Periodic orbits close to elliptic tori and applications to the three-body problem

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Abstract: We prove, under suitable non-resonance and non-degeneracy "twist" conditions, a Birkhoff-Lewis type result showing the existence of infinitely many periodic solutions, with larger and larger minimal period, accumulating onto elliptic invariant tori (of Hamiltonian systems). We prove the applicability of this result to the spatial planetary three-body problem in the small eccentricity-inclination regime. Furthermore, we find other periodic orbits under some restrictions on the period and the masses of the "planets". The proofs are based on averaging theory, KAM theory and variational methods[1].

Keywords: Periodic Orbits, Nearly-Integrable Hamiltonian Systems, Elliptic Tori, Averaging Theory, KAM Theory, Lyapunov-Schmidt Reduction, Variational Methods, Three-body Problem.

Contents

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Introduction

A classical problem in Hamiltonian dynamical systems concerns, since the researches of Poincaré, the existence of periodic orbits in the vicinity of invariant submanifolds.

Poincaré wrote: *"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13*[2] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut."* ([P], Tome 1, ch. III, a. 36).

This conjecture was often quoted by Birkhoff as a main motivation for his works. In the thirties, Birkhoff and Lewis [B]-[BL]-[L] established the existence of infinitely many periodic solutions in a neighborhood of an elliptic equilibrium whose linear frequencies are sufficiently *non-resonant*. This Theorem also requires a non-degeneracy - *"twist"*- condition, involving finitely many Taylor coefficients of the Hamiltonian at the equilibrium, and implying the system to be genuinely non-linear. In addition, if the Hamiltonian is sufficiently smooth, KAM theory ensures, in a neighborhood of the equilibrium small enough, the existence of invariant Lagrangian tori filling up a set of positive Lebesgue measure, see [Pö]. Furthermore, close to any KAM torus, it has been proved in [CZ] the existence of infinitely many others periodic orbits with larger and larger minimal period accumulating to the torus itself (as a consequence, the closure of the periodic orbits has positive Lebesgue measure). The result of [CZ] is proved considering the normal form Hamiltonian which describes the dynamics near each torus, checking the "twist condition", and, then, applying the Birkhoff-Lewis type Theorem of [Mo].

Actually, Birkhoff and Lewis established also the existence of infinitely many periodic solutions close to a non-constant periodic elliptic solution.

The question of the existence of periodic orbits with larger and larger minimal period clustering to *elliptic lower dimensional* invariant tori (of dimension greater than one) has not been yet investigated. An invariant torus is called elliptic, or linearly stable, if the linearized system along the torus, possesses purely imaginary eigenvalues.

A main motivation for studying such problem is Celestial Mechanics, which, indeed, inspired the whole development of KAM theory (Arnold [A] devoted one of the fundamental papers of this theory to the planar n-body problem).

Consider, in particular, the *non-planar planetary three body problem*, namely one "star" and two "planets" interacting through a Newtonian gravitational field in the three dimensional space. The masses of the planets are regarded as small parameters. According to Poincaré and Delaunay this system is described by a nearly-integrable Hamiltonian on a eight dimensional phase space, equipped with real-analytic action-angle variables. Such system turns out to be *properly degenerate*, i.e. the integrable limit (in which the three body-problem is described by two decoupled and integrable two-body systems) depends only on two action variables. Hence, in the integrable limit all bounded

[2] The Hamilton's equations.

motions lie on two-dimensional invariant tori supporting periodic or quasi-periodic motions according the ratio between the two frequencies (related, by Kepler's law, to the major semi-axis of the two limiting Keplerian ellipes) is a rational or irrational number. Furthemore, in the small eccentricity-inclination regime (of astronomical interest) these unperturbed two-tori are elliptic: the spatial planetary three body-problem calls for a perturbation theory for continuing *elliptic lower-dimensional* tori.

In the last years, an exhaustive perturbation theory for elliptic tori has indeed been developed by many authors, see [M], [E], [W], [K], [Pö1], [Pö2], [Bo] and [XJ]. The persistence of elliptic tori is ensured requiring "Melnikov non-resonance conditions" among the frequencies and further non-degeneracy conditions.

In light of these results, the spatial planetary three body problem (in the small eccentricity-inclination regime) has been recently reexamined in [BCV], where the persistence of two-dimensional elliptic invariant tori -with diophantine frequencies- has been proved, provided the Keplerian major semi-axes belong to a two dimensional set of positive measure.

Previous works on the spatial planetary three-body problem are [JM], for large inclinations (in this case the two-tori are hyperbolic), and [LR]-[R], for maximal dimensional tori (the proofs are computer assisted).

In the present paper we first prove, under suitable non-resonance and non-degeneracy "twist" conditions, a general Birkhoff-Lewis type result showing the existence of infinitely many periodic solutions, with larger and larger minimal period, accumulating onto elliptic lower dimensional invariant tori, see *Theorem 1.1* for the precise statement.

Furthermore we prove the applicability of Theorem 1.1 to the spatial planetary three-body problem, showing the existence of infinitely many periodic solutions accumulating on the elliptic KAM tori of [BCV], see *Theorem 0.2*. Such periodic orbits revolve around the star close to Keplerian ellipses with small eccentricities and small non-zero mutual inclinations. The verification of the "twist" condition (Lemma 5.1) is based on a KAM analysis.

Finally, in *Theorems 0.3, 0.4* we prove the existence of other periodic solutions of the spatial planetary three body-problem, the "periodic analogue" of the elliptic tori of [BCV], see *Theorem 6.1* for a more general statement.

We now present a simplified version of our results.

Periodic orbits accumulating on elliptic tori. As already said, the persistence of elliptic invariant tori is ensured assuming "Melnikov non-resonance conditions" among the frequencies. In particular, under the "second order Melnikov conditions" (see precisely (4)), the surviving tori are still elliptic and the normal form Hamiltonian describing the dynamics in a neighborhood is

$$\mathcal{H}_*(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) = \omega \cdot \mathcal{I}_* + \Omega Z_* \cdot \overline{Z}_* + \sum_{2|k|+|a+\overline{a}|\geq 3} R^*_{k,a,\overline{a}}(\varphi_*) \mathcal{I}_*^k Z_*^a \overline{Z}_*^{\overline{a}} \ ,$$

where $(\mathcal{I}_*, \varphi_*) \in \mathbf{R}^n \times \mathbf{T}^n$ and $(Z_*, \overline{Z}_*) \in \mathbf{C}^{2m}$. In these coordinates $\mathcal{T} := \{\mathcal{I}_* = 0, \varphi_* \in \mathbf{T}^n, Z_* = \overline{Z}_* = 0\}$ is the invariant and elliptic torus. $\omega \in \mathbf{R}^n$ are the torus frequencies and $\Omega \in \mathbf{R}^m$ the elliptic one's.

We refer to [JV] and [BHS] for many results concerning the dynamics of an Hamiltonian system close to an elliptic torus, in particular for the existence, under appropiate non-resonance and non-degeneracy conditions, of Cantor families of other invariant tori of dimensions greater than n.

Also for proving the existence of periodic orbits accumulating onto the elliptic torus $\mathcal{T}$ we need to assume suitable non-resonance and non-degeneracy "twist" conditions. Roughly (see Theorem 1.1 for the detailed statement) we have:

Theorem 0.1 *Under suitable non-resonance and non-degeneracy conditions, the Hamiltonian system generated by $\mathcal{H}_*$ affords infinitely many periodic solutions, with larger and larger minimal period, accumulating on the invariant elliptic torus $\mathcal{T}$.*

The precise Theorem 1.1 and a description of its proof are given in Section 1. Theorem 1.1 implies, in particular, the result of [CZ] for maximal dimensional tori.

The planetary, spatial three body-problem. Let $\varepsilon > 0$ denote the small parameter measuring the ratios between the masses of the planets and the mass of the star (see (92)). The existence of two-dimensional elliptic invariant tori has been proved in [BCV]. Roughly (see Theorem 5.2 for a precise statement and proof) their result states:

Theorem ([BCV]) *The spatial planetary three-body problem affords, for ε small enough, a family $\mathcal{F}$ of two-dimensional, elliptic invariant tori, travelled with diophantine quasi-periodic frequencies, provided the osculating Keplerian major semi-axes belong to a two-dimensional set of density close to one (as ε tends to 0).*

We will prove, in section 5, that:

Theorem 0.2 *The spatial planetary three-body problem affords, for ε small enough, infinitely many periodic solutions, with larger and larger minimal period, accumulating onto each elliptic invariant torus of the family $\mathcal{F}$.*

The proof of Theorem 0.2 boils down to check that the non-resonance and non-degeneracy twist conditions of Theorem 1.1 are fulfilled for the spatial planetary three-body problem. This task is accomplished by estimates based on a careful KAM analysis.

Finally, in section 6, we prove the existence of other periodic orbits of the spatial planetary three-body problem, the "periodic analogue" of the elliptic tori of [BCV]. More precisely, let consider the *integrable* (i.e. $\varepsilon = 0$) three-body problem when the two planets revolve around the star along circular orbits without interacting (*decoupled* two-body systems). Some of these motions will be periodic with, say, minimal period T. Taking into account the mutual interaction between the planets (i.e. $\varepsilon > 0$) we prove the existence of, at least, two, slightly deformed, T-periodic orbits. The parameter ε must belong to a suitable interval of values depending on the period T.

Theorem 0.3 *There exist $T_0 > 0$ and functions $0 < \underline{\varepsilon}(T) < \overline{\varepsilon}(T)$, defined for $T \in [T_0, \infty)$ and decreasing to zero as $T \to +\infty$ such that, for any circular periodic orbits of the decoupled three-body problem with minimal period $T \geq T_0$, for all $\underline{\varepsilon}(T) \leq \varepsilon \leq \overline{\varepsilon}(T)$, there exist at least two, ε-close, geometrically distinct T-periodic orbits of the spatial planetary three-body problem.*

The following Theorem can be seen as an ε-fixed version of Theorem 0.3, requiring, for fixed small values of the masses of the planets, restrictions on the period T.

Theorem 0.4 *There exist $\varepsilon_1 > 0$ and functions $0 < \underline{T}(\varepsilon) < \overline{T}(\varepsilon)$, defined for $\varepsilon \in (0, \varepsilon_1]$ and tending to infinity as $\varepsilon \to 0$, such that, for all $0 < \varepsilon \leq \varepsilon_1$ and for any circular periodic orbit of the decoupled three-body problem with minimal period $\underline{T}(\varepsilon) \leq T \leq \overline{T}(\varepsilon)$, there exist at least two, ε-close, geometrically distinct T-periodic orbits of the spatial planetary three-body problem.*

A more general Theorem implying Theorems 0.3 and 0.4 is given in Theorem 6.1.

The paper is organized as follows. In Section 1, we give the detailed statement of Theorem 0.1, namely Theorem 1.1, and we discuss its proof. Section 2 collects some number theoretical Lemmata used in Section 4 for defining the "non-resonant" periods T. Section 3 is devoted to a normal form averaging result. The proof of Theorem 1.1 is addressed in Section 4. In Section 5, after recalling the Poincaré-Delaunay formulation of the spatial planetary three-body problem, we prove, first, a finer version of the KAM result of [BCV] and, finally, Theorem 0.2. Section 6 contains the proof of Theorems 0.3, 0.4 and 6.1.

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Notations: We denote by $O(\xi)$ a real analytic function whose norm is bounded by $C\xi$, for a suitable constant $C > 0$ and $\forall\, 0 < \xi \leq \xi_0$. $\mathrm{Mat}(n \times n, \mathbf{R})$ (resp. $\mathrm{Mat}(n \times n, \mathbf{C})$) is the set of $n \times n$ matrices with real (resp. complex) coefficients and $\mathbf{1}_n$ the $n \times n$ identity matrix. B_r denotes the (closed) ball of radius r centered at 0, B_r^n the closed ball of radius r centered at 0 using the $|\cdot|_2$ norm, and D_ρ^d the complex d-ball of radius ρ centered at 0. $\sharp A$ is the cardinality of the set A. gcd is the greatest common divisor and lcm the least common multiple. We set $e_i := (0, \ldots, 1, \ldots, 0) \in \mathbf{Z}^n$ the i-th unit versor and $|a|_1 := \sum_{i=1} |a|_1$ norm. Through the paper C_i, c_i, const. will denote positive constants possibly depending n, m, ω, Ω, γ, τ, r, s, ρ, R^*, etc.

1 Periodic orbits accumulating on elliptic tori

We now give the precise statement concerning the existence of periodic orbits close to elliptic tori (Theorem 1.1). Let consider Hamiltonians of the form

$$\mathcal{H}_*(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) = \omega \cdot \mathcal{I}_* + \Omega Z_* \cdot \overline{Z}_* + \sum_{2|k|+|a+\overline{a}| \geq 3} R^*_{k,a,\overline{a}}(\varphi_*) \mathcal{I}_*^k Z_*^a \overline{Z}_*^{\overline{a}}, \tag{1}$$

where $(\mathcal{I}_*, \varphi_*) \in \mathbf{R}^n \times \mathbf{T}^n$ are action-angle variables and $(Z_*, \overline{Z}_*) \in \mathbf{C}^{2m}$ are called the normal (or elliptic) coordinates. The phase space $\mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m \times \mathbf{C}^m$ is equipped with the symplectic form[3] $d\mathcal{I}_* \wedge d\varphi_* + \mathrm{i} dZ_* \wedge d\overline{Z}_*$.

$\omega \in \mathbf{R}^n$ is the frequency vector and $\Omega :=$diag$(\Omega_1, \ldots, \Omega_m)$ is the $m \times m$ diagonal matrix of the normal (or elliptic) frequencies. Hence $\Omega Z_* \cdot \overline{Z}_*$ denotes $\sum_{1 \le j \le m} \Omega_j Z_{*j} \overline{Z}_{*j}$. We will often identify the diagonal matrix Ω with the vector $(\Omega_1, \ldots, \Omega_m) \in \mathbf{R}^m$. For the multi-indexes $k \in \mathbf{N}^n$, $a, \overline{a} \in \mathbf{N}^m$ we define $|k| := \sum_{i=1}^n k_i$, $|a| := \sum_{j=1}^m a_j$, $|\overline{a}| := \sum_{j=1}^m \overline{a}_j$.

The Hamiltonian $\mathcal{H}_*$ is assumed to be real analytic for $(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) \in D^n_{r_*} \times \mathbf{T}^n_{s_*} \times D^{2m}_{\rho_*} \subset \mathbf{C}^{2n+2m}$ for some positive constants r_*, s_*, ρ_*. [4] Real analytic means that $\mathcal{H}_*$ is real analytic in the real and imaginary parts of Z_*. The functions $R^*_{k,a,\overline{a}}(\varphi_*)$ can be expanded in Fourier series as

$$R^*_{k,a,\overline{a}}(\varphi_*) = \sum_{\ell \in \mathbf{Z}^n} R^*_{k,a,\overline{a},\ell}\, e^{\mathrm{i}\ell\cdot\varphi_*} \, . \tag{2}$$

Since $\mathcal{H}_*(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) \in \mathbf{R}$ for all $(\mathcal{I}_*, \varphi_*) \in \mathbf{R}^n \times \mathbf{T}^n$ and for all $Z_* \in \mathbf{C}^m$, the Taylor-Fourier coefficient $R^*_{k,a,\overline{a},\ell}$ satisfy

$$\overline{R^*_{k a,\overline{a},\ell}} = R^*_{k,\overline{a},a,-\ell} \, . \tag{3}$$

The frequency vector (ω, Ω) is assumed to satisfy the "second order Melnikov non-resonance conditions"

$$|\omega \cdot \ell + \Omega \cdot h| \ge \frac{\gamma}{1 + |\ell|^\tau} \, , \qquad \forall\ \ell \in \mathbf{Z}^n \ , h \in \mathbf{Z}^m \ , |h| \le 2 \ , (\ell, h) \neq (0,0) \tag{4}$$

for some positive constants γ, τ. We will use such condition in order to perform the averaging procedure of Section 3.

By condition (4), the frequency ω is rationally independent[5] (actually Diophantine), while (ω, Ω) can possess some resonance relations which can be described as follows. Eventually reordering the frequencies Ω_j, there exists $0 \le \hat{m} \le m$ such that[6] $\hat{\omega} := (\omega, \Omega_{\hat{m}+1}, \ldots, \Omega_m)$ is rationally independent on $\mathbf{Z}^{\hat{n}}$, where $\hat{n} := n + m - \hat{m}$, but $(\hat{\omega}, \Omega_j)$ is rationally dependent for all $1 \le j \le \hat{m}$ ($\hat{m}$ is the number of resonances). This means that there exist $M_j \in \mathbf{N}^+$ and $a_j \in \mathbf{Z}^{\hat{n}}$ such that[7]

$$M_j \Omega_j = a_j \cdot \hat{\omega} \qquad \text{with} \quad \gcd(a_{j1}, \ldots, a_{j\hat{n}}, M_j) = 1, \qquad \forall 1 \le j \le \hat{m}. \tag{5}$$

The Hamiltonian system[8]

$$\dot{\mathcal{I}}_* = -\partial_{\varphi_*}\mathcal{H}_* \ , \quad \dot{\varphi}_* = \partial_{\mathcal{I}_*}\mathcal{H}_* \ , \quad \dot{Z}_* = \mathrm{i}\partial_{\overline{Z}_*}\mathcal{H}_* \ , \quad \dot{\overline{Z}}_* = -\mathrm{i}\partial_{Z_*}\mathcal{H}_* \tag{6}$$

[3]We denote the imaginary unit by i (not to be confused with i often used as an index).

[4]We have used the following standard notations: for a given set $A \subset \mathbf{R}^d$, $d \in \mathbf{N}^+$ and $\delta, s > 0$, we denote by $A^d_\delta := \{z \in \mathbf{C}^d \mid \mathrm{dist}(z, A) < \delta\}$ and $\mathbf{T}^d_s := \{z \in \mathbf{C}^d \mid |\mathrm{Im}\ z_j| < s,\ \forall\ 1 \le j \le d\}$. Moreover $D^d_\delta := \{z \in \mathbf{C}^d \mid |z| < \delta\}$.

[5]i.e. $\omega \cdot \ell \neq 0,\ \forall \ell \in \mathbf{Z}^n$.

[6]if $\hat{m} = m$ we set $\hat{\omega} := \omega$.

[7]We denote by gcd and lcm the greatest common divisor and the least common multiple, respectively.

[8]Setting $Z_* = (P_* + \mathrm{i}Q_*)/\sqrt{2}$, $\overline{Z}_* = (P_* - \mathrm{i}Q_*)/\sqrt{2}$ with $(P_*, Q_*) \in \mathbf{R}^{2m}$ the last two equations are equivalent to $\dot{Q}_* = \partial_{P_*}\mathcal{H}, \dot{P}_* = -\partial_{Q_*}\mathcal{H}$.

possesses the elliptic invariant torus

$$\mathcal{T} := \left\{ (\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*) \in \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^{2m} \,\middle|\, \mathcal{I}_* = 0,\ Z_* = \overline{Z}_* = 0 \right\}$$

supporting the quasi-periodic solutions $(0, \varphi_{*0} + \omega t, 0, 0)$.

We define the symmetric "twist" matrix $\mathcal{R} \in \mathrm{Mat}(n \times n, \mathbf{R})$

$$\mathcal{R}_{i,i'} := (1 + \delta_{i,i'}) R^*_{e_i + e_{i'},0,0,0} - \sum_{\substack{1 \le j \le m \\ \ell \in \mathbf{Z}^n}} \frac{1}{\omega \cdot \ell + \Omega_j} \left(R^*_{e_i,e_j,0,\ell} R^*_{e_{i'},0,e_j,-\ell} + R^*_{e_i,0,e_j,-\ell} R^*_{e_{i'},e_j,0,\ell} \right) , \tag{7}$$

where $R^*_{k,a,\overline{a},\ell}$ are the Fourier coefficients, introduced in (2), of $R^*_{k,a,\overline{a}}(\varphi_*)$ and $\delta_{i,i'} := 1$ if $i = i'$ and 0 if $i \neq i'$. We also define the matrix $\mathcal{Q} \in \mathrm{Mat}(m \times n, \mathbf{R})$ as

$$\begin{aligned} \mathcal{Q}_{ji} \ := \ & R^*_{e_i,e_j,e_j,0} - \sum_{\substack{1 \le i' \le n \\ \ell \in \mathbf{Z}^n}} \frac{\ell_{i'}}{\omega \cdot \ell + \Omega_j} \left(R^*_{e_i,e_j,0,\ell} R^*_{e_{i'},0,e_j,-\ell} + R^*_{e_i,0,e_j,-\ell} R^*_{e_{i'},e_j,0,\ell} \right) \\ & - \sum_{\substack{1 \le j' \le m \\ \ell \in \mathbf{Z}^n}} \frac{1}{\omega \cdot \ell + \Omega_{j'}} \left(R^*_{0,e_j,e_j+e_{j'},-\ell} R^*_{e_i,e_{j'},0,\ell} + R^*_{0,e_j+e_{j'},e_j,\ell} R^*_{e_i,0,e_{j'},-\ell} \right) . \end{aligned} \tag{8}$$

for $1 \le j \le m,\ 1 \le i \le n$. We stress that $\mathcal{R}$ and $\mathcal{Q}$ are *real* matrices by (3).

We now give the precise statement of Theorem 0.1.

Theorem 1.1 *Let the Hamiltonian $\mathcal{H}_*$ in* (1) *be real analytic in $D^n_{r_*} \times \mathbf{T}^n_{s_*} \times D^{2m}_{\rho_*}$ and satisfy condition* (4)*. Assume the "twist" condition* $\det \mathcal{R} \neq 0$ *and one of the following "non-resonance" conditions:*

- (a) *one of the three cases below holds:*
  - *(i)* $m = 1, 2$ (low number of elliptic directions),
  - *(ii)* $\hat{m} = 0$ (no resonances among (ω, Ω)),
  - *(iii)* $M_j \ge \hat{m} \ge 1, \quad \forall 1 \le j \le \hat{m}$ *;*
- (b) $\alpha := \min_{1 \le j \le m} |(\Omega - \mathcal{Q}\mathcal{R}^{-1}\omega)_j| > 0$ *;*

where $\hat{m}$ and M_j are defined in (5)*.*

Then, there exist $\eta_0, C_0, C_1 > 0$ [9] *such that: $\forall \eta \in (0, \eta_0]$, there exist an open set of periods $\mathcal{A}_\eta \subseteq [\frac{1}{\eta^2}, \frac{1}{\eta^2} + C_0]$ with measure greater than $1/C_0$ such that $\forall T \in \mathcal{A}_\eta$ there exist $k := k(T) \in \mathbf{Z}^n$, $\widetilde{\omega} := \widetilde{\omega}(T) \in \mathbf{R}^n$, with $\widetilde{\omega} T = 2\pi k$, $|\widetilde{\omega} - \omega| \le C_1 \eta^2$, and at least n geometrically distinct T-periodic solutions $\zeta_\eta(t) = (\mathcal{I}_{*\eta}(t),\ \varphi_{*\eta}(t),\ Z_{*\eta}(t),\ \overline{Z}_{*\eta}(t))$ of the Hamiltonian system* (6) *satisfying*

- (i) $\sup_{t \in \mathbf{R}} \left(|\mathcal{I}_{*\eta}(t)| + |Z_{*\eta}(t)| + |\overline{Z}_{*\eta}(t)| \right) \le C_1 \eta^2$ *;*

[9] Such constants may depend on n, m, ω, Ω, γ, τ, r_*, s_*, ρ_*, $R^*_{k,a,\overline{a},\ell}$, $\hat{m}$, $\mathcal{Q}$, α.

- (ii) $\sup_{t\in\mathbf{R}} |\varphi_{*\eta}(t) - (\varphi_{*\eta}(0) + \widetilde{\omega} t)| \leq C_1 \eta$.

