariant. The first part will follow from the first resolvent identity:

$$(H_\lambda - z)^{-1} - (\hat{H}_\lambda - z)^{-1} \;=\; (H_\lambda - z)^{-1}\lambda V\chi_L(\hat{H}_\lambda - z)^{-1}\,, \tag{3.34}$$

where $1-\chi_L$ is a smooth characteristic function of the ball of radius L, centered at the origin, and $\hat{H}_\lambda := H_\lambda - \lambda V\chi_L$. Let us compare now K_λ and $\hat{K}_\lambda$, where the latter is the expression of the Kubo formula computed for $\hat{H}_\lambda$:

$$\begin{aligned}K_\lambda - \hat{K}_\lambda = \mathrm{Tr}(P-\hat{P})\,&[[P\,,\Lambda_1]\;,[P\,,\Lambda_2]]\\ &+ \mathrm{Tr}\,\hat{P}\left[\left[(P-\hat{P})\,,\Lambda_1\right]\;,[P\,,\Lambda_2]\right]\\ &\qquad + \mathrm{Tr}\,\hat{P}\left[\left[\hat{P}\,,\Lambda_1\right]\;,\left[(P-\hat{P})\,,\Lambda_2\right]\right]\;.\end{aligned} \tag{3.35}$$

Here we use the concise notation P instead of P_λ and $\hat{P} = \chi(\hat{H}_\lambda \le E_F)$. The idea is that $[P\,,\Lambda_1]\,[P\,,\Lambda_2]$ is basically supported near the origin, while $P-\hat{P}$ is essentially supported outside the ball of radius $\sqrt{L}$, hence their product is small. This idea can be materialized in the following fashion: Since

$$P - \hat{P} = \int_\Gamma \mathrm{d}z\hat{R}_z\lambda V\chi_L R_z$$

and

$$\begin{aligned}&\|(1-\chi_{\sqrt{L}})R_z\chi_L\|\\ &\quad\le\; \|(1-\chi_{\sqrt{L}}) < x >^{2N}\|\,\| < x >^{-2N} R_z < x >^{2N}\|\,\| < x >^{-2N}\chi_L\|\\ &\quad\le\; \frac{1}{\mathrm{dist}(z,\sigma(H_\lambda))}C_N L^{-N}\,,\end{aligned}$$

we get the bound

$$\|(P-\hat{P})\,[[P\,,\Lambda_1]\;,[P\,,\Lambda_2]]\,\|_1 \;\le\; \|\chi_{\sqrt{L}}\,[P\,,\Lambda_1]\,[P\,,\Lambda_2]\,\|_1 \;+\; C_N L^{-N}\,.$$

On the other hand,

$$\begin{aligned} &\|\chi_{\sqrt{L}}\,[P\,,\Lambda_1]\,[P\,,\Lambda_2]\,\|_1 \\ &\qquad \leq \|\chi_{\sqrt{L}} < x >^{-2N}\,\|\,\| < x >^{2N}\,[P\,,\Lambda_1]\,[P\,,\Lambda_2]\,\|_1 \;\leq\; C_N L^{-N}\,, \end{aligned} \tag{3.36}$$

where we used Proposition 3 (note that $\dot{B}_0(s) = \dot{P}(s) = g(s)\,[P(s)\,,\Lambda_1]$).

Hence the trace norm of the first contribution in Eq. (3.35) is bounded by $C_N L^{-N}$. Literally the same bounds holds whenever one replace P by $\hat{P}$, therefore all contribution on the r.h.s. of Eq. (3.35) are bounded by the same bound, and choosing L large enough, one can make difference $K_\lambda - \hat{K}_\lambda$ arbitrarily small.

Next, we want to show that $\hat{K}_\lambda$ is independent of λ. For this purpose let us compute its derivative with respect to λ. It is convenient to rewrite $\hat{K}_\lambda$ as

$$\hat{K}_\lambda = \mathrm{Tr}\left[\hat{P}\Lambda_1\hat{P}\,,\hat{P}\Lambda_2\hat{P}\right]\,.$$

Since the expression under the trace is a commutator, so that the derivative of $\hat{K}_\lambda$ is zero if (a) the operator inside the trace is trace class for any coupling in the vicinity of λ and we can interchange the trace and the derivative; (b) if $\partial_\lambda\hat{P}$ is trace class. The first item is a consequence of the bounds

$$\|\left[\hat{P}\,,\Lambda_1\right]\,H^N(s) < x_1 >^N\,\| \leq C_N\,; \quad \|\left[\hat{P}\,,\Lambda_2\right]\,H^N(s) < x_2 >^N\,\| \leq C_N\,,$$

which follow from Proposition 3 and from the complete symmetry between x_1 and x_2. To prove the second one, observe that

$$\partial_\lambda\hat{P} = -\int_\Gamma \mathrm{d}z\hat{R}_z V(1-\chi_L)\hat{R}_z\,.$$

Since $\hat{R}_z\hat{H}$ is uniformly bounded for all $z \in \Gamma$, the trace class condition can be seen from the boundness of $HV(1-\chi_L)H^{-1}$ and Proposition 2. Since the difference between $\hat{K}_\lambda$ and K_λ can be made arbitrarily small, we conclude the result. □

4. Phase space bounds and Trace Estimates

In this section we will prove Propositions 1-3.

