FAMILIES OF PERIODIC SOLUTIONS OF REVERSIBLE PDEs

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1 Introduction

Some families of periodic solutions of PDEs can be constructed using KAM theory; however a different approach leading to stronger results and simpler proofs is available. Such an approach is based on the Lyapunov–Schmidt decomposition combined with a suitable analysis of small denominators. Its main advantage is the elimination of the second Melnikov condition, a fact that allows the application to problems with periodic boundary conditions and to equations in more than one space dimension. Most of the general theory has been developed for equations that are of second order in time; we will mainly deal with this case. Moreover, we will concentrate on problems involving small denominators and only briefly report on some important results of a different kind.

2 An abstract theorem for nonresonant PDEs

Let $\{X_s\}$ be a scale of Hilbert spaces with norms $\|\cdot\|_s$ and scalar product $\langle\cdot;\cdot\rangle_s$. Let A be a (linear) morphism of the scale, and assume that there exists a Hilbert basis $\{\varphi_j\}_{j=1}^{\infty}$ such that

$$A\varphi_j = \omega_j^2\varphi_j \ , \quad \omega_j > 0$$

Fix s and let $g : X_s \supset \mathcal{U} \to X_s$, where $\mathcal{U}$ is a neighbourhood of the origin, be a C^∞ nonlinear operator having at the origin a zero of second order. We are interested in families of small amplitude periodic solutions of the equation

$$\ddot{x} + Ax = g(x) \ . \tag{2.1}$$

Example 2.1. The nonlinear wave equation with periodic boundary conditions:

$$w_{tt} - w_{xx} + V(x)w = f(x,w) \ , \tag{2.2}$$

$$w(x,t) = w(x+2\pi,t) \ , \quad w_x(x,t) = w_x(x+2\pi,t) \ , \tag{2.3}$$

where the potential V and the nonlinearity f are periodic of period 2π in x and $f(x,w) = O(|w|^2)$. The frequencies ω_j of the linearized system are the square roots of the periodic eigenvalues of the Sturm Liouville operator $-\partial_{xx}+V(x)$, that we assume to be positive. In this case $X_s=H^s(\mathbb{T})$. Provided $s>1/2$, a smooth f induces a smooth operator from X_s to itself. $\square$

Example 2.2. The nonlinear plate equation in the d dimensional cube:

$$w_{tt}+\Delta\Delta w+aw=f(w)\ ,\quad x\in\mathcal{Q} \tag{2.4}$$

$$w\big|_{\partial\mathcal{Q}}=\Delta w|_{\partial\mathcal{Q}}=0\ , \tag{2.5}$$

where $a>0$ and

$$\mathcal{Q}:=\left\{x=(x_1,...,x_d)\in\mathbb{R}^d\ :\ 0<x_i<\pi\right\}$$

Assume $f(w)=O(|w|^3)$, then the eigenfunctions of the linearized system are given by $\varphi_n=\sin(n_1x_1)\sin(n_2x_2)...\sin(n_dx_d)$ and the corresponding frequencies are $\omega_n=\sqrt{(n_1^2+...+n_d^2)^2+a}$ where $n\in\mathbb{Z}^d$ and $n_i\geq 1$. To fit the above scheme we order the basis in lexicographic order. Here $X_0=L^2(\mathcal{Q})$, and $X_s=D((\Delta\Delta)^s)\subset H^{4s}$ endowed by the graph norm. If the nonlinearity f is smooth and odd (i.e. $f(-w)=-f(w)$) then it defines a smooth map from X_s to itself for any $s>[d/2]/4$. $\square$

In the linear approximation ($g\equiv 0$) the general solution of (2.1) is the superposition of the linear normal modes, i.e. of the families of periodic solutions

$$x^{(j)}(t)=(a_j\cos(\omega_jt)+b_j\sin(\omega_jt))\varphi_j\ . \tag{2.6}$$

Fix one of the families, say $x^{(1)}$; to ensure its persistence in the nonlinear problem we make the following assumptions:

H1) (Nonresonance) For small enough $\gamma>0$ there exists a closed set $W_\gamma\subset\mathbb{R}^+$ having ω_1 as an accumulation point both from the right and from the left such that, for any $\omega\in W_\gamma$ one has

$$|\omega l-\omega_j|\geq\frac{\gamma}{l}\ ,\quad\forall l\geq 1\ ,\ \forall j\geq 2\quad\square \tag{2.7}$$