In particular, the closure of the family of periodic orbits ζ_η of (6) *contains the elliptic torus $\mathcal{T}$. Moreover the minimal period T_{min} of ζ_η satisfies $T_{min} \geq T^{1/(\tau+1)}/C_1 = O(\eta^{-2/(\tau+1)})$.*

Idea of the proof of Theorem 1.1. Since (ω, Ω) satisfy the second order Melnikov non-resonance conditions (4), in view of an averaging procedure (Proposition 3.1), the normal form Hamiltonian $\mathcal{H}_*$ is, in a suitable set of coordinates $(I, \phi, z, \overline{z}) \in \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^{2m}$, and sufficiently close to the torus $\mathcal{T}$, a small perturbation of the integrable Hamiltonian

$$H_{int} := \omega \cdot I + \frac{\eta^2}{2} \mathcal{R} I \cdot I + \Omega z \overline{z} + \eta^2 \mathcal{Q} I \cdot z\overline{z},$$

where $\eta > 0$ is a small rescaling parameter measuring the distance from $\mathcal{T}$. The Hamiltonian system generated by H_{int} possesses the elliptic tori $\mathcal{T}(I_0) := \{I = I_0,\ \phi \in \mathbf{T}^n,\ z = 0\}$ supporting the linear flow $t \to \{I_0,\ \phi_0 + (\omega + \eta^2 \mathcal{R} I_0)t, 0\}$. On the normal space the dynamics is described by $\dot{z} = \mathrm{i}\Omega_\eta(I_0) z$ where $\Omega_\eta(I_0) := \Omega + \eta^2 \mathcal{Q} I_0$ is the vector of the *"shifted elliptic frequencies"*. Our task is to find periodic solutions bifurcating from the ones of H_{int}.

The system is properly nonlinear by the "twist condition" $\det \mathcal{R} \neq 0$ (see remarks 4.3 and 4.4 for a comment). Such condition involves only finitely many Taylor coefficients $R^*_{k,a,\overline{a}}$ of the normal form Hamiltonian $\mathcal{H}_*$ and ensures, in particular, that the frequencies $\omega + \eta^2 \mathcal{R} I_0$ vary with the actions I_0. When

$$\widetilde{\omega} := \omega + \eta^2 \mathcal{R} I_0 \in \frac{1}{T} 2\pi \mathbf{Z}^n,$$

$\mathcal{T}(I_0)$ is a *completely resonant* torus, supporting the family of T-periodic motions $\mathcal{P} := \{I(t) = I_0,\ \phi(t) = \phi_0 + \widetilde{\omega} t,\ z(t) = 0\}$ (completely resonant frequencies $\widetilde{\omega} \in (2\pi/T)\mathbf{Z}^n$ always exists for some $T = O(\eta^{-2})$ and $I_0 = O(1)$, see (69), (70)). The family $\mathcal{P}$, diffeomorphic to $\mathbf{T}^n$, will not persist in its entirety for the complete Hamiltonian system due to resonances among the oscillations.

The *key point* to continue some periodic solutions of the family $\mathcal{P}$ is to choose properly the "1-dimensional parameter" T (the period) and the actions I_0: the period T and the "shifted elliptic frequencies" $\Omega_\eta(I_0)$ must satisfy a suitable *non-resonance property*, see (68).

In Lemma 4.1 (resp. 4.2) we actually show the existence, assuming condition (a) (resp. (b)) in Theorem 1.1, of "non resonant" periods T. Note that in condition (a) the "coupling matrix" $\mathcal{Q}$ does not play any role (in particular for $m = 2$ which is the case of the spatial three body problem). If conditions (a)-(ii)-(iii) are not verified, the system possesses "lower order" resonances (see equation (5)), which are an obstruction for finding "non-resonant" periods T. In this case one needs, in order to "move away" from the resonances, to evaluate the matrix $\mathcal{Q}$ and check condition (b). Conditions (a)-(b) are, indeed, sharp: if violated, it is not possible, in general, to find any "non-resonant" period T, see remark 4.1.

After the previous construction, the proof of Theorem 1.1 is based on a Lyapunov-Schmidt reduction, variational in nature, inspired to [ACE]-[AB]. The non-resonance property (68) and the "twist condition" $\det \mathcal{R} \neq 0$, allow to build, by means of the Contraction Mapping Theorem, a suitable family of pseudo-T-periodic solutions branching off the family $\mathcal{P}$, see Lemma 4.3. Finally, by a variational argument, we will select at least n ($=$ cat $\mathbf{T}^{n-1}$) geometrically distinct T-periodic solutions from them.

Finally, we recall that, for strictly convex nearly integrable Hamiltonian systems, [BK] have proved, via a variational argument, the existence of periodic orbits with arbitrarily large minimal periods. We have not tried to extend their approach in the present case, since the unperturbed Hamiltonian H_{int} is not-strictly convex (it is linear in the elliptic action variables $z\overline{z}$). Moreover the previously described phenomenon of resonances between the period T and the "shifted elliptic frequencies" is hardly recognizable by a purely variational approach (which works well for finding periodic orbits near maximal dimensional tori and problably near hyperbolic one's).

2 Study of the resonances and technical Lemmata

In this section we first collect some number theoretical Lemmata that will be used in section 4 for defining the "non-resonant periods" T (Lemma 2.3 will be used in Lemma 4.1).

Lemma 2.1 *Let $a_1, \ldots, a_n \in \mathbf{Z}$, $M \in \mathbf{N}$ with $\gcd(a_1, \ldots, a_n, M) = 1$. Then, $\forall\, b \in \mathbf{Z}$, the congruence $a_1 k_1 + \ldots + a_n k_n \equiv b \mod M$ has M^{n-1} solutions.*

Proof: Recall a well known result from number theory: let $a, b, M \in \mathbf{Z}$ and $c := \gcd(a, M)$; then the congruence $ak \equiv b \mod M$, $k \in \mathbf{Z}$, has solution if and only if[10] $c|b$ and in this case it has exactly c solutions. It means that there exist integers $0 \leq k_1 < \ldots < k_c \leq M-1$ such that $ak_h - b \in M\mathbf{Z}$ $\forall 1 \leq h \leq c$, but $ak - b \notin M\mathbf{Z}$ $\forall\, k \in \{0, \ldots, M-1\}$, $k \neq k_h$ $\forall 1 \leq h \leq c$.

We now prove the Lemma by induction over n. The case $n = 1$ is the above mentioned result. We now suppose the statement true for n and prove it for $n+1$. Let $c := \gcd(a_{n+1}, M)$. Our congruence is equivalent to $a_{n+1}k_{n+1} \equiv (b - \sum_{i=1}^n a_i k_i) \mod M$ which has exactly c solutions if and only if $c|(b - \sum_{i=1}^n a_i k_i)$. Hence we have only to prove that the number of integers $0 \leq k_1, \ldots, k_n \leq M-1$ for which $c|(b - \sum_{i=1}^n a_i k_i)$ is exactly $M^n/c = c^{n-1}(M/c)^n$. This amounts to prove that the number of integer solutions of $\sum_{i=1}^n a_i k_i \equiv b \mod c$, is exactly c^{n-1}. This is true, by the inductive hypotesis, if $\gcd(a_1, \ldots, a_n, c) = 1$. The last equality is actually true since, by hypotesis, $\gcd(a_1, \ldots, a_{n+1}, M) = 1$; hence if $d|a_1, \ldots, a_n, c$, since $c|a_{n+1}, M$, then $d|a_{n+1}, M$ and it follows $d = 1$. $\square$

[10] $c|b$ means that there exists $n \in \mathbf{Z}$ such that $b = cn$.

Lemma 2.2 *Let $\hat{m}, \hat{n} \in \mathbf{N}^+$. Let $\hat{\omega} \in \mathbf{R}^{\hat{n}}$ rationally independent. Let $\hat{\Omega} \in \mathbf{R}^{\hat{m}}$ with $M_j \hat{\Omega}_j = a_j \cdot \hat{\omega}$ for $M_j \in \mathbf{N}$, $a_j \in \mathbf{Z}^{\hat{n}}$. Suppose that $\gcd(a_{j1}, \ldots, a_{j\hat{n}}, M_j) = 1$. Let $M := \mathrm{lcm}\,(M_1, \ldots, M_{\hat{m}})$ and $K := \{0, \ldots, M-1\}^{\hat{n}}$. Suppose $M_j \geq \hat{m}$, $\forall\, 1 \leq j \leq \hat{m}$. Moreover, in the case $\hat{m} = 1$, suppose also $M_1 \geq 2$. Then there exists $k \in K$ such that $a_j \cdot k / M_j \notin \mathbf{Z}$, $\forall\, 1 \leq j \leq \hat{m}$.*

Proof: Let $\Lambda_j := \{k \in K \text{ s.t. } a_j \cdot k \in M_j \mathbf{Z}\}$, $\Lambda_j^* := \{k \in K_j \text{ s.t. } a_j \cdot k \in M_j \mathbf{Z}\}$, where $K_j := \{0, \ldots, M_j - 1\}^{\hat{n}}$. Since $M_j | M$ we have[11] $\sharp \Lambda_j = (M/M_j)^{\hat{n}} \sharp \Lambda_j^*$. Using Lemma 2.1 with the substitutions $n \to \hat{n}$, $b \to 0$, $a_i \to a_{ji}$, $M \to M_j$ we obtain that $\sharp \Lambda_j^* = M_j^{\hat{n}-1}$. Hence we have $\sharp \Lambda_j = M^{\hat{n}} / M_j$. Observing that $\{0\} \in \Lambda_j$ $\forall\, 1 \leq j \leq \hat{m}$, we have

$$\sharp \bigcup_{j=1}^{\hat{m}} (\Lambda_j \setminus \{0\}) \leq \sum_{j=1}^{\hat{m}} \sharp (\Lambda_j \setminus \{0\}) = \sum_{j=1}^{\hat{m}} \left(\frac{M^{\hat{n}}}{M_j} - 1 \right) = -\hat{m} + M^{\hat{n}} \sum_{j=1}^{\hat{m}} \frac{1}{M_j}. \qquad (9)$$

If $\hat{m} \geq 2$, since $M_j \geq \hat{m}$ by hypothesis, then $-\hat{m} + M^{\hat{n}} \sum_{j=1}^{\hat{m}} \frac{1}{M_j} \leq -\hat{m} + M^{\hat{n}} < -1 + M^{\hat{n}}$. If $\hat{m} = 1$, since $M_1 = M \geq 2$ by hypothesis, $-\hat{m} + M^{\hat{n}} \sum_{j=1}^{\hat{m}} \frac{1}{M_j} = -1 + M^{\hat{n}-1} < -1 + M^{\hat{n}}$. In both cases, from (9), we get

$$\sharp \bigcup_{j=1}^{\hat{m}} (\Lambda_j \setminus \{0\}) < M^{\hat{n}} - 1 = \sharp (K \setminus \{0\})$$

and the conclusion of the Lemma follows from $\sharp \cup_{j=1}^{\hat{m}} \Lambda_j < \sharp K$. $\square$

In view of the next Lemma we recall the definition of the ergodization time of a torus with linear flow. For any vector $\xi \in \mathbf{R}^{\hat{n}}$ the ergodization time $T_{\mathrm{erg}}(\xi, \varepsilon)$ required to fill $\mathbf{T}^{\hat{n}}$ within $\varepsilon > 0$ is defined as

$$T_{\mathrm{erg}}(\xi, \varepsilon) := \inf \Big\{ t \in \mathbf{R}_+ \;\Big|\; \forall x \in \mathbf{R}^{\hat{n}},\ \forall\, 1 \leq i \leq \hat{n},\ \mathrm{dist}(x_i, A_i + [0, t]\xi_i + \mathbf{Z}) \leq \varepsilon \Big\} \quad (10)$$

where A is some point of $\mathbf{R}^{\hat{n}}$. $T_{\mathrm{erg}}(\xi, \varepsilon)$ is clearly independent of the choice of A.

If (ω, Ω) are rationally independent or are resonant only at a sufficiently high order, namely if $(\omega, \Omega) \cdot p \neq 0$, $\forall p \in \mathbf{Z}^{n+m}$ with $0 < |p| \leq a/\delta$ for some constant $a := a_{n,m}$, then the trajectories of the linear flow $\{(\omega, \Omega)t\}_{t \in \mathbf{R}}$ will make an arbitrarily fine δ-net of $\mathbf{T}^{n+m}$, see Theorem 4.1 of [BBB]. This non-resonance assumption on (ω, Ω) is clearly sufficient to prove (13)-(14) below. In the present case, however, the weaker non-resonance assumptions (11), (12) are sufficient.

Lemma 2.3 *Let $M_j, a_j, \hat{m}$ as in (5) and suppose that*

$$M_j \geq \hat{m} \qquad \forall\, 1 \leq j \leq \hat{m}\ . \qquad (11)$$

In the case $\hat{m} = 1 = M_1$ suppose also

$$\exists\ n + 1 \leq i_* \leq \hat{n} = n + m - 1 \qquad \text{such that} \qquad a_{1 i_*} \neq 0\ . \qquad (12)$$

[11] We denote by $\sharp A$ the cardinality of the set A.

$\forall 0 < \delta \le 1/(2\beta)$, where $\beta := 2\max_{1\le j\le\hat m}|a_j|_1$, and $\forall\, t_0 > 0$, there exists $\tau \ge t_0$ such that

$$\begin{aligned} \mathrm{dist}(\omega_i\tau,\mathbf{Z}) &\le \delta \qquad \forall\, 1\le i\le n\ , & (13)\\ \mathrm{dist}(\Omega_j\tau,\mathbf{Z}) &\ge \min\left\{\frac{1}{2\beta}\ ,\ \frac{1}{4\max_{1\le j\le\hat m} M_j}\right\} =: d_0 \qquad \forall\, 1\le j\le m\ . & (14)\end{aligned}$$

Moreover $\tau - t_0 \le T_{\rm erg}(\hat\omega/M,\delta/M)$ where $M := \mathrm{lcm}\,(M_1,\ldots,M_{\hat m})$.
If (ω,Ω) are rationally independent, namely $\hat m = 0$, (13)–(14) are still verified setting $\beta := 2$, $d_0 := 1/4$, $M := 1$.

Proof: We first consider the cases $\hat m \ge 2$ or $\hat m = 1 < M_1$. We are in the hypoteses of Lemma 2.2. Hence there exists $k \in K := \{0,\ldots,M-1\}^{\hat n}$ such that $a_j\cdot k/M_j \notin \mathbf{Z}$, $\forall\, 1\le j\le\hat m$. Let[12]

$$x := \Big\langle \frac{1}{M}\Big(k + \frac{1}{\beta}\sum_{i=n+1}^{\hat n} e_i\Big) - \frac{\hat\omega}{M} t_0\Big\rangle \in [-1/2,1/2)^{\hat n}\ .$$

Since $\hat\omega/M$ is rationally independent, its linear flow ergodizes the torus $\mathbf{R}^{\hat n}/\mathbf{Z}^{\hat n}$. Let $t_0 \le \tau \le t_0 + T_{\rm erg}(\hat\omega/M,\delta/M)$ the first instant for which $\exists h \in \mathbf{Z}^{\hat n}$ with $|\frac{\hat\omega}{M}(\tau - t_0) - x - h|_\infty \le \delta/M$. If $y := \hat\omega\tau - k - \frac{1}{\beta}\sum_{i=n+1}^{\hat n} e_i - Mh'$ then, by a suitable choice of $h' \in \mathbf{Z}^{\hat n}$, we obtain by the construction above that $|y|_\infty \le \delta$. Hence, by definition of $\hat\omega$, $\forall\, 1\le i\le n$, $\omega_i\tau = \hat\omega_i\tau = y_i + k_i - Mh'_i$ and (13) holds.

Moreover, from the resonance relation (5), $\forall\, 1\le j\le\hat m$,

$$\Omega_j\tau = \frac{a_j\cdot\hat\omega\tau}{M_j} = \frac{a_j\cdot y}{M_j} + \frac{a_j\cdot k}{M_j} + \frac{1}{\beta M_j}\sum_{i=n+1}^{\hat n} a_{ji} + \frac{M}{M_j} a_j\cdot h'\ . \tag{15}$$

We observe that $a_j\cdot k/M_j \notin \mathbf{Z}$ implies $\mathrm{dist}(a_j\cdot k/M_j,\mathbf{Z}) \ge 1/M_j$. Also, $|a_j\cdot y| \le |a_j|_1\delta$, $|\sum_{i=n+1}^{\hat n} a_{ji}| \le |a_j|_1$. Hence, collecting these observations and (15), we have

$$\mathrm{dist}(\Omega_j\tau,\mathbf{Z}) \ge \frac{1}{M_j} - \frac{|a_j|_1\delta}{M_j} - \frac{|a_j|_1}{\beta M_j} \ge \frac{1}{4M_j} \qquad \forall\, 1\le j\le\hat m, \tag{16}$$

recalling also that $0 < \delta \le (1/2\beta)$ and the definition of β. Finally, by definition of $\hat\omega$, for $\hat m + 1 \le j \le m$, we have $\Omega_j\tau = \hat\omega_{n+j-\hat m}\tau = y_{n+j-\hat m} + k_{n+j-\hat m} + 1/\beta + Mh'_{n+j-\hat m}$, which implies

$$\mathrm{dist}(\Omega_j\tau,\mathbf{Z}) \ge \frac{1}{\beta} - \delta \ge \frac{1}{2\beta} \qquad \forall\, \hat m+1\le j\le m\ , \tag{17}$$

and (14) follows from (16)-(17).

[12] We define $\langle(y_1,\ldots,y_n)\rangle := (\langle y_1\rangle,\ldots,\langle y_n\rangle)$ where the function $\langle\cdot\rangle : \mathbf{R}\to[-1/2,1/2)$ is defined as $\langle y\rangle := y$ for $y\in[-1/2,1/2)$ and it is 1-periodically extended for $y\in\mathbf{R}$. Note that, for $y \ne (1/2)+\mathbf{Z}$, $\langle y\rangle = y - [y]$ where $[y]$ is the closest integer to y.

We now consider the case $\hat{m} = 1$, $M_1 = 1$. We have $M = M_1 = 1$. We set $a := a_1$ to simplify the notation. Let

$$x := \Big\langle \sum_{i=n+1}^{\hat{n}} b_i e_i - \hat{\omega} t_0 \Big\rangle ,$$

where

$$b_i := \begin{cases} \dfrac{\operatorname{sign}(a_i)}{2\sum_{i=n+1}^{\hat{n}} |a_i|} & \text{if} \quad a_i \neq 0 , \\ \dfrac{1}{2|a|_1} & \text{if} \quad a_i = 0 . \end{cases}$$

Note that $|b_i| \geq 1/(2|a|_1)$ for all i. Proceeding exactly as above we find $t_0 \leq \tau \leq t_0 + T_{\text{erg}}(\hat{\omega}, \delta)$ and an $h' \in \mathbf{Z}^{\hat{n}}$ such that, defining

$$y := \hat{\omega}\tau - \sum_{i=n+1}^{\hat{n}} b_i e_i - h',$$

we have $|y|_\infty \leq \delta$ and (13) holds. The proof of (14) is slightly different. For $j = 1$ we have

$$\Omega_1 \tau = a \cdot \hat{\omega}\tau = a \cdot y + a \cdot \sum_{i=n+1}^{\hat{n}} b_i e_i + a \cdot h' = a \cdot y + \frac{1}{2} + a \cdot h' ,$$

where, in the last equality we have used (12) abd the definition of b_i. Since $a \cdot h' \in \mathbf{Z}$ and $|a \cdot y| \leq |a|_1 \delta \leq 1/4$ we have

$$\operatorname{dist}(\Omega_1 \tau, \mathbf{Z}) \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4} . \tag{18}$$

Instead, for $2 \leq j \leq m$ we have $\Omega_j \tau = \hat{\omega}_{n+j-1}\tau = y_{n+j-1} + b_{n+j-1} + h'_{n+j-1}$ which implies

$$\operatorname{dist}(\Omega_j \tau, \mathbf{Z}) \geq b_{n+j-1} - \delta \geq \frac{1}{2|a|_1} - \delta = \frac{1}{\beta} - \delta \geq \frac{1}{2\beta} \qquad \forall\, 2 \leq j \leq m , \tag{19}$$

and (14) follows from (18)-(19).

We finally consider the case $\hat{m} = 0$, which is the simplest one since the linear flow of $\hat{\omega} = (\omega, \Omega)$ ergodizes the whole torus $\mathbf{T}^{n+m}$. Let define

$$x := \Big\langle \frac{1}{2} \sum_{i=n+1}^{n+m} e_i - \hat{\omega} t_0 \Big\rangle.$$

There exists $t_0 < \tau < t_0 + T_{\text{erg}}(\hat{\omega}, \delta)$ and $h \in \mathbf{Z}^{n+m}$ such that $|\hat{\omega}(\tau - t_0) - x - h|_\infty \leq \delta$, with $0 < \delta \leq 1/4$. Arguing as before we get $|\hat{\omega}\tau - (1/2)\sum_{i=n+1}^{n+m} e_i - h'|_\infty \leq \delta$ for a suitable $h' \in \mathbf{Z}^{n+m}$ and the estimates (13)-(14) follow. The Lemma is proved. $\square$

The next two Lemmata, whose proof is omitted, will be used in the proofs of Theorems 1.1 and 0.2, for constructing the pseudo periodic solutions through the Contraction Mapping Theorem.

Lemma 2.4 *Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces, $L : Y \longrightarrow X$ be a linear bounded operator and $P : X \longrightarrow Y$ be a C^1 map. Assume that*

$$\delta_0 \geq 2\,|L(P(0))|_X \tag{20}$$

and

$$\sup_{x \in B_{\delta_0}} |DP(x)| \leq \frac{1}{2\,|L|}\,. \tag{21}$$

Then, there exists a unique $x^\star \in B_{\delta_0}$ such that $x^\star = L(P(x^\star))$.

The next Lemma defines a suitable "Green operator" L associated to the linear system (2.5) below.

Lemma 2.5 *Let $T > 0$, $\Omega \in \mathrm{Mat}(m \times m, \mathbf{R})$, $M \in \mathrm{Mat}(n \times n, \mathbf{R})$ and define $\mathcal{M} := \mathbf{1}_m - e^{\mathrm{i}\Omega T} \in \mathrm{Mat}(m \times m, \mathbf{C})$. Assume that M and $\mathcal{M}$ are invertible. Let*

$$\begin{aligned} Y &:= C\Big([0,T],\, \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m\Big)\,, && (22)\\ X &:= \Big\{(J,\psi,z) \in Y \text{ s.t. } \psi(0) = \psi(T) = 0\,, z(0) = z(T)\,\Big\}. && (23) \end{aligned}$$

X and Y, endowed with the sup-norm $\|(\hat J,\hat\psi,\hat z)\| := \sup_{t\in[0,T]}(|\hat J(t)|,\, |\hat\psi(t)|,\, |\hat z(t)|)$, are Banach spaces.

For any $(\hat J,\hat\psi,\hat z) \in Y$ define the constants

$$\begin{aligned} \alpha(\hat J,\hat\psi) &:= -\frac{1}{T}\Big(\int_0^T\int_0^s \hat J(\theta)\,d\theta\,ds + M^{-1}\int_0^T \hat\psi(\theta)\,d\theta\Big) \in \mathbf{R}^n\,,\\ \beta(\hat z) &:= \mathcal{M}^{-1}e^{\mathrm{i}\Omega T}\int_0^T e^{-\mathrm{i}\Omega\theta}\hat z(\theta)\,d\theta \in \mathbf{C}^m \end{aligned}$$

and the linear "Green" operator $L : Y \longrightarrow X$ by

$$L\begin{pmatrix}\hat J\\ \hat\psi\\ \hat z\end{pmatrix} := \begin{pmatrix} \alpha(\hat J,\hat\psi) + \int_0^t \hat J(s)\,ds \\[2mm] M\alpha t + M\,\int_0^t\int_0^s \hat J(\theta)\,d\theta\,ds + \int_0^t \hat\psi(\theta)\,d\theta \\[2mm] e^{\mathrm{i}\Omega t}\Big(\beta(\hat z) + \int_0^t e^{-\mathrm{i}\Omega\theta}\hat z(\theta)\,d\theta\Big)\end{pmatrix}. \tag{24}$$

The Green operator L satisfies

$$|L| \leq C\left(|M^{-1}| + |M|\,T^2 + |M|\,|M^{-1}|\,T + |\mathcal{M}^{-1}|T\right), \tag{25}$$

for a suitable constant $C > 0$ depending on $\sup_{t\in[0,T]}|e^{\mathrm{i}\Omega t}|$, n and m.