Proof of Proposition 1. a) It is enough to check the bounds for $m \geq 0$. The proof is then by induction over m. For $m = 0$ both bounds i) and ii) are obvious. We assume now that i) and ii) hold true for all $m \leq M - 1/2$, and we prove the statements for $m = M$. If $M = 1/2$, i) is clear. For $M \geq 1$ we have, using the concise notation $H \equiv H_t$,

$$H^{M-1/2}\mathbf{p_A}H^{-M} = H^{M-3/2}\mathbf{p_A}H^{-M+1} + H^{M-3/2}[H,\mathbf{p_A}]H^{-M}. \tag{4.1}$$

Here $[\mathbf{p_A}, H] = iB\tilde{\mathbf{p}}_\mathbf{A} + i\nabla W(t)$, where $\tilde{\mathbf{p}}_\mathbf{A} = (p_{A,y}, -p_{A,x})$ if $\mathbf{p_A} = (p_{A,x}, p_{A,y})$. It follows that

$$\begin{aligned} H^{M-1/2}\mathbf{p_A}H^{-M} = {} & H^{M-3/2}\mathbf{p_A}H^{-M+1} + iBH^{M-3/2}\tilde{\mathbf{p}}_\mathbf{A}H^{-M} \\ & + iH^{M-3/2}\nabla W(t)H^{-M} \end{aligned} \tag{4.2}$$

and thus that

$$\|H^{M-1/2}\mathbf{p_A}H^{-M}\| \leq (1+B)\,\|H^{M-3/2}\mathbf{p_A}H^{-M+1}\| + \|H^{M-1}\nabla W(t)H^{-M+1}\|.$$

Applying the induction hypothesis with $m = M-1$ (since $W(t) \in H_{2m,\infty}(\mathbb{R}^2)$ we have $\nabla W(t) \in H_{2m-1,\infty}(\mathbb{R}^2) \subset H_{2m-2,\infty}(\mathbb{R}^2)$) it follows that

$$\|H^{M-1/2}\mathbf{p_A} H^{-M}\| \leq C_M$$

for a constant C_M depending only on B and on $D_M = \sup_{t\in\mathbb{R}} \|W(t)\|_{2M,\infty}$. This proves part i). As for part ii) for $m = M$, we consider first the case $M = 1/2$. Then we have

$$H^{1/2} f(x) H^{-1/2} = f(x) + \frac{1}{\pi}\int_0^\infty \frac{ds}{\sqrt{s}} \frac{H^{1/2}}{H+s}[f(x), H]\frac{1}{H+s}. \tag{4.3}$$

Since $[f(x), H] = 1/2(\mathbf{p_A}\cdot\nabla f + \nabla f\cdot \mathbf{p_A})$ we find

$$\begin{aligned} \|H^{1/2} f(x) H^{-1/2}\| \leq \|f(x)\| &+ \frac{1}{2\pi}\int_0^\infty \frac{ds}{\sqrt{s}} \|\frac{H}{H+s}\| \|H^{-1/2}\mathbf{p_A}\| \|\nabla f\| \|\frac{1}{H+s}\| \\ &+ \frac{1}{2\pi}\int_0^\infty \frac{ds}{\sqrt{s}} \|\frac{H^{1/2}}{H+s}\| \|\nabla f\| \|\mathbf{p_A} H^{-1/2}\| \|\frac{H^{1/2}}{H+s}\| \\ \leq C\|f\|_{1,\infty} \end{aligned} \tag{4.4}$$

where the constant C depends only on $D_0 = \sup_{t\in\mathbb{R}} \|W(t)\|_{0,\infty}$. This proves ii) if $M = 1/2$. Now we assume $M \geq 1$. In this case we have

$$\begin{aligned} H^M f(x) H^{-M} &= H^{M-1} f(x) H^{-M+1} + H^{M-1}[H, f(x)]H^{-M} \\ &= H^{M-1} f(x) H^{-M+1} + \frac{1}{2} H^{M-1}\left(\mathbf{p_A}\cdot\nabla f + \nabla f\cdot\mathbf{p_A}\right) H^{-M}, \end{aligned} \tag{4.5}$$

which implies that

$$\begin{aligned} \|H^M f(x) H^{-M}\| \leq \|H^{M-1} f(x) H^{-M+1}\| &+ \|H^{M-1/2}\nabla f H^{-M+1/2}\| \\ \times \frac{1}{2}&\left(\|H^{M-1}\mathbf{p_A} H^{-M+1/2}\| + \|H^{M-1/2}\mathbf{p_A} H^{-M}\|\right). \end{aligned} \tag{4.6}$$

Using the induction hypothesis with $m = M - 1/2$ and $m = M-1$, and using i) for $m = M$ (which was proven above) we find

$$\|H^M f(x) H^{-M}\| \leq C_M \|f\|_{2M,\infty}, \tag{4.7}$$

where C_M depends on B and on $D_M = \sup_{t\in\mathbb{R}} \|W\|_{2M,\infty}$. Here we used that $\|\nabla f\|_{2M-1,\infty} \leq \|f\|_{2M,\infty}$. This completes the proof of the claim a).