H2) (Nondegeneracy) Let $g_r(x)$ be the first non vanishing Taylor (homogeneous) polynomial of g; assume $r\geq 3$ and define

$$\beta_0:=\begin{cases}\langle g_r(\varphi_1),\varphi_1\rangle_0 & \text{if}\quad r\quad\text{is}\quad\text{odd}\\ \langle g_{r+1}(\varphi_1),\varphi_1\rangle_0 & \text{if}\quad r\quad\text{is}\quad\text{even}\end{cases} \tag{2.8}$$

Assume $\beta_0\neq 0$. $\square$

Denoting $\xi_1(\omega_1t)=\cos(\omega_1t)\varphi_1$ one has

Theorem 2.3. *Suppose the assumptions H1,H2 hold, then there exists a set $\mathcal{E}\subset\mathbb{R}$ having zero as an accumulation point, a positive ω_*, and a family of periodic solutions $\{x_\epsilon(t)\}_{\epsilon\in\mathcal{E}}$ of (2.1) with frequencies $\{\omega^\epsilon\}_{\epsilon\in\mathcal{E}}$ fulfilling*

$$\sup_t\|x_\epsilon(t)-\epsilon\xi_1(t\omega^\epsilon)\|_s\leq C\epsilon^r\ ,\quad|\omega^\epsilon-\omega_1|\leq C\epsilon^{r-1}\ . \tag{2.9}$$

Moreover the set $\mathcal{E}$ is in one to one correspondence either with $W_\gamma\cap[\omega_1,\omega_1+\omega_*)$, if $\beta_0<0$, or with $W_\gamma\cap(\omega_1-\omega_*,\omega_1]$, if $\beta_0>0$.

Proof. We consider only the case of odd r, the general case can be obtained by a slightly different treatment of the forthcoming equation ω. Look for an X_s valued function $q(t)$ periodic of period 2π and reversible (namely s.t. $q(t) = q(-t)$) and for a positive ω close to ω_1 such that $q(\omega t)$ is a solution of (2.1). They must satisfy the equation

$$L_\omega q = g(q)\ ,\ L_\omega := \omega^2 \frac{d^2}{dt^2} + A\ . \tag{2.10}$$

which will be considered as an ω dependent functional equation in the space $\mathcal{H} \subset H^1(\mathbb{T}, X_s)$ containing the reversible periodic functions. Equation (2.10) is studied using the Lyapunov–Schmidt decomposition, namely by decomposing it into an equation on $\mathrm{Ker}L_{\omega_1} \equiv \mathrm{span}(\xi_1)$ and an equation on its complement R. Precisely, denote by Q the projector on ξ_1 and by P the projector on R, make the Ansatz $q = \epsilon\xi_1 + \epsilon^r u$, with $u \in R$: Then (2.10) is equivalent to the system

$$\omega^2 = \omega_1^2 + \beta\epsilon^{r-1} \tag{2.11}$$

$$L_\omega u = Pg_r(\xi_1) + PG(\epsilon, u) \tag{2.12}$$

$$-\beta\xi_1 = Qg_r(\xi_1) + QG(\epsilon, u) \tag{2.13}$$

for the unknowns (ϵ, u, β): Here G contains all higher order corrections and $\omega \in W_\gamma$ is a parameter. The equations (2.11), (2.12) and (2.13) are called the ω, the P and the Q equation respectively.

First one solves the P equation (2.12). To this end one has to invert the linear operator $L_\omega\big|_R$. Its eigenfunctions are given by $\cos(lt)\varphi_j$, and the corresponding eigenvalues are

$$\lambda_{jl} = -l^2\omega^2 + \omega_j^2 = (l\omega + \omega_j)(\omega_j - l\omega)\ ,\ j \geq 2\ ,\ l \geq 1$$

By (2.7), $|\lambda_{jl}| > C\gamma$. So $(L_\omega\big|_R)^{-1}$ exists and is bounded. Applying this operator to the P equation and using the implicit function theorem one obtains a smooth function $u(\epsilon)$ that depends parametrically on $\omega \in W_\gamma$ and solves the P equation.