Setting $(J,\psi,z) = L(\hat J,\hat\psi,\hat z)$, then $(J,\psi,z) \in C^1$ and

$$\begin{cases} \dot J &= \hat J,\\ \dot\psi - M\,J &= \hat\psi,\\ \dot z - \mathrm{i}\Omega z &= \hat z\,. \end{cases}$$

The straightforward proof is omitted. We only point out that $L(Y) \subseteq X$ by definition of the constants $\alpha(\hat J,\hat\psi)$ and $\beta(\hat z)$.

3 Normal form around an elliptic torus

In order to study the dynamics of the Hamiltonian system (6) in a small neighborhood of $\mathcal{T}$ it is a convenient device to perform the following rescaling

$$\mathcal{I}_* := \eta^2 \mathcal{I}\,, \qquad \varphi_* := \varphi\,, \qquad Z_* := \eta Z\,, \qquad \overline{Z}_* := \eta \overline{Z}, \tag{26}$$

where $\eta > 0$ is a positive small parameter. Note that the linear tranformation (26) preserves the torus $\mathcal{T}$ and that a domain of order one in the new variables $(\mathcal{I}, \varphi, Z, \overline{Z})$ correspond to a domain in the old variables $(\mathcal{I}_*, \varphi_*, Z_*, \overline{Z}_*)$ that shrinks towards $\mathcal{T}$ for η tending to zero.

The new Hamiltonian $\mathcal{H}(\mathcal{I},\varphi, Z, \overline{Z}) = \eta^{-2}\mathcal{H}_*(\eta^2\mathcal{I}, \varphi, \eta Z, \eta\overline{Z})$ writes

$$\mathcal{H}(\mathcal{I},\varphi, Z, \overline{Z}) = \omega \cdot \mathcal{I} + \Omega Z \cdot \overline{Z} + \sum_{2|k|+|a+\overline{a}| \geq 3} \eta^{2|k|+|a+\overline{a}|-2} R^*_{k,a,\overline{a}}(\varphi) \mathcal{I}^k Z^a \overline{Z}^{\overline{a}} \tag{27}$$

and it is analytic on

$$D^n_{r_*/\eta^2} \times \mathbf{T}^n_{s_*} \times D^{2m}_{\rho_*/\eta}\,.$$

In order to find periodic solutions with large period close to the elliptic torus $\mathcal{T}$ we will apply in section 4 a finite dimensional reduction of Lyapunov-Schmidt type. For this purpose (see remark 4.2) we need first to perform a symplectic change of variables which eliminates from the Hamiltonian $\mathcal{H}$ defined in (27) the following terms[13]

$$\eta \sum_{\substack{1 \leq i \leq n \\ |a+\overline{a}|=1}} R^*_{e_i,a,\overline{a}}(\varphi) \mathcal{I}_i Z^a \overline{Z}^{\overline{a}}\ , \tag{28}$$

$$\eta^2 \sum_{|k|=2, \ell \neq 0} R^*_{k,0,0,\ell} \mathcal{I}^k e^{\mathrm{i}\ell\cdot\varphi}\ , \tag{29}$$

$$\eta^2 \sum_{\substack{1 \leq i \leq n \\ |a+\overline{a}|=2 \\ a \neq \overline{a} \text{ or } \ell \neq 0}} R^*_{e_i,a,\overline{a},\ell} \mathcal{I}_i Z^a \overline{Z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\varphi}\ , \tag{30}$$

$$\eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5 \\ |a+\overline{a}|=1}} R^*_{k,a,\overline{a}}(\varphi) \mathcal{I}^k Z^a \overline{Z}^{\overline{a}}\ . \tag{31}$$

This task will be accomplished in the next Proposition. The term (29) will be "averaged out" since ω is diophantine; the terms (28) and (31) using the first order Melnikov non-resonance conditions (namely conditions (4) for $|h| \leq 1$), and, finally, the term (30) using the second order Melnikov non-resonance conditions (4).

Proposition 3.1 (Averaging) *Let the Hamiltonian $\mathcal{H}$, defined in* (27)*, be real analytic on $D^n_r \times \mathbf{T}^n_s \times D^{2m}_\rho$ and satisfy the second-order Melnikov non-resonance conditions* (4)*. Then, for η small enough, depending on n, m, ω, Ω, γ, τ, r, s, ρ, there exists an analytic canonical change of coordinates $\Phi : (I, \phi, z, \overline{z}) \to (\mathcal{I}, \varphi, Z, \overline{Z})$, η-close to the identity,*

$$\Phi\ :\ D^n_{r/2} \times \mathbf{T}^n_{s/2} \times D^{2m}_{\rho/2} \longrightarrow D^n_r \times \mathbf{T}^n_s \times D^{2m}_\rho\,,$$

[13] For short, in the sequel we will often omit the summation over $\ell \in \mathbf{Z}^n$.

transforming the Hamiltonian $\mathcal{H}$ into the Hamiltonian[14]

$$\begin{aligned} H = \mathcal{H}\circ\Phi &= \omega\cdot I + \Omega z\overline{z} + \eta \sum_{|a+\overline{a}|=3} R^*_{0,a,\overline{a}}(\phi) z^a \overline{z}^{\overline{a}} + \\ &+ \eta^2 \Big[\frac{1}{2}\mathcal{R} I\cdot I + \mathcal{Q} I\cdot z\overline{z} + \sum_{|a+\overline{a}|=4} R_{0,a,\overline{a}}(\phi) z^a\overline{z}^{\overline{a}}\Big] + \\ &+ \eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5\\ |a+\overline{a}|=3,5}} R_{k,a,\overline{a}}(\phi) I^k z^a \overline{z}^{\overline{a}} + O(\eta^4) \ , \end{aligned} \tag{32}$$

for suitable $R_{k,a,\overline{a}} \in \mathbf{C}$, $R_{k,a,\overline{a}}(\phi)$ analytic on $\mathbf{T}^n_{s/2}$, and where $\mathcal{R}$ is the symmetric twist matrix defined in (7) *and $\mathcal{Q} \in \mathrm{Mat}(m\times n, \mathbf{R})$ is defined in* (8)*.*

Proof: We rewrite the Hamiltonian (27) in the form

$$\mathcal{H}(\mathcal{I},\varphi, Z, \overline{Z}) = \sum_{j\geq 0} \eta^j R^{(j)}_*(\mathcal{I},\varphi, Z,\overline{Z}) \tag{33}$$

with

$$R^{(0)}_* := \omega\cdot\mathcal{I} + \Omega Z\cdot\overline{Z} \qquad \text{and} \qquad R^{(j)}_* := \sum_{2|k|+|a+\overline{a}|=j+2} R^*_{k,a,\overline{a},\ell} \mathcal{I}^k Z^a \overline{Z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\varphi} \tag{34}$$

where we have expanded in Fourier series the functions $R^*_{k,a,\overline{a}}(\varphi)$ as in (2).

We want to define a canonical change of variables Φ from the new variables $\zeta := (I,\phi,z,\overline{z})$ to the old ones $\mathcal{Z} := (\mathcal{I},\varphi,Z,\overline{Z})$, as the flow at time 1 of a suitable Hamiltonian χ. Precisely $\Phi := \Phi^1_\chi$, where $\Phi^t_\chi(\zeta_0) := \zeta(t)$ is the unique solution of the Hamiltonian system

$$\dot I(t) = -\partial_\phi\chi(\zeta(t)) \ , \quad \dot\phi(t) = \partial_I\chi(\zeta(t)) \ , \quad \dot z(t) = \mathrm{i}\partial_{\overline{z}}\chi(\zeta(t)) \ , \quad \dot{\overline{z}}(t) = -\mathrm{i}\partial_z\chi(\zeta(t)) \tag{35}$$

with initial conditions $\zeta(0) = \zeta_0$.

The Lie operator (Poisson brackets), acting on a function $g := g(\zeta)$, is defined as

$$L_\chi g := \{g,\chi\} := \partial_\phi g\partial_I\chi - \partial_I g\partial_\phi\chi + \mathrm{i}\partial_z g\partial_{\overline{z}}\chi - \mathrm{i}\partial_{\overline{z}} g\partial_z\chi. \tag{36}$$

For every integer $j\geq 0$ we also set

$$L^0_\chi g := g \ , \qquad L^j_\chi g := L_\chi L^{j-1}_\chi g \ . \tag{37}$$

The new Hamiltonian $H = \mathcal{H}\circ\Phi$ can be developed, for all $j_0 \in \mathbf{N}^+$, as

$$H := \mathcal{H}\circ\Phi = \sum_{j=0}^{j_0} \frac{1}{j!} L^j_\chi \mathcal{H} + \frac{1}{j_0!}\int_0^1 (1-\xi) L^{j_0+1}_\chi \mathcal{H}\circ\Phi^\xi_\chi \, d\xi \ . \tag{38}$$

[14]We denote $\mathcal{Q}I\cdot z\overline{z} := \sum_{\substack{1\leq j\leq m\\ 1\leq i\leq n}} \mathcal{Q}_{ji} I_i z_j \overline{z}_j$.

We look for a Hamiltonian χ of the form

$$\chi = \sum_{j=1}^{j_0} \eta^j \chi^{(j)} \ , \quad \text{where} \quad \chi^{(j)} := \chi^{(j)}(I,\phi,z,\overline{z}) := \sum_{2|k|+|a+\overline{a}|=j+2} \chi_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \tag{39}$$

will be chosen later on. For the sake of simplicity we will often *omit* in the notation the summation over $\ell \in \mathbf{Z}^m$. Inserting (39) in (38) we obtain

$$H = \sum_{d=0}^{j_0} \eta^d R^{(d)} + \sum_{d=1}^{j_0} \eta^d \sum_{j=0}^{j_0} \frac{1}{j!} \sum_{h=0}^{d-1} \sum_{\substack{h+i_1+\ldots+i_j=d \\ 1\le i_1,\ldots i_j \le d}} L_{\chi^{(i_1)}} \ldots L_{\chi^{(i_j)}} R_*^{(h)} + \ O(\eta^{j_0+1}) \tag{40}$$

Denoting $[\ \cdot\ ]_d$ the d-th order in η, we obtain

$$[H]_1 = R_*^{(1)} + L_{\chi^{(1)}} R_*^{(0)} \tag{41}$$

and

$$[H]_d = R_*^{(d)} + L_{\chi^{(d)}} R_*^{(0)} + \sum_{j=0}^{j_0} \frac{1}{j!} \sum_{h=0}^{d-1} \sum_{\substack{h+i_1+\ldots+i_j=d \\ 1\le i_1,\ldots i_j \le d-1}} L_{\chi^{(i_1)}} \ldots L_{\chi^{(i_j)}} R_*^{(h)} \,. \tag{42}$$

Observe that

$$\sum_{\substack{2|k|+|a+\overline{a}|=h+2 \\ 2|k'|+|a'+\overline{a}'|=h'+2}} \{ I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \ , \ I^{k'} z^{a'} \overline{z}^{\overline{a}'} e^{\mathrm{i}\ell'\cdot\phi} \} = \tag{43}$$

$$\sum_{2|k''|+|a''+\overline{a}''|=h+h'+2} c_{k'',a'',\overline{a}'',\ell''} I^{k''} z^{a''} \overline{z}^{\overline{a}''} e^{\mathrm{i}\ell''\cdot\phi} \ , \tag{44}$$

for suitable constants $c_{k'',a'',\overline{a}'',\ell''}$, which are explicitly given by the following formula:

$$\begin{aligned} \{ I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \ , \ I^{k'} z^{a'} \overline{z}^{\overline{a}'} e^{\mathrm{i}\ell'\cdot\phi} \} \ &= \ \mathrm{i}\Big[\sum_{i=1}^{n} (\ell_i k'_i - \ell'_i k_i) I^{k+k'-e_i} z^{a+a'} \overline{z}^{\overline{a}+\overline{a}'} + \\ &+ \ \sum_{j=1}^{m} (a_j \overline{a}'_j - \overline{a}_j a'_j) I^{k+k'} z^{a+a'-e_j} \overline{z}^{\overline{a}+\overline{a}'-e_j} \Big] e^{\mathrm{i}(\ell+\ell')\cdot\phi} \,. \end{aligned} \tag{45}$$

Thus, we obtain that

$$L_{\chi^{(i_1)}} \ldots L_{\chi^{(i_j)}} R_*^{(h)} = \sum_{2|k|+|a+\overline{a}|=h+i_1+\ldots+i_j+2} c_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \tag{46}$$

for suitable constants $c_{k,a,\overline{a},\ell}$. Hence,

$$[H]_d = L_{\chi^{(d)}} R_*^{(0)} + \sum_{2|k|+|a+\overline{a}|=d+2} R_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \tag{47}$$

for suitable $R_{k,a,\overline{a},\ell}$ with

$$R_{k,a,\overline{a},\ell} := R_{k,a,\overline{a},\ell}(\chi^{(1)}, \ldots, \chi^{(d-1)}, R_*^{(0)}, \ldots, R_*^{(d)}) \ ; \tag{48}$$

we note that, by means of (41) and (47)

$$2|k| + |a + \overline{a}| = 3 \qquad \Longrightarrow \qquad R_{k,a,\overline{a},\ell} = R^*_{k,a,\overline{a},\ell} \ , \tag{49}$$

(recall also the setting in (34)). We also evaluate

$$L_{\chi^{(d)}} R_*^{(0)} = \{\omega \cdot I + \Omega z \overline{z}, \chi^{(d)}\} = \sum_{2|k|+|a+\overline{a}|=d+2} -\mathrm{i}(\omega \cdot \ell + \Omega \cdot (a - \overline{a})) \chi_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \ . \tag{50}$$

We define the following "resonant" set:

$$S := S_1 \cup S_2 \cup S_3 \subset \Big\{(k, a, \overline{a}, \ell) \quad \text{s.t.} \quad 3 \leq 2|k| + |a + \overline{a}| \leq 5 \ , \quad \ell \in \mathbf{Z}^n\Big\} \ , \tag{51}$$

with $S_2 = S_2^0 \cup S_2^1 \cup S_2^2$ and

$$\begin{array}{rcl} S_1 & := & \{k = 0,\ |a + \overline{a}| = 3,\ \ell \in \mathbf{Z}^n\} \\ S_2^0 & := & \{k = 0,\ |a + \overline{a}| = 4,\ \ell \in \mathbf{Z}^n\} \\ S_2^1 & := & \{k = e_i,\ a = \overline{a} = e_j,\ 1 \leq i \leq n,\ 1 \leq j \leq m,\ \ell = 0\} \\ S_2^2 & := & \{|k| = 2,\ a = \overline{a} = 0,\ \ell = 0\} \\ S_3 & := & \{2|k| + |a + \overline{a}| = 5,\ |a + \overline{a}| = 3, 5,\ \ell \in \mathbf{Z}^n\} \ . \end{array}$$

Let $j_0 = 3$. $\forall\ 1 \leq d \leq 3$, $\forall (k, a, \overline{a}, \ell)$ such that $2|k| + |a + \overline{a}| = d + 2, \ell \in \mathbf{Z}^n$, one can check, by condition (4), that

$$(k, a, \overline{a}, \ell) \notin S \quad \Longrightarrow \quad |\omega \cdot \ell + \Omega(a - \overline{a})| \geq \frac{\gamma}{1 + |\ell|^\tau} \ , \tag{52}$$

and hence we can define

$$\chi_{k,a,\overline{a},\ell} := \begin{cases} 0 & \text{if } \ (k, a, \overline{a}, \ell) \in S \\ -\mathrm{i}[\omega \cdot \ell + \Omega(a - \overline{a})]^{-1} R_{k,a,\overline{a},\ell} & \text{otherwise} \end{cases} \tag{53}$$

In light of this construction, using (47) and (50), we have[15]

$$[H]_d = \Pi_{S_d}\Big(\sum_{2|k|+|a+\overline{a}|=d+2} R_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \Big) = \sum_{(k,a,\overline{a},\ell)\in S_d} R_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \ . \tag{54}$$

Define

$$R_{k,a,\overline{a}} := R_{k,a,\overline{a},0} \ , \qquad R_{k,a,\overline{a}}(\phi) := \sum_{\ell\in\mathbf{Z}^n} R_{k,a,\overline{a},\ell} e^{\mathrm{i}\ell\cdot\phi} \ .$$

By recurrence, using (47), it is possible to evaluate the terms $R_{k,a,\overline{a}}$ explicitely.

[15] We denote by $\Pi_{\mathcal{S}}$, where $\mathcal{S} \subset \mathbf{N}^{n+2m} \times \mathbf{Z}^n$, the projection on $\mathcal{S}$ *i.e.*

$$\Pi_{\mathcal{S}}\Big(\sum_{k,a,\overline{a},\ell} c_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \Big) := \sum_{(k,a,\overline{a},\ell)\in\mathcal{S}} c_{k,a,\overline{a},\ell} I^k z^a \overline{z}^{\overline{a}} e^{\mathrm{i}\ell\cdot\phi} \ .$$

¿From (42) and (49) we have

$$[H]_2 = \Pi_{S_2}\left(\frac{1}{2}\{\{R_*^{(0)},\chi^{(1)}\},\chi^{(1)}\} + \{R_*^{(1)},\chi^{(1)}\} + R_*^{(2)}\right) .$$

Noting that

$$\{R_*^{(0)},\chi^{(1)}\} = -R_*^{(1)} + \Pi_{S_1} R_*^{(1)} ,$$

we have

$$[H]_2 = \Pi_{S_2}\left(\frac{1}{2}\{R_*^{(1)} + \Pi_{S_1}R_*^{(1)},\chi^{(1)}\} + R_*^{(2)}\right) .$$

In order to prove (32) we only need to show that

$$\frac{1}{2}\mathcal{R}I\cdot I = \Pi_{S_2^2}[H_2] = \frac{1}{2}\Pi_{S_2^2}\left(\{R_*^{(1)},\chi^{(1)}\}\right) + \Pi_{S_2^2}R_*^{(2)} , \tag{55}$$

$$\begin{aligned} \mathcal{Q}I\cdot z\overline{z} &= \Pi_{S_2^1}[H_2] = \Pi_{S_2^1}\left(\frac{1}{2}\{R_*^{(1)} + \Pi_{S_1}R_*^{(1)},\chi^{(1)}\}\right) + \Pi_{S_2^1}R_*^{(2)} \\ &= \Pi_{S_2^1}\left(\left\{\frac{1}{2}\Pi_{S_1^c}R_*^{(1)} + \Pi_{S_1}R_*^{(1)},\chi^{(1)}\right\}\right) + \Pi_{S_2^1}R_*^{(2)} , \end{aligned} \tag{56}$$

where $S_1^c := \{k = e_i,\ |a+\overline{a}| = 1,\ \ell\in\mathbf{Z}^n,\ 1\le i\le n\}$.

We first prove (55). Using (39), (53) and (45), we have

$$\begin{aligned} \Pi_{S_2^2}[H_2] &= \sum_{j=1}^m \sum_{\substack{|k|=|k'|=1\\ |a+\overline{a}|=|a'+\overline{a}'|=1\\ a+a'=\overline{a}+\overline{a}'=e_j}} \frac{1}{2}\cdot\frac{R^*_{k,a,\overline{a},\ell}R^*_{k',a',\overline{a}',-\ell}}{-\omega\cdot\ell+\Omega(\overline{a}-a)}(a_j\overline{a}_j' - \overline{a}_j a_j')I^{k+k'} \\ &+ \sum_{|k|=2} R^*_{k,0,0,0}I^k \\ &= \sum_{j=1}^m \sum_{\substack{1\le i,i'\le n\\ |a+\overline{a}|=|a'+\overline{a}'|=1\\ a+a'=\overline{a}+\overline{a}'=e_j}} \frac{1}{2}\cdot\frac{R^*_{e_i,a,\overline{a},\ell}R^*_{e_{i'},a',\overline{a}',-\ell}}{-\omega\cdot\ell+\Omega(\overline{a}-a)}(a_j\overline{a}_j' - \overline{a}_j a_j')I_iI_{i'} \\ &+ \sum_{i,i'=1}^n \frac{1+\delta_{i,i'}}{2}R^*_{e_i+e_{i'},0,0,0}I_iI_{i'} \\ &= \frac{1}{2}\sum_{i,i'=1}^n\sum_{j=1}^m\left(\frac{R^*_{e_i,e_j,0,\ell}R^*_{e_{i'},0,e_j,-\ell}}{-\omega\cdot\ell-\Omega_j} - \frac{R^*_{e_i,0,e_j,\ell}R^*_{e_{i'},e_j,0,-\ell}}{-\omega\cdot\ell+\Omega_j}\right) \\ &+ \sum_{i,i'=1}^n \frac{1+\delta_{i,i'}}{2}R^*_{e_i+e_{i'},0,0,0}I_iI_{i'} = \frac{1}{2}\mathcal{R}I\cdot I , \end{aligned} \tag{57}$$

where $\delta_{i,i'} = 1$ if $i = i'$ and 0 if $i\ne i'$. Here, we observe that, since $|a+\overline{a}| = |a'+\overline{a}'| = 1$, if $a+a' = \overline{a}+\overline{a}' = e_j$, then $(a,\overline{a},a',\overline{a}') = (e_j,0,0,e_j)$ or $(0,e_j,e_j,0)$. So (55) directly follows from (57).

We now prove (56). Using again (39), (53) and (45), we have

$$
\begin{aligned}
\Pi_{S_2^1}[H_2] \;&=\; -\mathrm{i}\,\Pi_{S_2^1}\Bigg(\Bigg\{\frac{1}{2}\Pi_{S_1^c}R_*^{(1)} + \Pi_{S_1}R_*^{(1)}\ , \\
&\qquad \sum_{\substack{1\le i'\le n\\ 1\le j'\le m}}\left(\frac{R^*_{e_{i'},e_{j'},0,\ell'}}{\omega\cdot\ell'+\Omega_{j'}}z_{j'} \;+\; \frac{R^*_{e_{i'},0,e_{j'},\ell'}}{\omega\cdot\ell'-\Omega_{j'}}\overline{z}_{j'}\right) I_{i'}e^{\mathrm{i}\ell'\cdot\phi}\Bigg\}\Bigg) + \Pi_{S_2^1}R_*^{(2)} \\
&= \frac{1}{2}\Pi_{S_2^1}\sum_{\substack{1\le i,i'\le n\\ 1\le j'\le m\\ |a+\overline{a}|=1}}(\ell_{i'}I_i+\ell_iI_{i'})R^*_{e_i,a,\overline{a},\ell}z^a\overline{z}^{\overline{a}}\left(\frac{R^*_{e_{i'},e_{j'},0,-\ell}}{-\omega\cdot\ell+\Omega_{j'}}z_{j'} + \frac{R^*_{e_{i'},0,e_{j'},-\ell}}{-\omega\cdot\ell'-\Omega_{j'}}\overline{z}_{j'}\right) \\
&\quad +\Pi_{S_2^1}\Bigg(\sum_{\substack{1\le i\le n\\ 1\le j,j'\le m}}\left\{R^*_{0,e_j,e_j+e_{j'},\ell}\overline{z}_{j'}z_j\overline{z}_je^{\mathrm{i}\ell\cdot\phi}\ ,\frac{R^*_{e_{i'},e_{j'},0,\ell'}}{\omega\cdot\ell'+\Omega_{j'}}z_{j'}I_{i'}e^{\mathrm{i}\ell'\cdot\phi}\right\} \\
&\qquad + \sum_{\substack{1\le i\le n\\ 1\le j,j'\le m}}\left\{R^*_{0,e_j+e_{j'},e_j,\ell}z_{j'}z_j\overline{z}_je^{\mathrm{i}\ell\cdot\phi}\ ,\frac{R^*_{e_{i'},0,e_{j'},\ell'}}{\omega\cdot\ell'-\Omega_{j'}}\overline{z}_{j'}I_{i'}e^{\mathrm{i}\ell'\cdot\phi}\right\}\Bigg) + \Pi_{S_2^1}R_*^{(2)} \\
&= \frac{1}{2}\sum_{\substack{1\le i,i'\le n\\ 1\le j\le m}}\ell_{i'}\Bigg(\frac{R^*_{e_i,0,e_j,\ell}R^*_{e_{i'},e_j,0,-\ell}}{-\omega\cdot\ell+\Omega_j} + \frac{R^*_{e_i,e_j,0,\ell}R^*_{e_{i'},0,e_j,-\ell}}{-\omega\cdot\ell-\Omega_j} \\
&\qquad\qquad + \frac{R^*_{e_i,0,e_j,\ell}R^*_{e_{i'},e_j,0,-\ell}}{-\omega\cdot\ell+\Omega_j} + \frac{R^*_{e_i,e_j,0,\ell}R^*_{e_{i'},0,e_j,-\ell}}{-\omega\cdot\ell-\Omega_j}\Bigg)I_iz_j\overline{z}_j \\
&\quad - \sum_{\substack{1\le i\le n\\ 1\le j,j'\le m}}\frac{1}{\omega\cdot\ell+\Omega_{j'}}\left(R^*_{0,e_j,e_j+e_{j'},-\ell}R^*_{e_i,e_{j'},0,\ell} + R^*_{0,e_j+e_{j'},e_j\ell}R^*_{e_i,0,e_{j'},-\ell}\right)I_iz_j\overline{z}_j \\
&\quad +\Pi_{S_2^1}\sum_{2|k|+|a+\overline{a}|=4}R^*_{k,a,\overline{a},\ell}I^kz^a\overline{z}^{\overline{a}}e^{\mathrm{i}\ell\cdot\phi}\ .
\end{aligned}
\tag{58}
$$

So (56) directly follows from (58).