b) We assume that $m, n \geq 0$, the other values can be treated similarly. We consider first the case $m \in \mathbb{N}$ (while n can also be half-integer) and we prove the claim by induction over m. For $m = 0$ the result is trivial. Now we assume it holds true for $n \in \mathbb{N}/2$ and $m \leq M-1$, and we prove it for $m = M$ (and all $n \in \mathbb{N}/2$). To this end we proceed by induction over n. For $n = 0$ the claim is again trivial. Thus we assume it holds also if $m = M$ and $n \leq N - 1/2$, for some $N \in \mathbb{N}/2$. Then we have

$$\begin{aligned} H^M \langle x_i\rangle^N H^{-M}\langle x_i\rangle^{-N} = \; & H^{M-1}\langle x_i\rangle^N H^{-M+1}\langle x_i\rangle^{-N} \\ &+ H^{M-1}[H, \langle x_i\rangle^N] H^{-M}\langle x_i\rangle^{-N} = H^{M-1}\langle x_i\rangle^N H^{-M+1}\langle x_i\rangle^{-N} \\ &+ 2iH^{M-1}\mathbf{p_A}\cdot\frac{x_i}{\langle x_i\rangle}\langle x_i\rangle^{N-1} H^{-M}\langle x_i\rangle^{-N} \\ &+ H^{M-1}(2 - \langle x_i\rangle^{-2})\langle x_i\rangle^{N-2} H^{-M}\langle x_i\rangle^{-N}. \end{aligned} \tag{4.8}$$

The first term on the r.h.s. of the last equation is bounded by induction assumption (with $m = M-1$ and $n = N$). The second term on the r.h.s. of (4.8) is also bounded. Indeed, for $N = 1/2$ this follows by part a) of the proposition. Otherwise we can write this term as

$$\left(H^{M-1}p_{A,i}H^{-M}\right)\left(H^{M}\frac{x_i}{\langle x_i\rangle}H^{-M}\right)\left(H^{M}\langle x_i\rangle^{N-1}H^{-M}\langle x_i\rangle^{-N}\right) \tag{4.9}$$

which is bounded, because $H^{M-1}p_{A,i}H^{-M}$ and $H^{M}x_i/\langle x_i\rangle H^{-M}$ are bounded by part a) of the proposition and because $H^{M}\langle x_i\rangle^{N-1}H^{-M}\langle x\rangle^{-N}$ is bounded by the induction assumption. The boundedness of the third term on the r.h.s. of (4.8) can be verified analogously. This proves part b) for all $m \in \mathbb{N}$, and $n \in \mathbb{N}/2$. To establish part b) for half integer values of m we can apply the same induction argument given by Eq. (4.8) and thus it only remains to prove the result for $m = 1/2$ and all $n \in \mathbb{N}/2$. We observe that

$$\begin{aligned}
H^{1/2}\langle x_i\rangle^{n}H^{-1/2}\langle x_i\rangle^{-n} &= 1 + \frac{1}{\pi}\int_0^\infty \frac{ds}{\sqrt{s}}\frac{H^{1/2}}{H+s}[\langle x_i\rangle^{n}, H]\frac{1}{H+s}\langle x_i\rangle^{-n} \\
= 1 + \frac{1}{\pi}\int_0^\infty \frac{ds}{\sqrt{s}}\frac{H^{1/2}}{H+s}&\left(p_{A,i}\frac{x_i}{\langle x_i\rangle}\langle x_i\rangle^{n-1} + \langle x_i\rangle^{n-1}\frac{x_i}{\langle x_i\rangle}\,p_{A,i}\right)\frac{1}{H+s}\langle x_i\rangle^{-n}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|H^{1/2}\langle x_i\rangle^{n}H^{-1/2}\langle x_i\rangle^{-n}\| \le\ & C_1 + C_2\int_0^\infty \frac{ds}{\sqrt{s}}\|\frac{H^{1/2}}{\sqrt{H+s}}\|\|(H+s)^{-1/2}p_{A,i}\| \\
&\times \|\frac{1}{H+s}\|\|(H+s)\langle x_i\rangle^{n-1}\frac{1}{H+s}\langle x_i\rangle^{-n}\| \\
&+ C_3\int_0^\infty \frac{ds}{\sqrt{s}}\|\frac{H^{1/2}}{H+s}\|\|\frac{1}{H+s}\|\|(H+s)\langle x_i\rangle^{n-2}\frac{1}{H+s}\langle x_i\rangle^{-n}\|.
\end{aligned} \tag{4.10}$$

Applying the proposition for $m = 1$ (which, as we already checked, holds true) and the fact that $\|(H+s)^{-1}\| \le (s+1)^{-1}$, last equation proves b) for $m = 1/2$ and for all $n \in \mathbb{N}/2$. □

Next we prove Proposition 2, which gives us a simple condition for an operator to be trace class.

Proof of Proposition 2. This result is based on two observations: the product of two Hilbert Schmidt operators is trace class (by the generalized Hölder inequality for trace ideals), and that the operator

$$\frac{1}{\sqrt{p^2+1}}\langle x\rangle^{-1}$$

is in $\mathcal{S}_3$ in $2D$ by the Birman-Solomyak Theorem [17]. The diamagnetic inequality and Proposition 1 imply that also

$$\frac{1}{\sqrt{p_A^2+1}}\langle x\rangle^{-1} \in \mathcal{S}_3\,,$$

where $\mathbf{p_A} = (\mathbf{p} - \mathbf{A}(\mathbf{x}))$. Since $H_\lambda + B \ge p_A^2$, also

$$\frac{1}{\sqrt{H_\lambda + B + 1}}\langle x\rangle^{-1} \in \mathcal{S}_3$$

in two dimensions, hence

$$\left(\frac{1}{\sqrt{H_\lambda + B + 1}}\langle x\rangle^{-1}\right)^3$$

is trace class. Using Proposition 1, one obtain that

$$\frac{1}{(H_\lambda + B + 1)^{3/2}}\frac{1}{\langle x\rangle^3}$$

is trace class, hence

$$AD \;=\; A\,\langle x\rangle^3 \left\{\frac{1}{\langle x\rangle^3}\frac{1}{(H_\lambda + B + 1)^{3/2}}\right\}(H_\lambda + B + 1)^{3/2}\,D$$

is trace class, using the bounds in Eq. (3.18) and the fact that the trace class is a two-side ideal upon multiplication by bounded operators. □