Inserting $u(\epsilon)$ in the Q equation one determines the parameter β as a function of ϵ: In particular one has $\beta(\epsilon) = C\beta_0$+higher order corrections, where $C > 0$. Inserting $\beta(\epsilon)$ in the ω equation one gets an equation for ϵ (remember that ω is fixed) which is a perturbation of the equation $\omega^2 - \omega_1^2 = C\beta_0\epsilon^{r-1}$. By the nondegeneracy this can be reduced to a fixed point equation for ϵ^{r-1} which is solved by the contraction mapping principle. $\square$

Remark 2.4. The theorem holds also in the case $r = 2$ but in this case the nondegeneracy condition takes a more complicated form. $\square$

Theorem 2.3 was proved in [Bam00]. The technique of Lyapunov–Schmidt decomposition was used for the first time to construct families of periodic solutions in PDEs by Craig and Wayne [CW93] who considered the model problem of the wave equation with periodic boundary conditions (see example 2.1); we will report on this work in Section 4.

Example 2.5. Consider the nonlinear wave equation with periodic boundary conditions (see example 2.1). Let ω_1 be such that $\omega_1 \neq \omega_j$ for any $j \neq 1$. Decompose V into its average a and a part $\tilde{V}$ of zero average, then condition H1 is satisfied if a belongs to an uncountable set which is dense in a neighbourhood of the origin (for the proof see lemma 3.1 of [BP02]). Condition H2 can be expressed in terms of the eigenfunctions of the Sturm Liouville operator. If it holds then theorem 2.3 applies and ensures persistence of the corresponding family of periodic orbits. Remark that, as a difference with respect to the case of Dirichlet boundary conditions the nonlinearity does not need to have some particular parity. □

Example 2.6. Consider the nonlinear plate equation (see example 2.2). In the case $d = 1$ all the frequencies are simple and the assumption H1 is satisfied if a is choosen in a subset of $\mathbb{R}^+$ having full measure. In the case $d > 1$, all the frequencies are multiple except the smallest one. Taking for ω_1 such a smallest frequency, H1 is fulfilled if a belongs to a dense uncountable subset of [0,1/4]. H2 is automatic provided the Taylor expansion of f at zero does not vanish at all orders (remember that $f(-w) = f(w)$). Then theorem 2.3 ensures persistence of the corresponding family of periodic orbits (for details see [BP02]). □

3 The resonant case

It is possible to generalize the above theorem to the case where the frequencies fulfill some resonance relations. We will consider only the Lagrangian case where $g = -\nabla H$.

Fix a frequency ω_1 of the linearized system, then the assumption H1 is substituted by

H1R) For any small enough γ there exists a closed set $W_\gamma \subset \mathbb{R}^+$ having ω_1 as an accumulation point both from the right and from the left and such that for any $\omega \in W_\gamma$ one has

$$\text{either} \qquad |\omega l - \omega_j| \geq \frac{\gamma}{l} , \qquad \text{or} \qquad l\omega_1 - \omega_j = 0 \tag{3.1}$$

□

To come to the nondegeneracy assumption define the resonant set by

$$\mathcal{I}_R := \{k \geq 1 \ : \ \exists l \geq 1 \ : \ l\omega_1 - \omega_k = 0\} \tag{3.2}$$

and let $\mathcal{N}$ be the space obtained by closing the space generated by $\{\varphi_k\}_{k \in \mathcal{I}_R}$ in the graph norm of $D(A)$. Remark that all solutions of the linearized system with initial datum in $\mathcal{N}$ and vanishing initial velocity are periodic of period $2\pi/\omega_1$. Let H_r be the first non vanishing Taylor coefficient of H and, for $x \in \mathcal{N}$, define the average of H_r by

$$\langle H_r \rangle(x) := \frac{\omega_1}{2\pi} \int_0^{2\pi/\omega_1} H_r(\cos(At)x)dt \ ;$$

consider the hypersurface $\mathcal{S} \subset \mathcal{N}$ of the points $x \in \mathcal{N}$ such that $\langle x; Ax\rangle_0 = 1$.

H2R) There exists a nondegenerate critical point x_0 of the functional $\langle H_r\rangle\big|_{\mathcal{S}}$, and the corresponding Lagrange multiplier does not vanish. $\square$

Denote by $\xi_0(\omega_1 t)$ the solution of the linearized system with initial datum x_0 and vanishing initial velocity.