The estimate on the new analyticity radii follows from (52) and from the fact that $\chi = O(\eta)$ (see (39)). □

4 Periodic orbits winding along the torus

In this section we prove the existence of periodic solutions of longer and longer (minimal) period shrinking closer and closer to the elliptic torus $\mathcal{T}$ (Theorem 1.1).

By Proposition 3.1 the Hamiltonian (27) can be transformed, thanks to the second order Melnikov non-resonance conditions (4), into the Hamiltonian

$$
H(I,\phi,z,\overline{z}) \;=\; \omega\cdot I + \Omega z\overline{z} + \eta\sum_{|a+\overline{a}|=3}R^*_{0,a,\overline{a}}(\phi)z^a\overline{z}^{\overline{a}}
$$

$$
\begin{aligned}
&+ \quad \eta^2\Big[\frac{1}{2}\mathcal{R}I\cdot I + \mathcal{Q}I\cdot z\overline{z} + \sum_{|a+\overline{a}|=4} R_{0,a,\overline{a}}(\phi) z^a \overline{z}^{\overline{a}}\Big] \\
&+ \quad \eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5 \\ |a+\overline{a}|=3,5}} R_{k,a,\overline{a}}(\phi) I^k z^a \overline{z}^{\overline{a}} + O(\eta^4)\ ,
\end{aligned} \tag{59}
$$

analytic on

$$
D^n_{r_*/(2\eta^2)} \times \mathbf{T}^n_{s_*/2} \times D^{2m}_{\rho_*/(2\eta)}\,.
$$

The Hamilton's equations of motion induced by the Hamiltonian (59)

$$
\dot{I} = -\partial_\phi H, \quad \dot{\phi} = \partial_I H, \quad \dot{z} = \mathrm{i}\partial_{\overline{z}} H, \quad \dot{\overline{z}} = -\mathrm{i}\partial_z H, \tag{60}
$$

can be written as

$$
\begin{aligned}
\dot{I} &= -\eta \sum_{|a+\overline{a}|=3} \partial_\phi R^*_{0,a,\overline{a}}(\phi) z^a \overline{z}^{\overline{a}} - \eta^2 \sum_{|a+\overline{a}|=4} \partial_\phi R_{0,a,\overline{a}}(\phi) z^a \overline{z}^{\overline{a}} + \\
&\quad -\eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5 \\ |a+\overline{a}|=3,5}} \partial_\phi R_{k,a,\overline{a}}(\phi) I^k z^a \overline{z}^{\overline{a}} + O(\eta^4)\,, \\
\dot{\phi}_i &= \omega_i + \eta^2\Big[(\mathcal{R}I)_i + \sum_{1\le j\le m} \mathcal{Q}_{ji} z_j \overline{z}_j\Big] + \\
&\quad \eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5 \\ |a+\overline{a}|=3,5}} R_{k,a,\overline{a}}(\phi) k_i I^{k-e_i} z^a \overline{z}^{\overline{a}} + O(\eta^4)\,, \\
\dot{z}_j &= \mathrm{i}\Omega_j z_j + \mathrm{i}\eta \sum_{|a+\overline{a}|=3} R^*_{0,a,\overline{a}}(\phi) \overline{a}_j z^a \overline{z}^{\overline{a}-e_j} + \\
&\quad \mathrm{i}\eta^2\Big[(\mathcal{Q}I)_j z_j + \sum_{|a+\overline{a}|=4} R_{0,a,\overline{a}}(\phi) \overline{a}_j z^a \overline{z}^{\overline{a}-e_j}\Big] + \\
&\quad \mathrm{i}\eta^3 \sum_{\substack{2|k|+|a+\overline{a}|=5 \\ |a+\overline{a}|=3,5}} R_{k,a,\overline{a}}(\phi) I^k \overline{a}_j z^a \overline{z}^{\overline{a}-e_j} + O(\eta^4)\,,
\end{aligned} \tag{61}
$$

for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Critical points of the Hamiltonian action functional

$$
\mathcal{A}(I(t), \phi(t), z(t), \overline{z}(t)) := \int_0^T I\cdot\dot{\phi} + \mathrm{i}z\dot{\overline{z}} - H(I(t), \phi(t), z(t), \overline{z}(t))\, dt, \tag{62}
$$

in a suitable space of T-periodic functions, are T-periodic solutions of (60).

4.1 The pseudo periodic solutions

We will find periodic solutions of the Hamiltonian system (60) close to periodic solutions of the integrable Hamiltonian

$$
H_{int} := \omega\cdot I + \frac{\eta^2}{2}\mathcal{R}I\cdot I + \Omega z\overline{z} + \eta^2 \mathcal{Q}I\cdot z\overline{z}. \tag{63}
$$

The manifold $\{z=0\}$ is invariant for the Hamiltonian system generated by H_{int}[16]

$$\dot I = 0, \qquad \dot\phi = \omega + \eta^2(\mathcal{R}I + \mathcal{Q}^T z\overline{z}), \qquad \dot z = \mathrm{i}(\Omega + \eta^2 \mathcal{Q}I)z \tag{64}$$

and it is completely filled up by the invariant tori $\mathcal{T}(I_0) := \{I = I_0,\ \phi \in \mathbf{T}^n,\ z = 0\}$. On $\mathcal{T}(I_0)$ the flow is $t \to \{I_0,\ \phi_0 + (\omega + \eta^2 \mathcal{R} I_0)t, 0\}$ and in its normal space it is determined by $\dot z = \mathrm{i}\Omega_\eta(I_0)z$ where $\Omega_\eta(I_0)$ is the $m \times m$ diagonal matrix with *real* coefficients associated to the vector of the "shifted elliptic frequencies"

$$\Omega_\eta(I_0) := \Omega + \eta^2 \mathcal{Q} I_0 \ , \tag{65}$$

For any $I_0 \in \mathbf{R}^n$, $\Omega_\eta(I_0)$ is a real matrix, since $\mathcal{Q}$ is real (see page 7).

For suitable $T > 0$, $I_0 \in \mathbf{R}^n$, $k \in \mathbf{Z}^n$, namely when

$$\widetilde\omega := \omega + \eta^2 \mathcal{R} I_0 = \frac{1}{T} 2\pi k \in \frac{1}{T} 2\pi \mathbf{Z}^n, \tag{66}$$

the torus $\mathcal{T}(I_0)$ is *completely resonant*, supporting the family of T-periodic motions

$$\mathcal{P} := \Big\{ I(t) = I_0, \quad \phi(t) = \phi_0 + \widetilde\omega t, \quad z(t) = 0 \Big\}. \tag{67}$$

The family $\mathcal{P}$ will not persist in its entirety for the complete Hamiltonian system (60). However, the non-resonance property (68) below, between the period T and the "shifted elliptic frequenies" $\Omega_\eta(I_0)$, is sufficient to prove the persistence of at least n geometrically distinct T-periodic solutions of the Hamiltonian system (60), close to $\mathcal{P}$. Precisely, the required *non resonance property* is

$$\mathcal{M} := \mathcal{M}(I_0, T) := \mathbf{1}_m - e^{\mathrm{i}\Omega_\eta(I_0)T} \quad \text{is invertible} \quad \text{and} \quad |\mathcal{M}^{-1}(I_0, T)| \le \text{ const.} \tag{68}$$

Our aim is then to find I_0 and T so that (66) and (68) hold: we will define $I_0 := I_0(T)$ in dependence on the "1-dimensional parameter" T in such a way that (66) is identically satisfied and then we will find T so that the non resonance property (68) holds. Moreover, for our perturbative arguments of Lemma 4.3, we want $I_0 = O(1)$ and $T < 2/\eta^2$.

Define, for $T \ge 1/\eta^2$,

$$\begin{aligned} I_0 \ &:= \ I_0(T) := -\frac{2\pi}{\eta^2 T} \mathcal{R}^{-1} \Big\langle \frac{\omega T}{2\pi} \Big\rangle \ , &(69)\\ k \ &:= \ k(T) = \frac{\omega T}{2\pi} - \Big\langle \frac{\omega T}{2\pi} \Big\rangle \ , &(70)\end{aligned}$$

where $\langle (x_1, \ldots, x_n) \rangle := (\langle x_1 \rangle, \ldots, \langle x_n \rangle)$ and the function $\langle \cdot \rangle : \mathbf{R} \to [-1/2, 1/2)$ is defined as $\langle x \rangle := x$ for $x \in [-1/2, 1/2)$ and it is 1-periodically extended for $x \in \mathbf{R}$. Notice that

[16]We recall the usual notation for the vector $\mathcal{Q}^T z\overline{z} \in \mathbf{R}^n$ whose components are $(\mathcal{Q}^T z\overline{z})_i := \sum_{j=1}^m \mathcal{Q}_{ji} I_i z_j \overline{z}_j$, $1 \le i \le n$ and for the $m \times m$ diagonal matrix $\Omega + \eta^2 \mathcal{Q} I := \mathrm{diag}\Big(\Omega_1 + \eta^2 (\mathcal{Q}I)_1, \ldots, \Omega_m + \eta^2 (\mathcal{Q}I)_m\Big)$.

$I_0 \in \mathbf{R}^n$ since $\mathcal{R}$ is a *real* matrix (see page 7). With the choice (69),(70), $\omega T+\eta^2 \mathcal{R} I_0(T) = 2\pi k$, and then (66) holds. In addition, for $T \geq 1/\eta^2$, $I_0(T) = O(1)$. Moreover

$$\Omega_\eta T = \Omega_\eta(I_0(T))T = \Omega T + \eta^2 \mathcal{Q} I_0 T = 2\pi \left(\Omega \frac{T}{2\pi} - \mathcal{Q}\mathcal{R}^{-1} \Big\langle \omega \frac{T}{2\pi} \Big\rangle \right). \tag{71}$$

In order to prove the non-resonance property (68), we note that

$$|\mathcal{M}^{-1}| \leq \frac{1}{\min\limits_{1\leq j\leq m} |e^{\mathrm{i}\Omega_{\eta j} T} - 1|} \leq \frac{1}{\min\limits_{1\leq j\leq m} |\sin(\Omega_{\eta j} T)|} \leq \frac{2}{\min\limits_{1\leq j\leq m} \mathrm{dist}(\Omega_{\eta j} T, 2\pi \mathbf{Z})} \,. \tag{72}$$

Lemma 4.1 *Suppose that condition* (*a*) *of Theorem 1.1 hold. Define, for* $\hat{m} \geq 1$,

$$d_0 := \min \left\{ \frac{1}{4 \max\limits_{1\leq j\leq \hat{m}} |a_j|_1}, \frac{1}{4 \max\limits_{1\leq j\leq \hat{m}} M_j} \right\}, \quad \delta := \min \left\{ \frac{d_0}{2 \max\limits_{1\leq j\leq m} |(\mathcal{Q}\mathcal{R}^{-1})_j|_1}, \frac{1}{4 \max\limits_{1\leq j\leq \hat{m}} |a_j|_1} \right\} \tag{73}$$

where $(\mathcal{Q}\mathcal{R}^{-1})_j$ *is the* j*-row of the matrix* $\mathcal{Q}\mathcal{R}^{-1}$, *and, for* $\hat{m} = 0$,

$$d_0 := \frac{1}{4} \,, \qquad \delta := \min \left\{ \frac{1}{8 \max\limits_{1\leq j\leq m} |(\mathcal{Q}\mathcal{R}^{-1})_j|_1} \,, \frac{1}{4} \right\}. \tag{74}$$

Let $M := \mathrm{lcm}\,(M_1, \ldots, M_{\hat{m}})$, *for* $\hat{m} \geq 1$, *and* $M = 1$ *for* $\hat{m} = 0$. *Finally let*[17]

$$T_e := T_{erg} \left(\frac{\hat{\omega}}{M}, \frac{\delta}{M} \right), \qquad \Theta := \min \left\{ \frac{\delta}{4 \max\limits_{1\leq i\leq n} |\omega_i|} \,, \frac{d_0}{8 \max\limits_{1\leq j\leq m} |\Omega_j|} \right\}. \tag{75}$$

Then $\forall\, t_0 \geq 0$, *there exists an interval* $\mathcal{J} \subset [t_0 - 2\pi\Theta, t_0 + 2\pi T_e + 2\pi\Theta]$ *of length, at least* $4\pi\Theta$, *such that* $\forall\, T \in \mathcal{J}$, $\mathcal{M}^{-1} := \mathcal{M}^{-1}(I_0(T), T)$ *satisfies*

$$|\mathcal{M}^{-1}| \leq \frac{4}{\pi d_0} \,. \tag{76}$$

Proof: In order to define the periods T we want to use Lemma 2.3. Let us verify that its hypothesis are fulfilled. If (ω, Ω) are rationally independent, i.e. $\hat{m} = 0$, we can apply it directly. If $\hat{m} = 1$ we observe that (11) holds by definition, since $M_j \geq 1 = \hat{m}$. If $\hat{m} = 1 = M_1$ we have also to show that (12) is satisfied. This is true: indeed, by contradiction, if (12) were false then $\Omega_1 = \sum_{i=1}^n a_{1i}\omega_i$ violating the *first* order Melnikov condition (namely (4) with $|h| \leq 1$). If $m = 1$ then $\hat{m} = 0, 1$ and we are in one of the previous cases. We finally consider the case $\hat{m} \geq 2$. We have to prove (11). Note first that it is verified for $m = 2$. Indeed in this case $\hat{m} = 2$ and, hence, $\hat{\omega} = \omega$. By definition $M_j \Omega_j = a_j \cdot \omega$ for $j = 1, 2$. Again by the *first* order Melnikov condition we have, for

[17] The ergodization time T_{erg} was defined in (10).

$j=1,2$, that $M_j \geq 2$ which implies $M_j \geq 2 = \hat{m}$ and (11) holds. All the remanent cases are covered by the hypotesis (iii). We can apply Lemma 2.3.

Let $t_0 \leq \tau \leq t_0 + T_e$ be the time found in Lemma 2.3 and consider the interval $\mathcal{J} := 2\pi\tau + 2\pi[-\Theta, \Theta] \subset [t_0 - 2\pi\Theta, t_0 + 2\pi T_e + 2\pi\Theta]$. $\forall\, T = 2\pi\tau + 2\pi\theta \in \mathcal{J}$ (i.e. $|\theta| \leq \Theta$) formula (71) becomes

$$\Omega_{\eta j} T = 2\pi\Big(\Omega_j(\tau+\theta) - (\mathcal{Q}\mathcal{R}^{-1})_j \langle \omega(\tau+\theta)\rangle\Big) \qquad \forall\, 1 \leq j \leq m. \tag{77}$$

By (13), $\forall\, 1 \leq i \leq n$ there exists a $k_i \in \mathbf{Z}$ such that $\omega_i \tau = k_i + \langle \omega_i \tau \rangle$ with $|\langle \omega_i \tau \rangle| \leq \delta$; moreover $|\omega_i \theta| \leq \delta/4$ by definition of Θ. Hence we have $|\omega_i(\tau+\theta) - k_i| \leq \delta + \delta/4 < 1/2$ being $\delta \leq 1/4$. So $\langle \omega_i(\tau+\theta)\rangle = \omega_i(\tau+\theta) - k_i$ and

$$|\langle \omega_i(\tau+\theta)\rangle| \leq \frac{5}{4}\delta\ , \qquad \forall\, 1 \leq i \leq n\ . \tag{78}$$

For $1 \leq j \leq m$, by (14) we have $\mathrm{dist}(\Omega_j \tau, \mathbf{Z}) \geq d_0$, hence, by the definition of Θ, $\mathrm{dist}(\Omega_j(\tau+\theta), \mathbf{Z}) \geq d_0 - d_0/8$. Collecting the previous inequalities and (78), from (77) we obtain

$$\mathrm{dist}(\Omega_{\eta j} T, 2\pi\mathbf{Z}) = 2\pi \mathrm{dist}\Big(\Omega_j(\tau+\theta) - (\mathcal{Q}\mathcal{R}^{-1})_j \langle \omega(\tau+\theta)\rangle, \mathbf{Z}\Big) \geq 2\pi\left(d_0 - \frac{d_0}{8} - \frac{5d_0}{8}\right) = \frac{\pi d_0}{2}$$

by definition of δ. Finally, we recall (72) to end the proof. $\square$

Lemma 4.2 *Let condition* (b) *of Theorem 1.1 hold, namely* $\alpha > 0$. *Let*

$$\theta := \min_{1 \leq i \leq n} \frac{1}{|\omega_i|} \qquad \text{and} \qquad d_1 := \min\Big\{\frac{\pi}{8m}, \frac{\pi\alpha\theta}{2nm}\Big\}.$$

Then $\forall\, t_0 \geq 0$ *there exists an open set* $\mathcal{A} \subset [t_0, t_0 + 4\pi\theta]$ *of measure, at least,* $\pi\theta/n$, *such that* $\forall\, T \in \mathcal{A}$, $\mathcal{M}^{-1} := \mathcal{M}^{-1}(I_0(T), T)$ *satisfies*

$$|\mathcal{M}^{-1}| \leq \frac{2}{d_1}\ . \tag{79}$$

Proof: The function $T \to \Omega_\eta(I_0(T))T$ is a piecewise smooth function with discontinuities at the points $T_{i,k} = \frac{2\pi}{\omega_i}(k + \frac{1}{2})$, $1 \leq i \leq n$, $k \in \mathbf{Z}$. Apart these points $\Omega_\eta(I_0(T))T$ is differentiable w.r.t. to T and has constant derivative $\Omega - \mathcal{Q}\mathcal{R}^{-1}\omega =: \xi \in \mathbf{R}^m$. By the definition of θ, in every interval of the type $(2\pi\theta(h-1/2), 2\pi\theta(h+1/2))$, $h \in \mathbf{Z}$, fall at most $n-1$ points of discontinuity and, hence, there exists an interval $\mathcal{J}_1 \subseteq (2\pi\theta(h-1/2), 2\pi\theta(h+1/2))$ of length, at least, $\Delta := 2\pi\theta/n$, in which $\Omega_\eta(I_0(T))T$ is smooth. Hence, on $\mathcal{J}_1$, by (71), $\Omega_\eta T = x + \xi T$ for a suitable $x \in \mathbf{R}^m$.

For $1 \leq j \leq m$, let define the sets

$$\mathcal{B}_j := \begin{cases} \{\ T \in \mathcal{J}_1 \ \text{ s.t. } \ \mathrm{dist}(\Omega_{\eta j} T, 2\pi\mathbf{Z}) > \dfrac{\pi}{8m}\ \} & \text{if} \quad |\xi_j|\Delta \geq \pi \\[2ex] \{\ T \in \mathcal{J}_1 \ \text{ s.t. } \ \mathrm{dist}(\Omega_{\eta j} T, 2\pi\mathbf{Z}) > \dfrac{\Delta}{4m}\alpha\ \} & \text{if} \quad |\xi_j|\Delta < \pi \end{cases}.$$

Remembering the definition $\alpha := \min_{1\le j\le m} |\xi_j|$, we note that $\mathrm{meas}(\mathcal{B}_j) \ge \Delta(1 - \frac{1}{2m})$, $\forall 1 \le j \le m$. Let $\mathcal{A} := \cap_{j=1}^m \mathcal{B}_j$, then $\mathcal{A} \subseteq \mathcal{J}_1$ and $\mathrm{meas}(\mathcal{J}_1) \ge \Delta/2$. By construction $\forall T \in \mathcal{A}$, $\mathrm{dist}(\Omega_{\eta j}T, 2\pi\mathbf{Z}) \ge d_1$, $\forall 1 \le j \le m$. Finally the Lemma follows from (72). $\square$

Remark 4.1 Our non resonance conditions (a)–(b) of Theorem 1.1 are sharp: if (a)–(b) are violated, it is not possible in general to find a period T is such a way that the matrix $\mathcal{M}$ defined in (68)–(71) is invertible (clearly it must be $m \ge 3$ and $\hat{m} \ge 1$). As an example, consider the Hamiltonian $\mathcal{H}_* =$

$$\omega \cdot \mathcal{I}_* + \Omega Z_* \cdot \overline{Z}_* + \frac{1}{2} |\mathcal{I}_*|^2 + \frac{1}{p} \sum_{j=1}^{\hat{m}} (a_{j1}\mathcal{I}_{*1} + a_{j2}\mathcal{I}_{*2}) Z_{*j} \overline{Z}_{*j} + \sum_{2|k|+|a+\overline{a}|\ge 6} R^*_{k,a,\overline{a}}(\varphi_*) \mathcal{I}_*^k Z_*^a \overline{Z}_*^{\overline{a}} \,,$$

with $m \ge 4$ ($\hat{m} \le m$) and

$$a_{j1} := \begin{cases} 1 & \text{if} \quad 1 \le j \le \hat{m} - 1 \\ 0 & \text{if} \quad j = \hat{m} \end{cases} , \qquad a_{j2} := \begin{cases} j & \text{if} \quad 1 \le j \le \hat{m} - 1 \\ 1 & \text{if} \quad j = \hat{m} \end{cases} .$$

The Hamiltonian $\mathcal{H}_*$ is in the form (1)–(5). We choose $\hat{m}$ so that $p := \hat{m} - 1$ is a prime integer with $3 \le p \le m - 1$ (we require $p \ne 1, 2$ since (ω, Ω) must satisfy the second order Melnikov non-resonance conditions (4) and (ω, Ω) are related by (80) below). Let also $\hat{\omega} = (\omega, \Omega_{\hat{m}+1}, \dots, \Omega_m)$ be any rationally independent vector in $\mathbf{R}^{\hat{n}}$ and define

$$\Omega_j := \frac{1}{p}(a_{j1}\omega_1 + a_{j2}\omega_2) \,, \quad \forall 1 \le j \le \hat{m} \,. \tag{80}$$

Note that, by (80), the equations (5) are fulfilled with $M_j = p$, $\forall 1 \le j \le \hat{m}$. Hence $M_j = p = \hat{m} - 1 < \hat{m}$ and condition $(a) - (iii)$ is violated. After the rescaling in (26), the new Hamiltonian is $\mathcal{H} = \omega \cdot \mathcal{I} + \Omega Z \cdot \overline{Z} + \eta^2(\frac{1}{2}|\mathcal{I}|^2 + \mathcal{Q}I \cdot Z\overline{Z}) + O(\eta^4)$ where $\mathcal{Q} \in \mathrm{Mat}(m \times n, \mathbf{R})$ is defined by

$$\mathcal{Q}_{ji} = \begin{cases} a_{ji}/p & \text{if} \quad 1 \le j \le \hat{m} \,, \quad i = 1, 2 \,, \\ 0 & \text{elsewhere} \,. \end{cases}$$

Since the Hamiltonian $\mathcal{H}$ does not contain terms of the form (28)–(29)-(30)-(31), $\mathcal{H}$ is yet in the normal form (32) (i.e. (59)) with $\mathcal{R} = \mathbf{1}_n$. Therefore $(\Omega - \mathcal{Q}\mathcal{R}^{-1}\omega)_j = (\Omega - \mathcal{Q}\omega)_j = 0$, $\forall 1 \le j \le \hat{m}$, by (80), and also condition (b) is violated. Now, whatever we choose $T \in \mathbf{R}$, $I_0 \in \mathbf{R}^n$, $n \in \mathbf{Z}^n$ such that $\omega + \eta^2 \mathcal{R} I_0 = 2\pi n/T$, i.e. (66) is satisfied, we get, substituing $I_0 = \eta^{-2}\mathcal{R}^{-1}((2\pi n/T) - \omega)$ into (65), $\Omega_\eta(I_0)T = (\Omega - \mathcal{Q}\omega)T + 2\pi\mathcal{Q}n$. Hence, $\forall 1 \le j \le m$

$$\Omega_{\eta j}T = \Omega_{\eta j}(I_0)T = 2\pi(\mathcal{Q}_{j1}n_1 + \mathcal{Q}_{j2}n_2) \,.$$

We claim that, for all $n_i \in \mathbf{Z}$, at least one $\Omega_{\eta j}T$ is an integer multiple of 2π and hence the matrix $\mathcal{M} := \mathbf{1}_m - e^{\mathrm{i}\Omega_\eta T}$ has a zero eigenvalue. In fact,

$$\mathcal{Q}_{j1}n_1 + \mathcal{Q}_{j2}n_2 = \begin{cases} (n_1 + n_2 j)/p & \text{if} \quad 1 \le j \le p \,, \\ n_2/p & \text{if} \quad j = \hat{m} \,. \end{cases}$$

Thus, if n_2 is a multiple of p, then $\Omega_{\eta\hat{m}}T \in 2\pi\mathbf{Z}$. Otherwise, let j^* be the unique solution of the linear congruence $n_2 j^* \equiv -n_1 \bmod p$ (recall that p is prime). In this case $\Omega_{\eta j^*}T \in 2\pi\mathbf{Z}$. In both cases $\mathcal{M}$ is not invertible.