In order to verify Proposition 3 we have to show that the operators $B_j(s)$, introduced in Section 3, decay in the energy and in the x_1 coordinate. Looking for example at the definition (3.6) of $B_1(s)$, we see that it contains the projection $P(s) = \chi(H(s) \leq E_F)$, which gives the necessary decay in the energy, and also the time derivative $\dot{P}(s) = ig(s)[P(s), \Lambda_1]$ which also gives the decay in the x_1 coordinate (because of the commutator $[P(s), \Lambda_1]$). In the proof below we explain how to make this argument precise and how to generalize it, by induction, to the operators $B_j(s)$, for $j \geq 2$.

Proof of Proposition 3. We proceed by induction over $j \in \mathbb{N}$. We first establish the inequality (3.20) for $j = 0$, namely that

$$\sup_s \|\dot{P}(s)\, H^N(s)\langle x_1\rangle^N\| < C_N\,. \tag{4.11}$$

In order to check the last inequality we use the simple identity $\dot{P}(s)H^N(s) = \partial_s(P(s)H^N(s)) - P(s)\partial_s(H^N(s))$. Since

$$\partial_s\left(H^N(s)\right) = g(s)\sum_{l=0}^{N-1} H^l(s)[H(s),\Lambda_1]H^{N-1-l}(s),$$

we find

$$\begin{aligned}\|\dot{P}(s)H^N(s)\langle x_1\rangle^N\| \leq{}& \|\partial_s(P(s)H^N(s))\langle x_1\rangle^N\| \\ &+ |g(s)|\sum_{k=0}^{N-1}\|P(s)H^k(s)[H(s),\Lambda_1]H^{N-1-k}(s)\langle x_1\rangle^N\|.\end{aligned} \tag{4.12}$$

Consider first the terms in the sum over k. Since $\partial_1\Lambda_1 = 0$ if $|x_1| > m$, for some $m > 0$, and because of the locality of $H(s)$, we have $[H(s),\Lambda_1]H^{N-1-l}(s) = [H(s),\Lambda_1]H^{N-1-l}(s)\chi(|x_1| \leq m)$. Since $\|\chi(|x_1| \leq m)\langle x_1\rangle^N\|$ and $\|P(s)H^N(s)\|$ are bounded for all $N \in \mathbb{N}$ and uniformly in s, we find

$$\|P(s)H^k(s)[H(s),\Lambda_1]H^{N-1-k}(s)\langle x_1\rangle^N\| \leq C_N\,\|H^{-N+k}[H(s),\Lambda_1]H^{N-1-k}(s)\|,$$

where the r.h.s. is finite by Proposition 1, part a). It remains to consider the first term on the r.h.s. of (4.12). Here we use the integral representation

$$P(s)H^N(s) = \int_\Gamma \mathrm{d}z z^N R_z(s)$$

and we find, since $\dot{R}_z(s) = -ig(s)R_z(s)[H(s),\Lambda_1]R_z(s)$, that

$$\|\partial_s(P(s)H^N(s))\langle x_1\rangle^N\| \le |g(s)| \int_\Gamma |dz||z|^N \|R_z(s)[H(s),\Lambda_1]\langle x_1\rangle^N\| \\ \times \|\langle x_1\rangle^{-N} R_z(s)\langle x_1\rangle^N\|.$$

The r.h.s. of the last equation is bounded because $[H(s),\Lambda_1] = [H(s),\Lambda_1]\chi(|x_1| \le m)$ and because, by Proposition 1, $\|\langle x_1\rangle^{-N} R_z(s)\langle x_1\rangle^N\| < C_N$, uniformly in s and in $z \in \Gamma$. This establishes Eq. (4.11). In order to prove Eq. (3.19) for $j = 1$ we use that, by virtue of (3.6),

$$\begin{aligned}\|B_1(s)H^N\langle x_1\rangle^N\| &\le \frac{1}{2\pi}\int_\Gamma |dz|\|R_z(s)[P(s),\dot{P}(s)]R_z(s)H^N(s)\langle x_1\rangle^N\| \\ &\le C\int_\Gamma |dz|\|\dot{P}(s)H^N(s)\langle x_1\rangle^N\| \\ &\times \Big(\|\langle x_1\rangle^{-N}P(s)R_z(s)\langle x_1\rangle^N\| + \|\langle x_1\rangle^{-N}R_z(s)\langle x_1\rangle^N\|\Big),\end{aligned} \tag{4.13}$$

and that the expression in the brackets is bounded by C_N. Indeed, it follows from Proposition 1, using an integral representation of $P(s)$ in terms of $R_z(s)$. In order to verify Eq. (3.20) for $j = 1$ we use that, by Proposition 1,

$$\|\langle x_1\rangle^{-N}H^{-N}(s)\Lambda_1 H^N(s)\langle x_1\rangle^N\| \le C_N, \tag{4.14}$$

and that $\dot{B}_1(s) = g(s)[\Lambda_1, B_1(s)] + \dot{g}(s)(g(s))^{-1}B_1(s)$.