Theorem 3.1. *[BP01] Suppose the assumptions H1R,H2R hold, then there exists a family of periodic solutions $\{x_\epsilon(t)\}_{\epsilon\in\mathcal{E}}$ of (2.1) with frequencies $\{\omega^\epsilon\}_{\epsilon\in\mathcal{E}}$ fulfilling*

$$\sup_t \|x_\epsilon(t) - \epsilon\xi_0(t\omega^\epsilon)\|_s \leq C\epsilon^r \ , \quad |\omega^\epsilon - \omega_1| \leq C\epsilon^{r-1} \ . \tag{3.3}$$

The set $\mathcal{E}$ has the same properties as in the nonresonant case.

The proof is obtained proceeding as in the nonresonant case. The only difference is that in this case the kernel of L_{ω_1} is no more one dimensional, but is isomorphic to $\mathcal{N}$ (the isomorphism being given by the map $x \mapsto \cos(At/\omega_1)x$). So the Q equation can be transformed into an equation in $\mathcal{N}$. This equation turns out to be a perturbation of the equation for the critical points of $\langle H_r\rangle\big|_{\mathcal{S}}$ and the nondegeneracy condition H2R allows to solve it by the implicit function theorem.

Applying the above theorem one can construct countably many families of periodic solutions of the ϕ^4 model

$$w_{tt} - w_{xx} = \pm w^3 + \text{higher order terms}$$

with Dirichlet boundary conditions, and also higher frequency periodic solutions of the nonlinear plate equation of example 2.2 (see [BP01, BP02], see also [LS88, Bou99b]).

In general it is difficult to check condition H2R. In the case of Hamiltonian systems with n degrees of freedom, $n < \infty$, topological arguments allow to avoid it. Indeed Weinstein Moser theorem (see [Wei73, Mos76]) ensures that close to a minimum of the energy there exist at least n periodic orbit on each surface of constant energy. In general they do not form regular families. A corresponding result for PDEs is not available at present; however there exists an *ad hoc* variational result for the wave equation

$$w_{tt} - w_{xx} = \pm w^p + \text{higher order terms} \ , \quad p \geq 2 \ . \tag{3.4}$$

which ensures that, having fixed $j \geq 1$, there exists a sequence of periodic orbits accumulating at zero, whose frequencies accumulate at j (which plays here the role of the $j-$th linear frequency). Such a result is due to Berti and Bolle [BB02].

Periodic solutions in the nonlinear wave equation

$$w_{tt} - w_{xx} + f(x,w) = 0 \ , \quad u(0,t) = u(\pi,t) = 0 \tag{3.5}$$

where constructed for the first time by Rabinowitz [Rab78] using global variational methods and a Lyapunov–Schmidt decomposition. Rabinowitz proved that, under suitable assumptions on f, equation (3.5) has at least one periodic solution with period $T = 2\pi p/q$, for any choice of the integers p and q. Remark that, when the period T is in rational ratio with 2π, the operator $L_\omega|_R$ has a compact inverse, i.e. there are no small denominators. The work [Rab78] was followed by a series of papers simplifying the proof and sharpening the result (see [Bre83] and references therein). In particular we mention the paper [BCN80] by Brezis, Coron and Nirenberg, where existence of periodic orbits is proved by a particularly simple method: the authors write a variational principle dual to the usual one and look for its critical points using the mountain pass lemma. It is remarkable that in this approach the Q equation becomes trivial.

4 Weakening the nonresonance condition

The main limitation of the results presented in Sections 2 and 3 rests in the nonresonance conditions H1 and H1R. Indeed such conditions are fulfilled with large probability (in a suitable parameter space) when $\omega_j \sim j^\nu$ with $\nu > 1$; when $\nu = 1$ the nonresonance conditions are satisfied typically on uncountable sets having zero measure, but when $\nu < 1$ they are only exceptionally satifyed (as in the plate equation). As a consequence the results of Sections 2 and 3 are not applicable to general equations in more than one space dimensions. Furthermore the method of Lyapunov–Schmidt decomposition can be extended to the case of reversible systems of first order in time, but the approach of Section 2 is no more applicable.