In the next Lemma we prove the existence of suitable pseudo T-periodic solutions of the Hamiltonian system (61) close to the manifold $\mathcal{P}$. Roughly, by the "twist condition" $\det\mathcal{R} \neq 0$ and the nonresonance property (68), the manifold $\mathcal{P}$ is "non-degenerate", i.e. the only T-periodic solutions of H_{int}, close to $\mathcal{P}$, are the set $\mathcal{P}$. This implies, by the Contraction Mapping Theorem, the existence of a manifold of pseudo T-periodic solutions ζ_{ϕ_0}, close to $\mathcal{P}$, diffeomorphic to $\mathbf{T}^n$. ζ_{ϕ_0} are solutions of (60) for all $t \in (0,T)$ and satisfy $\phi(T) = \phi(0) = \phi_0$, $z(T) = z(0)$ but it may happen that $I(T) \neq I(0)$.

Lemma 4.3 *Assume that Condition (a) or (b) of Theorem 1.1 holds. Then, there exist $\eta_0, C_0, C > 0$, such that: $\forall \eta \in (0, \eta_0]$, there exist an open set $\mathcal{A}_\eta \subseteq [\frac{1}{\eta^2}, \frac{1}{\eta^2}+C_0]$ of measure greater than $1/C_0$ such that $\forall T \in \mathcal{A}_\eta$ and $\forall \phi_0 \in \mathbf{T}^n$ there exists a unique function*

$$\zeta_{\phi_0} := \Big(I_{\phi_0}, \phi_{\phi_0}, z_{\phi_0}\Big) \in C\Big([0,T],\, \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m\Big) \cap C^1\Big((0,T),\, \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m\Big),$$

smooth in ϕ_0, such that

- *(i) $\zeta_{\phi_0}(t)$ solves (60) for all $t \in (0,T)$;*
- *(ii) ζ_{ϕ_0} satisfies $\phi_{\phi_0}(0) = \phi_{\phi_0}(T) = \phi_0$, $z_{\phi_0}(0) = z_{\phi_0}(T)$;*
- *(iii) $\sup_{t\in[0,T]} \Big(|I_{\phi_0}(t) - I_0| + |\phi_{\phi_0}(t) - \phi_0 - \widetilde{\omega} t| + |z_{\phi_0}(t)|\Big) \leq C\eta^2$.*

where $I_0 := I_0(T)$ is defined in (69) and $\widetilde{\omega} := \widetilde{\omega}(T) := \omega + \eta^2 \mathcal{R} I_0$.

Proof: In order to define the set $\mathcal{A}_\eta \subseteq [\frac{1}{\eta^2}, \frac{1}{\eta^2} + C_0]$ of "non-resonant" periods T, we use Lemmata 4.1 or 4.2 (according whether condition (a) or (b) of Theorem 1.1 holds) with $t_0 := \eta^{-2}$. $\forall T \in \mathcal{A}_\eta$ we look for a solution ζ_{ϕ_0} of (61) (i.e. (60)) of the form

$$I_{\phi_0}(t) = I_0 + \eta J(t), \quad \phi_{\phi_0}(t) = \phi_0 + \widetilde{\omega} t + \eta \psi(t), \quad z_{\phi_0}(t) = \eta w(t), \tag{81}$$

for suitable functions $(J,\, \psi,\, w) : [0,T] \to \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m$ satisfying $\psi(0) = \psi(T) = 0$ and $w(0) = w(T)$. The condition $\psi(0) = 0$ is a "tranversality condition": we impose the "correction" (J, ψ, w) to belong to a supplementary linear space to the tangent space of the unperturbed manifold $\mathcal{P}$ (see [BB] for a discussion of the different supplementary spaces). The functions $(J,\, \psi,\, w)$ must satisfy the system

$$\begin{aligned} \dot{J} &= O(\eta^3)\,, && (82)\\ \dot{\psi} - \eta^2 \mathcal{R} J &= O(\eta^3)\,, && (83)\\ \dot{w}_j - \mathrm{i}\Omega_{\eta j} w_j &= \mathrm{i}\eta^2 \sum_{|a+\overline{a}|=3} R^*_{0,a,\overline{a}}(\phi_0 + \widetilde{\omega} t)\overline{a}_j w^a \overline{w}^{\overline{a}-e_j} + O(\eta^3), \quad j = 1, \ldots, m && (84) \end{aligned}$$

where $\Omega_\eta = \Omega_\eta(I_0(T)) \in \mathrm{Mat}(m \times m, \mathbf{R})$ is the real diagonal matrix defined in (65).

In order to find (J, ψ, w) we use Lemmata 2.4 and 2.5. In connection with the notation of Lemma 2.5 we have here $M = \eta^2 \mathcal{R}$ and $\mathcal{M} = \mathbf{1}_m - e^{\mathrm{i}\Omega_\eta T}$. Let L denote the corresponding Green operator in (24) and $P := P(J, \psi, w; \phi_0)$ the right hand side of

(82),(83),(84). It is sufficient to find a fixed point $(J, \psi, w) \in X$, space defined in (23), of

$$(J, \psi, w) = L(P(J, \psi, w; \phi_0)) \ . \tag{85}$$

If T is obtained in Lemma 4.1, resp. Lemma 4.2, we have, by (79), resp. (76), that $|\mathcal{M}^{-1}| \leq 2/d_1$, resp. $|\mathcal{M}^{-1}| \leq 4/\pi d_0$. Moreover $|M| = O(\eta^2)$, $|e^{i\Omega_\eta}| = |\mathrm{diag}\{e^{i\Omega_{\eta j}}\}| \leq m$, $T = O(\eta^{-2})$ and, hence, from (25), we get the estimate on the Green operator $|L| \leq C'\eta^{-2}$ for some $C' > 0$. Moreover $P(0; \phi_0) = O(\eta^3)$. Then, for a constant C large enough we have $\delta_0 := C\eta \geq 2|L(P(0))|$. Finally, since $\sup_{B_{\delta_0}} |DP| = O(\eta^3)$, for η small enough, we verify also (21). Applying Lemma 2.4 in the ball B_{δ_0} we prove the existence of a solution ζ_{ϕ_0} satisfying $||\zeta_{\phi_0}|| \leq C\eta$, and hence, by (81), we get the estimate (iii).

Since $\zeta_{\phi_0} \in X$ solves the integral system (85), $\zeta_{\phi_0}(t)$ is actually C^1 for all $t \in (0, T)$ and solves (60).

Finally, since the operator $\phi_0 \to LP(J, \psi, w; \phi_0)$ is smooth, by the Implicit Function Theorem, we also deduce that the function $\phi_0 \to \zeta_{\phi_0}$ is C^1. $\square$

Remark 4.2 The terms (28), (29), (30), (31), that have been "averaged out" in Proposition 3.1, could not have been treated perturbatively in the previous Lemma. Indeed (28) (resp. (31)) would have given rise to a term $O(1)$ (resp. $O(\eta^2)$) in (84); (29) to a term $O(\eta)$ in (82); and (30) to a term $O(\eta^2)$ in (84). Roughly speaking, since the period $T \geq 1/\eta^2$, only terms of magnitude $o(\eta^2)$ can be dealt as perturbations.

Remark 4.3 The invertibility of the twist matrix $\mathcal{R}$ has been used in two ways: to modulate the frequency ω yielding the existence of nearby resonant frequencies, see (66), and to imply that the manifold $\mathcal{P}$ is non-degenerate (together with the non-resonance property (68)). Weaker conditions could in principle be used (if the elliptic variables were absent think to [CZ1]). However, in this case, when $\mathcal{R}$ is degenerate one is forced to study the higher order terms in the normal form of Proposition 3.1 making the perturbative analysis much more difficult. Moreover in the three body problem application we will prove that $\mathcal{R}$ is invertible, see Lemma 5.1.

Remark 4.4 When $m = 0$, namely in the case of *maximal* tori, the twist matrix $\mathcal{R}$ reduces simply to the matrix of the second derivatives of $\mathcal{H}_*$ w.r.t. the action variables $\mathcal{I}_*$. This is not the case when $m \neq 0$. Consider the Hamiltonian $\mathcal{H}_* = \omega\mathcal{I}_* + \Omega Z_* \overline{Z}_* + \mathcal{I}_*(Z_* + \overline{Z}_*) + \mathcal{I}_*^2/\Omega$ where $(\mathcal{I}_*, \varphi_*) \in \mathbf{R} \times \mathbf{T}$, $(Z_*, \overline{Z}_*) \in \mathbf{C}^2$ and $\Omega > 0$. The twist matrix in (7) turns out to be $\mathcal{R} = 0$ although the second derivative of $\mathcal{H}_*$ w.r.t. $\mathcal{I}_*$ equals to $2/\Omega \neq 0$. Heuristically, for *lower* dimensional elliptic tori, the twist matrix $\mathcal{R}$ takes into account, with the second addendum in the r.h.s. of (7), the interaction between the torus variables $(\mathcal{I}_*, \varphi_*)$ and the elliptic variables Z_*. Finally observe that the symplectic map $\mathcal{I}_* = I$, $\varphi_* = \phi + (z - \overline{z})/(\mathrm{i}\Omega)$, $Z_* = z - I/\Omega$ (which can be found through the averaging procedure of section 3) transforms $\mathcal{H}$ into the *isochronous* Hamiltonian $H = \omega I + \Omega z\overline{z}$.

4.2 The variational principle

We now define the "reduced Hamiltonian action functional" $\mathcal{E} : \mathbf{T}^n \to \mathbf{R}$ as the Hamiltonian action functional $\mathcal{A}$ evaluated on the pseudo T-periodic solutions ζ_{ϕ_0} obtained in Lemma 4.3, namely[18]

$$\mathcal{E}(\phi_0) := \mathcal{A}(I_{\phi_0}, \phi_{\phi_0}, z_{\phi_0}, \overline{z}_{\phi_0}) = \int_0^T I_{\phi_0} \cdot \dot{\phi}_{\phi_0} + \mathrm{i} z_{\phi_0} \cdot \dot{\overline{z}}_{\phi_0} - H(I_{\phi_0}, \phi_{\phi_0}, z_{\phi_0}, \overline{z}_{\phi_0})\, dt\,. \quad (86)$$

By Lemma 4.3 the reduced action functional $\mathcal{E}$ is smooth in ϕ_0.

Critical points of $\mathcal{E} : \mathbf{T}^n \to \mathbf{R}$ give rise to T-periodic solutions of the Hamiltonian system (60), according to the following Lemma:

Lemma 4.4 $\partial_{\phi_0}\mathcal{E}(\phi_0) = I_{\phi_0}(T) - I_{\phi_0}(0)$. *Hence, if $\phi^\star \in \mathbf{T}^n$ is a critical point of $\mathcal{E}$, then $\zeta_{\phi^\star}(t)$ can be extended to a T-periodic solution of the Hamiltonian system* (60).

Proof: Differentiating w.r.t. ϕ_0 in (86) we get

$$\begin{aligned} \partial_{\phi_0}\mathcal{E}(\phi_0) &= \int_0^T \partial_{\phi_0} I_{\phi_0} \cdot \dot{\phi}_{\phi_0} + I_{\phi_0} \cdot \partial_{\phi_0}\dot{\phi}_{\phi_0} + \mathrm{i}\partial_{\phi_0} z_{\phi_0} \dot{\overline{z}}_{\phi_0} + \mathrm{i} z_{\phi_0} \partial_{\phi_0} \dot{\overline{z}}_{\phi_0} \\ &- \partial_I H \partial_{\phi_0} I_{\phi_0} - \partial_\phi H \partial_{\phi_0}\phi_{\phi_0} - \partial_z H \partial_{\phi_0} z_{\phi_0} - \partial_{\overline{z}} H \partial_{\phi_0} \overline{z}_{\phi_0}\; dt \\ &= I_{\phi_0}(T)\partial_{\phi_0}\phi_{\phi_0}(T) - I_{\phi_0}(0)\partial_{\phi_0}\phi_{\phi_0}(0) + \mathrm{i}\partial_{\phi_0}\overline{z}_{\phi_0}(T) z_{\phi_0}(T) - \mathrm{i}\partial_{\phi_0}\overline{z}_{\phi_0}(0) z_{\phi_0}(0), \end{aligned}$$

by an integration by parts, and since $\zeta_{\phi_0}(t)$ satisfies the Hamilton's equations (60) in $(0, T)$. Moreover, since $\forall \phi_0 \in \mathbf{T}^n$, $\phi_{\phi_0}(T) = \phi_{\phi_0}(0) = \phi_0$ and $\overline{z}_{\phi_0}(T) = \overline{z}_{\phi_0}(0)$, deriving w.r.t. ϕ_0, we get $\partial_{\phi_0}\phi_{\phi_0}(T) = \partial_{\phi_0}\phi_{\phi_0}(0) = \mathbf{1}_n$ and $\partial_{\phi_0}\overline{z}_{\phi_0}(T) = \partial_{\phi_0}\overline{z}_{\phi_0}(0)$. This implies $\partial_{\phi_0}\mathcal{E}(\phi_0) = I_{\phi_0}(T) - I_{\phi_0}(0)$. Hence, if $\phi^\star \in \mathbf{T}^n$ is a critical point of $\mathcal{E}$, $\zeta_{\phi^\star}(T) = \zeta_{\phi^\star}(0)$ and we deduce that $\zeta_{\phi^\star}(t)$ can be extended to a T-periodic solution of the Hamiltonian system (60). $\square$

The following Lemma, which is a consequence of the autonomy of the Hamiltonian H, holds.

Lemma 4.5 $\forall \phi_0 \in \mathbf{T}^n$ *there exists $V(\phi_0, \eta) \in \mathbf{R}^n$ with $V(\phi_0, \eta) = \widetilde{\omega} + O(\eta^3)$ such that*

$$V(\phi_0, \eta) \cdot \partial_{\phi_0}\mathcal{E}(\phi_0) = 0, \qquad \forall \phi_0 \in \mathbf{T}^n. \quad (87)$$

Proof: Since $\zeta_{\phi_0} = (I_{\phi_0}, \phi_{\phi_0}, z_{\phi_0})$ satisfies the Hamiltonian system (60) in $(0, T)$ and $\phi_{\phi_0}(T) = \phi_{\phi_0}(0) = \phi_0$, $z_{\phi_0}(T) = z_{\phi_0}(0)$, then

$$H(I_{\phi_0}(T), \phi_0, z_{\phi_0}(0), \overline{z}_{\phi_0}(0)) = H(I_{\phi_0}(0), \phi_0, z_{\phi_0}(0), \overline{z}_{\phi_0}(0)). \quad (88)$$

By the mean value Theorem there exists ξ_{ϕ_0} in the segment between $[I_{\phi_0}(T), I_{\phi_0}(0)]$ such that

$$\partial_I H(\xi_{\phi_0}, \phi_0, z_{\phi_0}(0), \overline{z}_{\phi_0}(0)) \cdot [I_{\phi_0}(T) - I_{\phi_0}(0)] = 0. \quad (89)$$

[18] $\mathcal{E}(\phi_0) \in \mathbf{R}$ since $\int_0^T \mathrm{Im}\ (\mathrm{i} z_{\phi_0} \dot{\overline{z}}_{\phi_0})\ dt = \int_0^T \mathrm{Re}\ (z_{\phi_0} \dot{\overline{z}}_{\phi_0})\ dt = \int_0^T \frac{1}{2}\frac{d}{dt}(z_{\phi_0}\overline{z}_{\phi_0})\ dt = 0$.

Define $V(\phi_0, \eta) := \partial_I H(\xi_{\phi_0}, \phi_0, z_{\phi_0}(0), \overline{z}_{\phi_0}(0))$. By Lemma 4.4 and (89), formula (87) follows. Finally, since $I_{\phi_0}(T) - I_0 = I_{\phi_0}(0) - I_0 = O(\eta^2)$, $z_{\phi_0}(0) = O(\eta^2)$ and $\partial_I H = \omega + \eta^2 \mathcal{R} I + \eta^2 \mathcal{Q}^T z\overline{z} + O(\eta^3)$, we deduce the estimate $V(\phi_0, \eta) = \widetilde{\omega} + O(\eta^3)$. $\square$

Proof of Theorem 1.1: By Lemma 4.4 the absolute minimum (and maximum) $\phi^\star \in \mathbf{T}^n$ gives rise to a T-periodic solution $\zeta_{\phi^\star}$ of (60). However one expects the existence of at least n geometrically distinct T-periodic solutions of (60), i.e. solutions not obtained one from each other simply by time-translations.

In order find multiple geometrically distinct periodic solutions of (60) we restrict the reduced action function $\mathcal{E}$ to the plane $E := [\widetilde{\omega}]^\perp$ orthogonal to the periodic flow $\widetilde{\omega} = (1/T)2\pi k$ with $k \in \mathbf{Z}^n$. The set $\mathbf{Z}^n \cap E$ is a lattice of E, (see for example Lemma 8.2 of [BBB]) and hence $\mathcal{E}$ can be defined on the quotient space $\Gamma := E/(\mathbf{Z}^n \cap E) \sim \mathbf{T}^{n-1}$.

A critical point $\phi^\star$ of $\mathcal{E} : \Gamma \to \mathbf{R}$ is a critical point of $\mathcal{E} : \mathbf{T}^n \to \mathbf{R}$. Indeed, since the tangent space $T_{\phi_0}\Gamma = [\widetilde{\omega}]^\perp$, then

$$\partial_{\phi_0} \mathcal{E}(\phi^\star) = \lambda(\phi^\star)\widetilde{\omega} \tag{90}$$

for some Lagrange multiplier $\lambda(\phi^\star) \in \mathbf{R}$. (90) and Lemma 4.5 imply that $\lambda(\phi^\star)(|\widetilde{\omega}|^2 + O(\eta^3)) = 0$ and so, for η small, $\lambda(\phi^\star) = 0$.

By the Lusternik-Schnirelman category theory, see for example [Am], since $\mathrm{cat}\Gamma = \mathrm{cat}\mathbf{T}^{n-1} = n$, we can define the n min-max critical values $c_1 \le c_2 \le \ldots \le c_n$ for the reduced action functional $\mathcal{E}_{|\Gamma}$. Let $\phi_1^\star, \ldots, \phi_n^\star \in \Gamma$ be n corresponding critical points. If the critical levels c_i are all distinct, the T-periodic solutions $\zeta_{\phi_i^\star}(t) = (I_0, \phi_i^\star + \widetilde{\omega}t, 0) + O(\eta^2)$ of (60) are geometrically distinct, since their actions $\mathcal{A}(\zeta_{\phi_i^\star}) = \mathcal{E}(\phi_i^\star) = c_i$ are different. On the other hand, if some min-max critical level c_i coincide, then $\mathcal{E}_{|\Gamma}$ possesses infinitely many critical points. However *not* all the corresponding T-periodic solutions of (60) are necessarily geometrically distinct, since two different critical points could belong to the same orbit. In any case, since one periodic solution can cross Γ at most a finite number of times, the existence of infinitely many geometrically distinct orbits of (60) follows[19].

Finally, under the inverse transformation of Φ, defined in Proposition 3.1, η-close to the identity and the inverse transformation of (26), we find the T-periodic solutions ζ_η of Theorem 1.1 satisfying the estimates (i)-(ii).

The statement on the minimal period descends from the following Proposition. $\square$

Proposition 4.1 *Let $\zeta(t) = (I(t), \phi(t), z(t))$ be a T-periodic solution of* (61) *with $C\eta^{-2} \ge T \ge T_0$ for some T_0 large, independent of η. Then the minimal period T_{min} of ζ satisfies $T_{min} \ge \mathrm{const}\ T^{1/(\tau+1)}$.*

Proof: Since ζ is T-periodic, $\phi(T) - \phi(0) = 2\pi k$ for some $k \in \mathbf{Z}^n$. We claim that $T_{min} \ge T/g$ where $g := \gcd(k_1, \ldots k_n)$. Indeed $T_{min} \ge T^\phi_{min}$ where T^ϕ_{min} is the minimal

[19] Non-degenerate critical points of the Poincaré-Melnikov primitive $\phi_0 \to \int_0^T (H - H_{int})(I_0, \phi_0 + \widetilde{\omega}t, 0)\, dt$ could be continued to solutions of (60). However, it is very difficult to compute this function, its critical points and check whether they are non-degenerate (if ever true).

period of $\phi(t)$. Moreover $T^{\phi}_{min} = T/n$ for some integer $n \geq 1$ and $\phi(T_{min}) - \phi(0) = 2\pi\widetilde{k}$ for some $\widetilde{k} \in \mathbf{Z}^n$. It follows that $\widetilde{k}n = k$ and so $n \leq g$, proving the claim.