For general $j > 1$ we have

$$B_j(s) \;=\; \frac{1}{2\pi}\int_\Gamma R_z(s)\left[P(s)\,,\dot{B}_{j-1}(s)\right]R_z(s)\mathrm{d}z + \; S_j(s) - 2P(s)S_j(s)P(s)\;,$$

and

$$S_j(s) \;=\; \sum_{m=1}^{j-1} B_m(s)B_{j-m}(s)\;. \tag{4.15}$$

Thus Eq. (3.19) follows directly by the induction Hypothesis and Proposition 1. Similarly, Eq. (3.20) can be proven using the induction hypothesis because $\dot{B}_j(s) = g(s)[\Lambda_1, B_j(s)] + \tilde{B}_j(s)$, where $\tilde{B}_j(s)$ has the same structure as $B_j(s)$. □

5. Propagation Estimate

In order to prove Proposition 5 we first need an auxiliary result, which ensures that the energy remains bounded during the physical evolution. We first learned about the existence of such bounds from Gian Michele Graf.

Lemma 5.1 (Energy boundedness)**.** *Suppose $H_\lambda(t)$ is as in Eq. (1.1) and let $U(t,s)$ be the time evolution generated by $H_\lambda(t)$. Then*

$$\sup_{s,t\in\mathbb{R}} \|H_\lambda^{-m/2}(s)U(s,t)H_\lambda^{m/2}(t)\| \le C_m \tag{5.1}$$

for all integer values of m.

Proof. We use the gauged transformed Hamiltonians $H(s) = e^{i\phi(s)\Lambda_1} H_\lambda e^{-i\phi(s)\Lambda_1}$ (where s is the scaled time) and the time evolution $U_\tau(s,t) = e^{i\phi(s)\Lambda_1} U(s,t) e^{-i\phi(t)\Lambda_1}$ generated by the Hamiltonians $H(s)$. Moreover since the $H(s)$ are constant for $s > 1$, it is enough to prove that

$$\sup_{s,t\in[0,1]} \|H^{-m/2}(s) U_\tau(s,t) H^{m/2}(t)\| \leq C_m. \tag{5.2}$$

We use the following identity:

$$H^{-\frac{1}{2}}(s) U_\tau(s,t) H^{\frac{1}{2}}(t) U_\tau(t,s) = 1 + \int_s^t dr H^{-\frac{1}{2}}(s) \frac{d}{dr} \left(U_\tau(s,r) H^{\frac{1}{2}}(r) U_\tau(r,s) \right).$$

Since $\partial_r H^{1/2}(r) = ig(r)[H^{1/2}(r), \Lambda_1]$ we can multiply both sides by $U_\tau(s,t)$ from the right to get

$$\begin{aligned} H^{-1/2}(s) U_\tau(s,t) H^{1/2}(t) &= U_\tau(s,t) \\ &+ \int_s^t \mathrm{d}r \, ig(r) H(s)^{-1/2} U_\tau(s,r) [H(r)^{1/2}, \Lambda_1] U_\tau(r,t) \,. \end{aligned} \tag{5.3}$$

Since $\|[H^{1/2}(r), \Lambda_1]\| < \infty$ (see Eq. (5.4) below with $k = 0$), last equation proves the proposition for $m = 1$. For larger values of m we use induction. Eq. (5.3) implies that

$$\begin{aligned} H^{-m/2}(s) U_\tau(s,t) H^{m/2}(t) &= H^{-(m-1)/2}(s) U_\tau(s,t) H^{(m-1)/2}(t) \\ &+ i \int_s^t \mathrm{d}r \dot{g}(r) H^{-m/2}(s) U_\tau(s,r) [H(r)^{1/2}, \Lambda_1] U_\tau(r,t) H^{(m-1)/2}(t) \,. \end{aligned}$$

The first term is bounded by the induction hypothesis. The second one is also bounded by the induction hypothesis and because of the bound:

$$\|H^{-k/2}(r) [H^{1/2}(r), \Lambda_1] H^{k/2}(r)\| \leq C_k \,. \tag{5.4}$$

To prove the last relation we use the integral representation

$$\sqrt{H} = \frac{2}{\pi} \int_0^\infty \mathrm{d}x \frac{H}{x^2 + H} \,, \tag{5.5}$$

which implies that (using the concise notation $H \equiv H(r)$):

$$H^{-k/2} [H^{1/2}, \Lambda_1] H^{k/2} = \frac{2}{\pi} \int_0^\infty \mathrm{d}x \, x^2 \, \frac{H^{-k/2}}{x^2 + H} [H, \Lambda_1] \frac{H^{k/2}}{x^2 + H} \,. \tag{5.6}$$

Now we commute one of the resolvent $(x^2 + H)^{-1}$ through the commutator $[H, \Lambda_1]$ and we find

$$\begin{aligned} H^{-k/2} [H^{1/2}, \Lambda_1] H^{k/2} &= \frac{2}{\pi} \int_0^\infty \mathrm{d}x \, x^2 \, \frac{H^{-k/2}}{(x^2 + H)^2} [H, \Lambda_1] H^{k/2} \\ &+ \frac{2}{\pi} \int_0^\infty \mathrm{d}x \, x^2 \, \frac{H^{-k/2}}{(x^2 + H)^2} [H, [H, \Lambda_1]] \frac{H^{k/2}}{x^2 + H}. \end{aligned} \tag{5.7}$$