In order to avoid such limitations one would like to be able to work with the weaker nonresonance condition "there exists a $\tau > 0$ such that $|l\omega - \omega_j| \geq \gamma/l^\tau$". This was done by Craig and Wayne [CW93] who used the Nash Moser theorem to solve the P equation. It turns out that the application of the Nash Moser theorem requires to construct and estimate the inverse of the linear operator describing the linearization of the P equation at an approximate solution. This is the main difficulty of Craig–Wayne's approach. To overcome it they use the techniques by Frölich and Spencer [FS83], performing a very careful analysis of small denominators. The method by Craig and Wayne was extended by Bourgain in order to construct periodic (and also quasi periodic) solutions in higher dimensional equations. The resulting method seems very general, but at present a theorem "ready for application" is not available. We present here the result obtained by Bourgain applying such a method to the nonlinear wave equation

$$w_{tt} - \Delta w + aw + w^3 = 0 \tag{4.1}$$

on $\mathbb{T}^d$. Fix a multiindex $n \in \mathbb{Z}^d$ different from zero, and let

$$\xi_n(\omega_n t, x) := \cos(n \cdot x + \omega_n t) \ , \quad \omega_n := \sqrt{n_1^2 + ... + n_d^2 + a}$$

be the corresponding symmetric reversible solution.

Theorem 4.1. *[Bourgain [Bou95]]. If a belongs to a subset of full measure of $\mathbb{R}^+$, then there exists a Cantor set $\mathcal{E}$ of positive measure, accumulating at* *zero, and a family of periodic solutions $\{w_\epsilon(t,x)\}_{\epsilon\in\mathcal{E}}$ of (4.1) with frequency ω^ϵ fulfilling*

$$|\epsilon\xi_n(\omega^\epsilon t,x)-w_\epsilon(t,x)|\leq C\epsilon^3\ ,\quad |\omega_n-\omega^\epsilon|\leq C\epsilon^2\ .$$

In the case $d=1$ the result was proved in [CW93]; subsequently, still in the case $d=1$ Kuksin introduced a simpler technique in order to find the "large measure result" of theorem 4.1 (see [Bou99a] pp.90–94).

The Craig–Wayne–Bourgain method allows to deal also with first order equations. For example it was applied to the Schrödinger equation in one [CW94] or two dimensions [Bou98].

5 The water wave problem

A particular problem that has attracted the attention of many researchers since the very beginning of the theory of PDEs is that of existence of standing water waves. The first rigorous proof of their existence was obtained only recently by Plotnikov and Toland [PT01]; we present here their result.

Consider a perfect fluid lying above a horizontal bottom, and confined between two parallel vertical walls. The fluid is subject to gravity, and atmospheric pressure acts at the free surface. This is a dynamical system governed by the Euler equations supplemented by appropriate boundary conditions. It was pointed out by Zakharov that this system is Hamiltonian (see [Zak68]). The corresponding Hamiltonian function is the energy of the fluid, and conjugated variables are given by the wave profile and the velocity potential at the free surface.

In the linear approximation the general solution is given by the superposition of the normal modes. The problem is to continue the normal modes to families of periodic solutions of the nonlinear system (the stending waves). Fix one of the normal modes, and denote by $\eta(t,x_1)$ the corresponding profile of the free surface (x_1 being the horizontal variable). Then it is possible to tune the depth h, the width l of the region occupied by the fluid and the gravitational constant g in such a way that the period of the solution is normalized to 2π and the linear frequencies fulfill a suitable nonresonance condition. Denote by (g_0,l_0,h_0) a choice of the parameters realizing such conditions, then one has

Theorem 5.1. *[Plotnikov & Toland [PT01]] There exists an infinite set $\mathcal{E}\subset\mathbb{R}$ having zero as an accumulation point and, for any $\epsilon\in\mathcal{E}$, a sequence g_ϵ, l_ϵ and a standing wave solution of the water wave problem with gravity g_ϵ in a box of width l_ϵ. Denote by η_ϵ the corresponding profile of the free surface, then one has*

$$|\eta_\epsilon(t,x_1)-\epsilon^2\eta(t,x_1)|<C\epsilon^3\ ,\quad |g_\epsilon-g_0|+|l_\epsilon-l_0|\leq C\epsilon\ .$$

The main difficulties in proving this result are as follows: first the linear frequencies behave as $\omega_n\sim n^{1/2}$ and therefore the nonresonance conditions that can be satisfied are quite weak. Second, one has that the mathematical formulation of the problem involves a nonlinear non local operator which is unbounded. To overcome such difficulties, Plotnikov and Toland use a Lagrangian description of the fluid motion and a Lyapunov–Schmidt approach to the nonlinear problem which ensues. The P equation is solved by Nash Moser theorem. The required invertibility of the linearized operator is obtained in two steps: first it is reduced to a suitable canonical form and then such a canonical form (which is essentially a perturbation of an operator involving derivatives and Hilbert transform) is studied in detail.

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