By the second equation in (61) we also have $\phi(T) - \phi(0) = \omega T + O(\eta^2 T)$ and then, letting $\hat{k} := k/g \in \mathbf{Z}^n$, $2\pi g\hat{k} = \omega T + O(\eta^2 T)$. We deduce that

$$T_{min} \geq \frac{T}{g} \geq \frac{2\pi}{|\omega|}|\hat{k}| - O\Big(\frac{\eta^2 T}{|\omega|g}\Big) \geq \frac{2\pi}{|\omega|}|\hat{k}| - O\Big(\frac{C}{|\omega|}\Big), \tag{91}$$

since $\eta^2 T \leq C$. Let now choose $h \in \mathbf{Z}^n \setminus \{0\}$ with $|h| \leq |\hat{k}|$ such that $\hat{k} \cdot h = 0$. Multiplying by h we get $0 = 2\pi g\hat{k} \cdot h = \omega \cdot hT + O(\eta^2 T|h|)$ ans so $\omega \cdot h = O(\eta^2|h|)$. Using the diophantinity of ω assumed in (4), namely $|\omega \cdot h| \geq \gamma(1+|h|)^{-\tau}$, $\forall h \in \mathbf{Z}^n$, we obtain $|h|^{\tau+1} \geq$ const $\eta^{-2} \geq$ const T. Hence $|\hat{k}|^{\tau+1} \geq |h|^{\tau+1} \geq$ const T and $|\hat{k}| \geq$ const $T^{1/(\tau+1)}$. The proposition follows from (91) for $T \geq T_0$ large enough. □

Remark 4.5 The periodic orbits ζ_η found in Theorem 1.1 are $O(\eta^2)$ close to the torus $\mathcal{T}$ since I_0, defined in (69), satisfies $I_0 = O(1)$. We could also try to find periodic solutions when e.g. $I_0 = O(1/\eta)$ and so $T = O(1/\eta)$. However the terms $O(I^k) = O(\eta^{-k})$ would be more difficult to control.

5 The planetary spatial three-body problem

In this section we will prove the existence of periodic orbits of the planetary non planar three body problem, with "small eccentricities" and "small mutual inclinations", accumulating onto two-dimensional elliptic invariant tori. We first discuss the classical Hamiltonian formulation of this problem which dates back to Delaunay and Poincaré. For a detailed treatment see the Appendix of [BCV].

The three massive points ("bodies") P_0, P_1, P_2, with masses m_0, m_1, m_2, interact one each other through gravity (with constant of gravitation 1). Assume that the masses of the bodies satisfy, for some $0 < \overline{\kappa} \leq 1$,

$$\overline{\kappa}\varepsilon \leq \frac{m_1}{m_0}\,,\ \frac{m_2}{m_0} \leq \varepsilon \leq 1\,. \tag{92}$$

The number $\varepsilon > 0$ is regarded as a small parameter: the point P_0 represents "the star" and the points P_1 and P_2 "the planets".

We now recall the classical definition of the "osculating ellipses" (at time t_0) of the two-body problems associated to the planets P_i ($i = 1, 2$) and the star P_0. Let $u^{(0)}$ and $u^{(i)}$ denote the coordinates of the points P_0 and P_i at time t_0 and let $\dot{u}^{(0)}$ and $\dot{u}^{(i)}$ denote the respective velocities. The "osculating plane" is defined as the plane spanned by $(u^{(i)} - u^{(0)})$ and $(\dot{u}^{(i)} - \dot{u}^{(0)})$; the "osculating ellipse" is defined as the Keplerian ellipse (lying on the osculating plane) defined by the Kepler solution, with initial data $(u^{(0)}, u^{(i)})$ and $(\dot{u}^{(i)} - \dot{u}^{(0)})$, of the two-body problem (P_0, P_i) obtained disregarding (for $t \geq t_0$) the third body P_j ($j \neq i$).

We assume that the eccentricities of such ellipses are small and that the intersection angle between the two planes containing the two osculating ellipses (usually referred as "mutual inclination") is also small. It is customary in celestial mechanics to denote the *major semi-axes* of such ellipses by a_i and their eccentricities by e_i. Let

$$\Lambda_i^* := \kappa_i^* \sqrt{a_i}\ , \qquad \kappa_i^* := \frac{m_i}{\varepsilon} \frac{1}{\sqrt{m_0(m_0+m_i)}}\ ,$$

(κ_i^* is a dimensionless constant satisfying $\frac{\overline{\kappa}}{\sqrt{2}} < \kappa_i^* < 1$). Since we are interested in small eccentricities, collisions are avoided by requiring that the major semi-axes $a_i = a_i(\Lambda^*) := (\Lambda_i^*/\kappa_i^*)^2$, $i = 1, 2$, are different, and different from zero. We, therefore, fix, once and for all,

$$0 < a_{\min} < a_{\max} \tag{93}$$

and, from now on, we shall consider (attaching the index 1 to the "inner planet") values of Λ^* in the set

$$\mathcal{L} := \{\Lambda^* \in \mathbf{R}^2 : a_{\min} \le a_1 < a_2 \le a_{\max}\}\ . \tag{94}$$

The following classical result follows from the Delaunay-Poincaré theory, see [BCV].

Theorem (Delaunay-Poincaré) *Fix $\Lambda_0^* \in \mathcal{L}$. There exists a symplectic set of variables*[20]

$$(I_0, \varphi_0, p_0, q_0) \in \mathbf{R}^2 \times \mathbf{T}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$$

where $I_0 = \Lambda^* \in \mathcal{I}$ and $\mathcal{I} \subset \mathcal{L} \subset \mathbf{R}^2$ is a suitable two-dimensional open cube centered at Λ_0^*, such that the Hamiltonian of the spatial planetary three-body problem takes the form

$$\mathcal{H}_0(I_0, \varphi_0, p_0, q_0) = h_0(I_0) + f_0(I_0, \varphi_0, p_0, q_0)\,, \tag{95}$$

with

$$\begin{aligned}
&h_0 := -\frac{1}{2}\sum_{i=1}^{2}\frac{\kappa_i}{I_{0i}^2}\,, \quad \kappa_i := \Big(\frac{m_i}{\varepsilon}\Big)^3 \frac{1}{m_0^2(m_0+m_i)}, \quad \Big(\frac{\bar{\kappa}^3}{2} < \kappa_i < 1\Big),\\
&f_0 := \varepsilon f_1(I_0, p_0, q_0) + \varepsilon f_2(I_0, \varphi_0, p_0, q_0)\,,\\
&f_1 := f_{1,0}(I_0) + \sum_{j=1}^{2} \overline{\Omega}_j(I_0)(p_{0j}^2 + q_{0j}^2) + \widetilde{f}_1(I_0, p_0, q_0)\ ,\\
&\int_{\mathbf{T}^2} f_2\, d\varphi_0 = 0\ , \qquad \sup_{\mathcal{I}^2_{\sigma_0}} |\widetilde{f}_1| \le \mathrm{const}|(p_0, q_0)|^4\ .
\end{aligned} \tag{96}$$

f_1 and f_2 are real-analytic and uniformly bounded on[21]

$$\mathcal{D}_0 := \mathcal{I}^2_{\sigma_0} \times \mathbf{T}^2_{s_0} \times D^4_{\rho_0} \subset \mathbf{C}^8\ , \tag{97}$$

[20] With symplectic form $dI_0 \wedge d\varphi_0 + dp_0 \wedge dq_0$.

[21] $D^d_\rho \subset \mathbf{C}^d$ denotes the open complex ball of radius ρ centered at the origin.

where σ_0, s_0, ρ_0 are suitable positive numbers. Moreover $\widetilde{f}_1$ is even in (p_0, q_0) and

$$\inf_{I_0\in\mathcal{I}} \overline{\Omega}_j(I_0) > \inf_{I_0\in\mathcal{I}^2_{\sigma_0}} |\overline{\Omega}_j(I_0)| > \text{ const } > 0 \ ,$$
$$\inf_{I_0\in\mathcal{I}} \Big(\overline{\Omega}_2(I_0) - \overline{\Omega}_1(I_0)\Big) > \inf_{I_0\in\mathcal{I}^2_{\sigma_0}} |\overline{\Omega}_2(I_0) - \overline{\Omega}_1(I_0)| > \text{ const } > 0\,. \tag{98}$$

We underline that the actions I_0 are simply $I_0 = \Lambda^*$ and we refer to [BCV]-Theorem 1.1 for the complete expressions of the other Delaunay-Poincaré variables (φ_0, p_0, q_0).

We remark that the eccentricities e_i are estimated as[22]

$$c^-_{\mathcal{I}} |(p_0, q_0)| \le e_i \le c^+_{\mathcal{I}} |(p_0, q_0)| \ , \qquad i = 1, 2 \tag{99}$$

for two suitable constants $0 < c^-_{\mathcal{I}} < c^+_{\mathcal{I}}$ (depending only on $\mathcal{I}$).

The Hamiltonian $\mathcal{H}_0$ in (95) describes a nearly-integrable, *properly degenerate*, system: the integrable Hamiltonian h_0 depends only on the two action variables (I_{01}, I_{02}). Such degeneracies are a typical feature of problems arising in celestial mechanics and the source of the main difficulties (the application of standard KAM theory for finding maximal tori require the Hamiltonian to be non-degenerate). The frequency vector $\nabla h_0(I_0)$ is independent of ε, and then the conjugated angles φ_0 may be regarded as "fast angles" and, in "first approximation" the $\mathcal{H}_0$-motions are governed by the averaged Hamiltonian $h_0+\varepsilon f_1$. By (96), for any fixed I_0, $\{\varphi_0 \in \mathbf{T}^2\} \times \{p_0 = q_0 = 0\}$ is an elliptic invariant torus for the averaged Hamiltonian $h_0+\varepsilon f_1$ run by the linear flow $\varphi_0 \to \varphi_0+\nabla(h_0+\varepsilon f_{1,0})(I_0)\, t$. These are the quasi-periodic motions that will persist, for ε small, and I_0 in a nearly full (two-dimensional) measure set, as proved in [BCV], see also Theorem 5.2.

In the next Proposition, through an appropiate averaging procedure, The Hamiltonian (95) is casted into a suitable normal form.

Proposition 5.1 *Fix $N \in \mathbf{N}^+$. There exists a $O(\sqrt{\varepsilon})$-close to the identity, real analytic, symplectic change of variables $(J, \psi, z, \overline{z}) \in U^2_{\tilde r} \times \mathbf{T}^2_{\tilde s} \times D^4_{\tilde\rho} \to (I_0, \varphi_0, p_0, q_0) \in \mathcal{D}_0 \subset \mathbf{C}^8$ transforming the Hamiltonian* (95) *into the real analytic Hamiltonian*

$$\mathcal{H}(J, \psi, z, \overline{z}) = h_\varepsilon(J) + \varepsilon\widetilde{\Omega}(J) z\overline{z} + \varepsilon g(J, z, \overline{z}; \varepsilon) + \varepsilon^N f(J, \psi, z, \overline{z}; \varepsilon) \tag{100}$$

where $h_\varepsilon := h_0 + O(\varepsilon)$,

$$\tilde r = \text{const}\,\sqrt{\varepsilon}, \qquad \tilde s = \text{const}, \qquad \tilde\rho = \text{const} \ , \tag{101}$$

$$U := \Big\{ J \in \mathcal{I} \quad : \quad |h_0'(J)\cdot \ell| \ge \alpha_0 \ , \forall\ \ell \in \mathbf{Z}^2 \ , 0 < |\ell| \le K \Big\} \subset \mathcal{I}, \tag{102}$$

is a closed set and

$$K := \frac{6(N-1)}{s_0} \log\frac{1}{\varepsilon} \ , \quad \alpha_0 := 2 \sup_{\mathcal{I}_{\sigma_0}} |h_0''| \tilde r K = \text{const}\sqrt{\varepsilon}\log\frac{1}{\varepsilon} \ , \tag{103}$$

[22] See (1.5),(C.10),(C.15),(C.17) of [BCV].

It results that

$$\mathrm{meas}(\mathcal{I} \setminus U) \le \text{const } \alpha_0 = O\Big(\sqrt{\varepsilon} \log \frac{1}{\varepsilon}\Big). \tag{104}$$

Moreover

$$\sup_{J \in U^2_{\tilde{r}}} |g(J, z, \bar{z})| \le \text{ const } \Big(|z| + |\bar{z}|\Big)^3, \tag{105}$$

$$\begin{aligned} &\inf_{J \in U} \widetilde{\Omega}_j(J) \ge \inf_{J \in U^2_{\tilde{r}}} |\widetilde{\Omega}_j(J)| \ge \chi_0 > 0, \\ &\inf_{J \in U} \Big(\widetilde{\Omega}_2(J) - \widetilde{\Omega}_1(J)\Big) \ge \inf_{J \in U^2_{\tilde{r}}} |\widetilde{\Omega}_2(J) - \widetilde{\Omega}_1(J)| \ge \chi_0 > 0, \end{aligned} \tag{106}$$

for some positive constant χ_0.

Proof: The Hamiltonian (100) has been deduced in [BCV] in the case $N = 3$ and on the smaller domain $U' := \{J \in \mathcal{I} \mid |h_0'(J) \cdot \ell| \ge \text{const}(\sqrt{\varepsilon} \log^{\tau+1}(1/\varepsilon))/(1 + |l|^\tau)$, $\forall \ell \in \mathbf{Z}^2 \setminus \{0\}\} \subset U$, see formula (2.22) of [BCV] and introduce the complex coordinates (170). Note also that the analyticity constant $\tilde{\rho} = \text{const}$ is bigger than $\tilde{\rho} = \log^{-1}(1/\varepsilon)$ given in [BCV] . For the proof of the Proposition see the Appendix. $\square$

The manifold $\{z = 0\}$ is invariant under the Hamiltonian system generated by the integrable Hamiltonian $h_\varepsilon(J) + \varepsilon \widetilde{\Omega}(J) z \bar{z}$,

$$\dot{J} = 0, \qquad \dot{\psi} = \partial_J h_\varepsilon(J) + \varepsilon \partial_J \widetilde{\Omega}(J) z \bar{z}, \qquad \dot{z} = \mathrm{i} \varepsilon \widetilde{\Omega}(J) z \tag{107}$$

and it is completely filled up by the invariant elliptic tori $\mathcal{T}(J_0) := \{J = J_0,\ \psi \in \mathbf{T}^2$, $z = 0\}$, supporting the linear flow $t \to \{J_0,\ \psi_0 + \omega_\varepsilon(J_0) t, 0\}$ with torus frequency $\omega_\varepsilon(J_0) := \partial_J h_\varepsilon(J_0)$. This family of 2-dimensional tori will not persist in its entirety for the complete Hamiltonian system generated by $\mathcal{H}$

$$\dot{J} = -\partial_\psi \mathcal{H}, \quad \dot{\psi} = \partial_J \mathcal{H}, \quad \dot{z} = \mathrm{i} \partial_{\bar{z}} \mathcal{H}, \quad \dot{\bar{z}} = -\mathrm{i} \partial_z \mathcal{H}, \tag{108}$$

due to resonances among the oscillations. However, the persistence of a set of positive (2-dimensional) measure of perturbed elliptic invariant tori -those with diophantine frequency $\omega_\varepsilon(J_0)$ - has been proved in [BCV], using the KAM Theorem of [Pö2].

In Theorem 0.2 we will prove the existence of an abundance of periodic solutions with larger and larger period accumulating on each perturbed elliptic torus, applying Theorem 1.1. For this aim, we will first reprove (Theorem 5.2) the existence of the [BCV]-elliptic tori, using the KAM Theorem of [Pö1] and furnishing also the KAM-normal form describing the dynamics in its neighborhood (it will be of the form considered in (1)). Moreover, for proving that the hypothesys of Theorem 1.1 are satisfied, we also need precise informations on the form of the KAM-transformation bringing into the KAM normal form, see Theorem 5.1-(*ii*).

For the existence result of the elliptic invariant tori (Theorem 5.2) and their surrounding periodic orbits (Theorem 0.2) it is sufficient, and we will assume, $N = 3$.

First of all we rewrite the Hamiltonian $\mathcal{H}$ in (100) in a form suitable to apply the KAM Theorem of [Pö1]. Introducing the coordinate $y \in \mathbf{R}^n$ around each torus $\mathcal{T}(J_0)$ in the usual way $J = J_0 + y$, the Hamiltonian $\mathcal{H}$ can be developed as

$$H(y, \psi, z, \overline{z}; \omega) := \mathcal{H}(J_0 + y, \psi, z, \overline{z}) = \mathcal{N} + P \tag{109}$$

where $\mathcal{N} := h_\varepsilon(J_0) + h'_\varepsilon(J_0) \cdot y + \varepsilon\widetilde{\Omega}(J_0) z\overline{z}$ and $P = P(y, \psi, z, \overline{z}; J_0) = P_1 + P_2 + P_3 + P_4$ with

$$\begin{aligned} P_1 &:= h_\varepsilon(J_0 + y) - h_\varepsilon(J_0) - h'_\varepsilon(J_0) \cdot y = O(|y|^2), \\ P_2 &:= \varepsilon\Big(\widetilde{\Omega}(J_0 + y) - \widetilde{\Omega}(J_0)\Big) z\overline{z} = O(\varepsilon|y||z||\overline{z}|), \\ P_3 &:= \varepsilon g(J_0 + y, z, \overline{z}) = O\Big(\varepsilon(|z| + |\overline{z}|)^3\Big), \\ P_4 &:= \varepsilon^3 f(J_0 + y, x, z, \overline{z}) = O(\varepsilon^3) \ . \end{aligned} \tag{110}$$

By the non-isocronicity property det $h''_\varepsilon(J_0) \neq 0$ there is a one-to-one correspondence between the actions $J_0 \in U \subset \mathbf{R}^2$ and the torus frequency $\omega := \omega(J_0) = h'_\varepsilon(J_0) \in \mathcal{O} \subset \mathbf{R}^2$ where

$$\mathcal{O} := \{ \, \omega = h'_\varepsilon(J_0) \ : \ J_0 \in U \, \} \, . \tag{111}$$

By (104) we have

$$\text{meas}\,(h'_\varepsilon(\mathcal{I}) \setminus \mathcal{O}) \le \text{const } \alpha_0 = O\Big(\sqrt{\varepsilon} \log\Big(\frac{1}{\varepsilon}\Big)\Big). \tag{112}$$

The frequencies $\omega := \omega(J_0)$ may be introduced as parameters: denoting $J_0 = J_0(\omega) := (h'_\varepsilon)^{-1}(\omega)$ its inverse function, the Hamiltonian (109) can be finally written in the form

$$H(y, \psi, z, \overline{z}; \omega) := \mathcal{N}(y, z, \overline{z}; \omega) + P(y, \psi, z, \overline{z}; \omega) \tag{113}$$

where $\mathcal{N}(y, z, \overline{z}; \omega) := e(\omega) + \omega \cdot y + \Omega(\omega) z\overline{z}$, $e(\omega) := h_\varepsilon(J_0(\omega))$, $\Omega(\omega) := \varepsilon\widetilde{\Omega}(J_0(\omega))$ and the perturbation $P(y, \psi, z, \overline{z}; \omega)$ is obtained by the one in (109) just replacing J_0 with $J_0(\omega) := (h'_\varepsilon)^{-1}(\omega)$. Recalling (101) the Hamiltonian H in (113) is real analytic in

$$(y, \psi, z, \overline{z}; \omega) \in U^2_{\widetilde{r}/2} \times \mathbf{T}^2_{\widetilde{s}} \times D^4_{\widetilde{\rho}} \times \mathcal{O}^2_{\widetilde{\delta}}, \qquad \text{with} \qquad \widetilde{\delta} := \text{const}\,\sqrt{\varepsilon} \tag{114}$$

(since $\omega \to J_0(\omega) := (h'_\varepsilon)^{-1}$ is analytic).

H is in a suitable form to apply the KAM Theorem of [Pö1] that we rewrite in the next subsection.

5.1 A KAM Theorem for elliptic tori

The KAM Theorem of [Pö1] applies to Hamiltonians like

$$H := H(y, \psi, z, \overline{z}; \omega) := \mathcal{N} + P = e(\omega) + \omega \cdot y + \Omega(\omega) z\overline{z} + P(y, \psi, z, \overline{z}; \omega)$$

where $(y,\psi,z,\overline{z}) \in \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{C}^m \times \mathbf{C}^m$ and $\omega \in \mathbf{R}^n$ is regarded as a parameter varying over a compact subset $\mathcal{O} \subseteq \mathbf{R}^n$.

The functions $P = P(y,\psi,z,\overline{z};\omega)$ and $\Omega(\omega)$ are real analytic on the complex domain $\mathcal{D}_{\overline{r},\overline{s}} \times \mathcal{O}_\delta$ where $\mathcal{D}_{\overline{r},\overline{s}} := D^n_{\overline{r}^2} \times \mathbf{T}^n_{\overline{s}} \times D^{2m}_{\overline{r}}$ and $\overline{r},\overline{s},\delta$ are suitable positive constants.

The size of the perturbation P is measured by the following norm. Taking the Fourier-Taylor expansions

$$P(y,\psi,z,\overline{z};\omega) = \sum_{\substack{\ell\in\mathbf{Z}^n\\ a,\overline{a}\in\mathbf{N}^m}} P_{a,\overline{a},\ell}(y;\omega)\, z^a\overline{z}^{\overline{a}}\, e^{\mathrm{i}\ell\cdot\psi} = \sum_{\ell\in\mathbf{Z}^n} P_\ell(y,z,\overline{z};\omega)\, e^{\mathrm{i}\ell\cdot\psi},$$

let define[23]

$$|||P|||_{\overline{r},\overline{s},\delta} := \sum_{\ell\in\mathbf{Z}^n} |\mathbf{M}P_\ell|_{\overline{r},\delta} e^{|\ell|\overline{s}}$$

where $\mathbf{M}P_\ell := \sum_{a,\overline{a}\geq 0} |P_{a,\overline{a},\ell}(y)| z^a\overline{z}^{\overline{a}}$ and $|\cdot|_{\overline{r},\delta}$ denotes the sup-norm over the $(y,z,\overline{z}) \in D^n_{\overline{r}^2} \times D^{2m}_{\overline{r}}$ and $\omega \in \mathcal{O}_\delta$. The following Theorem follows from [Pö1].

Theorem 5.1 (Pöschel [Pö1]) *Fix $\tau > n-1$. Suppose*

$$\sup_{\omega\in\mathcal{O}_\delta} |\partial_\omega \Omega(\omega)| \leq M \tag{115}$$

for some $0 < M < +\infty$, and that the non-resonance condition

$$|\Omega(\omega)\cdot h| \geq \alpha\ , \qquad \forall\ 1 \leq |h| \leq 2,\ h \in \mathbf{Z}^n, \qquad \forall \omega \in \mathcal{O}, \tag{116}$$

is satisfied for some $\alpha > 0$. Then, there exists a positive constant $\kappa := \kappa(\overline{s})$ such that, if P is sufficiently small,

$$\mathcal{P} := |||P|||_{\overline{r},\overline{s},\delta} \leq \frac{\kappa}{M+1}\alpha\overline{r}^2 \leq \frac{\delta\overline{r}^2}{16}\ , \tag{117}$$

then:

(i) there exist a normal form $\mathcal{N}_* := e_* + \omega\cdot y_* + \Omega_*(\omega) z_*\overline{z}_*$, a Cantor set $\mathcal{O}(\alpha) \subset \mathcal{O}$ on which

$$|\omega\cdot\ell + \Omega_*(\omega)\cdot h| \geq \frac{\alpha}{2(1+|\ell|^\tau)}\ , \quad \forall \ell \in \mathbf{Z}^n\ ,\ |h| \leq 2\ ,\ |\ell|+|h| \neq 0\ ,\ \omega \in \mathcal{O}(\alpha)\ , \tag{118}$$

and a transformation $\mathcal{F} : \mathcal{D}_{\overline{r}/2,\overline{s}/2} \times \mathcal{O}(\alpha) \longrightarrow \mathcal{D}_{\overline{r},\overline{s}} \times \mathcal{O}_\delta$ real analytic and symplectic for each ω and Whitney smooth in ω, such that

$$H_* := H \circ \mathcal{F} = \mathcal{N}_* + R_* \quad \textit{with} \quad R_* := \sum_{2|k|+|a+\overline{a}|\geq 3} R^*_{ka\overline{a}}(\psi_*) y_*^k z_*^a \overline{z}_*^{\overline{a}}; \tag{119}$$

[23] The relation with the usual Fourier norm (used in the Appendix) $\|P\|_{\overline{r},\overline{s},\delta} := \sum_{\ell\in\mathbf{Z}^n} |P_\ell|_{\overline{r},\delta} e^{|\ell|\overline{s}}$ is $|||P|||_{\overline{r}/2,\overline{s},\delta} \leq 2^{2m}\|P\|_{\overline{r},\overline{s},\delta}$ and $|||P|||_{\overline{r},\overline{s},\delta} \geq 2^{2m}\|P\|_{\overline{r},\overline{s},\delta}$.