The first term on the r.h.s. of the last equation equals $H^{-(k+1)/2}[H,\Lambda_1]H^{k/2}$ and is bounded, by Proposition 1, a) - because $[H,\Lambda_1] = 1/2(\mathbf{p_A}\cdot\nabla\Lambda_1+\nabla\Lambda_1\cdot\mathbf{p_A})$. On the other hand the norm of the second term on the r.h.s. of (5.7) is bounded by

$$\frac{2}{\pi}\int_0^\infty \mathrm{d}x\,\|\frac{x^2}{x^2+H}\|\,\|\frac{H}{x^2+H}\|\,\|H^{-k/2-1}[H,[H,\Lambda_1]]H^{k/2}\|\,\|\frac{1}{x^2+H}\| \tag{5.8}$$

which is finite, because $\|x^2(x^2+H)^{-1}\|$ and $\|H(x^2+H)^{-1}\|$ are bounded by 1, and $\|(x^2+H)^{-1}\|\le(C+x^2)^{-1}$ (this term ensures the convergence of the integral), and by Proposition 1, $\|H^{-k/2-1}[H,[H,\Lambda_1]]H^{k/2}\|\le C_k$ (here we capitalize on the fact that $[H,[H,\Lambda_1]]$ is quadratic in $\mathbf{p_A}$). This establishes (5.4). $\square$

Proof of Proposition 5. We only consider the case $t_1=t$ and $t_2=0$: if $t_2\neq 0$ the proof is identical. We proceed by induction over n. If $n=0$ the claim follows by Lemma 5.1. Now we take some $N\in\mathbb{N}/2$, we assume the proposition holds true for all $n=0,1/2,1,\ldots,N-1/2$ and for all $m\in\mathbb{N}/2$, and we prove it for $n=N$. To this end we use induction over m. First of all we have to prove the proposition for $n=N$ and $m=0$. In the following we denote $H_t=H_\lambda(t)$. We have

$$\begin{aligned}
\|\langle x_i\rangle^N U(t,0)H_0^{-N}\varphi\|^2 &= \langle\varphi, H_0^{-N}U(0,t)\langle x_i\rangle^{2N}U(t,0)H_0^{-N}\varphi\rangle\\
&= \langle\varphi,H_0^{-N}\langle x_i\rangle^{2N}H_0^{-N}\varphi\rangle + \int_0^t ds\langle\varphi,H_0^{-N}U(0,s)[iH(s),\langle x_i\rangle^{2N}]U(s,0)H_0^{-N}\varphi\rangle\\
&= \|\langle x_i\rangle^N H_0^{-N}\varphi\|^2\\
&\quad+\int_0^t ds\langle\varphi,H_0^{-N}U(0,s)\left\{2N\langle x_i\rangle^{2N-1}\tfrac{x_i}{\langle x_i\rangle}\,p_{A,i}+h.c\right\}U(s,0)H_0^{-N}\varphi\rangle.
\end{aligned}\tag{5.9}$$

The first term on the r.h.s. of the last equation can be estimated as

$$\|\langle x_i\rangle^N H_0^{-N}\varphi\|\le\|\langle x_i\rangle^N H_0^{-N}\langle x_i\rangle^{-N}H_0^N\|\|H_0^{-N}\langle x_i\rangle^N\varphi\|\le C\|\langle x_i\rangle^N\varphi\|, \tag{5.10}$$

by Proposition 1. Now we note that

$$\begin{aligned}
\langle\varphi,&H_0^{-N}U(0,s)\left\{2N\langle x_i\rangle^{2N-1}\frac{x_i}{\langle x_i\rangle}\,p_{A,i}+h.c\right\}U(s,0)H_0^{-N}\varphi\rangle\\
&\le C_1\sum_{i=1,2}\|\langle x_i\rangle^{N-1/2}U(s,0)H_0^{-N}\varphi\|\|\langle x_i\rangle^{N-1/2}p_{A,i}U(s,0)H_0^{-N}\varphi\|.
\end{aligned}\tag{5.11}$$

The first factor on the r.h.s. can be bounded using the induction assumption with $n=N-1/2$ and $m=0$. In the second factor, on the other hand, we commute $p_{A,i}$ to the left. We get

$$\begin{aligned}
\|\langle x_i\rangle^{N-1/2}p_{A,i}U(s,0)H_0^{-N}\varphi\|\le\|p_{A,i}\langle x_i\rangle^{N-1/2}U(s,0)H_0^{-N}\varphi\|\\
+(n-1/2)\|\langle x_i\rangle^{N-3/2}U(s,0)H_0^{-N}\varphi\|
\end{aligned}\tag{5.12}$$

To handle the first term on the r.h.s. of the last equation we write

$$\begin{aligned}
\|p_{A,i}\langle x_i\rangle^{N-1/2}U(s,0)H_0^{-N}\varphi\|\le\|p_{A,i}H_s^{-1/2}\|\|H_s^{1/2}\langle x_i\rangle^{N-1/2}H_s^{-1/2}\langle x_i\rangle^{-N+1/2}\|\\
\times\|\langle x_i\rangle^{N-1/2}H_s^{1/2}U(s,0)\varphi\|,
\end{aligned}\tag{5.13}$$

where the factor $H_s^{1/2}\langle x_i\rangle^{N-1/2}H_s^{-1/2}\langle x_i\rangle^{-N+1/2}$ is bounded, uniformly in s, by Proposition 1, part b). Now we insert the last equation into (5.12) and we substitute the result into (5.11). Then we use the induction assumption for $n=N-1/2$ and

$m = 1/2$ and for $n = N - 3/2$ and $m = 0$ (in order to bound the contribution of the second term on the r.h.s. of (5.12)), and we get

$$\begin{aligned}\langle \varphi, H_0^{-N} U(0,s) &\left\{ 2N \langle x_i \rangle^{2N-1} \frac{x_i}{\langle x_i \rangle} \, p_{A,i} + h.c \right\} U(s,0) H_0^{-N} \varphi \rangle \\ &\leq C (s^{N-1/2} + 1)^2 \| \langle x_i \rangle^{N-1/2} \varphi \|^2\end{aligned} \tag{5.14}$$

Inserting this equation and (5.10) in the r.h.s. of (5.9) and performing the integration over s we find

$$\| \langle x_i \rangle^N U(t,0) H_0^{-N} \varphi \|^2 \leq C (t^{2N} + 1) \| \langle x_i \rangle^N \varphi \|^2 \tag{5.15}$$

which proves the proposition for $n = N$ and $m = 0$.