(ii) $\mathcal{F}$ *has the following form (omitting the dependence on* ω*)*

$$\begin{array}{rcl} y & = & y_* + Y(y_*, \psi_*, z_*, \overline{z}_*) \\ \psi & = & \psi_* + X(\psi_*) \\ z & = & z_* + Z(\psi_*, z_*, \overline{z}_*) \\ \overline{z} & = & \overline{z}_* + \overline{Z}(\psi_*, z_*, \overline{z}_*) \end{array} \tag{120}$$

where

$$Y := \sum_{2|k|+|a+\overline{a}|\le 2} Y_{ka\overline{a}}(\psi_*) y_*^k z_*^a \overline{z}_*^{\overline{a}}, \qquad Z := \sum_{|a+\overline{a}|\le 1} Z_{a\overline{a}}(\psi_*) z_*^a \overline{z}_*^{\overline{a}}, \tag{121}$$

and, denoting by $\|\cdot\|_* := \sup_{\mathcal{D}_{\overline{r}/2,\overline{s}/2}} |\cdot|$,

$$\|Y\|_* \ , \quad \frac{\overline{r}^2}{\overline{s}}\|X\|_* \ , \quad \overline{r}\|Z\|_* \ , \quad \overline{r}\|\overline{Z}\|_* \le \mathrm{const}\frac{\mathcal{P}}{\alpha}; \tag{122}$$

(iii) if M *is sufficiently small, i.e., if*

$$M < 1/4 \ , \tag{123}$$

then

$$\mathrm{meas}\Big(\mathcal{O} \setminus \mathcal{O}(\alpha)\Big) = O(\alpha d^{n-1}) \tag{124}$$

where d *is the exterior diameter of* $\mathcal{O}$*.*

By Theorem 5.1 the Hamiltonian system generated by H possesses a family of n-dimensional elliptic invariant tori $\mathcal{T} := \{(y, \psi, z) = (Y(0, \psi_*, 0, 0),\ \psi_*,\ z(\psi_*, 0, 0))\}$ traveled with frequencies ω, for each frequency vector in the Cantor set $\mathcal{O}_\alpha$. The dynamics near each torus $\mathcal{T}$ is described by the normal form (119), which in turn coincide with (1).

Let us make some comments on Theorem 5.1. Point (i) is Theorem A of [Pö1] (we have fixed the constant ρ in the statement of Theorem A as $\rho := \overline{s}/4$).

Concerning point (ii), formula (120) follows from Section 4 of [Pö1] (see, in particular, page 574 of [Pö1]). Formula (122) follows from Section 7 of [Pö1] (see in particular the last estimate on page 592).

Point (iii) follows from Theorem B of [Pö1]. Indeed, from (115), (116) and (123) $\mathcal{O}$ is *essentially non-resonant* according to the definition at page 565 of [Pö1]. For finite dimensional systems this reduces to verify only that $\mathcal{O}$ is *non-resonant* (see again page 565 of [Pö1]), namely

$$\min_{\omega\in\mathcal{O}} |\ell\cdot\omega + \Omega(\omega)\cdot h| \ge \alpha \tag{125}$$

for all $0 < |h| \le 2$ and $\ell \in \mathcal{K}_h$, the closed convex hull of the gradient set $\mathcal{G}_h := \{\partial_\omega(\Omega(\omega)\cdot h) \ : \ \omega \in \mathcal{O}\}$. Since, by (115) and $|h| \le 2$, [24] $\mathcal{K}_h \subseteq B^n_{2M}$, taking $M < 1/4$ as in (123), the unique integer vector $l \in \mathcal{K}_h$ is $l = 0$. Hence condition (125) must be verified for $\ell = 0$ only, and this is condition (116).

[24] $B^n_R \subset \mathbf{R}^n$ is the closed ball of radius R and center 0 using the $|\cdot|_2$ norm.

5.2 Abundance of periodic solutions in the three-body problem

In this section we prove the existence of periodic orbits accumulating on elliptic 2-dimensional tori of the three body-problem.

First of all we show that the KAM-Theorem 5.1 applies to the Hamiltonian H in (113) reproving the existence of elliptic invariant tori in the spatial planetary three body problem as in [BCV]. We assume the frequency parameter $\omega \in \mathbf{R}^2$ to vary over the compact subset $\mathcal{O}$.

By (114) and (101), the Hamiltonian H in (113) is real analytic on $\mathcal{D}_{\overline{r},\overline{s}} \times \mathcal{O}_\delta := D^2_{\overline{r}^2} \times \mathbf{T}^2_{\overline{s}} \times D^4_{\overline{r}} \times \mathcal{O}_\delta$ for

$$\overline{r} := \sqrt{c_0 \varepsilon} \ , \qquad \overline{s} := \text{const} \ , \qquad \delta := \text{const}\sqrt{\varepsilon} \ , \tag{126}$$

where c_0 is a small constant which will be determined later on (we restrict the domain of H w.r.t. to the larger one defined in (114) because with the choice (126) the smallness KAM condition (117) is satisfied, see (131), (132)).

Applying the KAM-Theorem 5.1, we get

Theorem 5.2 *Fix $\tau > n-1$. For c_0 and ε small enough, there exists a Cantor set $\mathcal{O}(\varepsilon) \subset \mathcal{O} \subset h'_\varepsilon(\mathcal{I})$, with*

$$\text{meas}(h'_\varepsilon(\mathcal{I}) \setminus \mathcal{O}(\varepsilon)) \leq \ \text{const} \sqrt{\varepsilon} \log \frac{1}{\varepsilon} \, , \tag{127}$$

such that, for any $\omega = \omega(J_0) \in \mathcal{O}(\varepsilon)$, there exists a symplectic trasformation

$$\Phi : \mathcal{D}_{\overline{r}/2, \overline{s}/2} \longrightarrow \mathcal{D}_{\overline{r},\overline{s}} \ ,$$

of the form (120), transforming the three-body Hamiltonian H in (113) into the normal form $H_* := H \circ \Phi$ as in (119). Moreover

$$|\omega \cdot \ell + \Omega_*(\omega) \cdot h| \geq \frac{\text{const}\, \varepsilon}{1 + |\ell|^\tau} \ , \qquad \forall \ell \in \mathbf{Z}^n \ , \ |h| \leq 2 \ , \ |\ell| + |h| \neq 0 \ , \ \omega \in \mathcal{O}(\varepsilon) \ . \tag{128}$$

In particular, the three-body problem Hamiltonian H possesses a family of 2-dimensional elliptic invariant tori traveled with frequencies ω.

Proof: Recalling that $\Omega(\omega) := \varepsilon \widetilde{\Omega}(J_0(\omega))$, it is easy to see that $\sup_{\omega \in \mathcal{O}_\delta} |\partial_\omega \Omega(\omega)| \leq \varepsilon C_0 := M$ for some positive constant C_0 and condition (115) holds true. Moreover, for $0 < \varepsilon < 1/(4C_0)$, $M < 1/4$ and also condition (123) is satisfied.

We claim that condition (116) holds, with

$$\alpha := \chi_0 \varepsilon, \tag{129}$$

namely,

$$|\Omega(\omega) \cdot h| \geq \alpha := \chi_0 \varepsilon, \qquad \forall 1 \leq |h| \leq 2, \ h \in \mathbf{Z}^2, \quad \forall \omega \in \mathcal{O}. \tag{130}$$

Indeed $\forall \omega \in \mathcal{O}$, $\Omega(\omega) = \varepsilon \widetilde{\Omega}(J_0)$ for some $J_0 \in U$. Moreover, by (106), it is easy to see that $|\widetilde{\Omega}(J_0) \cdot h| \geq \chi_0$, $\forall 1 \leq |h| \leq 2$, $h \in \mathbf{Z}^2$, $J_0 \in U$ and the claim follows.

It remains to check the smallness condition (117), i.e. $\mathcal{P} \leq \varepsilon \overline{r}^2$, since $\kappa = O(1)$ and $\alpha := \chi_0 \varepsilon$. Note that, from (110) and (126),

$$\mathcal{P} \leq \text{const}(\overline{r}^4 + \varepsilon \overline{r}^4 + \varepsilon \overline{r}^3 + \varepsilon^3) \leq \text{const}(\overline{r}^4 + \varepsilon^3). \tag{131}$$

Hence, by (131), in order to check condition (117), it is sufficient that

$$\overline{r}^4 + \varepsilon^3 \leq \text{const}\ \varepsilon \overline{r}^2. \tag{132}$$

Since $\overline{r} := \sqrt{c_0 \varepsilon}$, (132) holds true for c_0 and ε small enough. By Theorem 5.1 the result follows.

¿From (124), since $d = O(1)$ and $\alpha = O(\varepsilon)$, meas $(\mathcal{O} \setminus \mathcal{O}(\varepsilon)) = O(\varepsilon)$. Moreover, since by (112), meas $(h'_\varepsilon(\mathcal{I}) \setminus \mathcal{O}) = O(\sqrt{\varepsilon} \log(1/\varepsilon))$, the measure estimate (127) follows. $\square$

For proving that each elliptic invariant torus found in Theorem 5.2 lies in the closure of periodic orbits of the three body problem, we will apply Theorem 1.1 to the normal form Hamiltonian $H_* := H \circ \Phi$ given in (119) (which has the form (1)) corresponding to the three body problem.

The crucial hypothesys to verify in Theorem 1.1 is the non-degeneracy of the twist matrix $\mathcal{R}$ defined in (7). We have to evaluate, for $1 \leq i, i' \leq n = 2$ and $1 \leq j \leq m = 2$

$$\begin{aligned} R^*_{e_i + e_{i'}, 0, 0}(\psi_*) &= \frac{1}{1 + \delta_{i,i'}} \frac{\partial^2 H_*}{\partial y_{*i} \partial y_{*i'}}(0, \psi_*, 0, 0) \ , \\ R^*_{e_i, e_j, 0}(\psi_*) &= \frac{\partial^2 H_*}{\partial y_{*i} \partial z_{*j}}(0, \psi_*, 0, 0) \ . \end{aligned} \tag{133}$$

$R^*_{e_i, 0, e_j}(\psi_*)$ can be computed as the $R^*_{e_i, e_j, 0}(\psi_*)$ and their calculation is omitted. Due to the special form (120)-(121) of the canonical transformation Φ, we have

$$\partial_{y_{*i}} (H \circ \Phi) = [(\partial_y H) \circ \Phi]\, \partial_{y_{*i}} y \qquad \text{with} \qquad \partial_{y_{*i}} y = e_i + Y_{e_i, 0, 0}(\psi_*) \ , \tag{134}$$

from which

$$\partial^2_{y_{*i'} y_{*i}} (H \circ \Phi) = \Big[\Big(\partial^2_{yy} H \Big) \circ \Phi\ \partial_{y_{*i'}} y \Big]\, \partial_{y_{*i}} y \tag{135}$$

$$\begin{aligned} \partial^2_{z_{*j} y_{*i}} (H \circ \Phi) &= \Big[\Big(\partial^2_{yy} H \Big) \circ \Phi\ \partial_{z_{*j}} Y + \Big(\partial^2_{zy} H \Big) \circ \Phi \Big(e_j + \partial_{z_{*j}} Z \Big) + \\ &+ \Big(\partial^2_{\overline{z} y} H \Big) \circ \Phi\ \partial_{z_{*j}} \overline{Z} \Big] \partial_{y_{*i}} y \ . \end{aligned} \tag{136}$$

We are now able to prove that

Lemma 5.1 *For c_0 and ε small enough, the twist matrix $\mathcal{R}$ of the planetary three body problem is invertible and $|\mathcal{R}^{-1}| = O(1)$.*

Proof: We need to evaluate (135)-(136) for $(y_*, \psi_*, z_*, \overline{z}_*) = (0, \psi_*, 0, 0) =: \star$. By (120), (122), (131) and since $r_0 := \sqrt{c_0 \varepsilon}$ and $\alpha := \chi_0\, \varepsilon$, it results

$$|y_i(\star)| = |Y_i(\star)| \le \mathrm{const}\, \frac{\mathcal{P}}{\alpha} \le \mathrm{const} \frac{\overline{r}^4 + \varepsilon^3}{\varepsilon} = \mathrm{const} \frac{c_0^2 \varepsilon^2 + \varepsilon^3}{\varepsilon} \le \mathrm{const}\ \varepsilon. \tag{137}$$

Moreover, by (122) and standard "Cauchy estimates"[25], we have

$$|\partial_{y_*} Y(\star)|\ ,\ \ \frac{1}{\overline{r}} |\partial_{z_*} Y(\star)|\ ,\ \ |\partial_{z_*} Z(\star)|\ ,\ \ |\partial_{z_*} \overline{Z}(\star)| \le \mathrm{const} \frac{\mathcal{P}}{\alpha \overline{r}^2} = O(c_0), \tag{138}$$

(for the second estimate note that $\partial_{z_*} Y$ is independent of y_* due to (121)).

¿From (135)-(136) we deduce, using (137), (138) and since $H = h_0(J_0 + y) + O(\varepsilon)$, with h_0 defined in (96),

$$\begin{aligned}
\partial^2_{y_{*i} y_{*i}} H_*(\star) &= \frac{-3\kappa_i}{((J_0)_i + y_i(\star))^4} + O_\sharp\left(\frac{\mathcal{P}}{\alpha \overline{r}^2}\right) + O_\sharp(\varepsilon) = \frac{-3\kappa_i}{(J_0)_i^4} + O_\sharp(c_0), \\
\partial^2_{y_{*i'} y_{*i}} H_*(\star) &= O_\sharp(c_0) + O_\sharp(\varepsilon) = O_\sharp(c_0) \qquad \text{if} \quad i \ne i', \\
\partial^2_{z_{*j} y_{*i}} H_*(\star), &= O_\sharp\left(\frac{\mathcal{P}}{\alpha \overline{r}}\right) + O_\sharp(\varepsilon) = O_\sharp(c_0^{3/2} \sqrt{\varepsilon}),
\end{aligned}$$

where $O_\sharp(c)$ denotes a function of ψ_* with sup-norm for $\psi_* \in \mathbf{T}_{\overline{s}/2}$ smaller than a constant multiplied by c. It follows

$$R^*_{2e_i,0,0}(\psi_*) = \frac{-3\kappa_i}{2(J_0)_i^4} + O_\sharp(c_0),\ R^*_{e_i+e_{i'},0,0}(\psi_*) = O_\sharp(c_0) \text{ if } i' \ne i,\ R^*_{e_i,e_j,0}(\psi_*) = O_\sharp(c_0^{3/2}\sqrt{\varepsilon})\ .$$

Evaluating the Fourier coefficients of the above function we obtain

$$R^*_{2e_i,0,0,0} = \frac{-3\kappa_i}{2(J_0)_i^4} + O(c_0)\ , \qquad R^*_{e_i+e_{i'},0,0,0} = O(c_0)\ ,\ \text{for } i' \ne i \tag{139}$$

$$|R^*_{e_i,e_j,0,\ell}|\ ,\ \ |R^*_{e_i,0,e_j,\ell}| \le \mathrm{const}\ c_0^{3/2} \sqrt{\varepsilon}\ e^{-|\ell|\overline{s}/4}. \tag{140}$$

¿From (128), $|\omega \cdot \ell + \Omega_{*j}(\omega)| \ge \mathrm{const}\ \varepsilon/(1 + |\ell|^\tau)$, $\forall \ell \in \mathbf{Z}^n$ and then, from (140) the second addendum in definition of the twist matrix $\mathcal{R}$ introduced in (7) is

$$\begin{aligned}
\sum_{\substack{1 \le j \le m \\ \ell \in \mathbf{Z}^n}} \frac{1}{\omega \cdot \ell + \Omega_{*j}(\omega)} \Big(R^*_{e_i,e_j,0,\ell} R^*_{e_{i'},0,e_j,-\ell} + R^*_{e_i,0,e_j,-\ell} R^*_{e_{i'},e_j,0,\ell} \Big) \ &= \\
O\Big(\sum_{\ell \in \mathbf{Z}^n} \frac{1 + |\ell|^\tau}{\varepsilon} c_0^3 \varepsilon\ e^{-\frac{|\ell|\overline{s}}{2}} \Big) \ &= \ O(c_0^3).
\end{aligned} \tag{141}$$

Finally (139) and (141) imply that the entries of the twist matrix $\mathcal{R}$ corresponding to the spatial three body-problem are

$$\mathcal{R}_{i,i} = \frac{-3\kappa_i}{2(J_0)_i^4} + O(c_0) + O(c_0^3)\ , \qquad \mathcal{R}_{i,i'} = O(c_0) + O(c_0^3) \text{ for } i' \ne i. \tag{142}$$

[25] Cauchy estimates allow to bound n-derivatives of analytic functions on a set A in terms of their sup-norm on larger domains $A \subset A'$ divided by $\mathrm{dist}(\partial A, \partial A')^n$.

By (142), for c_0 small enough, the matrix $\mathcal{R}$ is invertible and $|\mathcal{R}^{-1}| = O(1)$. $\square$

We can finally prove the abundance of periodic solutions in the spatial planetary three body problem.

Proof of Theorem 0.2: Since the number of elliptic variables z is $m = 2$ condition (a)-(i) of Theorem 1.1 holds. Moreover, by Lemma 5.1, the twist matrix $\mathcal{R}$ is invertible with $|\mathcal{R}^{-1}| = O(1)$, and then we can apply Theorem 1.1, proving the existence of an abundance of periodic solutions of the three body problem. Theorem 0.2 is finally proved. $\square$

6 Periodic orbits near resonant elliptic tori

In this section we study the persistence, for $\varepsilon > 0$, of the circular decoupled periodic motions of the planets around the star, once suitable conditions on the period and the masses of the bodies are satisfied.

Let consider $J_0 \in \mathcal{I}$, $T > 0$ and $k = (k_1, k_2) \in \mathbf{Z}^2$ with $\gcd(k_1, k_2) = 1$, such that

$$\omega := h_0'(J_0) = \frac{2\pi k}{T} \ , \tag{143}$$

where h_0 is the integrable Hamiltonian defined in (96). The decoupled three-body problem possesses the family of (circular) periodic solutions

$$\check{\zeta}_{\psi_0}(t) := (J_0,\, \psi_0 + \omega t,\, 0,\, 0),$$

with minimal period T and parametrized by $\psi_0 \in \mathbf{T}^2$.

Since h_0 is properly degenerate the persistence of these motions for $\varepsilon > 0$ can not be established without further informations on the perturbation f_0. For continuing some of these solutions we will exploit the normal form $\mathcal{H}$ defined in (100) (with $N = 4$).

First of all, through a Lyapunov-Schmidt reduction similar to the one of Lemma 4.3, we obtain the existence of suitable pseudo-T-periodic solutions.

Lemma 6.1 *There exists $c_1, \varepsilon_1 > 0$, such that, for all $T > 0$, $k \in \mathbf{Z}^2$ with $\gcd(k_1, k_2) = 1$, $0 < \varepsilon \le \min\left\{\varepsilon_1, \frac{1}{c_1 T}\right\}$, $J_0 \in U$ satisfying $\omega := h_0'(J_0) = 2\pi k/T$, and $\forall\ \psi_0 \in \mathbf{T}^2$, there exists a unique function*

$$\zeta_{\psi_0} = (J_{\psi_0}, \psi_{\psi_0}, z_{\psi_0}) \in C^0([0,T], \mathbf{R}^2 \times \mathbf{T}^2 \times \mathbf{C}^2) \cap C^1((0,T), \mathbf{R}^2 \times \mathbf{T}^2 \times \mathbf{C}^2),$$

smooth in ψ_0, so that, $\forall t \in (0,T)$, $\zeta_{\psi_0}(t)$ is a solution of the Hamiltonian system (108)[26] *with $\psi_{\psi_0}(0) = \psi_{\psi_0}(T) = \psi_0$, $z_{\psi_0}(0) = z_{\psi_0}(T)$ and*

$$\sup_{t \in [0,T]} |\zeta_{\psi_0}(t) - \hat{\zeta}_{\psi_0}(t)| \le c_1 \varepsilon^2 \ ,$$

where $\hat{\zeta}_{\psi_0}(t) := (J_\varepsilon,\, \psi_0 + \omega t,\, 0,\, 0)$ and $J_\varepsilon = J_0 + O(\varepsilon)$ is the unique solution of $h_\varepsilon'(J_\varepsilon) = \omega$.

[26]For the Hamiltonian $\mathcal{H}$ defined in (100) with $N = 4$.

Proof: We first note that, since det $h_0''(J) \neq 0$ (see (95)–(96)), by the Implicit Function Theorem, J_ε exists, is unique and is ε-close to $J_0 \in U$. Moreover, for ε_1 small enough, $J_\varepsilon \in \mathbf{R}^2 \cap U_{\mathrm{const}\,\varepsilon} \subset U_{\widetilde{r}/4}$ (since $\widetilde{r} = O(\sqrt{\varepsilon})$).

Define X and Y as in Lemma 2.5, with $n = m = 2$. Let also $\Omega := \varepsilon\widetilde{\Omega}(J_\varepsilon)$, $M := \partial_J^2 h_\varepsilon(J_\varepsilon)$ and $\mathcal{M} := \mathbf{1}_2 - e^{\mathrm{i}\varepsilon\widetilde{\Omega}(J_\varepsilon)T}$. Notice that, if $0 < \varepsilon T \le 1/c_1$ is small enough (namely c_1 is large enough), then $|\mathcal{M}^{-1}|$ is bounded by a constant (independent on ε).

We look for T-periodic solutions of the Hamiltonian system (108) of the form $\zeta = \hat{\zeta}_{\psi_0} + (\widetilde{J}, \widetilde{\psi}, \widetilde{z}, \overline{\widetilde{z}})$. Set

$$P\begin{pmatrix} \widetilde{J} \\ \widetilde{\psi} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} -\varepsilon^4 \partial_\psi f(\star) \\ \\ -\partial_J^2 h_\varepsilon(J_\varepsilon)\,\widetilde{J} + \partial_J h_\varepsilon(J_\varepsilon + \widetilde{J}) - \partial_J h_\varepsilon(J_\varepsilon) + \varepsilon\partial_J\widetilde{\Omega}(J_\varepsilon + \widetilde{J})\widetilde{z}\cdot\overline{\widetilde{z}} + \\ + \Big(\varepsilon\partial_J g + \varepsilon^4 \partial_J f\Big)(\star) \\ \\ \mathrm{i}\Big[\varepsilon\Big(\widetilde{\Omega}(J_\varepsilon + \widetilde{J}) - \widetilde{\Omega}(J_\varepsilon)\Big)\cdot\widetilde{z} + \Big(\varepsilon\partial_{\overline{z}} g + \varepsilon^4 \partial_{\overline{z}} f\Big)(\star)\Big] \end{pmatrix},$$

where the star above denotes, for short, $\star := (J_\varepsilon + \widetilde{J}, \psi_0 + \omega t + \widetilde{\psi}, \widetilde{z}, \overline{\widetilde{z}})$. We note that $P(0) = O(\varepsilon^4)$ by (105). We want to apply Lemma 2.4. From (25) of Lemma 2.5 we find $|L| = O(1/(c_1^2\varepsilon^2))$, hence $\delta_0 := 2\,|L(P(0))| = O(\varepsilon^2/c_1^2)$ and $\sup_{x\in B_{\delta_0}} |DP(x)| \le \mathrm{const}\,(\delta_0 + \varepsilon^4)$. Then, taking c_1 big enough, we can apply Lemma 2.4 proving the existence and uniqueness of a $\zeta^\star_{\psi_0} \in B_{\rho_0}$ so that

$$\zeta^\star_{\psi_0} = L(P(\zeta^\star_{\psi_0})). \tag{144}$$

Let $\zeta_{\psi_0}(t) := \hat{\zeta}_{\psi_0} + \zeta^\star_{\psi_0}$. By means of (144) and (2.5), it follows that ζ_{ψ_0} satisfies the Hamilton equations (108) in $(0, T)$ and the boundary conditions $\psi_{\psi_0}(0) = \psi_{\psi_0}(T) = \psi_0$, $z_{\psi_0}(0) = z_{\psi_0}(T)$. As usual, by the Implicit Function Theorem, $\psi_0 \to \zeta_{\psi_0}$ is smooth. $\square$

Finally, critical points of the "reduced action functional" $\mathcal{E} : \mathbf{T}^2 \to \mathbf{R}$ defined by

$$\mathcal{E}(\psi_0) := \int_0^T J_{\psi_0}\cdot\dot{\psi}_{\psi_0} + \mathrm{i} z_{\psi_0}\cdot\dot{\overline{z}}_{\psi_0} - \mathcal{H}(\zeta_{\psi_0})\,dt$$

give rise, arguing as in Lemma 4.4, to periodic solutions of (108). As in the proof of Theorem 1.1 (see page 27) we deduce the existence of at least two geometrically distinct periodic solutions of (108) (corresponding to the points of maximum ψ_0^+ and minimum ψ_0^- of $\mathcal{E}$).

We can finally state the following result which, in particular, will imply Theorems 0.3 and 0.4 :

Theorem 6.1 *There exists $c_1, \varepsilon_1 > 0$, such that, for all $T > 0$, $k \in \mathbf{Z}^2$ with $gcd(k_1, k_2) = 1$, $0 < \varepsilon \le \min\Big\{\varepsilon_1, \frac{1}{c_1 T}\Big\}$, $J_0 \in U$ satisfying $\omega := h_0'(J_0) = 2\pi k/T$, there exist at least two geometrically distinct T-periodic solutions $\zeta_{\psi_0^\pm} = (J_{\psi_0^\pm}, \psi_{\psi_0^\pm}, z_{\psi_0^\pm})$ of the Hamiltonian system (108)[27] with*

$$\sup_{t\in\mathbf{R}} |\zeta_{\psi_0^\pm}(t) - \hat{\zeta}_{\psi_0^\pm}(t)| \le c_1\varepsilon^2\ ,$$

[27]For the Hamiltonian $\mathcal{H}$ defined in (100) with $N = 4$.

where $\hat\zeta_{\psi_0^\pm}(t) := (J_\varepsilon,\, \psi_0^\pm + \omega t,\, 0,\, 0)$ *and* $J_\varepsilon = J_0 + O(\varepsilon)$ *is the unique solution of* $h'_\varepsilon(J_\varepsilon) = \omega$ $= 2\pi k/T$. *Note that* $\sup_{t\in[0,T]} |\zeta_{\psi_0^\pm}(t) - (J_0,\, \psi_0^\pm + \omega t,\, 0,\, 0)| = O(\varepsilon)$.

Remark 6.1 The condition $0 < \varepsilon T < 1/c_1$ (for c_1 large enough) required in the Theorem is sharp. Indeed, for $\varepsilon T = O(1)$ some further resonance phenomenon can appear, destroying *any* periodic solution: note that $e^{\mathrm{i}\varepsilon\widetilde\Omega(J_\varepsilon)T} - \mathbf{1}_2$ could be degenerate. It is for reaching this estimate that we take $N = 4$ in the Hamiltonian (100).

We now show that Theorems 0.3 and 0.4 follow by Theorem 6.1.
Consider $J_0 \in \mathcal{I}$, $T > 0$, $k = (k_1, k_2) \in \mathbf{Z}^2$ with $\gcd(k_1, k_2) = 1$, such that $\omega := h'_0(J_0) = 2\pi k/T$, i.e. (143) holds. Under suitable restrictions on ε and T we shall prove that $J_0 \in U$ where $U = U(\varepsilon) \subset \mathcal{I}$ is the domain defined in (102).

First of all we claim that there exists a constant $c_2 > 0$ large enough such that

$$\varepsilon \ge e^{-T/c_2} \quad \Longrightarrow \quad \omega\cdot\ell \neq 0\ , \qquad \forall\, 0 < |\ell| \le K \tag{145}$$

where $K = K(\varepsilon) = \text{const}\ \log(1/\varepsilon)$ is defined in (103). In fact, since $\gcd(k_1, k_2) = 1$,

$$\omega\cdot\ell = 0 \quad \Longrightarrow \quad k\cdot\ell = 0 \quad \Longrightarrow \quad \ell \in \mathbf{Z}(-k_2, k_1) \Longrightarrow \quad |\ell| \ge |k|. \tag{146}$$

Moreover

$$|k| = T\frac{|h'_0(J_0)|}{2\pi} \ge c_{\mathcal{I}} T \tag{147}$$

where $c_{\mathcal{I}} := (1/2\pi)\min_{J_0\in\mathcal{I}} |h'_0(J_0)| > 0$ (note that $h'_0(J_0) = (\kappa_1 J_{01}^{-3}, \kappa_2 J_{02}^{-3}) \neq 0$ and $\mathcal{I}$ is compact).

¿From (146)-(147) and the hypothesis $\varepsilon \ge e^{-T/c_2}$, we obtain that $k\cdot\ell = 0$, $\ell \neq 0$, imply $|\ell| \ge c_{\mathcal{I}} T > c_{\mathcal{I}} c_2 \log(1/\varepsilon)$. For c_2 large enough, $c_{\mathcal{I}} c_2 \log(1/\varepsilon) > K = \text{const}\log(1/\varepsilon)$, and we conclude that $k\cdot\ell \neq 0$ for all $0 < |\ell| \le K$. The claim is proved.

It follows that $\forall\, 0 < |\ell| \le K$, $k\cdot\ell \ge 1$ and hence $|\omega\cdot\ell| = 2\pi|k\cdot\ell|T^{-1} \ge 2\pi T^{-1}$. Moreover, if $\varepsilon\log^2(1/\varepsilon) \le c_3 T^{-2}$, then $2\pi T^{-1} \ge 2\pi\sqrt{\varepsilon/c_3}\log(1/\varepsilon) \ge \alpha = \text{const}\sqrt{\varepsilon}\log(1/\varepsilon)$ (the constant α is defined in (103)) for a suitable $c_3 > 0$ large enough.

We have proved that conditions $\varepsilon \ge e^{-T/c_2}$ and $\varepsilon\log^2(1/\varepsilon) \le c_3 T^{-2}$ imply $J_0 \in U$. In conclusion, defining the functions

$$\underline{\varepsilon}(T) := \min\left\{e^{-T/c_2},\ (c_1 T)^{-1},\ \varepsilon_1\right\} \quad \text{and} \quad \overline{\varepsilon}(T) := \min\left\{F(c_3 T^{-2}),\ (c_1 T)^{-1},\ \varepsilon_1\right\}\ , \tag{148}$$

where F is the inverse of $G(\varepsilon) := \varepsilon\log^2(1/\varepsilon)$ and c_1, ε_1 are defined in Lemma 6.1, we have proved:

Lemma 6.2 *Let* $T > 0$, $k = (k_1, k_2) \in \mathbf{Z}^2$, *with gcd*$(k_1, k_2) = 1$, *and* $J_0 \in \mathcal{I}$ *such that* $\omega := h'_0(J_0) = \frac{2\pi k}{T}$ *holds. Then*

$$\underline{\varepsilon}(T) \le \varepsilon \le \overline{\varepsilon}(T) \qquad \Longrightarrow \qquad J_0\,\in\, U\ . \tag{149}$$

We can finally deduce Theorems 0.3 and 0.4.

Proof of Theorem 0.3: It is a direct consequence of Lemma 6.2 and of Theorem 6.1 taking $\underline{\varepsilon}(T)$, $\overline{\varepsilon}(T)$ as in (148) and $T_0 > 0$ as the last instant for which $\underline{\varepsilon}(T_0) = \overline{\varepsilon}(T_0)$. $\square$

Proof of Theorem 0.4: Define

$$\underline{T}(\varepsilon) := \min\left\{c_2 \log\frac{1}{\varepsilon}\ ,\frac{1}{c_1\varepsilon_1}\right\} \qquad \text{and} \qquad \overline{T}(\varepsilon) := \min\left\{\frac{\sqrt{c_3}}{\sqrt{\varepsilon}\log(1/\varepsilon)}\ ,\frac{1}{c_1\varepsilon_1}\right\}$$

with ε_1, c_2, c_3 defined above. Theorem 0.4 is a direct consequence of Lemma 6.2 and of Theorem 6.1. $\square$

7 Appendix: Proof of Proposition 5.1

We first recall the following Averaging Theorem, the proof of which can be found in [BCV]. We first introduce some notations. For $r, \rho, s > 0$ and $U \subset \mathbf{R}^2$ we denote the complex set $W_{r,\rho,s} := U_r \times \mathbf{T}^2 \times D^4_\rho$. For a function $f := f(I, \varphi, p, q)$ real analytic for $(I, \varphi, p, q) \in W_{r,\rho,s}$ we denote by $\|f\|_{r,\rho,s}$ its "sup-Fourier" norm given by

$$\|f\|_{r,\rho,s} := \sum_{k\in\mathbf{Z}^2}\left(\sup_{(I,p,q)\in U_r\times D^4_\rho}|f_k(I,p,q)|\right)e^{|k|s}\ .$$

Proposition 7.1 (Averaging Theorem) *Let $\mathcal{H}_0 := h_0(I_0)+f_0(I_0, \varphi_0, p_0, q_0)$ be a real-analytic Hamiltonian on $W_{r,\rho,s}$. Assume that there exist $\alpha_0, K > 0$, satisfying $Ks \geq 6$, such that*

$$|h_0'(I_0)\cdot k| \geq \alpha_0/2\ , \qquad \forall\, k \in \{k\in\mathbf{Z}^2:\ 0<|k|\leq K\}\ ,\quad \forall\, I_0\in U_r\ . \tag{150}$$

Assume, also, that, if $d := \min\{rs, \rho^2\}$, then

$$\|f_0\|_{r,\rho,s} < \frac{\alpha_0 d}{c\ Ks}\ , \tag{151}$$

where $c > 1$ is a suitable (universal) constant. Then, there exists a real-analytic symplectic transformation

$$\Psi : (I', \varphi', p', q') \in W_{r/2,\rho/2,s/6} \mapsto (I_0, \varphi_0, p_0, q_0) = \Psi(I', \varphi', p', q') \in W_{r,\rho,s} \tag{152}$$

and a real-analytic function $g_0 = g_0(I', p', q')$ such that

$$\mathcal{H}' := \mathcal{H}_0 \circ \Psi = h_0 + g_0 + f_*\ , \tag{153}$$

and the following bounds hold[28]*:*

$$\sup_{(I',p',q')\in U_{r/2}\times V_{\rho/2}}|g_0(I',p',q') - f_{0,0}(I',p',q')| \leq \frac{c}{\alpha_0 d}\ \|f\|^2_{r,\rho,s}\ , \tag{154}$$

[28] $f_{0,0}$ denotes the 0-Fourier coefficient of f_0, i.e., its φ-average.

$$\|f_*\|_{r/2,\rho/2,s/6} \le e^{-Ks/6}\ \|f\|_{r,\rho,s}\ . \tag{155}$$

Furthermore, for each $(I',\varphi',p',q') \in W_{r/2,\rho/2,s/6}$*,* $(I_0,\varphi_0,p_0,q_0) = \Psi(I',\varphi',p',q')$ *satisfies*

$$s\ |I_0 - I'|,\quad r\ |\varphi_0-\varphi'|,\quad \rho\ |p_0-p'|,\quad \rho\ |q_0-q'| \le \frac{c\|f\|_{r,\rho,s}}{\alpha_0}\ . \tag{156}$$

Let, now, $\mathcal{H}_0 = h_0 + f_0$ be as in (95). and U as in (102). The estimate (104) directly follows by the definition of U. Next, let us choose the sets and the parameters involved in Proposition 7.1 as follows:

$$K := \frac{6(N-1)}{s_0}\log\frac{1}{\varepsilon}\ ,\quad \alpha_0 := 2\sup_{\mathcal{I}_{\sigma_0}}|h_0''|rK\ ,\quad r = c_*\sqrt{\varepsilon}\le\sigma_0\ ,\quad s := s_0\ ,\quad \rho := \rho_0\ , \tag{157}$$

where c_* is a suitable large constant to be fixed later and σ_0, s_0, ρ_0 were defined in (97). Moreover we better specify the definition of U in (102):

$$U := \Big\{ J\in\mathcal{I}\quad :\quad |h_0'(J)\cdot\ell| \ge \alpha_0\ , \forall\ \ell\in\mathbf{Z}^2\ , 0<|\ell|\le K\Big\} \subset \mathbf{R}^2\ .$$

The estimate (104) directly follows by the previous definition. Notice that, from these definitions, there follows (for ε small enough) that

$$\alpha_0 = \ \text{const}\,\sqrt{\varepsilon}\log\frac{1}{\varepsilon}\ ,\qquad d = \ \text{const}\, r\ ,\qquad \frac{\alpha_0 d}{cKs} = \ \text{const}\,\varepsilon c_*^2\ , \tag{158}$$

(clearly, in the last evaluation, "const" does not involve c_*).

Now, it is not difficult to check that, choosing c_* big enough and letting ε be small enough, assumptions (150) and (151) are met. In fact, observing that f_0 in (95)-(96) is such that

$$\|f_0\|_{r,\rho,s} \le \ \text{const}\,\varepsilon\ ,$$

(151) follows from last equality in (158), by choosing c_* large enough. As for (150), observe that for any point in $I_0\in U_r$ there is a point $I_0^*\in U$ at distance less than r from it. Hence by the definitions of α_0 and r and by Cauchy estimates[29], for any $I_0\in U_r$ and any $0<|k|\le K$,

$$\begin{aligned}
|h_0'(I_0)\cdot k| &\ge |h_0'(I_0^*)\cdot k| - |h_0'(I_0^*) - h_0'(I_0)|\ |k| \\
&\ge \alpha_0 - \sup_{\mathcal{I}_{\sigma_0}}|h_0''|\ r\ K \\
&= \alpha_0/2\ ,
\end{aligned}$$

which proves also (150). Thus, by Proposition 7.1, there exists a real analytic symplectic transformation

$$\Psi : (I',\varphi',p',q') \in \mathcal{D}_1 := U_{\frac{r}{2}}\times\mathbf{T}^2_{\frac{s}{6}}\times D^4_{\frac{\rho}{2}} \to (I_0,\varphi_0,p_0,q_0)\in U_r\times\mathbf{T}^2_{s_0}\times D^4_{\rho_0}\subset\mathcal{D}_0,$$

[29] "Cauchy estimates" allow to bound n-derivatives of analytic functions on a set A in terms of their sup-norm on larger domains $A'\supset A$ divided by $\text{dist}\,(\partial A,\partial A')^n$.

such that

$$|I'-I_0| \le \text{ const } \frac{\varepsilon}{\alpha_0} = \text{ const } \frac{\sqrt{\varepsilon}}{\log\frac{1}{\varepsilon}}$$

$$|p'-p_0| \ , \ |q'-q_0| \le \text{ const } \frac{\varepsilon}{\alpha_0 \rho} = \text{ const } \frac{\sqrt{\varepsilon}}{\log\frac{1}{\varepsilon}} \ , \tag{159}$$

and which casts the Hamiltonian $\mathcal{H}_0$ into $\mathcal{H}' := \mathcal{H}_0 \circ \Psi$ with

$$\mathcal{H}'(I',\varphi',p',q') := h(I') + g(I',p',q') + f_*(I',\varphi',p',q') \,, \tag{160}$$

where (since, by (96), $f_{0,0}$ coincides with $\varepsilon f_1(I,p,q)$)

$$\begin{aligned} \sup_{(I',p',q')\in U_{r/2}\times V_{\rho/2}} |g_0 - \varepsilon f_1| &\le \text{ const } \frac{\varepsilon^2}{\alpha_0 r} = \text{ const } \frac{\varepsilon}{\log(1/\varepsilon)} \ , \\ \|f_*\|_{r/2,\rho/2,s/6} &\le \text{ const } \varepsilon e^{-Ks/6} \le \text{ const } \varepsilon^N \ . \end{aligned} \tag{161}$$

Thus, setting $g_0 =: \varepsilon \overline{g}$, $f_* =: \varepsilon^N \overline{f}$, we see that $\mathcal{H}'$ can be rewritten as

$$\begin{aligned} \mathcal{H}' &:= h(I') + \varepsilon\overline{g}(I',p',q') + \varepsilon^N \overline{f}(I',\varphi',p',q') \,, \\ \overline{g} &= f_1(I',p',q') + \frac{1}{\log(1/\varepsilon)} \overline{f}_1(I',p',q') \end{aligned} \tag{162}$$

with $\overline{f}$ and $\overline{f}_1$ real-analytic on $\mathcal{D}_1$ (compare (157)).

We now look for *elliptic equilibria of the Hamiltonian* $\overline{g}$ in (162). Set

$$G(I',p',q') := \Big(\partial_{p'}\overline{g}(I',p',q'),\ \partial_{q'}\overline{g}(I',p',q')\Big) \,.$$

Recalling (162) and the definition of f_1 in (96), we see that, by (98),

$$G(I',0,0)\Big|_{\varepsilon=0} = 0 \quad \text{and} \quad \det \partial_{(p',q')} G(I',0,0)\Big|_{\varepsilon=0} = 16(\overline{\Omega}_1\overline{\Omega}_2)^2 > 0 \,, \ \forall\ I' \in U_{r/2}.$$

Therefore, by the Implicit Function Theorem, we infer that, for any $I' \in U_{r/2}$ and for ε small enough, there exist real-analytic functions, so that

$$I' \in U_{r/2} \to \Big(p'(I',\varepsilon),\ q'(I',\varepsilon)\Big) \in B_{\text{const}/\log\frac{1}{\varepsilon}} \subset B_{\rho/2} \,,$$

and

$$\partial_{p'}\overline{g}\Big(I',p'(I',\varepsilon),q'(I',\varepsilon)\Big) = 0 = \partial_{q'}\overline{g}\Big(I',p'(I',\varepsilon),q'(I',\varepsilon)\Big) \,. \tag{163}$$

For ε small enough, we can consider the following analytic symplectic transformation, which leaves fixed the I'-variable and is $O(\frac{1}{\log\frac{1}{\varepsilon}})$−close to the identity[30],

$$\Phi' : (J',\psi',v',u') \in U_{r/2} \times \mathbf{T}^2_{s/7} \times D_{\rho/3} \ \mapsto \ (I',\varphi',p',q') \in U_{r/2} \times \mathbf{T}^2_{s/6} \times D^4_{\rho/2} \,,$$

[30] Φ' has generating function $J' \cdot \varphi' + \Big(v' + p'(J',\varepsilon)\Big) \cdot \Big(q' - q'(J',\varepsilon)\Big)$.

given by

$$\begin{array}{rcl} I' & = & J' \,, \\ \varphi' & = & \psi' + p'(J',\varepsilon)\, \partial_{I'} q'(J',\varepsilon) + \partial_{I'} q'(J',\varepsilon)\, v' - \partial_{I'} p'(J',\varepsilon)\, u' \,, \\ p' & = & v' + p'(J',\varepsilon) \,, \\ q' & = & u' + q'(J',\varepsilon) \,. \end{array}$$

In view of (163), the new Hamiltonian $\hat{\mathcal{H}} := \mathcal{H}' \circ \Phi'$ has the form

$$\hat{\mathcal{H}}(J',\psi',v',u') = h(J') + \varepsilon \hat{g}(J',v',u') + \varepsilon^N \hat{f}(J',\psi',v',u') \,,$$

with $\hat{f}$ and $\hat{g}$ analytic in $U_{r/2} \times \mathbf{T}^2_{s/7} \times D_{\rho/3}$ and

$$\partial_{v',u'} \hat{g}(J',0,0) = \partial_{p',q'} \overline{g}(I', p'(I',\varepsilon), q'(I',\varepsilon)) = 0 \,, \qquad \forall\ I' \in U_{r/2} \;.$$

Also, the eigenvalues of the symplectic quadratic part of $\hat{g}$ are given by $\pm \mathrm{i} \hat{\Omega}_j(J')$, for $j = 1, 2$, where

$$\hat{\Omega}_j \in \mathbf{R} \qquad \text{and} \qquad |\hat{\Omega}_j - \overline{\Omega}_j| \le \mathrm{const} \frac{1}{\log \frac{1}{\varepsilon}} \,. \tag{164}$$

Thus, by a well known result by Weierstrass on the symplectic diagonalization of quadratic Hamiltonians, we can find an analytic transformation $O(\frac{1}{\log \frac{1}{\varepsilon}})$−close to the identity

$$\check{\Phi} : (J,\psi,v,u) \in U_{r/2} \times \mathbf{T}^2_{s/8} \times D^4_{\rho/4} \ \mapsto\ (J',\psi',v',u') \in U_{r/2} \times \mathbf{T}^2_{s/7} \times D^4_{\rho/3} \,,$$

so that $J' = J$ and the quadratic part of $\hat{g}$ becomes, simply, $\sum_{j=1}^2 \hat{\Omega}_j(J)\, (u_j^2 + v_j^2)$. Whence, the Hamiltonian $\hat{\mathcal{H}}$ takes the form $\check{\mathcal{H}} := \hat{\mathcal{H}} \circ \check{\Phi}$, with

$$\check{\mathcal{H}}(J,\psi,v,u) = h_\varepsilon(J) + \varepsilon \sum_{j=1}^{2} \hat{\Omega}_j(J)\, (u_i^2 + v_i^2) + \varepsilon \widetilde{g}(J,v,u) + \varepsilon^N \widetilde{f}(J,\psi,v,u) \,, \tag{165}$$

where

$$h_\varepsilon(J) := h_0(J) + \varepsilon \hat{g}(J,0,0) \ , \tag{166}$$

$\widetilde{g}$, $\widetilde{f}$, $\hat{\Omega}_j$ are real-analytic for (J,ψ,v,u) in

$$\mathcal{D}_2 := U_{r/2} \times \mathbf{T}^2_{s/8} \times D^4_{\rho/4} \tag{167}$$

and

$$\sup_{J \in U_{r/2}} |\widetilde{g}(J,v,u)| \le\ \mathrm{const}\, |(v,u)|^3 \,. \tag{168}$$

Because of (164), the non-degeneracy condition (98) implies (for ε small enough)

$$\begin{aligned} &\inf_{J \in U} \hat{\Omega}_i \ge \inf_{J \in U_{r/2}} |\hat{\Omega}_i| \ge\ \mathrm{const}\ > 0 \ , \\ &\inf_{J \in U} \left(\hat{\Omega}_2 - \hat{\Omega}_1 \right) \ge \inf_{J \in U_{r/2}} |\hat{\Omega}_2 - \hat{\Omega}_1| \ge\ \mathrm{const}\ > 0 \,. \end{aligned} \tag{169}$$

Setting $\widetilde{\Omega}_j := 2\hat{\Omega}_j$ for $j = 1, 2$, introducing complex coordinates

$$z = \frac{v + \mathrm{i}u}{\sqrt{2}}, \qquad \overline{z} = \frac{v - \mathrm{i}u}{\sqrt{2}} \tag{170}$$

and defining

$$g(J, z, \overline{z}) := \widetilde{g}\left(J, \frac{z + \overline{z}}{\sqrt{2}}, \frac{z - \overline{z}}{\mathrm{i}\sqrt{2}}\right) \quad \text{and} \quad f(J, \psi, z, \overline{z}) := \widetilde{f}\left(J, \psi, \frac{z + \overline{z}}{\sqrt{2}}, \frac{z - \overline{z}}{\mathrm{i}\sqrt{2}}\right),$$

we obtain (100). Finally (101),(105),(106), follows from (157),(168),(169), respectively.

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