Now we assume that the proposition holds true for $n \leq N - 1/2$ and all $m \in \mathbb{N}/2$ and also for $n = N$ and $\mathbb{N}/2 \ni m \leq M - 1/2$, for some $M \in \mathbb{N}/2$, and we verify it for $n = N$ and $m = M$. We compute

$$\begin{aligned}\| \langle x_i \rangle^N & H_t^M U(t,0) H_0^{-N-M} \varphi \|^2 \\ &= \langle \varphi, H_0^{-N-M} U(0,t) H_t^M \langle x_i \rangle^{2N} H_t^M U(t,0) H_0^{-N-M} \varphi \rangle \\ &= \langle \varphi, H_0^{-N} \langle x_i \rangle^{2N} H_0^{-N} \varphi \rangle \\ &\quad + \int_0^t ds \frac{d}{ds} \langle \varphi, H_0^{-N-M} U(0,s) H_s^M \langle x_i \rangle^{2N} H_s^M U(s,0) H_0^{-N-M} \varphi \rangle \\ &= \langle \varphi, H_0^{-N} \langle x_i \rangle^{2N} H_0^{-N} \varphi \rangle \\ &\quad + \int_0^t ds \, \langle \varphi, H_0^{-N-M} U(0,s) H_s^M [iH(s), \langle x_i \rangle^{2N}] H_s^M U(s,0) H_0^{-N-M} \varphi \rangle \\ &\quad + \int_0^t ds \, \langle \varphi, H_0^{-N-M} U(0,s) \left\{ (\tfrac{d}{ds} H_s^M) \langle x_i \rangle^{2N} H_s^M + h.c \right\} U(s,0) H_0^{-N-M} \varphi \rangle.\end{aligned} \tag{5.16}$$

The first term on the r.h.s. of the last equation can be bounded, using Proposition 1, part b), by

$$\begin{aligned}\langle \varphi, H_0^{-N} \langle x_i \rangle^{2N} H_0^{-N} \varphi \rangle &= \| \langle x_i \rangle^N H_0^{-N} \varphi \|^2 \\ &\leq \| \langle x_i \rangle^N H_0^{-N} \langle x_i \rangle^{-N} H_0^N \|^2 \| H_0^{-N} \|^2 \| \langle x_i \rangle^N \varphi \|^2 \\ &\leq C \| \langle x_i \rangle^N \varphi \|^2.\end{aligned} \tag{5.17}$$

The second term on the r.h.s. of (5.16) can be controlled in the same way as we did with the second term on the r.h.s. of (5.9). We find, applying the induction assumption for $n = N - 1/2$ and $m = M$

$$\begin{aligned}\int_0^t ds \, \langle \varphi, H_0^{-N-M} U(0,s) H_s^M [iH(s), \langle x_i \rangle^{2N}] H_s^M U(s,0) H_0^{-N-M} \varphi \rangle& \\ \leq C (t^{2N} + 1) \| \langle x_i \rangle^{N-1/2} \varphi \|^2.&\end{aligned} \tag{5.18}$$

We consider now the third term on the r.h.s. of (5.16), and we first assume that $M \in \mathbb{N}$. Then we have

$$\frac{d}{ds} H_s^M = \sum_{j=1}^M H_s^{j-1} \dot{H}_\lambda(s) H_s^{m-j} \tag{5.19}$$

where $\dot{H}_\lambda(s) = 1/\tau^2\dot{g}(s/\tau)\Lambda_1$. Note here that $\dot{H}_\lambda(s) = 0$ if $s > \tau$. From last equation it follows that

$$(\frac{d}{ds}H_s^M)\langle x_i\rangle^{2N}H_s^M + h.c. = \frac{\dot{g}(s/\tau)}{\tau^2}\sum_{j=1}^{M}\left(H_s^{j-1}\Lambda_1 H_s^{M-j}\langle x_i\rangle^{2N}H_s^M + h.c.\right). \tag{5.20}$$

To handle this term we note that, for any $j = 1, 2, \dots M$, there is a constant $D < \infty$ such that

$$H_s^{j-1}\Lambda_1 H_s^{M-j}\langle x_i\rangle^{2N}H_s^M + h.c. \leq D H_s^{M-1/2}\langle x_i\rangle^{2N}H_s^{M-1/2}. \tag{5.21}$$

To prove the last equation we set $A = H_s^{M-1/2}\langle x_i\rangle^{2N}H_s^{M-1/2}$. First of all we note that $A \geq 0$ and that

$$\|\frac{1}{\sqrt{A}}H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{M/2-1/4}\| < \infty.$$

In fact for any $\varphi \in \mathcal{H}$ we have

$$\begin{aligned}
&\|\frac{1}{\sqrt{A}}H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{M/2-1/4}\varphi\|^2 \\
&\quad= \langle\varphi, H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{M/2-1/4}\, A^{-1}\, H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{M/2-1/4}\varphi\rangle \\
&\quad= \langle\varphi, H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{-M/2+1/4}\langle x_i\rangle^{2N}H_s^{-M/2+1/4}\langle x_i\rangle^N H_s^{M/2-1/4}\varphi\rangle \\
&\quad\leq \|H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{-M/2+1/4}\langle x_i\rangle^N\|^2\|\varphi\|^2 \\
&\quad\leq C\|\varphi\|^2,
\end{aligned} \tag{5.22}$$

where we used Proposition 1, part b). Now, if we denote by B the operator on the l.h.s. of (5.21) we have

$$B = \sqrt{A}\frac{1}{\sqrt{A}}B\frac{1}{\sqrt{A}}\sqrt{A} \tag{5.23}$$

and (5.21) follows if we prove that $A^{-1/2}BA^{-1/2}$ is bounded. Since we know that $A^{-1/2}H_s^{M/2-1/4}\langle x_i\rangle^N H_s^{M/2-1/4}$ is bounded, it is enough to prove the boundedness of

$$\begin{aligned}
&H_s^{-M/2+1/4}\langle x_i\rangle^{-N}H_s^{-M/2+1/4}BH_s^{-M/2+1/4}\langle x_i\rangle^{-N}H_s^{-M/2+1/4} \\
&= H_s^{-M/2+1/4}\langle x_i\rangle^{-N}H_s^{-M/2+j-3/4}\Lambda_1 H_s^{M-j}\langle x_i\rangle^{2N}H_s^{M/2+1/4}\langle x_i\rangle^{-N}H^{-M/2+1/4} \\
&= C_1\langle x_i\rangle^{-N}H_s^{-M+j-1/2}\Lambda_1 H_s^{M-j}\langle x_i\rangle^{2N}H_s^{1/2}\langle x_i\rangle^{-N}C_2,
\end{aligned} \tag{5.24}$$

where, by Proposition 1, the operators $C_1 = H_s^{-M/2+1/4}\langle x_i\rangle^{-N}H_s^{M/2-1/4}\langle x_i\rangle^N$ and $C_2 = \langle x_i\rangle^N H^{M/2-1/4}\langle x_i\rangle^{-N}H^{-M/2+1/4}$ are bounded. Using part b) of that statement we can exchange the operators $\langle x_i\rangle^{-N}$ with the powers of the Hamiltonian H_s once again. The operator on the r.h.s. of the last equation can thus be written as

$$\begin{aligned}
\tilde{C}_1 H_s^{-M+j-1/2}\langle x_i\rangle^{-N}\Lambda_1 H_s^{M-j}\langle x_i\rangle^N H_s^{1/2}\tilde{C}_2 &= \tilde{C}_1 H_s^{-M+j-1/2}\Lambda_1 H_s^{M-j+1/2} \\
&\quad\times H_s^{-M+j-1/2}\langle x_i\rangle^{-N}H_s^{M-j}\langle x_i\rangle^N H_s^{1/2}\tilde{C}_2,
\end{aligned}$$

where the operators $\tilde{C}_1$ and $\tilde{C}_2$ are bounded. Because of Proposition 1, part a), the operator $H_s^{-M+j-1/2}\Lambda_1 H_s^{M-j+1/2}$ is bounded. So, if we exchange the operators $H_s^{-M+j-1/2}$ and $\langle x_i\rangle^{-N}$, using Proposition 1, part b), the operators on the r.h.s. of the last equation becomes

$$\tilde{\tilde{C}}_1\langle x_i\rangle^{-N}H_s^{-1/2}\langle x_i\rangle^N H_s^{1/2}\tilde{C}_2, \tag{5.25}$$

for some bounded operator $\tilde{\tilde{C}}_1$. But now, because of Proposition 1, also the operator $\langle x_i\rangle^{-N}H_s^{-1/2}\langle x_i\rangle^N H_s^{1/2}$ is bounded. This establishes that the operator $A^{-1/2}BA^{-1/2}$ is bounded and completes the proof of Eq. (5.21).

Plugging Eq. (5.21) into the r.h.s. of (5.20) we observe that the third term on the r.h.s. of (5.16) is bounded by

$$\begin{aligned}
&\int_0^t ds\, \langle\varphi, H_0^{-N-M}U(0,s)\left\{(\tfrac{d}{ds}H_s^M)\langle x_i\rangle^{2N}H_s^M + h.c\right\}U(s,0)H_0^{-N-M}\varphi\rangle \\
&\leq \frac{C}{\tau^2}\int_0^t ds\chi(s\leq\tau)\langle\varphi, H_0^{-N-M}U(0,s)H_s^{M-1/2}\langle x_i\rangle^{2N}H_s^{M-1/2}U(s,0)H_0^{-N-M}\varphi\rangle \\
&\leq \frac{C}{\tau^2}\int_0^{\min(t,\tau)}(s^{2N}+1)\|\langle x_i\rangle^N\varphi\|^2 \leq \frac{C}{\tau^2}\left(\min(t,\tau)^{2N+1}+1\right)\|\langle x_i\rangle^N\varphi\|^2 \\
&\leq C(t^{2N-1}+1)\|\langle x_i\rangle^N\varphi\|^2\,,
\end{aligned} \tag{5.26}$$

where we used the induction assumption for $n = N$ and $m = M - 1/2$. A similar result can also be proved if $M \in \mathbb{N}/2$ is not an integer. Inserting (5.26), (5.18) and (5.17) into (5.16) we finally find that

$$\|\langle x_i\rangle^N H_t^M U(t,0)H_0^{-N-M}\varphi\|^2 \leq C(t^{2N}+1)\|\langle x_i\rangle^N\varphi\|^2\,. \tag{5.27}$$

$\square$

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